Chapter 2

Sobolev spaces

2.1 Weakly differentiable functions

In this first paragraph we introduce the notion of weak derivatives and define the Sobolev spaces of weakly differentiable functions. This generalizes the notion of differentiability to a class of functions which are not differentiable in the classical sense.

We will sometimes refer to distributions and the notion of derivatives in the sense of distributions, which are assumed to be known. However, we will recall all the required definitions and results to make this chapter self-contained.

2.1.1 One derivative in L^p in dimension one

We begin with the one dimensional case. Let I be an open and non-empty interval of $\mathbb{R}$. The key observation behind the definition of the weak derivative is the integration by parts. For $u \in C^1(I)$ and $\phi \in C_0^\infty(I)$ we have

$$\int_I u'\phi \, dx = -\int_I u\phi' \, dx. \tag{2.1}$$

The right-hand side makes sense even when u is not differentiable. This is how we define the function u' which appears in the left-hand side.

Definition 2.1. Let $u \in L^1_{\text{loc}}(I)$. We say that u has a weak derivative in $L^1_{\text{loc}}(I)$ if there exists $v \in L^1_{\text{loc}}(I)$ such that

$$\forall \phi \in C_0^\infty(I), \quad -\int_I u\phi' \, dx = \int_I v\phi \, dx. \tag{2.2}$$

In this case we denote by u' this function v.

Definition 2.2. We denote by $W^{1,p}(I)$ the set of functions $u \in L^p(I)$ with a weak derivative $u' \in L^p(I)$. We also write $H^1(I)$ for $W^{1,2}(I)$.

Of course, if u is differentiable in the usual sense, then the derivatives in the usual and in the weak senses coincide. However u and u' are not necessarily in $L^p(I)$, so u is not necessarily in $W^{1,p}(I)$ (if I is a compact interval, then continuous functions are integrable and hence, in this particular case, continuously differentiable functions on I belong to $W^{1,p}(I)$).

The weak derivative is just the derivative in the sense of distributions. A function $u \in L^1_{\text{loc}}(I)$ defines a distribution T_u on I. This distribution has a derivative $T'_u \in \mathcal{D}'(I)$. Saying that the derivative of u belongs to $L^1_{\text{loc}}(I)$ means that T'_u is the distribution defined by a function in $L^1_{\text{loc}}(I)$. In other words, for some $v \in L^1_{\text{loc}}(I)$ we have $T'_u = T_v$ in $\mathcal{D}'(I)$. Then a function $u \in L^p(I)$ belongs to $W^{1,p}(I)$ if and only if $u' \in L^p(I)$, where u' is understood in the sense of distributions.

Notice also that a function v satisfying (2.2) is necessarily unique (up to equality almost everywhere), so there is no ambiguity in the definition of u'.

Example 2.3. We consider on $]0,1[$ the function $u : x \mapsto x^{-\frac{1}{4}}$. Then u belongs to $L^2(]0,1[)$ but its derivative $u' : x \mapsto -\frac{1}{4}x^{-\frac{5}{4}}$ is not in $L^2(]0,1[)$, so u is not in $H^1(]0,1[)$. We similarly consider $u : x \mapsto x^{-\frac{1}{4}}$ on $]1,+\infty[$. Then $u' \in L^2(]1,+\infty[)$ but $u \notin L^2(]1,+\infty[)$, so $u \notin H^1(]1,+\infty[)$.

On the other hand, the function $u \mapsto x^{\frac{3}{4}}$ belongs to $H^1(]0,1[)$ and $x \mapsto x^{-\frac{3}{4}}$ belongs to $H^1(]1,+\infty[)$.

***Exercise* 1.** Let $p \in [1,+\infty]$ and $\alpha \in \mathbb{R}$. Does the function $x \mapsto x^\alpha$ belongs to $W^{1,p}(]0,1[)$? $W^{1,p}(]1,+\infty[)$? $W^{1,p}(]0,+\infty[)$?

These first examples concern functions differentiable in the usual sense. But $W^{1,p}(I)$ contains functions which are only differentiable in the weak sense.

Example 2.4. We consider on $]-1,1[$ the map $u : x \mapsto |x|$. Then its derivative in the sense of distributions is given by

$$u' : x \mapsto \begin{cases} -1 & \text{if } x < 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Then u belongs to $W^{1,p}(]-1,1[)$ for any $p \in [1,+\infty]$.

A function in $L^p(I)$ can have a derivative in the sense of distributions which is not a function.

Example 2.5. The Heaviside function

$$H : x \mapsto \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x < 0, \end{cases}$$

belongs to $L^p(]-1,1[)$. However, there is no function $v \in L^p(]-1,1[)$ such that

$$\forall \phi \in C_0^\infty(]-1,1[), \quad \int_{-1}^1 v\phi\,dx = -\int_{-1}^1 H\phi'\,dx = \phi(0),$$

so H has no weak derivative in $L^p(]-1,1[)$, and hence it does not belong to $W^{1,p}(]-1,1[)$. With the vocabulary of distributions the derivative of H in the sense of distributions is the Dirac distribution $H' = \delta$, and δ is not associated to any function in $L^p(]-1,1[)$. Similarly, a piecewise C^1 function on I which is not continuous cannot be in $W^{1,p}(I)$ (with the vocabulary of distributions, by the jump formula the derivative in the sense of distributions of such a function involves Dirac distributions and cannot belong to $L^p(I)$).

More generally, we can prove that a function in $W^{1,p}(I)$ is necessarily continuous on I. We recall that a function in $L^p(I)$ or in $W^{1,p}(I)$ is in fact a equivalence class of functions which are pairwise almost everywhere equal. When we say that $u \in W^{1,p}(I)$ is continuous, this means that one of the representatives of u is continuous.

Proposition 2.6. *Let $p \in [1,+\infty]$ and $u \in W^{1,p}(I)$. Then u has a representative $\tilde{u} \in \mathcal{L}^p(I)$ such that, for $x, y \in I$,*

$$\tilde{u}(y) - \tilde{u}(x) = \int_x^y u'(s)\,ds.$$

In particular $\tilde{u}$ is continuous. If $p > 1$ then $\tilde{u}$ is even $\frac{p-1}{p}$-Hölder continuous on I (when $p = +\infty$ this means that $\tilde{u}$ is Lipschitz continuous). Moreover, if I is not bounded and if $p \in]1,+\infty[$ then $\tilde{u}$ goes to 0 at infinity.

Finally, for all $p \in [1,+\infty]$, $\tilde{u}$ is bounded and hence $u \in L^\infty(I)$.

For the proof we recall the following results (prove them as an exercice if not already known).

Lemma 2.7. *Let $u \in L^1_{\text{loc}}(I)$ be such that*

$$\forall \phi \in C_0^\infty(I), \quad \int_I u\phi'\,dx = 0.$$

There exists a constant α such that $u = \alpha$ almost everywhere.

Lemma 2.8. *Let $w \in L^1_{\text{loc}}(I)$ and $x_0 \in I$. Then the map*

$$v \mapsto \int_{x_0}^{x} w(s)\,ds$$

is well defined, it is continuous on I, and

$$\forall \phi \in C_0^\infty(I), \quad \int_I v\phi'\,dx = -\int_I w\phi\,dx.$$

Now we can prove Proposition 2.6:

Proof. We fix $x_0 \in I$. For $x \in I$ we set

$$v(x) = \int_{x_0}^{x} u'(s)\,ds.$$

This makes sense since $u' \in L^p(I) \subset L^1_{\text{loc}}(I)$. Then, by Lemma 2.8, v is continuous and its derivative in the sense of distributions is u'. By Lemma 2.7, there exists a constant α such that $u - v = \alpha$ almost everywhere. We set $\tilde{u} = v + \alpha$.

For $x, y \in I$ we have

$$\tilde{u}(y) - \tilde{u}(x) = v(y) - v(x) = \int_x^y u'(s)\,ds.$$

If $p = 1$ then for some $x_0 \in I$ we have $|\tilde{u}(y)| \leqslant |\tilde{u}(x_0)| + \|u'\|_{L^1(I)}$.

If $p = +\infty$ then $|\tilde{u}(y) - \tilde{u}(x)| \leqslant |y - x|\,\|u'\|_{L^\infty(I)}$, so $\tilde{u}$ is $\|u'\|_{L^\infty(I)}$-Lipschitz continuous. If $p \in]1, +\infty[$, we have by the Hölder inequality

$$|\tilde{u}(y) - \tilde{u}(x)| \leqslant \left| \int_x^y |u'(s)|\,ds \right| \leqslant |y - x|^{\frac{p-1}{p}} \left(\int_I |u'(s)|^p\,ds \right)^{\frac{1}{p}}.$$

This proves that $\tilde{u}$ is $\frac{p-1}{p}$-Hölder continuous, and in particular uniformly continuous. All the statements of the proposition follow. □

***Exercise* 2.** Let $\alpha \in \mathbb{R}$. For $x \in \mathbb{R}$ we set

$$u_\alpha(x) = \begin{cases} x^\alpha e^{-x} & \text{if } x \geqslant 0, \\ -|x|^\alpha e^{-|x|} & \text{if } x < 0. \end{cases}$$

1. Prove that $u_\alpha \in C^1(\mathbb{R})$ if $\alpha > 1$.
2. Prove that $u_\alpha \in H^1(\mathbb{R})$ if $\alpha > \frac{1}{2}$.

***Exercise* 3.** Let $u \in L^p(I)$. Prove that $u \in W^{1,p}(I)$ if and only if there exists $v \in L^p(I)$ such that

$$\forall \phi \in C_0^1(I), \quad \int_I u\phi' = -\int_I v\phi.$$

***Exercise* 4.** **1.** Let $u_+ \in C_0^1([0, +\infty[)$. For $x \in \mathbb{R}$ we set

$$u(x) = \begin{cases} u_+(x) & \text{if } x \geqslant 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Does u belong to $H^1(\mathbb{R})$? to $C^1(\mathbb{R})$?
2. Same questions with

$$u(x) = \begin{cases} u_+(x) & \text{if } x \geqslant 0, \\ u_+(-x) & \text{if } x < 0, \end{cases}$$

3. Same questions with

$$u(x) = \begin{cases} u_+(x) & \text{if } x \geqslant 0, \\ -3u_+(-x) + 4u_+(-x/2) & \text{if } x < 0. \end{cases}$$

2.1.2 General definitions

The above definitions can be extended in any dimension $d \in \mathbb{N}^*$ and we can consider any order $k \in \mathbb{N}$ of derivatives.

Definition 2.9. Let Ω be an open subset of $\mathbb{R}^d$. For $p \in [1, +\infty]$ and $k \in \mathbb{N}$ we set

$$W^{k,p}(\Omega) = \left\{ u \in L^p(\Omega) \, : \, \partial^\alpha u \in L^p(\Omega) \text{ for all } \alpha \in \mathbb{N}^d \text{ with } |\alpha| \leqslant k \right\},$$

where $\partial^\alpha u$ is the derivative of u in the sense of distributions. In other words, a function $u \in L^p(\Omega)$ belongs to $W^{k,p}(\Omega)$ if for all $\alpha \in \mathbb{N}^d$ such that $|\alpha| \leqslant k$ there exists $v_\alpha \in L^p(\Omega)$ such that

$$\forall \phi \in C_0^\infty(\Omega), \qquad \int_\Omega u \, \partial^\alpha \phi \, dx = (-1)^{|\alpha|} \int_\Omega v_\alpha \, \phi \, dx. \tag{2.3}$$

In this case v_α is unique (up to equality almost everywhere) and we set $\partial^\alpha u = v_\alpha$. We also set $H^k(\Omega) = W^{k,2}(\Omega)$.

Remark 2.10. By the Riesz Theorem and by density of $C_0^\infty(\Omega)$ in $L^2(\Omega)$, a function $u \in L^1_{\text{loc}}(\Omega)$ belongs to $H^k(\Omega)$ is and only if for all $\alpha \in \mathbb{N}^d$ with $|\alpha| \leqslant k$ there exists $C_\alpha > 0$ such that

$$\forall \phi \in C_0^\infty(\Omega), \qquad \left| \int_\Omega u \partial^\alpha \phi \, dx \right| \leqslant C_\alpha \left\| \phi \right\|_{L^2(\Omega)}.$$

Example 2.11. Let $\alpha > 0$. For $x \in B(0,1) \setminus \{0\}$ we set $u(x) = |x|^{-\alpha}$. Then $u \in L^p(B(1))$ if and only if $\alpha p < d$. On the other hand u is of class C^1 on $B(1) \setminus \{0\}$ and $\nabla u(x) = -\alpha \, |x|^{-\alpha-2} x$ for all $x \in B(0,1) \setminus \{0\}$. Thus $\nabla u \in L^p(B(1))$ if and only if $(\alpha+1)p < d$. This proves that if $\alpha \geqslant \frac{d}{p} - 1$ then u is not in $W^{1,p}(B(0,1))$. Now assume that $\alpha < \frac{d}{p} - 1$.

Let $\phi \in C_0^\infty(B(0,1))$. Since $u \in L^1(B(0,1))$ we have by the dominated convergence theorem

$$-\int_{B(0,1)} |x|^{-\alpha} \nabla \phi \, dx = -\lim_{\varepsilon \to 0} \int_{B(0,1) \setminus B(0,\varepsilon)} |x|^{-\alpha} \nabla \phi \, dx.$$

For $\varepsilon \in]0,1[$ we have by the Green formula

$$-\int_{B(0,1) \setminus B(0,\varepsilon)} |x|^{-\alpha} \nabla \phi \, dx = -\int_{S(0,\varepsilon)} |x|^{-\alpha} \phi \nu \, d\sigma(x) - \alpha \int_{B(0,1) \setminus B(0,\varepsilon)} |x|^{-\alpha-2} x \phi \, dx,$$

where $S(0,\varepsilon)$ is the sphere of radius ε. On the one hand we have

$$\left| \int_{S(0,\varepsilon)} |x|^{-\alpha} \phi \nu \, d\sigma(x) \right| \leqslant \|\phi\|_\infty |S(0,1)| \, \varepsilon^{d-1-\alpha} \xrightarrow[\varepsilon \to 0]{} 0,$$

and on the other hand,

$$\int_{B(0,1) \setminus B(0,\varepsilon)} |x|^{-\alpha-2} x \phi \, dx \xrightarrow[\varepsilon \to 0]{} \int_{B(0,1)} |x|^{-\alpha-2} x \phi \, dx.$$

This proves that the map $x \mapsto -\alpha \, |x|^{-\alpha-2} x$ is the gradient of u on $B(0,1)$ in the weak sense. And hence $\nabla u \in L^p(B(0,1))$. Finally we have proved that $u \in W^{1,p}(B(0,1))$ if and only if $(\alpha+1)p < d$.

***Exercise* 5.** Let $d \geqslant 2$. We denote by B the unit ball in $\mathbb{R}^d$. Let $u \in C^1(\overline{B} \setminus \{0\})$ such that ∇u (well defined on $B \setminus \{0\}$) is in $L^1_{\text{loc}}(B)$.

1. For $\varepsilon \in]0,1]$ we denote by $B(\varepsilon)$ the ball of radius ε. Prove that

$$\int_{B \setminus B(\varepsilon)} \frac{\varepsilon^{d-1}}{|x|^{d-1}} \nabla u(x) \, dx \xrightarrow[\varepsilon \to 0]{} 0.$$

2. For $\varepsilon \in]0,1]$ we denote by $S(\varepsilon)$ the sphere of radius ε. We set $S = S(1)$. Prove that

$$\int_{S(\varepsilon)} |u| \leqslant \varepsilon^{d-1} \int_S |u| + \int_{B \setminus B(\varepsilon)} \frac{\varepsilon^{d-1}}{|x|^{d-1}} |\nabla u| \, dx.$$

3. Prove that $u \in L^1_{\mathrm{loc}}(B)$.
4. Prove that for $j \in [\![1,d]\!]$ and $\phi \in C_0^\infty(B)$ we have

$$\forall \phi \in C_0^\infty(B), \quad -\int_B u\,\partial_j\phi\,dx = \int_B \partial_j u\,\phi\,dx$$

***Exercise* 6.** Does the map $x \mapsto \ln(|\ln(|x|)|)$ belong to $W^{1,d}(B(0,1))$ (we recall that $B(0,1)$ is the unit ball of $\mathbb{R}^d$) ?

Example 2.11 and Exercise 6 show that the results of Proposition 2.6 are only valid in dimension 1. In higher dimensions, a function in $W^{1,p}(\Omega)$ is not necessarily continuous.

In the following proposition we give some basic properties for the set $W^{k,p}(\Omega)$ (the proof is left as an exercice). We define $C_0^\infty(\overline{\Omega})$ as the restrictions to $\overline{\Omega}$ of functions in $C_0^\infty(\mathbb{R}^d)$.

Proposition 2.12. *Let $p \in [1,+\infty]$, $k \in \mathbb{N}^*$ and $\alpha = (\alpha_1,\dots,\alpha_d) \in \mathbb{N}^d$ with $|\alpha| \leqslant k$. Let $u \in W^{k,p}(\Omega)$.*

(i) We have $\partial^\alpha u \in W^{k-|\alpha|,p}(\Omega)$ and for $\beta \in \mathbb{N}^d$ with $|\beta| \leqslant k-|\alpha|$ we have $\partial^\beta(\partial^\alpha u) = \partial^{\alpha+\beta}u$.

(ii) Let ω be an open subset of Ω. Then the restriction $u_{|\omega}$ of u on ω belongs to $W^{k,p}(\omega)$ and $\partial^\alpha(u_\omega) = (\partial^\alpha u)_{|\omega}$.

(iii) Let $\chi \in C_0^\infty(\overline{\Omega})$. Then $\chi u \in W^{k,p}(\Omega)$ and

$$\partial^\alpha(\chi u) = \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \partial^\beta \chi \partial^{\alpha-\beta} u,$$

where we have set

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}, \qquad \alpha! = \alpha_1!\dots\alpha_d!.$$

When $\Omega = \mathbb{R}^d$ and $p = 2$ we can use the Fourier transform to give a simple characterisation of $H^k(\mathbb{R}^d)$. Notice that in Definition 2.9 we can see the derivatives of u in the sense of tempered distributions. This means that we can replace $C_0^\infty(\mathbb{R}^d)$ by $\mathcal{S}(\mathbb{R}^d)$ in (2.3).

Proposition 2.13. *Let $k \in \mathbb{N}^*$ and $u \in L^2(\mathbb{R}^d)$. Then $\partial^\alpha u \in L^2(\Omega)$ if and only if the map $\xi \mapsto (i\xi)^\alpha \hat{u}(\xi)$ belongs to $L^2(\mathbb{R}^d)$ (and, in this case, the latter is the Fourier transform of the former). Then $u \in H^k(\mathbb{R}^d)$ if and only if*

$$\int_{\mathbb{R}^d} \left(1+|\xi|^2\right)^k |\hat{u}(\xi)|^2\,d\xi < +\infty. \tag{2.4}$$

Proof. Let $u \in L^2(\mathbb{R}^d)$. For $\phi \in \mathcal{S}(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} (iy)^\alpha \hat{u}\phi\,dy = \int_{\mathbb{R}^d} u\widehat{(iy)^\alpha\phi}\,dy = (-1)^{|\alpha|}\int_{\mathbb{R}^d} u\partial^\alpha\hat{\phi}\,dy. \tag{2.5}$$

Assume that $\partial^\alpha u \in L^2(\mathbb{R}^d)$ for all $\alpha \in \mathbb{N}^d$ with $|\alpha| \leqslant k$. Then for such an α (2.5) gives

$$\int_{\mathbb{R}^d} (iy)^\alpha \hat{u}\phi\,dy = \int_{\mathbb{R}^d} \partial^\alpha u\hat{\phi}\,dy = \int_{\mathbb{R}^d} \widehat{\partial^\alpha u}\phi\,dy,$$

so the map $y \mapsto (iy)^\alpha \hat{u}(y)$ belongs to $L^2(\mathbb{R}^d)$, and it is the Fourier transform of $\partial^\alpha u$.

Conversely, let $u \in L^2(\mathbb{R}^d)$ be such that (2.4) holds. For $\alpha \in \mathbb{N}^d$ with $|\alpha| \leqslant k$ (2.5) applied with $\check{\phi}$ gives

$$\left|\int_{\mathbb{R}^d} u\partial^\alpha\phi\,dy\right| = \left|\int_{\mathbb{R}^d} y^\alpha\hat{u}\check{\phi}\,dy\right| \leqslant \frac{\|y^\alpha\hat{u}\|_{L^2(\mathbb{R}^d)}}{(2\pi)^{\frac{d}{2}}} \|\phi\|_{L^2(\mathbb{R}^d)}.$$

This proves that $\partial^\alpha u \in L^2(\mathbb{R}^d)$ and the proof is complete. □

Remark 2.14. If $u \in L^2(\mathbb{R}^d)$ is such that Δu belongs to $L^2(\mathbb{R}^d)$, then u belongs to $H^2(\mathbb{R}^d)$. This remark does not hold on a general domain (see Remark 3.17 below).

In (2.4), k is the number of derivatives in $L^2(\mathbb{R}^d)$. In particular it is an integer. But it makes sense to write the same condition with any real exponent. This is a way to define derivatives of real order, which will turn out to be useful. By Proposition 2.13, the following definition coincides with the previous one when $s \in \mathbb{N}$.

Definition 2.15. Let $s \geqslant 0$. We define $H^s(\mathbb{R}^d)$ as the set of functions $u \in L^2(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} \left(1 + |\xi|^2\right)^s |\hat{u}(\xi)|^2 \, d\xi < +\infty.$$

***Exercise* 7.** Let $p \in [1, +\infty]$ and $u \in W^{1,p}(\mathbb{R}^d)$. Let $\rho \in C_0^\infty(\mathbb{R}^d)$. We recall that $(\rho * u) \in C^\infty(\mathbb{R}^d)$. Prove that for $j \in [\![1, d]\!]$ we have

$$\partial_j(\rho * u) = \rho * (\partial_j u).$$

Deduce that $(\rho * u) \in W^{1,p}(\mathbb{R}^d)$.

***Exercise* 8.** Let $u \in W^{1,\infty}(\mathbb{R}^d)$. Let B be a compact subset of $\mathbb{R}^d$. Let $\rho \in C_0^\infty(\mathbb{R}^d, \mathbb{R}_+)$ such that $\int_{\mathbb{R}^d} \rho \, dx = 1$. For $\varepsilon > 0$ and $x \in \mathbb{R}^d$ we set $\rho_\varepsilon(x) = \rho(x/\varepsilon)$.
1. Prove that there exists a sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ going to 0 such that if we set $u_n = \rho_{\varepsilon_n} * u$ for all $n \in \mathbb{N}$ then $u_n(x)$ goes to $u(x)$ for almost all $x \in B$.
2. Prove that for all $n \in \mathbb{N}$ we have $\|\nabla u_n\|_{L^\infty(\mathbb{R}^d)} \leqslant \|\nabla u\|_{L^\infty(\mathbb{R}^d)}$.
3. Prove that for almost all $x, y \in B$ we have $|u(x) - u(y)| \leqslant \|\nabla u\|_{L^\infty(\mathbb{R}^d)} |x - y|$.
4. Prove that u has a representative which is $\|\nabla u\|_{L^\infty(\mathbb{R}^d)}$-Lipschitz (and in particular continuous).

2.2 Topology on the Sobolev spaces

In this section we define the norms on the Sobolev spaces we have just defined, and we give the properties of these new functional spaces. In the particular case of Sobolev spaces on the Euclidean space, we prove that smooth functions are dense in the Sobolev space, and we show on some examples how this important result is used to generalize some properties known for regular functions. The density of smooth functions in the general case will be discussed in the following section.

2.2.1 Banach spaces

Let Ω be an open subset of $\mathbb{R}^d$. Let $p \in [1, +\infty]$ and $k \in \mathbb{N}$. For $u \in W^{k,p}(\Omega)$ we set

$$\|u\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha| \leqslant k} \|\partial^\alpha u\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}}. \tag{2.6}$$

This defines a norm on $W^{k,p}(\Omega)$. We could also consider the quantity

$$\sum_{|\alpha| \leqslant k} \|\partial^\alpha u\|_{L^p(\Omega)}, \tag{2.7}$$

which defines an equivalent norm on $W^{k,p}(\Omega)$.

On $H^k(\Omega)$ we define an inner product by setting, for $u, v \in H^k(\Omega)$,

$$\langle u, v \rangle_{H^k(\Omega)} = \sum_{|\alpha| \leqslant k} \langle \partial^\alpha u, \partial^\alpha v \rangle_{L^2(\Omega)}. \tag{2.8}$$

The corresponding norm is exactly (2.6) with $p = 2$.

Remark 2.16. With the notation of Proposition 2.12, we observe that

$$\|\partial^\alpha u\|_{W^{k-|\alpha|,p}(\Omega)} \leqslant \|u\|_{W^{k,p}(\Omega)},$$

for $\omega \subset \Omega$ we have

$$\|u\|_{W^{k,p}(\omega)} \leqslant \|u\|_{W^{k,p}(\Omega)},$$

and for $\chi \in C_0^\infty(\overline{\Omega})$ there exists $C_\chi > 0$ independant of u such that

$$\|\chi u\|_{W^{k,p}(\Omega)} \leqslant C_\chi \|u\|_{W^{k,p}(\Omega)}.$$

Theorem 2.17. *Let $k \in \mathbb{N}$ and $p \in [1, +\infty]$. The Sobolev space $W^{k,p}(\Omega)$, endowed with the norm (2.7) or (2.6), is a Banach space. In particular, $H^k(\Omega)$ with the inner product (2.8) is a Hilbert space.*

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $W^{k,p}(\Omega)$. The sequences $(\partial^\alpha u_n)_{n\in\mathbb{N}}$ for $|\alpha| \leqslant k$ are Cauchy sequences in $L^p(\Omega)$. Since $L^p(\Omega)$ is complete by the Riesz-Fisher theorem, there exist $v_\alpha \in L^p(\Omega)$ for $|\alpha| \leqslant k$ such that $\partial^\alpha u_n$ goes to v_α. For $|\alpha| \leqslant k$ and $\phi \in C_0^\infty(\Omega)$ we have

$$(-1)^{|\alpha|} \int_\Omega v_0 \, \partial^\alpha \phi \, dx = (-1)^{|\alpha|} \lim_{n\to+\infty} \int_\Omega u_n \, \partial^\alpha \phi \, dx = \lim_{n\to+\infty} \int_\Omega \partial^\alpha u_n \, \phi \, dx = \int_\Omega v_\alpha \, \phi \, dx.$$

This proves that in the sense of distributions we have $\partial^\alpha v_0 = v_\alpha \in L^p(\Omega)$. Then $v_0 \in W^{k,p}(\Omega)$ and

$$\|u_n - v_0\|_{W^{k,p}(\Omega)}^p = \sum_{|\alpha|\leqslant k} \|\partial^\alpha u_n - v_\alpha\|_{L^p(\Omega)}^2 \xrightarrow[n\to+\infty]{} 0.$$

Thus the sequence $(u_n)_{n\in\mathbb{N}}$ has a limit in $W^{k,p}(\Omega)$. This proves that $W^{k,p}(\Omega)$ is complete. □

The proofs of the following two results are omitted (see [Brézis]).

Theorem 2.18. *If $p \in]1, +\infty[$ then $W^{k,p}(\Omega)$ is reflexive.*

Theorem 2.19. *If $p \in [1, +\infty[$ then $W^{k,p}(\Omega)$ is separable.*

Proposition 2.20. *Let $s \geqslant 0$. Then the map*

$$(u, v) \mapsto \left(\int_{\mathbb{R}^d} \left(1 + |\xi|^2\right)^s \hat{u}(\xi) \overline{\hat{v}(\xi)} \, d\xi \right)^{\frac{1}{2}}$$

defines a scalar product on $H^s(\mathbb{R}^d)$. When s is an integer, this norm is equivalent to the norm defined by (2.6) with $p = 2$.

2.2.2 Approximation by smooth functions

We know that for $p \in [1, +\infty[$ the set $C_0^\infty(\Omega)$ of smooth and compactly supported functions on the open set Ω is dense in $L^p(\Omega)$. In this paragraph we will see in what sense we can approach functions in $W^{k,p}(\Omega)$ by smooth functions.

More precisely, we prove the density of smooth functions in the Sobolev spaces when $\Omega = \mathbb{R}^d$. This will not be the case in general domains. Since the closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$ will play an important role in applications, we introduce the following notation.

Definition 2.21. For $k \in \mathbb{N}$ and $p \in [1, +\infty[$ we denote by $W_0^{k,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$. We also set $H_0^k(\Omega) = W_0^{1,2}(\Omega)$.

***Exercise* 9.** For $x \in]-1, 1[$ we set $u(x) = 1$. Prove that for $p \in [1, +\infty]$ there is no sequence $(u_n)_{n\in\mathbb{N}}$ in $C_0^\infty(]-1, 1[)$ which goes to u in $W^{1,p}(]-1, 1[)$.

As in $L^p(\mathbb{R}^d)$, the proofs will rely on regularization by convolution with a sequence of mollifiers. Let $\rho \in C_0^\infty(\mathbb{R}^d, [0, 1])$ be supported in $B(0, 1)$ and such that $\int_{\mathbb{R}^d} \rho \, dx = 1$. For $n \in \mathbb{N}^*$ and $x \in \mathbb{R}^d$ we set $\rho_n(x) = n^d \rho(nx)$.

Lemma 2.22. *Let Ω be an open subset of $\mathbb{R}^d$. Let $n \in \mathbb{N}^*$ and let ω be an open subset of Ω such that $B\left(x, \frac{1}{n}\right) \subset \Omega$ for all $x \in \omega$. Let $\rho_n \in C_0^\infty(\mathbb{R}^d)$ be as above and let . Let $u \in W^{k,p}(\Omega)$. Then $\rho_n * u \in C^\infty(\mathbb{R}^d) \cap W^{k,p}(\omega)$ and for $|\alpha| \leqslant k$ we have in the weak sense on ω*

$$\partial^\alpha(\rho_n * u) = \rho_n * (\partial^\alpha u).$$

Proof. We prove the case $k = 1$, and the general case follows by induction. Let $j \in [\![1, d]\!]$ and $\phi \in C_0^\infty(\omega)$. We have

$$-\int_\omega (\rho_n * u)(x) \partial_j \phi(x) \, dx = -\int_{B(0,\frac{1}{n})} \rho_n(y) \int_\omega u(x - y) \partial_j \phi(x) \, dx \, dy.$$

For $y \in B\big(0, \frac{1}{n}\big)$ the map $x \mapsto u(x-y)$ belongs to $W^{1,p}(\omega)$, so

$$-\int_\omega (\rho_n * u)(x) \partial_j \phi(x)\,dx = \int_{B(0,\frac{1}{n})} \rho_n(y) \int_\omega \partial_j u(x-y)\phi(x)\,dx\,dy = \int_\omega (\rho_n * \partial_j u)(x)\phi(x)\,dx.$$

The conclusion follows. □

Notice that the lemma applies in particular with $\omega = \Omega = \mathbb{R}^d$.
Given $v \in W^{k,p}(\mathbb{R}^d)$ we set for $n \in \mathbb{N}^*$

$$v_n = \rho_n * v.$$

Then $v_n \in C^\infty(\mathbb{R}^d) \cap W^{k,p}(\mathbb{R}^d)$ and for $\alpha \in \mathbb{N}^d$ with $|\alpha| \leqslant k$ we have

$$\partial^\alpha v_n = \rho_n * (\partial^\alpha v). \tag{2.9}$$

In particular,

$$\|v_n - v\|_{W^{k,p}(\mathbb{R}^d)} \xrightarrow[n\to+\infty]{} 0.$$

Moreover, if v is compactly supported then $v_n \in C_0^\infty(\mathbb{R}^d)$ for all $n \in \mathbb{N}^*$.

Statement (2.9) can be seen as a particular case on $\mathbb{R}^d$ of the more general following result. For the following two proofs we also consider $\chi \in C_0^\infty(\mathbb{R}^d)$ supported in the ball $B(0,2)$ of radius 2 and equal to 1 on $B(0,1)$. Then for $m \in \mathbb{N}^*$ and $x \in \mathbb{R}^d$ we set $\chi_m(x) = \chi(\frac{x}{m})$.

Theorem 2.23. *Let $p \in [1,+\infty[$. Then $C_0^\infty(\mathbb{R}^d)$ is dense in $W^{k,p}(\mathbb{R}^d)$. In other words we have $W_0^{k,p}(\mathbb{R}^d) = W^{k,p}(\mathbb{R}^d)$.*

Proof. Let $u \in W^{k,p}(\mathbb{R}^d)$ and $\varepsilon > 0$. Let $\alpha \in \mathbb{N}^d$ with $|\alpha| \leqslant k$. By Proposition 2.12 we have $\chi_m u \in W^{k,p}(\mathbb{R}^d)$ for all $m \in \mathbb{N}^*$ and

$$\|\partial^\alpha(\chi_m u) - \partial^\alpha u\|_{L^p(\mathbb{R}^d)} \leqslant \sum_{0\leqslant\beta\leqslant\alpha} \binom{\alpha}{\beta} \left\|\partial^{\alpha-\beta}(\chi_m - 1)\partial^\beta u\right\|_{L^p(\mathbb{R}^d)}.$$

By the dominated convergence theorem we have for $\beta \leqslant \alpha$

$$\left\|\partial^{\alpha-\beta}\chi_m \partial^\beta u\right\|_{L^p(\mathbb{R}^d)}^p \leqslant \left\|\partial^{\alpha-\beta}(\chi_m - 1)\right\|_{L^\infty(\mathbb{R}^d)} \int_{|x|\geqslant m} \left|\partial^\beta u(x)\right|^p\,dx \xrightarrow[m\to+\infty]{} 0,$$

so there exists $m \in \mathbb{N}^*$ such that

$$\|u - \chi_m u\|_{W^{k,p}(\mathbb{R}^d)} \leqslant \frac{\varepsilon}{2}.$$

We set $v = \chi_m u$, and for $n \in \mathbb{N}^*$ we set $v_n = \rho_n * v$. Then $v_n \in C_0^\infty(\mathbb{R}^d)$ and for all $\alpha \in \mathbb{N}^d$ with $|\alpha| \leqslant k$ we have by Lemma 2.22

$$\|\partial^\alpha v_n - \partial^\alpha v\|_{L^p(\mathbb{R}^d)} = \|\rho_n * (\partial^\alpha v) - \partial^\alpha v\|_{L^p(\mathbb{R}^d)} \xrightarrow[n\to+\infty]{} 0.$$

Then, if for $n \in \mathbb{N}^*$ large enough we set $u_\varepsilon = v_n$ we have $u_\varepsilon \in C_0^\infty(\mathbb{R}^d)$ and

$$\|u_\varepsilon - v\|_{W^{k,p}(\mathbb{R}^d)} \leqslant \frac{\varepsilon}{2},$$

so finally

$$\|u_\varepsilon - u\|_{W^{k,p}(\mathbb{R}^d)} \leqslant \varepsilon.$$ □

Remark 2.24. For any $\varepsilon > 0$ the function u_ε constructed in the previous proof is such that $\|u_\varepsilon\|_{L^\infty(\mathbb{R}^d)} \leqslant \|u\|_{L^\infty(\mathbb{R}^d)}$.

The conclusion of Theorem 2.23 does not hold in a general domain Ω. In other words, $W_0^{k,p}(\Omega) \neq W^{k,p}(\Omega)$ (see Exercise 9 and Proposition 2.42 below). However we have the following weaker result of approximation by regular functions on any compact subset of Ω. For a result of approximation on the whole domain Ω we refer to Proposition 2.32 below.

Theorem 2.25. *Let $p \in [1,+\infty[$ and $k \in \mathbb{N}$. Let Ω be an open subset of $\mathbb{R}^d$. Let $u \in W^{k,p}(\Omega)$. There exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $C_0^\infty(\mathbb{R}^d)$ such that $u_{n|\Omega}$ goes to u in $L^p(\Omega)$ and for any open bounded subset ω such that $\overline{\omega} \subset \Omega$ we have*

$$\left\| u_{n|\omega} - u_{|\omega} \right\|_{W^{k,p}(\omega)} \xrightarrow[n\to+\infty]{} 0.$$

Proof. For $x \in \mathbb{R}^d$ we set

$$v(x) = \begin{cases} u(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^d \setminus \Omega. \end{cases}$$

Then we set $v_n = \rho_n * v \in C^\infty(\mathbb{R}^d)$ and $u_n = \chi_n v_n$. We have

$$\|u_n - u\|_{L^p(\Omega)} \leqslant \|u_n - v\|_{L^p(\mathbb{R}^d)} \leqslant \|\chi_n(\rho_n * v) - \chi_n v\|_{L^p(\Omega)} + \|\chi_n v - v\|_{L^p(\Omega)} \xrightarrow[n\to+\infty]{} 0.$$

Let $N \in \mathbb{N}$ be so large that $B\big(x, \frac{1}{N}\big) \subset \Omega$ and $\chi_N = 1$ on $B\big(x, \frac{1}{N}\big)$ for all $x \in \omega$. Then for $|\alpha| \leqslant k$ we have by Lemma 2.22

$$\|\partial^\alpha(u_n - u)\|_{L^p(\omega)} = \|\partial^\alpha(v_n - v)\|_{L^p(\omega)} = \|\rho_n * (\partial^\alpha v) - \partial^\alpha v\|_{L^p(\omega)} \xrightarrow[n\to+\infty]{} 0.$$

This proves that χv_n goes to u in $W^{k,p}(\omega)$. □

***Exercise* 10.** Let Ω be a bounded subset of $\mathbb{R}^d$ and $p \in [1,+\infty[$. Prove that $W_0^{1,p}(\Omega)$ is the closure of $C_0^k(\Omega)$ in $W^{1,p}(\Omega)$.

2.2.3 Examples of properties proved by density

It is not always convenient to prove results about differentiation in the weak sense, and most of the properties of Sobolev spaces are proved by density. We first prove the result for regular functions (smooth, or of class C^k for a property in $W^{k,p}$), and then the general case is deduced by density.

Here we give some examples of results which are already known for regular functions and which can be extended in the suitable Sobolev spaces by density.

We begin with the integration by parts on $\mathbb{R}^d$.

Proposition 2.26 (Green Formula on $\mathbb{R}^d$)**.** *Let Ω be an open subset of $\mathbb{R}^d$ and $u, v \in H_0^1(\Omega)$. For $j \in [\![1,d]\!]$ we have*

$$\int_\Omega (\partial_j u) v \, dx = -\int_\Omega u(\partial_j v) \, dx.$$

Proof. Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be sequences in $C_0^\infty(\mathbb{R}^d)$ which go to u and v in $H^1(\mathbb{R}^d)$. The Green formula for smooth and compactly supported functions gives, for all $n \in \mathbb{N}$,

$$\int_\Omega (\partial_j u_n) v_n \, dx = -\int_\Omega u_n(\partial_j v_n) \, dx.$$

Taking the limit $n \to +\infty$ gives the result. □

We continue with the product of differentiable functions. If u and v are continuously differentiable, then so is the product uv. The same result holds for weak derivatives. Notice that in this result and the following we do not take functions in $W_0^{1,p}(\Omega)$. The approximation by regular functions is given by Theorem 2.25.

Proposition 2.27 (Differentiation of a product)**.** *Let Ω be an open subset of $\mathbb{R}^d$. Let $p \in [1,+\infty]$ and $u, v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$. Then $uv \in W^{1,p}(\Omega)$ and, for $j \in [\![1,d]\!]$,*

$$\partial_j(uv) = (\partial_j u)v + u(\partial_j v). \tag{2.10}$$

Proof. Assume that $p < +\infty$. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $C_0^\infty(\mathbb{R}^d)$ as given by Theorem 2.25. After extraction of a subsequence if necessary, we can assume that $u_n(x)$ tends to $u(x)$ for almost all $x \in \mathbb{R}^d$. By Remark 2.24, we can also assume that $\|u_n\|_{L^\infty(\mathbb{R}^d)} \leqslant \|u\|_{L^\infty(\mathbb{R}^d)}$ for all $n \in \mathbb{N}$. By Proposition 2.12, we have $u_n v \in W^{1,p}$ for all $n \in \mathbb{N}$ and, for $j \in [\![1,d]\!]$ and $\phi \in C_0^\infty(\mathbb{R}^d)$,

$$-\int_{\mathbb{R}^d} u_n v \partial_j \phi \, dx = \int_{\mathbb{R}^d} \big((\partial_j u_n) v + u_n (\partial_j v)\big) \phi \, dx.$$

The limit $n \to +\infty$ yields (2.10). In particular $\partial_j(uv) \in L^p(\mathbb{R}^d)$, and the proof is complete if $p < +\infty$.

Now assume that $p = +\infty$. Then uv and $(\partial_j u)v + u(\partial_j v)$ are in $L^\infty(\mathbb{R}^d)$. Let $\phi \in C_0^\infty(\mathbb{R}^d)$. Let $\chi \in C_0^\infty(\mathbb{R}^d)$ be equal to 1 on a neighborhood of $\operatorname{supp}(\phi)$. Then χu and χv are in $W^{1,p}(\mathbb{R}^d)$ for any $p \in [1, +\infty[$ so

$$\begin{aligned} -\int_{\mathbb{R}^d} uv\partial_j \phi \, dx = -\int_{\mathbb{R}^d} \chi u \, \chi v \partial_j \phi \, dx &= \int_{\mathbb{R}^d} \big(\partial_j(\chi u)\, \chi v + \chi u \, \partial_j(\chi v)\big)\, \phi \, dx \\ &= \int_{\mathbb{R}^d} \big((\partial_j u) v + u(\partial_j v)\big) \phi \, dx. \end{aligned}$$

This proves (2.10) and concludes the proof. □

Then we discuss the chain rule, which will be important in particular for changes of variables.

Proposition 2.28 (Chain rule)**.** *Let Ω_1 and Ω_2 be two open subsets in $\mathbb{R}^d$, and let $\Phi = (\Phi_1, \dots, \Phi_d) : \Omega_1 \to \Omega_2$ be a diffeomorphism of class C^1. We assume that* $\operatorname{Jac}(\Phi)$ *and* $\operatorname{Jac}(\Phi^{-1})$ *are bounded on Ω_1 and Ω_2, respectively. Let $p \in [1, +\infty]$. Then for $u \in W^{1,p}(\Omega_2)$ we have $u \circ \Phi \in W^{1,p}(\Omega_1)$ and for $j \in [\![1,d]\!]$,*

$$\partial_j(u \circ \Phi) = \sum_{k=1}^d \big((\partial_k u) \circ \Phi\big) \partial_j \Phi_k.$$

In particular there exists $C_\Phi > 0$ such that

$$\|u \circ \Phi\|_{W^{1,p}(\Omega_2)} \leqslant C_\Phi \|u\|_{W^{1,p}(\Omega_1)}.$$

Proof. Assume that $p < +\infty$. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $C_0^\infty(\mathbb{R}^d)$ which goes to u in the sense of Theorem 2.25. Let $\psi \in C_0^\infty(\Omega_1)$, $K_1 = \operatorname{supp}(\psi)$ and $K_2 = \Phi(K_1)$. Then K_2 is a compact of Ω_2. For $n \in \mathbb{N}$ and $\psi \in C_0^\infty(\mathbb{R}^d)$ we have

$$-\int_{\mathbb{R}^d} (u_n \circ \Phi) \partial_j \psi \, dx = \sum_{k=1}^d \int_{\mathbb{R}^d} (\partial_k u_n \circ \Phi) \partial_j \Phi_k \, \psi \, dx. \tag{2.11}$$

By the change of variables $y = \Phi(x)$ we have

$$\begin{aligned} \|u_n \circ \Phi - u \circ \Phi\|_{L^p(K_1)}^p &= \int_{K_1} |(u_n - u)(\Phi(x))|^p \, dx \\ &= \int_{K_2} |(u_n - u)(y)|^p \left|J\Phi^{-1}(y)\right| \, dy \\ &\leqslant \left\|J\Phi^{-1}\right\|_{L^\infty(K_2)} \|u_n - u\|_{L^p(K_2)}^p \\ &\xrightarrow[n\to+\infty]{} 0. \end{aligned}$$

We similarly have, for all $j, k \in [\![1,d]\!]$,

$$\begin{aligned} &\|(\partial_j u_n \circ \Phi)\partial_k \Phi_j - (\partial_j u \circ \Phi)\partial_k \Phi_j\|_{L^p(K_1)} \\ &\qquad \leqslant \|\partial_k \Phi_j\|_{L^\infty(K_1)} \|(\partial_j u_n \circ \Phi) - (\partial_j u \circ \Phi)\|_{L^p(K_1)} \xrightarrow[n\to+\infty]{} 0. \end{aligned}$$

We take the limit $n \to +\infty$ in (2.11) and conclude when $p < +\infty$. The case $p = +\infty$ follows as in the proof of Proposition 2.27. □

We finish this paragraph with the characterisation of $H^1(\mathbb{R}^d)$ by the difference quotients. We recall that the classical notion of differentiability is defined by looking at the limit at each point of the difference quotient. The following result gives a link between this point of view and the weak derivative.

For $u \in L^2(\mathbb{R}^d)$ and $h \in \mathbb{R}^d \setminus \{0\}$ we define the diffence quotient $D_h u \in L^2(\mathbb{R}^d)$ by

$$D_h u(x) = \frac{u(x+h) - u(x)}{|h|}. \tag{2.12}$$

Notice that for $u, v \in L^2(\mathbb{R}^d)$ we have

$$\int_{R^d} (D_h u) v \, dx = \int_{\mathbb{R}^d} u(D_{-h} v) \, dx. \tag{2.13}$$

Proposition 2.29. *Let $u \in L^2(\Omega)$.*

(i) *Assume that $u \in H^1(\Omega)$. Then for $h \in \mathbb{R}^d \setminus \{0\}$ we have*

$$\|D_h u\|_{L^2(\mathbb{R}^d)} \leqslant \|\nabla u\|_{L^2(\mathbb{R}^d)} .$$

(ii) *Assume that there exists $C > 0$ such that for all $h \in \mathbb{R}^d \setminus \{0\}$ we have*

$$\|D_h u\|_{L^2(\Omega)} \leqslant C.$$

Then $u \in H^1(\mathbb{R}^d)$ and for all $j \in [\![1, d]\!]$ we have

$$\left\| \frac{\partial u}{\partial x_j} \right\|_{L^2(\mathbb{R}^d)} \leqslant C.$$

Proof. • Assume that $u \in C_0^\infty(\mathbb{R}^d)$. Then for $x \in \mathbb{R}^d$ we have by the Cauchy-Schwarz inequality

$$|u(x+h) - u(x)|^2 \leqslant \left(\int_0^1 |\nabla u(x+th)| \, |h| \, dt \right)^2 \leqslant |h|^2 \int_0^1 |\nabla u(x+th)|^2 \, dt.$$

Then, by the Fubini Theorem and the change of variables $y = x + th$,

$$\int_{\mathbb{R}^d} |u(x+h) - u(x)|^2 \, dx \leqslant |h|^2 \int_{t=0}^1 \int_{x \in \mathbb{R}^d} |\nabla u(x+th)|^2 \, dx \, dt \leqslant |h|^2 \|\nabla u\|_{L^2(\mathbb{R}^d)}^2 .$$

This gives the first property.

• We denote by $(e_1, \dots, e_d)$ the canonical basis of $\mathbb{R}^d$. Let $j \in [\![1, d]\!]$. Let $\phi \in C_0^\infty(\mathbb{R}^d)$. For $t \in \mathbb{R}^*$ we have

$$\left| \left\langle u, D_{-te_j} \phi \right\rangle_{L^2(\mathbb{R}^d)} \right| = \left| \left\langle D_{te_j} u, \phi \right\rangle_{L^2(\mathbb{R}^d)} \right| \leqslant C \|\phi\|_{L^2(\mathbb{R}^d)} .$$

On the other hand, $u \in L^1_{\text{loc}}(\mathbb{R}^d)$ so by the dominated convergence theorem we have

$$\left| - \langle u, \partial_j \phi \rangle_{L^2(\mathbb{R}^d)} \right| = \left| \lim_{t \to 0} \left\langle u, D_{-te_j} \phi \right\rangle_{L^2(\mathbb{R}^d)} \right| \leqslant C \|\phi\|_{L^2(\mathbb{R}^d)} .$$

By the Riesz Theorem there exists $v_j \in L^2(\mathbb{R}^d)$ such that

$$\forall \phi \in C_0^\infty(\mathbb{R}^d), \quad - \langle u, \partial_j \phi \rangle_{L^2(\mathbb{R}^d)} = \langle v_j, \phi \rangle_{L^2(\mathbb{R}^d)} .$$

This proves that $u \in H^1(\mathbb{R}^d)$ with, for all $j \in [\![1, d]\!]$,

$$\left\| \frac{\partial u}{\partial x_j} \right\|_{L^2(\mathbb{R}^d)} = \|v_j\|_{L^2(\mathbb{R}^d)} \leqslant C.$$

The proof is complete. □

***Exercise* 11. 1.** Prove that the first statement of Proposition 2.29 holds in $W^{1,p}(\mathbb{R}^d)$ for any $p \in [1, +\infty[$.

2. Prove that the second statement of Proposition 2.29 holds in $W^{1,p}(\mathbb{R}^d)$ for any $p \in]1, +\infty[$.

2.3 Sobolev spaces on domains with boundary

In the previous section we have given some properties of the Sobolev spaces on $\mathbb{R}^d$, or local properties in general domains. In this section we look more carefully at the behavior of functions in Sobolev spaces at the boundary of the domain.

The model case will be the half space

$$\mathbb{R}^d_+ = \{x = (x_1, \dots, x_d) \in \mathbb{R}^d \, : \, x_1 > 0\}.$$

This is the simplest case because the boundary $\partial\mathbb{R}^d_+ = \{0\} \times \mathbb{R}^{d-1}$ is flat. Then, if the open subset Ω of $\mathbb{R}^d$ is sufficiently regular, the boundary $\partial\Omega$ can be locally straightened out and, with a partition of unity and a change of variables for each part, the problem on Ω is reduced to a problem far from the boundary (where we can apply the results on $\mathbb{R}^d$) and a finite number of problems on $\mathbb{R}^d_+$.

It is the purpose of this section to make these ideas clearer and to deduce some results for the Sobolev spaces on bounded subsets.

2.3.1 Regular domains

Let $k \in \mathbb{N}^* \cup \{\infty\}$. We recall that an open subset Ω of $\mathbb{R}^d$ is said to be of class C^k if for any $w \in \partial\Omega$ there exist an orthonormal basis $\beta = (\beta_1, \dots, \beta_d)$ of $\mathbb{R}^d$, an open subset $\mathcal{O}$ of $\mathbb{R}^{d-1}$, $a, b \in \mathbb{R}$ with $a < b$ and an application $\varphi : \mathcal{O} \to]a, b[$ of class C^k such that $\mathcal{U}$ defined by

$$\mathcal{U} = \left\{ \sum_{j=1}^d x_j \beta_j, (x_2, \dots, x_d) \in \mathcal{O}, x_1 \in]a, b[\right\}$$

is a neighborhood of w in $\mathbb{R}^d$ and

$$\Omega \cap \mathcal{U} = \left\{ \sum_{j=1}^d x_j \beta_j, x' = (x_2, \dots, x_d) \in \mathcal{O}, x_1 \in \big]\varphi(x'), b\big[\right\}.$$

In particular, in $\mathcal{U}$ the boundary $\partial\Omega$ is the graph of φ in the basis β. We can always construct the basis β with the vectors of the canonical basis $(e_1, \dots, e_d)$, possibly in a different order. For $x' = (x_2, \dots, x_d) \in \mathcal{O}$ we set

$$\tilde{\varphi}(x') = \varphi(x')\beta_1 + \sum_{j=2}^d x_j \beta_j.$$

Then $\partial\Omega \cap \mathcal{U}$ is also the image of $\mathcal{O}$ by $\tilde{\varphi}$.

Given $w \in \partial\Omega \cap \mathcal{V}$ and $x' = (x_2, \dots, x_d) \in \mathcal{O}$ such that $w = \tilde{\phi}(x')$, the outward normal derivative to Ω at point w is defined by

$$\nu(w) = \frac{-\beta_1 + \sum_{j=2}^d \partial_j \varphi(x')\beta_j}{\sqrt{1 + |\nabla\varphi(x')|^2}}. \tag{2.14}$$

The is the only vector such that $\nu(w) \perp T_w(\partial\Omega)$, $|\nu(w)| = 1$ and, for some $t_0 > 0$,

$$w + t\nu(w) \begin{cases} \in \Omega, & \forall t \in]-t_0, 0], \\ \notin \Omega, & \forall t \in]0, t_0[. \end{cases}$$

We define on $\partial\Omega$ the topology and the corresponding Borel σ-algebra inherited from the usual structure on $\mathbb{R}^d$. We define the Lebesgue measure of a Borel set $B \in \partial\Omega \cap \mathcal{U}$ as follows:

$$\sigma(B) = \int_{\mathcal{O}} \mathbb{1}_B(\tilde{\varphi}(x)) \sqrt{1 + |\nabla\varphi(x')|^2} \, dx'.$$

Thus, if f is an integrable function on $\partial\Omega \cap \mathcal{U}$ we have

$$\int_{\partial\Omega\cap\mathcal{U}} f\,d\sigma = \int_{\mathcal{O}} f(\tilde{\varphi}(x))\sqrt{1+|\nabla\varphi(x')|^2}\,dx'.$$

Then we can define Lebesgue spaces on $\partial\Omega$ as on any measure space.

For $x = \sum_{j=1}^d x_j\beta_j \in \mathcal{U}$ we set

$$\Phi(x) = \big(x_1 - \varphi(x_2,\dots,x_d)\big)e_1 + \sum_{j=2}^d x_j e_j.$$

Then Φ is of classe C^k and it is injective. So it defines a bijection on its image denoted by $\mathcal{W}$. Then $\mathcal{W}$ is open in $\mathbb{R}^d$ and the inverse Φ^{-1} of Φ is of class C^k on $\mathcal{W}$ (Φ defines a diffeomorphism of class C^k from $\mathcal{U}$ to $\mathcal{W}$). Moreover we have

$$\Phi(\mathcal{U}\cap\Omega) = \mathcal{W}\cap\mathbb{R}^d_+.$$

Notice also that for $x' \in \mathcal{O}$ we have $\Phi(\tilde{\varphi}(x')) = (0,x')$, and then $\mathcal{W}_j \cap \partial\mathbb{R}^d_+ = \{0\}\times\mathcal{O}$.

The interest of this change of variables is to transform a function supported in $\Omega\cap\mathcal{U}$ to a function on $\mathbb{R}^d_+$, where the properties of Sobolev spaces are easier.

Notice that if Ω is bounded then its boundary $\partial\Omega$ is compact. This is not necessary but it will simplify the discussion (an unbounded open subset can also have a compact boundary, but we will not consider this situation here).

Now let Ω be a bounded open subset of $\mathbb{R}^d$ of class C^k for some $k \geqslant 1$. There exist $N \in \mathbb{N}^*$, open subsets $\mathcal{U}_1,\dots,\mathcal{U}_N,\mathcal{W}_1,\dots,\mathcal{W}_N$ of $\mathbb{R}^d$ and diffeomorphisms $\Phi_j : \mathcal{U}_j \to \mathcal{W}_j$ of class C^k such that $\partial\Omega \subset \bigcup_{j=1}^N \mathcal{U}_j$ and for all $j \in [\![1,N]\!]$ we have $\Phi(\Omega\cap\mathcal{U}_j) = \mathbb{R}^d_+ \cap \mathcal{W}_j$.

If we set $\Omega = \mathcal{U}_0$ then $\bigcup_{j=0}^N \mathcal{U}_j$ is an open cover of $\overline{\Omega}$. We consider a corresponding partition of unity $(\chi_j)_{0\leqslant j\leqslant N}$ ($\chi_j \in C_0^\infty(\mathbb{R}^d,[0,1])$ is supported in $\mathcal{U}_j$ for all $j \in [\![0,N]\!]$ and $\sum_{j=0}^N \chi_j = 1$ on $\overline{\Omega}$).

For $u \in W^{1,p}(\Omega)$ we set $u_j = \chi_j u$ for all $j \in [\![0,N]\!]$. Then $u = \sum_{j=0}^N u_j$, $u_j \in W^{1,p}(\Omega)$ for all $j \in [\![0,N]\!]$, u_0 is supported in a compact subset of Ω, and u_j is supported in a compact subset of $\overline{\Omega}\cap\mathcal{U}_j$ for all $j \in [\![1,N]\!]$. In particular, the extension of u_0 by 0 on $\mathbb{R}^d$ is in $W^{1,p}(\mathbb{R}^d)$, and $(u_j\circ\Phi^{-1})$ belongs to $W^{1,p}(\mathbb{R}^d_+\cap\mathcal{W}_j)$ (and can be extended by 0 to a function in $W^{1,p}(\mathbb{R}^d_+)$) for all $j \in [\![1,N]\!]$.

We will use this setting to prove results for Sobolev spaces on Ω.

2.3.2 Extension

We begin with a result of extension. In order to deduce results in $W^{1,p}(\Omega)$ from results on $W^{1,p}(\mathbb{R}^d)$ it is natural to extend functions in $W^{1,p}(\Omega)$ to functions in $W^{1,p}(\mathbb{R}^d)$ (notice that in the proof of Theorem 2.25 we were able to prove results on $W^{k,p}(\omega)$ for $\omega \subset\subset \Omega$ precisely because we had a function with a nice behavior on a bigger domain).

It is clear, at least in dimension 1, than extending functions by 0 outside Ω does not always give a function in $W^{1,p}(\mathbb{R}^d)$. However, we have seen in Exercise 4 that in dimension 1 we can indeed extend a function in $H^1(\mathbb{R}^*_+)$ to a function in $H^1(\mathbb{R})$. We generalize this observation to the case of a function in $W^{1,p}(\mathbb{R}^d_+)$ and then, by the argument described above, to the case of a function in $W^{1,p}(\Omega)$ for a regular bounded open subset Ω.

Proposition 2.30. *Let $p \in [1,+\infty]$. For $u \in L^p(\mathbb{R}^d_+)$ and $x = (x_1,\dots,x_d) = (x_1,x') \in \mathbb{R}^d$ we set*

$$(Pu)(x) = \begin{cases} u(x_1,x') & \text{if } x_1 > 0,\\ u(-x_1,x') & \text{if } x_1 < 0.\end{cases}$$

Then $Pu \in L^p(\mathbb{R}^d)$ *and* $\|Pu\|_{L^p(\mathbb{R}^d)} = 2^{\frac{1}{p}} \|u\|_{L^p(\mathbb{R}^d_+)}$. *For* $u \in W^{1,p}(\mathbb{R}^d_+)$ *we have* $Pu \in W^{1,p}(\mathbb{R}^d)$ *with*

$$\partial_1(Pu) = \tilde{P}(\partial_1 u) \quad \text{and} \quad \partial_j(Pu) = P(\partial_j u), \quad 2 \leqslant j \leqslant d,$$

where

$$(\tilde{P}v)(x) = \begin{cases} v(x_1, x') & \text{if } x_1 > 0, \\ -v(-x_1, x') & \text{if } x_1 < 0. \end{cases}$$

In particular, P *defines a continuous extension from* $W^{1,p}(\mathbb{R}^d_+)$ *to* $W^{1,p}(\mathbb{R}^d)$.

Proof. • We set $\mathbb{R}^d_- = \mathbb{R}^d \backslash \overline{\mathbb{R}^d_+}$. It is easy to see that $\|Pu_{|\mathbb{R}^d_-}\|^p_{L^p(\mathbb{R}^d_-)} = \|u\|^p_{L^p(\mathbb{R}^d_+)}$ if $p < +\infty$, so $Pu \in L^p(\mathbb{R}^d)$ with $\|Pu\|^p_{L^p(\mathbb{R}^d)} = 2\|u\|^p_{L^p(\mathbb{R}^d_+)}$. If $p = +\infty$ we have $\|Pu\|_{L^\infty(\mathbb{R}^d)} = \|u\|_{L^\infty(\mathbb{R}^d_+)}$.

• For $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ we set $\sigma(x) = (-x_1, x_2, \dots, x_d)$. Let $j \in [\![2, d]\!]$. Let $\phi \in C_0^\infty(\mathbb{R}^d)$. If ϕ is supported in $\mathbb{R}^d_+$ we have

$$-\int_{\mathbb{R}^d} Pu\, \partial_j \phi \, dx = -\int_{\mathbb{R}^d_+} u\, \partial_j \phi \, dx = \int_{\mathbb{R}^d_+} \partial_j u \, \phi \, dx = \int_{\mathbb{R}^d} P(\partial_j u)\, \phi \, dx.$$

If ϕ is supported in $\mathbb{R}^d_-$ then, similarly,

$$\begin{aligned} -\int_{\mathbb{R}^d} Pu\, \partial_j \phi \, dx &= -\int_{\mathbb{R}^d_-} (u \circ \sigma)\, \partial_j \phi \, dx = -\int_{\mathbb{R}^d_+} u\, (\partial_j \phi \circ \sigma)\, dx \\ &= -\int_{\mathbb{R}^d_+} u\, \partial_j (\phi \circ \sigma)\, dx = \int_{\mathbb{R}^d_+} \partial_j u\, (\phi \circ \sigma)\, dx = \int_{\mathbb{R}^d_-} ((\partial_j u \circ \sigma) \phi \, dx \\ &= \int_{\mathbb{R}^d} P(\partial_j u)\, \phi \, dx. \end{aligned}$$

We consider the general case. Let $\chi \in C_0^\infty(\mathbb{R}, [0,1])$ be even, equal to 1 on [-1,1] and supported in]-2,2[. For $n \in \mathbb{N}$ and $x \in \mathbb{R}^d$ we set $\chi_n(x) = \chi(nx_1)$. Since $(1 - \chi_n)\phi$ is supported outside $\partial \mathbb{R}^d_+$ we have

$$-\int_{\mathbb{R}^d} Pu(1 - \chi_n) \partial_j \phi \, dx = -\int_{\mathbb{R}^d} Pu \partial_j \big((1 - \chi_n)\phi \big)\, dx = \int_{\mathbb{R}^d_+} P(\partial_j u)(1 - \chi_n) \phi \, dx.$$

By the dominated convergence theorem this yields

$$-\int_{\mathbb{R}^d} Pu\, \partial_j \phi \, dx = \int_{\mathbb{R}^d} P(\partial_j u) \phi \, dx.$$

This proves that in the weak sense we have $\partial_j(Pu) = P(\partial_j u)$. In particular $\partial_j(Pu) \in L^p(\mathbb{R}^d)$ with $\|\partial_j(Pu)\|_{L^p(\mathbb{R}^d)} = 2^{\frac{1}{p}} \|\partial_j u\|_{L^p(\mathbb{R}^d_+)}$.

• We proceed similarly for the first partial derivative. We observe that for $\phi \in C_0^\infty(\mathbb{R}^d)$ we now have $\partial_1(\phi \circ \sigma) = -(\partial_1 \phi) \circ \sigma$, so if ϕ is supported outside $\partial \mathbb{R}^d_+$ we now have

$$-\int_{\mathbb{R}^d} Pu\, \partial_1 \phi \, dx = \int_{\mathbb{R}^d} \tilde{P}(\partial_1 u) \phi \, dx.$$

On the other hand $(1 - \chi_n)$ does not commute with the partial derivative ∂_1. But the additional term is estimated as follows. Let $R > 0$ be such that ϕ is supported in $\mathbb{R} \times B_{d-1}(0, R)$ ($B_{d-1}(0, R)$ is the ball of radius R in $\mathbb{R}^{d-1}$). Since χ is even we have

$$\begin{aligned} \left| \int_{\mathbb{R}^d} Pu\, \partial_1 \chi_n \, \phi \, dx \right| &= \left| \int_{\mathbb{R}^d_+} u\, \partial_1 \chi_n \, (\phi - \phi \circ \sigma)\, dx \right| \\ &\leqslant n \|\chi'\|_\infty \int_{x_1=0}^{\frac{2}{n}} \int_{x' \in B_{d-1}(0,R)} \left| u(x_1, x') \right| \left| \phi(x_1, x') - \phi(-x_1, x') \right| \, dx' \, dx_1 \\ &\leqslant 4 \|\chi'\|_\infty \|\partial_1 \phi\|_\infty \int_{x_1=0}^{\frac{2}{n}} \int_{x' \in B_{d-1}(0,R)} \left| u(x_1, x') \right| \, dx' \, dx_1 \\ &\xrightarrow[n \to +\infty]{} 0. \end{aligned}$$

The conclusion follows as above. □

Theorem 2.31. *Let Ω be an open bounded subset of class C^1 in $\mathbb{R}^d$. Let $p \in [1,+\infty]$. Let $\mathcal{O}$ be an open subset of $\mathbb{R}^d$ such that $\overline{\Omega} \subset \mathcal{O}$. Then there exists a bounded linear operator $P : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^d)$ (which is also bounded for the norm of $L^p(\Omega)$) such that Pu is supported in $\mathcal{O}$ and $(Pu)_{|\Omega} = u$ for all $u \in W^{1,p}(\Omega)$.*

Proof. Let $u \in W^{1,p}(\Omega)$. We use the notation introduced in Paragraph 2.3.1. Without loss of generality we can assume that $\mathcal{U}_j \subset \mathcal{O}$ and $\mathcal{W}_j$ is symmetric with respect to $\partial\mathbb{R}^d_+$ for all $j \in [\![1,N]\!]$ (for instance $\mathcal{W}_j$ is a ball centered on $\partial\mathbb{R}^d_+$). We denote by v_0 the extension of u_0 by 0 on $\mathbb{R}^d$. We have $\|v_0\|_{W^{1,p}(\mathbb{R}^d)} = \|u_0\|_{W^{1,p}(\Omega)}$. Let $j \in [\![1,N]\!]$. We denote by $\tilde{v}_j$ the extension of $u_j \circ \Phi_j^{-1}$ on $\mathcal{W}_j$ given by Proposition 2.30. It is supported in a compact subset of $\mathcal{W}_j$, and $\tilde{v}_j \circ \Phi_j$ is compactly supported in $\mathcal{U}_j$. Then we denote by v_j the extension by 0 of $\tilde{v}_j \circ \Phi_j$ on $\mathbb{R}^d$. By Propositions 2.28 and 2.30 and Remark 2.16 there exist constants $C_\Phi, C_{\Phi^{-1}}, C_P, C_{\chi_j} > 0$ independant of u such that

$$\begin{aligned}\|v_j\|_{W^{1,p}(\mathbb{R}^d)} &= \|\tilde{v}_j \circ \Phi_j\|_{W^{1,p}(\mathcal{U}_j)} \leqslant C_\Phi \|\tilde{v}_j\|_{W^{1,p}(\mathcal{W}_j)} \leqslant C_P C_\Phi \left\|u_j \circ \Phi^{-1}\right\|_{W^{1,p}(\mathcal{W}_j \cap \mathbb{R}^d_+)} \\ &\leqslant C_P C_\Phi C_{\Phi^{-1}} \|u_j\|_{W^{1,p}(\mathcal{U}_j \cap \Omega)} \leqslant C_P C_\Phi C_{\Phi^{-1}} C_{\chi_j} \|u\|_{W^{1,p}(\Omega)}.\end{aligned}$$

Finally we set $Pu = \sum_{j=0}^N v_j$, and $Pu \in W^{1,p}(\mathbb{R}^d)$ satisfies all the required properties. □

We recall that given an open subset Ω of $\mathbb{R}^d$, we denote by $C_0^\infty(\overline{\Omega})$ the set of restrictions to Ω of functions in $C_0^\infty(\mathbb{R}^d)$. When $k = 1$, the following result follows from Theorem 2.31 (applied with $\mathcal{O} = \mathbb{R}^d$) and Theorem 2.23. For the general case we can extend Theorem 2.31 to the case $k \geqslant 2$, we can also give a direct proof.

Proposition 2.32. *Let $p \in [1,+\infty[$ and $k \in \mathbb{N}$. Let Ω be equal to $\mathbb{R}^d_+$ or be a bounded open subset of $\mathbb{R}^d$. Let $u \in W^{k,p}(\mathbb{R}^d_+)$. There exists a sequence $(u_n)_{n\in\mathbb{N}}$ of functions in $C_0^\infty(\overline{\Omega})$ such that*

$$\|u_n - u\|_{W^{k,p}(\Omega)} \xrightarrow[n\to+\infty]{} 0.$$

Proof. We prove the case Ω bounded. The case $\Omega = \mathbb{R}^d_+$ is more direct and is left as an exercice. Let $w \in \partial\Omega$. We use the notation of Paragraph 2.3.1. Let $u \in W^{1,p}(\Omega)$ be supported in $\overline{\Omega} \cap \mathcal{U}$. We denote by $\tilde{u}$ the extension of u by 0 on $\mathbb{R}^d$. For $\tau > 0$ we set

$$\mathcal{U}_\tau = \left\{ \sum_{j=1}^d x_j \beta_j, (x_2, \dots, x_d) \in \mathcal{O}, x_1 \in]a, b-\tau[\right\}.$$

There exists $\tau_0 > 0$ such that $\operatorname{supp}(u) \subset \mathcal{U}_\tau$. For $\tau \in]0,\tau_0]$ and $x \in \mathcal{U}_\tau$ we set

$$u_\tau(x) = u(x + \tau\beta_1).$$

We extend u_τ by 0 on $\mathcal{U} \backslash \mathcal{U}_\tau$. The restriction of u_τ to $\mathcal{U} \cap \Omega$ is in $W^{k,p}(\mathcal{U} \cap \Omega)$ and the derivatives of u_τ up to order k are the translations of the corresponding derivatives of u:

$$\partial^\alpha(u_\tau) = (\partial^\alpha u)_\tau.$$

By continuity in $L^p(\mathbb{R}^d)$ of the translation we have

$$\|u_\tau - u\|^p_{W^{k,p}(\mathcal{U}\cap\Omega)} = \sum_{|\alpha|\leqslant k} \|\partial^\alpha u_\tau - \partial^\alpha u\|^p_{L^p(\mathcal{U}\cap\Omega)} = \sum_{|\alpha|\leqslant k} \|v^\alpha_\tau - v^\alpha\|^p_{L^p(\mathbb{R}^d)} \xrightarrow[\tau\to 0]{} 0,$$

where we have denoted by v^α_τ and v^α the extensions by 0 of $\partial^\alpha u_\tau$ and $\partial^\alpha u$ (then $v^\alpha_\tau(x) = v^\alpha(x + \tau\beta_1)$ almost everywhere on $\mathbb{R}^d$).

Now let $\tau \in]0,\tau_0]$ be fixed. There exists $\eta_0 > 0$ if we set

$$\mathcal{V} = \bigcup_{x \in \operatorname{supp}(u_\tau)\cap\Omega} B(x,\eta_0),$$

then for all $y \in \mathcal{V}$ we have $y + \tau\beta_1 \in \mathcal{U} \cap \Omega$. Let $\rho \in C_0^\infty(\mathbb{R}^d,[0,1])$ be supported in $B(0,1)$ and such that $\int_{\mathbb{R}^d} \rho = 1$. For $\eta \in]0,\eta_0]$ and $x \in \mathbb{R}^d$ we set $\rho_\eta(x) = \eta^{-d}\rho(x/\eta)$. For $\eta \in]0,\eta_0]$ we set $u^\eta_\tau = \rho_\eta * u_\tau$. Its restriction to $\mathcal{U} \cap \Omega$ belongs to $C_0^\infty(\overline{\Omega})$. Since $u_\tau \in W^{k,p}(\mathcal{V})$ we can prove as in the proof of Theorem 2.23 that

$$\|u^\eta_\tau - u_\tau\|_{W^{k,p}(\mathcal{U}\cap\Omega)} \xrightarrow[\eta\to 0]{} 0.$$

It remains to chose $\tau > 0$ small enough and then $\eta > 0$ small enough to conclude. □

2.4 Sobolev inequalities

In this section we prove some inclusions between Sobolev spaces. The inclusions between Lebesgue spaces are already known. In particular we know that $L^p(\mathbb{R}^d)$ is never included in $L^q(\mathbb{R}^d)$ if $p \neq q$. The purpose here is to prove that if we add information about the derivatives then we get better results. In particular we will prove (continuous) inclusions of the form $W^{1,p}(\mathbb{R}^d) \subset L^q(\mathbb{R}^d)$ for suitable pairs (p,q).

As for Lebesgue spaces, we get stronger results on a bounded domain Ω. In this case we will prove compact inclusions. For instance, $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$. This means that if a sequence of functions in $H^1(\Omega)$ is bounded, then it has a convergent subsequence for the $L^2(\Omega)$ norm. This result will be of great importance for the analysis of PDEs. We will already use this fact in the following section (see the proof of Theorem 2.49).

2.4.1 Morrey's inequality

We have already seen in Proposition 2.6 that if I is an interval of $\mathbb{R}$ then for any $p \in [1,+\infty]$ we have $W^{1,p}(I) \subset L^\infty(I)$ (and that a function in $W^{1,p}(I)$ is also Hölder-continuous if $p > 1$). This result only holds in dimension 1. Indeed if $1 \leqslant p < d$ then for $\alpha \in \left] -\frac{d}{p}+1, 0\right[$ the function $x \mapsto |x|^\alpha$ belongs to $W^{1,p}(B(0,1))$ but not to $L^\infty(B(0,1))$. The purpose of the following theorem is to prove that we recover a result analogous to Proposition 2.6 if $p > d$.

***Exercise* 12.** Find $u \in W^{1,d}(\mathbb{R}^d)$ such that $u \notin L^\infty(\mathbb{R}^d)$.

Theorem 2.33 (Morrey's inequality)**.** *Let $p \in]d,+\infty[$. We have*

$$W^{1,p}(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d) \cap C^{0,1-\frac{d}{p}}(\mathbb{R}^d)$$

and there exists $C > 0$ such that, for $u \in W^{1,p}(\mathbb{R}^d)$,

$$\|u\|_{L^\infty(\mathbb{R}^d)} \leqslant C \|u\|_{W^{1,p}(\mathbb{R}^d)},$$

$$\forall x_1, x_2 \in \mathbb{R}^d, \quad |u(x_1) - u(x_2)| \leqslant C |x_1 - x_2|^{1-\frac{d}{p}} \|\nabla u\|_{L^p(\mathbb{R}^d)}.$$

In dimension 1, we have used the fundamental theorem of calculus to compare $u(x)$ to $u(x_0)$ for some fixed x_0. It gave a one-dimensional integral which was controled by the norm of u'. In higher dimension we can still write the fundamental theorem of calculus for regular functions but the corresponding one-dimensional integral is not controled by the d-dimensional integral which defines the norm of ∇u. The trick in the following proof is to compare $u(x)$ to the mean value of u on an open subset of $\mathbb{R}^d$. This will give a d-dimensional integral controled as stated in the theorem.

Proof. • We consider $u \in C_0^\infty(\mathbb{R}^d)$. The general case will follow by density. Let $x \in \mathbb{R}^d$ and let $\mathcal{O}$ be an open subset of $\mathbb{R}^d$. We set

$$\delta(x,\mathcal{O}) = \sup_{y \in \mathcal{O}} |y - x|.$$

For $y \in \mathcal{O}$ and $h = (h_1, \dots, h_d) = y - x$ we have

$$\begin{aligned}
|u(y) - u(x)| &\leqslant \int_0^1 \left| \frac{d}{dt} u(x+th) \right| dt \\
&\leqslant \int_0^1 \sum_{j=1}^d |h_j| \, |\partial_j u(x+th)| \, dt \\
&\leqslant \delta(x,\mathcal{O}) \sum_{j=1}^d \int_0^1 |\partial_j u(x+th)| \, dt.
\end{aligned}$$

For $t \in]0,1]$ we set

$$t(\mathcal{O} - x) = \{t(y-x), y \in \mathcal{O}\}.$$

If we set

$$u_{\mathcal{O}} = \frac{1}{|\mathcal{O}|} \int_{\mathcal{O}} u(y)\, dy,$$

then we have

$$\begin{aligned} |u(x) - u_{\mathcal{O}}| &\leqslant \frac{1}{|\mathcal{O}|} \int_{y\in\mathcal{O}} |u(x) - u(y)|\ dy \\ &\leqslant \frac{\delta(x,\mathcal{O})}{|\mathcal{O}|} \int_{h\in(\mathcal{O}-x)} \int_0^1 \sum_{j=1}^d |\partial_j u(x+th)|\ dt\, dh \\ &\leqslant \frac{\delta(x,\mathcal{O})}{|\mathcal{O}|} \int_0^1 \frac{1}{t^d} \sum_{j=1}^d \int_{\eta\in t(\mathcal{O}-x)} |\partial_j u(x+\eta)|\ d\eta\, dt. \end{aligned}$$

By the Hölder inequality we have for $t \in [0,1]$

$$\begin{aligned} \sum_{j=1}^d \int_{t(\mathcal{O}-x)} |\partial_j u(x+\eta)|\ d\eta &\leqslant \sum_{j=1}^d \left(\int_{t(\mathcal{O}-x)} |\partial_j u(x_1+\eta)|^p\ d\eta \right)^{\frac{1}{p}} |t(\mathcal{O}-x)|^{\frac{p-1}{p}} \\ &\leqslant t^{\frac{d(p-1)}{p}} |\mathcal{O}|^{\frac{p-1}{p}} \|\nabla u\|_{L^p(\mathbb{R}^d)}, \end{aligned}$$

so

$$|u(x) - u_{\mathcal{O}}| \leqslant \delta(x,\mathcal{O})\, |\mathcal{O}|^{-\frac{1}{p}} \|\nabla u\|_{L^p(\mathbb{R}^d)} \int_0^1 t^{-\frac{d}{p}}\, dt = \frac{\delta(x,\mathcal{O})\, |\mathcal{O}|^{-\frac{1}{p}} \|\nabla u\|_{L^p(\mathbb{R}^d)}}{1-\frac{d}{p}}. \tag{2.15}$$

- Now let $x_1, x_2 \in \mathbb{R}^d$ and let $\mathcal{O}$ be the open ball with diameter $[x_1, x_2]$. We have $\delta(x_1,\mathcal{O}) = \delta(x_2,\mathcal{O}) = |x_1 - x_2|$ and $|\mathcal{O}| = \frac{c_d}{2^d} |x_1 - x_2|^d$ where c_d is the size of the unit ball in $\mathbb{R}^d$. Thus

$$|u(x_1) - u(x_2)| \leqslant |u(x_1) - u_{\mathcal{O}}| + |u(x_2) - u_{\mathcal{O}}| \leqslant \frac{2^{1+\frac{d}{p}} c_d^{-\frac{1}{p}}}{1-\frac{d}{p}} |x_1 - x_2|^{1-\frac{d}{p}} \|\nabla u\|_{L^p(\mathbb{R}^d)}.$$

This gives the second statement. Now for $x \in \mathbb{R}^d$ we apply (2.15) with $\mathcal{O} = B(x,1)$, the ball of center x and radius 1. The Hölder inequality gives

$$|u_{\mathcal{O}}| \leqslant c_d^{-\frac{1}{p}} \|u\|_{L^p(\mathbb{R}^d)},$$

so

$$|u(x)| \leqslant c_d^{-\frac{1}{p}} \left(\|u\|_{L^p(\mathbb{R}^d)} + \frac{1}{1-\frac{d}{p}} \|\nabla u\|_{L^p(\mathbb{R}^d)} \right).$$

This completes the proof. □

2.4.2 Gagliardo-Nirenberg Inequality

In this paragraph we consider the case $p \leqslant d$. This is particularly interesting for the common case $p = 2$.

We want to prove that if we control ∇u in some Lebesgue space, then we can control u in another Lebesgue space. Assume that there exists $q \in [1, +\infty[$ and $C > 0$ such that

$$\forall v \in C_0^\infty(\mathbb{R}^d), \quad \|v\|_{L^q(\mathbb{R}^d)} \leqslant C \|\nabla v\|_{L^p(\mathbb{R}^d)}. \tag{2.16}$$

Let $u \in W^{1,p}(\mathbb{R}^d) \setminus \{0\}$. For $\lambda > 0$ and $x \in \mathbb{R}^d$ we set $u_\lambda(x) = u(\lambda x)$. Then for all $\lambda > 0$ we have

$$\lambda^{-\frac{d}{q}} \|u\|_{L^q(\mathbb{R}^d)} = \|u_\lambda\|_{L^q(\mathbb{R}^d)} \leqslant C \|\nabla u_\lambda\|_{L^p(\mathbb{R}^d)} = C\lambda^{1-\frac{d}{p}} \|u\|_{L^p(\mathbb{R}^d)}.$$

Letting λ go to 0 or to $+\infty$ we see that we necessarily have

$$-\frac{d}{q} = 1 - \frac{d}{p}. \tag{2.17}$$

In the following theorem we prove that if (2.17) holds then we indeed have (2.16). For $p \in [1, d[$ we define $p^* \in [1, +\infty[$ by

$$p^* = \frac{pd}{d-p}, \quad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{d}. \tag{2.18}$$

Notice that we have $p^* > p$ and $p^* \to +\infty$ if $p \to d$.

Theorem 2.34 (Gagliardo-Nirenberg-Sobolev inequality)**.** *Let $p \in [1, d[$ and let p^* be defined by (2.18). There exists $C > 0$ such that for all $u \in C_0^1(\mathbb{R}^d)$ we have*

$$\|u\|_{L^{p^*}(\mathbb{R}^d)} \leqslant C \|\nabla u\|_{L^p(\mathbb{R}^d)}.$$

Proof. • Let $u \in C_0^1(\mathbb{R}^d)$. For $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ and $j \in [\![1, d]\!]$ we have

$$|u(x)| = \left| \int_{-\infty}^{x_j} \partial_j u(x_1, \dots, x_{j-1}, t, x_{j+1}, \dots, x_d)\, dt \right| \leqslant v_j(\tilde{x}_j)^{d-1}$$

where $\tilde{x}_j = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_d)$ and

$$v_j(\tilde{x}_j) = \left(\int_{\mathbb{R}} |\nabla u(x_1, \dots, x_{j-1}, t, x_{j+1}, \dots, x_d)|\, dt \right)^{\frac{1}{d-1}}.$$

This gives

$$|u(x)|^{\frac{d}{d-1}} \leqslant \prod_{j=1}^{d} v_j(\tilde{x}_j).$$

Now we prove by induction on $d \geqslant 2$ that if we set

$$v : x \in \mathbb{R}^d \mapsto \prod_{j=1}^{d} v_j(\tilde{x}_j),$$

then we have

$$\|v\|_{L^1(\mathbb{R}^d)} \leqslant \prod_{j=1}^{d} \|\tilde{v}_j\|_{L^{d-1}(\mathbb{R}^{d-1})}. \tag{2.19}$$

The case $d = 2$ is easy. Assume that (2.19) is true up to the dimension $d-1$ for some $d \geqslant 3$. We fix $x_1 \in \mathbb{R}$ and see v as a function of $x' = (x_2, \dots, x_d)$. By the Hölder inequality we have

$$\int_{\mathbb{R}^{d-1}} v(x_1, x')\, dx' \leqslant \|\tilde{v}_1\|_{L^{d-1}(\mathbb{R}^{d-1})} \left(\int_{\mathbb{R}^{d-1}} \prod_{j=2}^{d} \tilde{v}_j(x_1, \tilde{x}'_j)^{\frac{d-1}{d-2}}\, dx_2 \dots dx_d \right)^{\frac{d-2}{d-1}},$$

where for $j \in [\![2, d]\!]$ we have set $\tilde{x}'_j = (x_2, \dots, x_{j-1}, x_{j+1}, \dots, x_d)$. The induction assumption gives

$$\int_{\mathbb{R}^{d-1}} \prod_{j=2}^{d} \tilde{v}_j(x_1, \tilde{x}'_j)^{\frac{d-1}{d-2}}\, dx' \leqslant \prod_{j=2}^{d} \left(\int_{\mathbb{R}^{d-2}} \tilde{v}_j(x_1, \tilde{x}'_j)^{d-1}\, d\tilde{x}'_j \right)^{\frac{1}{d-2}}$$

and hence

$$\int_{\mathbb{R}^{d-1}} v(x_1, x')\, dx' \leqslant \|\tilde{v}_1\|_{L^{d-1}(\mathbb{R}^{d-1})} \prod_{j=2}^{d} \left(\int_{\mathbb{R}^{d-2}} \tilde{v}_j(x_1, \tilde{x}'_j)^{d-1}\, d\tilde{x}'_j \right)^{\frac{1}{d-1}}.$$

After integration over $x_1 \in \mathbb{R}$ we get, by the Hölder inequality,

$$\int_{\mathbb{R}^d} v(x)\, dx \leqslant \|\tilde{v}_1\|_{L^{d-1}(\mathbb{R}^{d-1})} \prod_{j=2}^{d} \left(\int_{\mathbb{R}^{d-1}} \tilde{v}_j(\tilde{x}_j)^{d-1}\, d\tilde{x}_j \right)^{\frac{1}{d-1}}.$$

This is (2.19). We deduce

$$\int_{\mathbb{R}^d} |u(x)|^{\frac{d}{d-1}}\, dx \leqslant \left(\int_{\mathbb{R}^d} |\nabla u(x)|\, dx \right)^{\frac{d}{d-1}},$$

which gives the result for $u \in C_0^1(\mathbb{R}^d)$ when $p = 1$.
• Let $\gamma > 1$. The case $p = 1$ applied to $|u|^{\gamma-1} u$ (still in $C_0^1(\mathbb{R}^d)$, with gradient $\gamma |u|^{\gamma-1} \nabla u$) gives

$$\left(\int_{\mathbb{R}^d} |u|^{\frac{\gamma d}{d-1}} \right)^{\frac{d-1}{d}} \leqslant \gamma \int_{\mathbb{R}^d} |u|^{\gamma-1} |\nabla u| \, dx \leqslant \gamma \left(\int_{\mathbb{R}^d} |u|^{\frac{(\gamma-1)d}{d-1}} \, dx \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^d} |\nabla u|^p \, dx \right)^{\frac{1}{p}}. \tag{2.20}$$

If we choose

$$\gamma = \frac{p(d-1)}{d-p} > 1$$

we have

$$\frac{\gamma d}{d-1} = \frac{(\gamma-1)p}{p-1} = \frac{dp}{d-p} = p^*,$$

and the conclusion follows for $u \in C_0^1(\mathbb{R}^d)$. The general case $u \in W^{1,p}(\mathbb{R}^d)$ follows by density. □

In Theorem 2.34 we have only used the fact that $\nabla u \in L^p(\mathbb{R}^d)$. If u is also in $L^p(\mathbb{R}^d)$ we have better conclusions. We know that $L^p(\mathbb{R}^d) \cap L^{p^*}(\mathbb{R}^d) \subset L^q(\mathbb{R}^d)$ (with continuous inclusion) for any $q \in [p, p^*]$. This is the first statement of the following theorem. The second statement is about the limit case $p = d$. Notice that Theorem 2.34 does not hold with $p = d$ and $p^* = +\infty$ (see Exercise 12), but for $u \in W^{1,d}(\mathbb{R}^d)$ we have a result similar to the case $p < d$.

Theorem 2.35. (i) *Let $p \in [1, d[$. Then for all $q \in [p, p^*]$ we have $W^{1,p}(\mathbb{R}^d) \subset L^q(\mathbb{R}^d)$ with continuous injection.*

(ii) *For all $q \in [d, +\infty[$ we have $W^{1,d}(\mathbb{R}^d) \subset L^q(\mathbb{R}^d)$ with continuous injection.*

Proof. We prove the second statement. We prove by induction on $\gamma \geqslant d-1$ that for $q \in \left[d, \frac{\gamma d}{d-1}\right]$ there exists $C_q > 0$ such that, for all $u \in C_c^1(\mathbb{R}^d)$,

$$\|u\|_{L^q(\mathbb{R}^d)} \leqslant C_q \|u\|_{W^{1,d}(\mathbb{R}^d)}. \tag{2.21}$$

The result will follow by density. (2.21) is clear when $\gamma = d-1$. We assume that it is proved up to $\gamma - 1$ for some $\gamma \geqslant d$. Let $u \in C_c^1(\mathbb{R}^d)$. We use estimate (2.20) from the previous proof with $p = d$. With the induction assumption this gives

$$\|u\|^{\gamma}_{L^{\frac{\gamma d}{d-1}}(\mathbb{R}^d)} \leqslant \gamma \|u\|^{\gamma-1}_{L^{\frac{(\gamma-1)d}{d-1}}(\mathbb{R}^d)} \|\nabla u\|_{L^d(\mathbb{R}^d)} \leqslant \gamma C^{\gamma-1}_{\frac{(\gamma-1)d}{d-1}} \|u\|^{\gamma}_{W^{1,d}(\mathbb{R}^d)}.$$

This gives (2.21) for $q = \frac{\gamma d}{d-1}$. The case $q \in \left[d, \frac{\gamma d}{d-1}\right]$ follows since u belongs to $L^d(\mathbb{R}^d)$. □

2.4.3 Sobolev embeddings on a bounded domain

So far we have only proved inclusions between Sobolev spaces on $\mathbb{R}^d$. Our purpose in this paragraph is to prove analogous results for Sobolev spaces on a bounded open subset Ω. For this, we will use the extension operator of Theorem 2.31 to deduce inequalities on Ω from their analogs on $\mathbb{R}^d$.

However, as said in introduction, we will get better results on Ω. For instance we recall that $L^p(\Omega) \subset L^q(\Omega)$ if $p > q$. This will automatically improve the result of Theorem 2.34 (in particular the discussion before Theorem 2.34 is not valid on a bounded domain).

Another very important difference between the case of $\mathbb{R}^d$ and the case of a bounded domain is that some inclusions will be not only continuous but also compact.

Definition 2.36. Let X and Y be Banach spaces. A bounded linear operator $T : X \to Y$ is said to be compact if for any bounded sequence $(u_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$, the sequence $(Tu_n)_{n\in\mathbb{N}}$ has a convergent subsequence in Y. Equivalently, T is compact if $\overline{T(B_X)}$ is compact in Y, where B_X is the unit ball in X.

Compactness for a set of functions is usually given by the Ascoli-Arzelá Theorem, which we recall now.

Theorem 2.37 (Ascoli-Arzelá Theorem)**.** *Let K be a compact metric space and let $\mathcal{F}$ be a bounded subset of $C(K,\mathbb{R})$. We assume that $\mathcal{F}$ is equicontinuous:*

$$\forall \varepsilon > 0, \exists \delta > 0, \forall f \in \mathcal{F}, \forall x, y \in K, \quad d(x,y) \leqslant \delta \implies |f(x) - f(y)| \leqslant \varepsilon.$$

Then the closure $\overline{\mathcal{F}}$ of $\mathcal{F}$ in $C(K)$ is compact.

The results of Theorem 2.35 and 2.33 are extended to bounded domains as follows.

Theorem 2.38. *Let Ω be a bounded open subset of class C^1 in $\mathbb{R}^d$. Let $p \in [1,+\infty]$. Then we have the following compact inclusions.*

(i) *If $p < d$ then for all $q \in [1, p^*[$ we have $W^{1,p}(\Omega) \subset\subset L^q(\Omega)$.*

(ii) *For all $q \in [d,+\infty[$ we have $W^{1,d}(\Omega) \subset\subset L^q(\Omega)$.*

(iii) *If $p > d$ then we have $W^{1,p}(\Omega) \subset\subset C^0(\overline{\Omega})$.*

In particular we always have $W^{1,p}(\Omega) \subset\subset L^p(\Omega)$.

Proof of Theorem 2.38. • We begin with the last case. By the extension Theorem 2.31, we can see functions in $W^{1,p}(\Omega)$ as functions in $W^{1,p}(\mathbb{R}^d)$ supported in some fixed compact of $\mathbb{R}^d$. If $p < +\infty$, the conclusion follows from the Morrey inequality (Theorem 2.33) and the Ascoli-Arzelá Theorem 2.37. Since $W^{1,+\infty}(\Omega)$ is continuously embedded in $W^{1,p}(\Omega)$ for any $p \in]d,+\infty[$, it is also compactly embedded in $C^0(\overline{\Omega})$.
• Assume that (i) is proved and let $q \in [d,+\infty[$. Then there exists $p \in [1,d[$ such that $q < p^*$. Then we have

$$W^{1,d}(\Omega) \subset W^{1,p}(\Omega) \subset\subset L^q(\Omega),$$

where the first inclusion is continuous (since Ω is bounded) and the second is compact by (i). Thus it only remains to prove (i).
• Let $q \in [1,p^*[$. We consider a sequence $(u_n)_{n\in\mathbb{N}}$ bounded in $W^{1,p}(\Omega)$. As above, we identify this sequence with a sequence (still denoted by $(u_n)_{n\in\mathbb{N}}$) bounded in $W^{1,p}(\mathbb{R}^d)$ such that the functions u_n are supported in the same bounded open subset $\mathcal{U}$. Let $\rho \in C_0^\infty(\mathbb{R}^d,\mathbb{R}_+)$ be supported in the unit ball and such that $\int_{\mathbb{R}^d} \rho = 1$. For $\varepsilon > 0$ and $x \in \mathbb{R}^d$ we set $\rho_\varepsilon = \frac{1}{\varepsilon^d}\rho\big(\frac{x}{\varepsilon}\big)$, and then $u_n^\varepsilon = \rho_\varepsilon * u_n \in C^\infty(\mathbb{R}^d)$. Let $\varepsilon > 0$. For $n \in \mathbb{N}$ and $x \in \mathbb{R}^d$ we have

$$|u_n^\varepsilon(x)| \leqslant \|\rho_\varepsilon\|_{L^\infty(\mathbb{R}^d)} \|u_n\|_{L^1(\mathcal{U})}$$

and

$$|\nabla u_n^\varepsilon(x)| \leqslant \|\nabla\rho_\varepsilon\|_{L^\infty(\mathbb{R}^d)} \|u_n\|_{L^1(\mathcal{U})},$$

so the sequence $(u_n^\varepsilon)_{n\in\mathbb{N}}$ is bounded in $C^0(\mathbb{R}^d)$ and uniformly equicontinuous. Moreover the functions u_n^ε are supported in a common bounded set $\mathcal{V}$ of $\mathbb{R}^d$, so by the Ascoli-Arzelá Theorem 2.37 there exists a subsequence $(u_{n_k}^\varepsilon)_{k\in\mathbb{N}}$ which converges uniformly in $\mathcal{V}$ and hence in $\mathcal{U}$. This gives

$$\limsup_{j,k\to+\infty} \left\|u_{n_j}^\varepsilon - u_{n_k}^\varepsilon\right\|_{L^q(\mathcal{U})} = 0.$$

• We already know that u_n^ε goes to u_n as $\varepsilon \to 0$ in $L^q(\mathcal{U})$ for all $n \in \mathbb{N}$. We prove that this convergence is uniform with respect to n. Let $v \in C_0^1(\mathbb{R}^d)$ be supported in $\mathcal{U}$. For $\varepsilon > 0$ we set $v_\varepsilon = \rho_\varepsilon * v$. Then for $x \in \mathbb{R}^d$ we have

$$v_\varepsilon(x) - v(x) = \int_{B(0,1)} \rho(y)\big(v(x-\varepsilon y) - v(x)\big)\,dy = -\varepsilon \int_{B(0,1)} \rho(y) \int_0^1 \nabla v(x - \varepsilon t y)\cdot y\,dt\,dy,$$

and hence

$$\begin{aligned}\|v_\varepsilon - v\|_{L^1(\mathcal{U})} &= \int_{\mathcal{U}} |v_\varepsilon(x) - v(x)|\,dx \leqslant \varepsilon \int_{B(0,1)} \rho(y) \int_0^1 \int_{\mathcal{U}} |\nabla v(x-\varepsilon t y)|\,dx\,dt\,dy \\ &\leqslant \varepsilon \|\nabla v\|_{L^1(\mathcal{U})}.\end{aligned} \tag{2.22}$$

By density, the same estimate holds for any $v \in W^{1,p}(\mathbb{R}^d)$ supported in $\mathcal{U}$ (note that if $v_m \in C^1(\mathbb{R}^d)$ goes to v in $W^{1,p}(\mathbb{R}^d)$ then $\rho_\varepsilon * v_n$ goes to $\rho_\varepsilon * v$ in $L^1(\mathbb{R}^d)$). Let $\theta \in]0,1[$ be such that

$$\frac{1}{q} = \theta + \frac{1-\theta}{p^*}.$$

By (2.22) applied with $v = u_n$ and the Gagliardo-Nirenberg inquality (Theorem 2.34 there exists $C > 0$ independant on u, n or ε such that

$$\|u_n^\varepsilon - u_n\|_{L^q(\mathcal{U})} \leqslant \|u_n^\varepsilon - u_n\|_{L^1(\mathcal{U})}^\theta \|u_n^\varepsilon - u_n\|_{L^{p^*}(\mathbb{R}^d)}^{1-\theta} \leqslant C\varepsilon^\theta \|\nabla u_n\|_{L^p(\mathcal{U})}.$$

This proves that u_n^ε goes to u_n in $L^q(\mathcal{U})$ as $\varepsilon \to 0$ uniformly with respect to $n \in \mathbb{N}$. Then for any $\eta > 0$ we get

$$\limsup_{j,k\to+\infty} \|u_{n_j} - u_{n_k}\|_{L^q(\mathcal{U})} \leqslant \eta.$$

Using a standard diagonal argument, we obtain a subsequence which goes to 0 in $L^q(\mathcal{U})$ and hence in $L^q(\Omega)$. □

***Exercise* 13.** Let $p \in [1, d[$. Prove that we have the continuous inclusion $W^{1,p}(B(0,1)) \subset L^{p^*}(B(0,1))$, but that this inclusion is not compact.

2.5 Traces

We recall that functions in the Sobolev spaces are not really functions, but equivalence classes of functions pairwise almost everywhere equal. In particular, for u in some Sobolev space $W^{k,p}(\Omega)$, it does not make sense to consider the value of u at some point $x_0 \in \Omega$.

We have seen in Proposition 2.6 that, in dimension 1, an element u of $W^{1,p}(I)$ has a continuous reprentative $\tilde{u}$. It is reasonnable to consider $\tilde{u}(x_0)$ as the value of u at x_0. Indeed, if $\tilde{v}$ is another representative of u then $\tilde{v}(x_0)$ can be far from $\tilde{u}(x_0)$, but for almost all $x \in I$ "close to x_0" then $\tilde{v}(x)$ is equal to $\tilde{u}(x)$ and hence "close to $\tilde{u}(x_0)$".

However, this possible definition only works in dimension 1, since in higher dimension an element of $W^{1,p}(\Omega)$ does not necessarily have a continuous representative.

In applications, it is not crucial to give the value of a function at a point, but we are interested in what happens at the boundary of the domain. This will be important for instance for integration by parts (Green formula in higher dimension), where the value of the function at the boundary appears. For regular domains, the boundary is a submanifold of dimension $(d-1)$. This is still of dimension 0 for the Lebesgue measure on Ω, but if $d \geqslant 2$ this is in some sense "bigger" than a point.

Our purpose in this section is the following. Given a regular open subset Ω of $\mathbb{R}^d$ and $u \in W^{1,p}(\Omega)$, we want to give a natural sense to the restriction of u on the boundary $\partial\Omega$, in such a way that if u belongs to $C^0(\overline{\Omega})$ then the new definition coincides with the usual one.

2.5.1 Trace

As explained in the previous section, we begin our analysis with the model case $\Omega = \mathbb{R}^d_+$ and then, using a partition of unity and changes of variables, we will give a more general result.

Proposition 2.39. *Let $p \in [1, +\infty[$. There exists $C > 0$ such that for $u \in C_0^\infty(\overline{\mathbb{R}^d_+})$ we have*

$$\|u(0,\cdot)\|_{L^p(\mathbb{R}^{d-1})}^p \leqslant C \|u\|_{W^{1,p}(\mathbb{R}^d_+)}^p.$$

For the proof we only have to integrate over $\mathbb{R}^{d-1}$ the one-dimensional case which is very close to Proposition 2.6:

Proof. For $x' \in \mathbb{R}^{d-1}$ we have

$$\left|u(0,x')\right|^p \leqslant p \int_0^{+\infty} \left|\partial_{x_1} u(s,x')\right| \left|u(s,x')\right|^{p-1} \, ds$$

so, by the Hölder and Young inequalities,

$$\begin{aligned}\left|u(0,x')\right|^p &\leqslant p \left(\int_0^{+\infty} \left|\partial_{x_1} u(s,x')\right|^p \, ds \right)^{\frac{1}{p}} \left(\int_0^{+\infty} \left|u(s,x')\right|^p \, ds \right)^{\frac{p-1}{p}} \\ &\leqslant \int_0^{+\infty} \left|\partial_{x_1} u(s,x')\right|^p \, ds + (p-1) \int_0^{+\infty} \left|u(s,x')\right|^p \, ds.\end{aligned}$$

After integration over $x' \in \mathbb{R}$ we get

$$\|u(0,\cdot)\|_{L^p(\mathbb{R}^{d-1})}^p \leqslant (p-1) \|u\|_{L^p(\mathbb{R}^d_+)}^p + \|\partial_{x_1} u\|_{L^p(\mathbb{R}^d)}^p ,$$

and the conclusion follows. □

Theorem 2.40. *Let Ω be an open subset of $\mathbb{R}^d$ of class C^1. Let $p \in [1,+\infty[$. There is a unique bounded linear operator*

$$\gamma_0 : W^{1,p}(\Omega) \to L^p(\partial\Omega)$$

such that

$$\forall u \in W^{1,p}(\Omega) \cap C^0(\overline{\Omega}), \quad \gamma_0(u) = u|_{\partial\Omega}.$$

Proof. Let $u \in C_0^\infty(\overline{\Omega})$. We use the notation of Paragraph 2.3.1. Let $j \in [\![1,N]\!]$. We have

$$\begin{aligned}\int_{\partial\Omega\cap\mathcal{U}_j} |u|^p \, d\sigma &= \int_{\mathcal{O}_j} \left|u(\tilde{\varphi}(x'))\right|^p \sqrt{1 + |\nabla\varphi(x')|^2} \, dx' \\ &\leqslant C_\varphi \int_{\mathcal{O}_j} \left|u(\tilde{\varphi}(x'))\right|^p \, dx' = C_\varphi \int_{\partial\mathbb{R}^d_+} \left|(u \circ \Phi^{-1})\right|^p \, dx',\end{aligned}$$

where $C_\varphi = \sup_{x'\in\mathcal{O}} \sqrt{1 + |\nabla\varphi(x')|^2}$ and $(u \circ \Phi^{-1})$ has been extended by 0 on $\mathbb{R}^d_+$. By Propositions 2.39 and 2.28 there exists $C_j > 0$ independant of u such that

$$\int_{\partial\Omega\cap\mathcal{U}_j} |u|^p \, d\sigma \leqslant C_\varphi C \left\|u \circ \Phi^{-1}\right\|_{\mathcal{W}_j\cap\mathbb{R}^d_+}^p \leqslant C_j \|u_j\|_{W^{1,p}(\Omega)}^p .$$

Then,

$$\left\|u_{|\partial\Omega}\right\|_{L^p(\Omega)} \leqslant \sum_{j=1}^N \left\|u_{j\,|\partial\Omega}\right\|_{L^p(\Omega)} \leqslant \sum_{j=1}^N C_j \|u_j\|_{W^{1,p}(\Omega)} .$$

Finally, there exists $C > 0$ such that for all $u \in C_0^\infty(\overline{\Omega})$ we have

$$\left\|u_{|\partial\Omega}\right\|_{L^p(\partial\Omega)} \leqslant C \|u\|_{W^{1,p}(\Omega)} .$$

Since $C_0^\infty(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$, the map $u \in C_0^\infty(\overline{\Omega}) \mapsto u_{|\partial\Omega} \in L^p(\partial\Omega)$ extends to a unique continuous map on $W^{1,p}(\Omega)$. Moreover, if $u \in W^{1,p}(\Omega) \cap C^0(\overline{\Omega})$ then the sequence $(u_n)_{n\in\mathbb{N}}$ given by the proof of Proposition 2.32 goes uniformly to u and hence the restriction of u_n goes to the restriction of u uniformly on $\partial\Omega$, and hence in $L^p(\partial\Omega)$. □

The following notation is motivated by Theorem 2.47 below:

Definition 2.41. When $p = 2$ we denote by $H^{1/2}(\partial\Omega)$ the range of $\gamma_0 : H^1(\Omega) \to L^2(\Omega)$.

We do not discuss the properties of $H^{1/2}(\partial\Omega)$ here. However we will use in the following chapter that even if γ_0 is not surjective, $H^{1/2}(\partial\Omega)$ is dense in $L^2(\partial\Omega)$.

Proposition 2.42. *Let Ω be an open subset of $\mathbb{R}^d$ of class C^1. Let $p \in [1,+\infty[$ and $u \in W^{1,p}(\Omega)$. Then we have*

$$\gamma_0(u) = 0 \quad \Longleftrightarrow \quad u \in W_0^{1,p}(\Omega).$$

Proof. • Assume that $u \in W_0^{1,p}(\mathbb{R}^d)$. Then there is a sequence $(u_n)_{n\in\mathbb{N}}$ in $C_0^\infty(\Omega)$ going to u in $W^{1,p}(\Omega)$. Since $\gamma_0(u_n) = 0$ for all $n \in \mathbb{N}$ and γ_0 is continuous, we have $\gamma_0(u) = 0$.
• For the converse, we consider the case $\Omega = \mathbb{R}^d_+$ and u supported in a bounded domain. Then, with a partition of unity and changes of variables as above, we get the general case. So let $u \in W^{1,p}(\mathbb{R}^d_+)$ such that $\gamma_0(u) = 0$. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $C_0^\infty(\overline{\mathbb{R}^d_+})$ which goes to u in $W^{1,p}(\mathbb{R}^d_+)$ (see Proposition 2.32). Let $n \in \mathbb{N}$ and $x_1 > 0$. For $x' \in \mathbb{R}^{d-1}$ we have by the Hölder inequality

$$\begin{aligned}\left|u_n(x_1,x')\right|^p &\leqslant \left(\left|u_n(0,x')\right| + \int_0^{x_1} \left|\nabla u_n(s,x')\right| ds\right)^p\\ &\leqslant 2^{p-1}\left|u_n(0,x')\right|^p + 2^{p-1}\left(\int_0^{x_1} \left|\nabla u_n(s,x')\right| ds\right)^p\\ &\leqslant 2^{p-1}\left|u_n(0,x')\right|^p + 2^{p-1}x_1^{p-1}\int_0^{x_1} \left|\nabla u_n(s,x')\right|^p ds,\end{aligned}$$

so for $\varepsilon > 0$

$$\int_0^\varepsilon \int_{\mathbb{R}^{d-1}} \left|u_n(x_1,x')\right|^p dx'\,dx_1 \leqslant 2^{p-1}\varepsilon \left\|\gamma_0(u_n)\right\|^p_{L^p(\mathbb{R}^d_+)} + 2^{p-1}\varepsilon^p \int_{\mathbb{R}^{d-1}}\int_0^\varepsilon \left|\nabla u_n(s,x')\right|^p ds\,dx'.$$

Taking the limit $n \to 0$ yields, by continuity of the trace,

$$\|u\|^p_{L^p(]0,\varepsilon[\times\mathbb{R}^{d-1})} \leqslant 2^{p-1}\varepsilon^p \|\nabla u\|^p_{L^p(]0,\varepsilon[\times\mathbb{R}^{d-1})}. \tag{2.23}$$

Let $\chi \in C^\infty(\mathbb{R}_+, [0,1])$, equal to 1 on $[0,1]$ and equal to 0 on $[2,+\infty[$. Then for $n \in \mathbb{N}^*$ and $x = (x_1,\dots,x_d) \in \mathbb{R}^d_+$ we set $\chi_n(x) = \chi(nx_1)$. For $n \in \mathbb{N}^*$ we set $u_n = (1-\chi_n)u$, so that $u_n \in C_0^\infty(\mathbb{R}^d_+)$. By the dominated convergence theorem, we have

$$\|u_n - u\|_{L^p(\mathbb{R}^d_+)} = \|\chi_n u\|_{L^p(\mathbb{R}^d_+)} \xrightarrow[n\to+\infty]{} 0.$$

For $n \in \mathbb{N}^*$ we have

$$\nabla(u_n - u) = (1-\chi_n)\nabla u - u\partial_1\chi_n.$$

The first term goes to 0 in $L^p(\mathbb{R}^d_+)$. For the second term we use (2.23) to write

$$\begin{aligned}\|u\partial_1\chi_n\|^p_{L^p} &= n^p \int_{x_1=\frac{1}{n}}^{\frac{2}{n}} \left|\chi'(nx_1)\right|^p \int_{x'\in\mathbb{R}^{p-1}} \left|u(x_1,x')\right|^p dx'\,dx_1\\ &\leqslant 2^{2p-1}\left\|\chi'\right\|_\infty \|\nabla u\|^p_{L^p(]0,\frac{2}{n}[\times\mathbb{R}^{d-1})}\\ &\xrightarrow[n\to+\infty]{} 0.\end{aligned}$$

This proves that

$$\|u_n - u\|_{W^{1,p}(\mathbb{R}^d_+)} \xrightarrow[n\to+\infty]{} 0,$$

and hence $u \in W_0^{1,p}(\mathbb{R}^d_+)$. □

***Exercise* 14.** Find an open domain Ω and $u \in W^{1,\infty}(\Omega)$ such that $u_{|\partial\Omega} = 0$ but u is not in the closure of $C_0^\infty(\Omega)$ in $W^{1,+\infty}(\Omega)$.

2.5.2 Normal derivative

Let Ω be a bounded open subset of class C^1 in $\mathbb{R}^d$. For the rest of this section we only consider the case $p = 2$.

Let $u \in H^2(\Omega)$. For $j \in [\![1,d]\!]$ the derivative $\partial_j u$ belongs to $H^1(\Omega)$ and hence has a trace on $\partial\Omega$. Then we set

$$\gamma_1(u) = \partial_\nu u = \sum_{j=1}^d \gamma_0(\partial_j u)\nu_j \in L^2(\partial\Omega),$$

where $\nu = (\nu_1, \dots, \nu_d)$ is the outward normal derivative (see (2.14)). Notice that if u belongs to $C^1(\overline{\Omega})$ then on $\partial\Omega$ we have

$$\partial_\nu u = \nabla u \cdot \nu.$$

This defines a continuous function γ_1 from $H^2(\Omega)$ to $L^2(\Omega)$. We can prove (see Theorem 2.47 below for the case $\Omega = \mathbb{R}^d_+$) that

$$\left\{\partial_\nu u, u \in H^2(\Omega)\right\} = H^{1/2}(\Omega).$$

2.5.3 Green Formula

As said above, one of the motivations for the definition of the traces is the generalization of the Green Formula to functions which are not regular in the usual sense. The following results are deduced from the regular analogs by density of regular functions and continuity of the traces. For $u \in W^{1,1}(\Omega)$ we can write $\int_{\partial\Omega} u \, d\sigma$ instead of $\int_{\partial\Omega} \gamma_0(u) \, d\sigma$ and $\int_{\partial\Omega} \partial_\nu u \, v \, d\sigma$ instead of $\int_{\partial\Omega} \gamma_1(u)\gamma_0(v) \, d\sigma$.

Theorem 2.43. *Let $u, v \in H^1(\Omega)$. Then for $j \in [\![1, d]\!]$ we have*

$$\int_\Omega u \, \partial_j v \, dx = \int_{\partial\Omega} uv \, d\sigma - \int_\Omega \partial_j u \, v \, dx$$

Theorem 2.44. *Let $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$. Then we have*

$$-\int_\Omega \Delta u \, v \, dx = -\int_{\partial\Omega} \partial_\nu u \, v \, d\sigma + \int_\Omega \nabla u \cdot \nabla v \, dx.$$

2.5.4 Appendix

In this additional paragraph we continue the discussion about traces and the Green formula. In particular we define, via the Green Formula, a normal derivative for functions which are not in $H^2(\Omega)$.

We have denoted by $H^{1/2}(\Omega) \subset L^2(\Omega)$ the range of the trace γ_0 defined on $H^1(\Omega)$. This is a vector space, which can be endowed with the following norm.

$$\|g\|_{H^{1/2}(\partial\Omega)} = \inf_{\substack{w \in H^1 \\ \gamma_0(w) = g}} \|w\|_{H^1(\Omega)}.$$

We notice that $H^1_g(\Omega) = \left\{w \in H^1(\Omega) \, : \, \gamma_0(w) = g\right\}$ is a nonempty (by definition of $H^{1/2}(\partial\Omega)$) and closed (since γ_0 is continuous) affine subspace (since γ_0 is linear) of the Hilbert space $H^1(\Omega)$, so by the Hilbert projection theorem there exists a unique $R(g) \in H^1_g(\Omega)$ such that

$$\|g\|_{H^{1/2}(\partial\Omega)} = \|R(g)\|_{H^1(\Omega)}.$$

Moreover $R(g)$ is the only solution in $H^1_g(\Omega)$ of

$$\forall v \in H^1_0(\Omega), \quad \langle R(g), v\rangle_{H^1(\Omega)} = 0.$$

From this we can deduce that the application which maps $g \in H^{1/2}(\Omega)$ to $R(g) \in H^1(\Omega)$ is linear, and then that $H^{1/2}(\partial\Omega)$ is a Banach space:

Proposition 2.45. *$H^{1/2}(\partial\Omega)$ is a Banach space.*

Proof. Let $(g_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $H^{1/2}(\Omega)$. Then $(R(g_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in $H^1(\Omega)$. Since $H^1(\Omega)$ is complete, $R(g_n)$ tends to some w in $H^1(\Omega)$. We set $g = \gamma_0(w) \in H^{1/2}(\partial\Omega)$. Then we have

$$\|g_n - g\|_{H^{1/2}(\partial\Omega)} = \|R(g - g_n)\|_{H^1(\Omega)} = \|R(g) - R(g_n)\|_{H^1(\Omega)} \xrightarrow[n\to+\infty]{} 0.$$

This proves that the sequence $(g_n)_{n\in\mathbb{N}}$ has a limit in $H^{1/2}(\partial\Omega)$, and hence that $H^{1/2}(\partial\Omega)$ is complete. □

We denote by $H^{-1/2}(\partial\Omega)$ the dual of $H^{1/2}(\partial\Omega)$.

Proposition 2.46. *Let $u \in H^1(\Omega)$ such that $\Delta u \in L^2(\Omega)$. Then the map*

$$g \in H^{1/2}(\Omega) \mapsto \int_\Omega \big(\Delta u\, v_g + \nabla u \cdot \nabla v_g\big)\, dx, \tag{2.24}$$

where $v_g \in H^1(\Omega)$ satisfies $\gamma_0(v_g) = g$ is well defined (the definition does not depend on the choice of v_g) and defines a continuous linear map on $H^{1/2}(\partial\Omega)$ which we denote by $\partial_\nu u$.

We recall that in a general domain Ω the assumptions that $u \in H^1(\Omega)$ and $\Delta u \in L^2(\Omega)$ do not imply that $u \in H^2(\Omega)$.

Proof. We first observe that if v_1 and v_2 in $H^1(\Omega)$ are such that $\gamma_0(w_1) = \gamma_0(v_2) = g$ then $v_1 - v_2$ belongs to $H^1_0(\Omega)$, so there exists a sequence $(\phi_n)_{n\in\mathbb{N}}$ in $C_0^\infty(\Omega)$ which goes to $v_1 - v_2$ in $H^1(\Omega)$. For all $n \in \mathbb{N}$ we have

$$\int_\Omega \big(\Delta u\, \phi_n + \nabla u \cdot \nabla \phi_n\big)\, dx = \langle \Delta u, \phi_n \rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} + \langle \nabla u, \nabla \phi_n \rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} = 0,$$

so, taking the limit $n \to +\infty$,

$$\int_\Omega \big(\Delta u\, v_1 + \nabla u \cdot \nabla v_1\big)\, dx = \int_\Omega \big(\Delta u\, v_2 + \nabla u \cdot \nabla v_2\big)\, dx.$$

This proves that the definition in (2.24) does not depend on the choice of v_g, and the map $\partial_\nu u$ is well-defined on $H^{1/2}(\partial\Omega)$.

For $g \in H^{1/2}(\partial\Omega)$ we have

$$\left| \int_\Omega \big(\Delta u\, v_g + \nabla u \cdot \nabla v_g\big)\, dx \right| \leqslant \Big(\|\Delta u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)} \Big) \|v_g\|_{H^1(\Omega)},$$

and hence

$$\left| \int_\Omega \big(\Delta u\, v_g + \nabla u \cdot \nabla v_g\big)\, dx \right| \leqslant \Big(\|\Delta u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)} \Big) \|g\|_{H^{1/2}(\partial\Omega)}.$$

This proves that the map $\partial_\nu u$ is continuous on $H^{1/2}(\partial\Omega)$. Since it is also linear, this defines an element of $H^{-1/2}(\partial\Omega)$. ☐

By definition, we have the following Green formula for $u, v \in H^1(\Omega)$ such that $\Delta u \in L^2$:

$$-\int_\Omega \Delta u\, v\, dx = -\langle \partial_\nu u, v \rangle_{H^{-1/2}(\partial\Omega),H^{1/2}(\partial\Omega)} + \int_\Omega \nabla u \cdot \nabla v. \tag{2.25}$$

We finish this section about traces by giving a general result on $\Omega = \mathbb{R}^d_+$ by means of the Fourier transform. This will in particular ensure that the two definitions of $H^{1/2}$ on $\mathbb{R}^{d-1} \simeq \partial\mathbb{R}^d_+$ are equivalent, and that the trace on $H^1(\Omega)$ and the normal trace on $H^2(\Omega)$ have the same range.

Theorem 2.47. *Let $k \in \mathbb{N}$ and $s > k + \frac{1}{2}$. Then the map*

$$\begin{cases} \mathcal{S}(\mathbb{R}^d) & \to & \mathcal{S}(\mathbb{R}^{d-1}) \\ u & \mapsto & \partial_1^k u(0, \cdot) \end{cases}$$

has a unique continuous extansion $\gamma_k : H^s(\mathbb{R}^d) \to H^{s-k-\frac{1}{2}}(\mathbb{R}^{d-1})$. Moreover, γ_k is surjective and there exists a continuous linear map $R_k : H^{s-k-\frac{1}{2}}(\mathbb{R}^{d-1}) \to H^s(\mathbb{R}^d)$ such that

$$\gamma_k \circ R_k = \mathrm{Id}_{H^{s-k-\frac{1}{2}}(\mathbb{R}^{d-1})}.$$

Proof. • We first observe that for $m \in \mathbb{N}$, $\eta > 0$ and $\sigma > \frac{m+1}{2}$ we have, with the change of variable $t = \sqrt{\eta}\theta$

$$\int_{\mathbb{R}} t^m \big(\eta + t^2\big)^{-\sigma} \, dt = \eta^{\frac{m+1}{2}-\sigma} C_{m,\sigma}, \quad \text{where} \quad C_{m,\sigma} = \int_{\mathbb{R}} \theta^m (1+\theta^2)^{-\sigma} \, d\theta. \tag{2.26}$$

• Let $\phi \in \mathcal{S}(\mathbb{R}^d)$. For $x' \in \mathbb{R}^{d-1}$ we have by the inversion formula

$$\partial_1^k \phi(0,x') = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{d-1}} e^{ix'\cdot\xi'} \left(\int_{\mathbb{R}} (i\xi_1)^k \hat\phi(\xi_1,\xi') \, d\xi_1 \right) d\xi',$$

so the Fourier transform (in $\mathbb{R}^{d-1}$) of $\partial_1^k \phi(0,\cdot)$ is given by

$$g : \xi' \mapsto \frac{1}{2\pi} \int_{\mathbb{R}} (i\xi_1)^k \hat\phi(\xi_1,\xi') \, d\xi_1. \tag{2.27}$$

By the Cauchy-Schwarz inequality and (2.26) applied with $\eta = 1 + |\xi'|^2$ we have, for all $\xi' \in \mathbb{R}^{d-1}$,

$$\begin{aligned} 4\pi^2 \left|g(\xi')\right|^2 &\leqslant \left(\int |\hat\phi(\xi_1,\xi')|^2 \big(1+\xi_1^2+|\xi'|^2\big)^s \, d\xi_1 \right) \left(\int_{\mathbb{R}} \xi_1^{2k} \big(1+\xi_1^2+|\xi'|^2\big)^{-s} \, d\xi_1 \right) \\ &\leqslant C_{2k,s} \big(1+|\xi'|^2\big)^{-(s-k-\frac12)} \int |\hat\phi(\xi_1,\xi')|^2 \big(1+\xi_1^2+|\xi'|^2\big)^s \, d\xi_1. \end{aligned}$$

Multiplying by $\big(1+|\xi'|^2\big)^{s-k-\frac12}$ and integrating over $\xi' \in \mathbb{R}^{d-1}$ gives

$$\left\| (\partial_1^k \phi)(0,\cdot) \right\|^2_{H^{s-k-\frac12}(\mathbb{R}^{d-1})} \leqslant \frac{C_{2k,s}}{4\pi^2} \left\| \phi \right\|^2_{H^s(\mathbb{R}^d)}.$$

This proves the first statement of the theorem.

• Now we prove that γ_k is surjective with a continuous right inverse. We begin with $v \in \mathcal{S}(\mathbb{R}^{d-1})$. Let $g \in \mathcal{S}(\mathbb{R}^{d-1})$ be the Fourier transform of v on $\mathbb{R}^{d-1}$. The expression (2.27) suggests to find f such that

$$g(\xi') = \frac{1}{2\pi} \int_{\mathbb{R}} (i\xi_1)^k f(\xi_1,\xi') \, d\xi_1. \tag{2.28}$$

Let $N > \frac12\big(s-k-\frac12\big)$. For $\xi = (\xi_1,\xi') \in \mathbb{R}^d$ we set

$$f(\xi) = \frac{2\pi}{C_{k,N+\frac12}} \frac{(-i)^k \big(1+|\xi'|^2\big)^N}{\big(1+|\xi|^2\big)^{N+\frac{k}{2}+\frac12}} g(\xi').$$

In particular, for all $\xi' \in \mathbb{R}^{d-1}$ the map $\xi_1 \mapsto (-i\xi_1)^k f(\xi_1,\xi')$ is integrable on $\mathbb{R}$ and (2.28) holds by (2.26). Moreover, by (2.26) again we have

$$\begin{aligned} &\int_{\mathbb{R}^d} (1+|\xi|^2)^s \left|f(\xi)\right|^2 \, d\xi \\ &\quad = \frac{4\pi^2}{C^2_{k,N+\frac12}} \int_{\mathbb{R}^{d-1}} (1+|\xi'|^2)^{2N} \left|g(\xi')\right|^2 \left(\int_{\mathbb{R}} (1+|\xi|^2)^{-(2N+k+1-s)} \, d\xi_1 \right) d\xi \\ &\quad = \frac{4\pi^2 C_{0,2N+k+1-s}}{C^2_{k,N+\frac12}} \int_{\mathbb{R}^{d-1}} (1+\left|\xi'\right|^2)^{s-k-\frac12} \left|g(\xi')\right|^2 \, d\xi' \end{aligned}$$

Then if we denote by u the inverse Fourier transform of f we obtain that $u \in H^s(\mathbb{R}^d)$ and

$$\left\| u \right\|^2_{H^s(\mathbb{R}^d)} \leqslant \frac{4\pi^2 C_{0,2N+k+1-s}}{C^2_{k,N+\frac12}} \left\| v \right\|^2_{H^{s-k-\frac12}(\mathbb{R}^{d-1})}. \tag{2.29}$$

Moreover (2.28) ensures that $\gamma_k(u) = v$. Thus we have defined a map $R_k : \mathcal{S}(\mathbb{R}^{d-1}) \to H^s(\mathbb{R}^d)$ such that $\gamma_k \circ R_k = \mathrm{Id}$. By (2.29), R_k extends to a continuous map from $H^{s-k-\frac12}(\mathbb{R}^{d-1})$ to $H^s(\mathbb{R}^d)$, and the proof is complete. □

2.6 Poincaré Inequality

In Theorem 2.34 we have given an estimate with the norm $\|\nabla u\|_{L^p(\mathbb{R}^d)}$ and not the full norm $\|u\|_{W^{1,p}(\mathbb{R}^d)}$. In application, and in particular for the analysis of second order PDEs, we will often be in the situation where we only control the norm of the gradient of the function and not the function itself.

It turns out that in some particular situations, the norm of the function is in fact controled by the norm of the gradient:

$$\|u\|_{L^p(\Omega)} \leqslant C \|\nabla u\|_{L^p(\Omega)}. \tag{2.30}$$

In this case, $\|\nabla u\|_{L^p(\Omega)}$ defines a norm on $W^{1,p}(\Omega)$, equivalent to $\|u\|_{W^{1,p}(\Omega)}$. An inequality like (2.30) is called a Poincaré inequality. This is the subject of this paragraph.

Before giving precise statements, we notice that a Poincaré inquality cannot hold in a space which contains constant functions. In an unbounded domain, troubles can come from slowly varying functions. For instance on $\mathbb{R}$ we consider for $n \in \mathbb{N}^*$ the function u_n defined by

$$u_n(x) = \begin{cases} 1 - \frac{|x|}{n} & \text{if } |x| \leqslant n, \\ 0 & \text{if } |x| > n. \end{cases}$$

Then we have $\|u\|_{L^2(\mathbb{R})}^2 = \frac{2n}{3}$ and $\|u'\|_{L^2(\mathbb{R})}^2 = \frac{2}{n}$. A Poincaré inequality cannot hold in $H^1(\mathbb{R})$.

In fact, we have discussed all the problems to prove a Poincaré inequality. Roughly speaking, on a bounded domain, and if we remove constant functions, a Poincaré inquality holds. The first way to remove constant functions is to consider only functions vanishing at the boundary.

We first recall that Lemma 2.7 also holds in higher dimension.

Proposition 2.48. *Let Ω be an open connected subset of $\mathbb{R}^d$. Let $u \in L^1_{\mathrm{loc}}(\Omega)$ be such that $\nabla u = 0$ (in the sense of distributions). Then there exists a constant α such that $u = \alpha$ almost everywhere.*

Proof. We proceed by induction on the dimension. The case $d = 1$ is already known. We assume that $d \geqslant 2$ and that the result is known up to the dimension $d - 1$.

It is enough to consider the case $\Omega = \prod_{j=1}^d]a_j, b_j[$. Let $\chi \in C_0^\infty(]a_1, b_1[)$ be such that $\int_{a_1}^{b_1} \chi(x_1)\, dx_1 = 1$. For $x' \in \Omega' = \prod_{j=2}^d]a_j, b_j[$ we set

$$v(x') = \int_{a_1}^{b_1} u(x_1, x')\chi(x_1)\, dx_1.$$

This defines a function $v \in L^1_{\mathrm{loc}}(\Omega')$. For $\psi \in C_0^\infty(\Omega')$ and $j \in [\![2, d]\!]$ we have

$$\begin{aligned} -\int_{\Omega'} v(x')\partial_j\psi(x')\, dx' &= -\int_\Omega u(x_1, x')\chi(x_1)\partial_j\psi(x')\, dx_1\, dx' \\ &= -\int_\Omega u(x_1, x')\partial_j\big(\chi(x_1)\psi(x')\big)\, dx_1\, dx' \\ &= 0. \end{aligned}$$

This proves that, in the sense of distributions, we have $\nabla v = 0$ on Ω' By the induction assumption there exists α such that $v = \alpha$ almost everywhere on Ω'.

Now let $\phi \in C_0^\infty(\Omega)$. For $x = (x_1, x') \in \Omega$ we set

$$\tilde{\phi}(x') = \int_{a_1}^{b_1} \phi(x_1, x')\, dx_1$$

and

$$\zeta(x) = \int_{a_1}^{x_1} \big(\phi(t, x') - \chi(t)\tilde{\phi}(x')\big)\, dt.$$

Then $\zeta \in C_0^\infty(\Omega)$ and $\phi = \partial_{x_1}\zeta + \chi \otimes \tilde\phi$, so

$$\int_\Omega u\phi\,dx = \int_{a_1}^{b_1}\int_{\Omega'} u(x_1,x')\chi(x_1)\tilde\phi(x')\,dx'\,dx_1 = \int_{\Omega'} v\tilde\phi\,dx' = \alpha\int_{\Omega'}\tilde\phi\,dx' = \alpha\int_\Omega \phi\,dx.$$

This proves that $u = \alpha$ almost everywhere on Ω. □

Now we can prove the Poincaré inequality.

Theorem 2.49. *Let Ω be an open bounded subset of $\mathbb{R}^d$. Let $p \in [1,+\infty[$. Then there exists $C > 0$ such that*

$$\forall u \in W_0^{1,p}(\Omega), \quad \|u\|_{L^p(\Omega)} \leqslant C\,\|\nabla u\|_{L^p(\Omega)}.$$

Proof. Assume by contradiction that the statement is not true. Then for all $n \in \mathbb{N}$ there exists $u_n \in W_0^{1,p}(\Omega)$ such that

$$\|u_n\|_{L^p(\Omega)} > n\,\|\nabla u_n\|_{L^p(\Omega)}.$$

Since this inequality can be divided by $\|u_n\|_{L^p(\Omega)}$ (which cannot be 0), we can assume without loss of generality that $\|u_n\|_{L^p(\Omega)} = 1$ for all $n \in \mathbb{N}$. Then

$$\|\nabla u_n\|_{L^p(\Omega)} \xrightarrow[n\to+\infty]{} 0, \tag{2.31}$$

and the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $W^{1,p}(\Omega)$. Since $W^{1,p}(\Omega)$ is compactly embedded in $L^p(\Omega)$ (see Theorem 2.38), there exists an increasing sequence $(n_k)_{k\in\mathbb{N}} \in \mathbb{N}^\mathbb{N}$ and $v \in L^p(\Omega)$ such that

$$\|u_{n_k} - v\|_{L^p(\Omega)} \xrightarrow[k\to+\infty]{} 0.$$

With (2.31), this implies that the sequence $(u_{n_k})_{k\in\mathbb{N}}$ is a Cauchy sequence in $W^{1,p}(\Omega)$. Since $W^{1,p}(\Omega)$ is complete (see Theorem 2.17), the sequence $(u_{n_k})_{k\in\mathbb{N}}$ has a limit in $W^{1,p}(\Omega)$. This limit is necessarily v. In particular v belongs to $W^{1,p}(\Omega)$, and by (2.31) we have $\nabla v = 0$. By Proposition 2.48, v is constant on each connected component of Ω. Since u_{n_k} belongs to $W_0^{1,p}(\Omega)$ for all $k \in \mathbb{N}$, we also have $v \in W_0^{1,p}(\Omega)$, so $v = 0$, which gives a contradiction with the fact that $\|u_{n_k}\|_{L^2(\Omega)} = 1$ for all $k \in \mathbb{N}$. □

Notice that the proof of Theorem 2.49 does not give any clue about the constant C of the inequality. We now give a similar result, with a more constructive proof. Moreover the open set Ω is only required to be bounded in one direction. This means that Ω is included in a strip of the form

$$\Omega \subset \left\{x \in \mathbb{R}^d, x\cdot e \in]a,b[\right\},$$

for some $e \in \mathbb{R}^d$, $|e| = 1$ and $a, b \in \mathbb{R}$.

Theorem 2.50 (Poincaré inequality)**.** *Let Ω be an open subset of $\mathbb{R}^d$, bounded in one direction. Let $p \in [1,+\infty[$. Then there exists $C_\Omega > 0$ such that, for all $u \in W_0^{1,p}(\Omega)$,*

$$\|u\|_{L^p(\Omega)} \leqslant C_\Omega\,\|\nabla u\|_{L^p(\Omega)}.$$

For instance, we can take $C_\Omega = (b-a)p$.

Proof. • It is enough to prove the estimate for $u \in C_0^\infty(\Omega)$. Then the result will follow by density of $C_0^\infty(\Omega)$ in $W_0^{1,p}(\Omega)$. We can extend u by 0, this gives a function in $C_0^\infty(\mathbb{R}^d)$ supported in Ω.

• We first consider the one-dimensional case. Then there exists $a, b \in \mathbb{R}$ such that $\Omega \subset]a,b[$. Then we can extend u as a function in $C_0^\infty(]a,b[)$ which vanishes outside Ω. Then for all $x \in]a,b[$ we have $(u(x)^p)' = pu'(x)u(x)^{p-1}$ so, by the Hölder inequality

$$|u(x)|^p \leqslant p\int_a^b \left|u'(s)\right| |u(s)|^{p-1}\,ds \leqslant p\,\|u\|_{L^p(\Omega)}^{p-1}\left\|u'\right\|_{L^p(\Omega)}.$$

After integration over $]a,b[$ this gives

$$\|u\|_{L^p(\Omega)}^p \leqslant (b-a)p\,\|u\|_{L^p(\Omega)}^{p-1}\left\|u'\right\|_{L^p(\Omega)},$$

and the conclusion follows after simplification by $\|u\|_{L^p(\Omega)}^{p-1}$.

• Now we consider the general case. Let $(f_1, \dots, f_d)$ be an orthonormal basis of $\mathbb{R}^d$ such that

$$\operatorname{supp}(u) \subset \left\{ y_1 f_1 + y' f' \ : \ y_1 \in]a, b[, y' \in \mathbb{R}^{d-1} \right\},$$

for some $a, b \in \mathbb{R}$, where for $y' = (y_2, \dots, y_d) \in \mathbb{R}^{d-1}$ we have set $y' f' = \sum_{j=2}^d y_j f_j$. By a change of variables and using the one-dimensional case we can write

$$\begin{aligned}
\int_\Omega |u(x)|^p \, dx &= \int_{y' \in \mathbb{R}^{d-1}} \int_{y_1 = a}^b \left| u(y_1 f_1 + y' f') \right|^p \, dy_1 \, dy' \\
&\leqslant ((b-a)p)^p \int_{y' \in \mathbb{R}^{d-1}} \int_{y_1 = a}^b \left| \frac{\partial}{\partial y_1} u(y_1 f_1 + y' f') \right|^p \, dy_1 \, dy' \\
&\leqslant ((b-a)p)^p \int_{y' \in \mathbb{R}^{d-1}} \int_{y_1 = a}^b \left| \nabla u(y_1 f_1 + y' f') \right|^p \, dy_1 \, dy' \\
&\leqslant ((b-a)p)^p \int_\Omega |\nabla u(x)|^p \, dx.
\end{aligned}$$

The conclusion follows with $C = (b-a)p$. □

It can be important in application to have an explicit constant for the Poincaré inequality. Computing the optimal constant for particular sets Ω requires more work, and we do not discuss this issue here, but we already have an upper bound.

After Theorem 2.50, the interest of the proof given for Theorem 2.49 is not clear. For the proof of Theorem 2.50 we have really used the fact that the function u vanishes at the boundary. While for the proof of Theorem 2.49 we have in fact only used the property that the only constant function is 0. The interest of the proof of Theorem 2.49 is that it can be used in any such situation. For instance, we give the following version of the Poincaré inequality.

For a bounded open subset Ω we define

$$\widetilde{W}^{1,p}(\Omega) = \left\{ u \in W^{1,p}(\Omega) \ : \ \int_\Omega u \, dx = 0 \right\}. \tag{2.32}$$

Notice that if Ω is connected then the only function $u \in \widetilde{W}^{1,p}(\Omega)$ such that $\nabla u = 0$ is $u = 0$.

Theorem 2.51 (Poincaré-Wirtinger inequality)**.** *Let Ω be an open, connected and bounded subset of $\mathbb{R}^d$. Let $p \in [1, +\infty]$ Then there exists $C > 0$ such that, for all $u \in \widetilde{W}^{1,p}(\Omega)$,*

$$\forall u \in \widetilde{W}^{1,p}(\Omega), \quad \|u\|_{L^p(\Omega)} \leqslant C \|\nabla u\|_{L^p(\Omega)}.$$

***Exercise* 15.** Let Ω be an open, bounded and connected subset of $\mathbb{R}^d$. Let $p \in [1, +\infty]$.

1. Prove Theorem 2.51.

2. For $u \in W^{1,p}(\Omega)$ we set

$$\mathcal{N}(u) = \|\nabla u\|_{L^p(\Omega)} + \left| \int_\Omega u \, dx \right|.$$

Prove that $\mathcal{N}$ is a norm on $W^{1,p}(\Omega)$, equivalent to the usual one.

2.7 The dual of $H_0^1(\Omega)$

Definition 2.52. We denote by $H^{-1}(\Omega)$ the dual space of $H_0^1(\Omega)$.

We recall that the dual space of $H_0^1(\Omega)$ is the set of continuous linear forms on $H_0^1(\Omega)$. It is endowed with the norm defined by

$$\|\varphi\|_{H^{-1}(\Omega)} = \sup_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{|\varphi(u)|}{\|u\|_{H_0^1(\Omega)}}.$$

We usually write $\langle \varphi, u\rangle$ (or $\langle \varphi, u\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}$) instead of $\varphi(u)$. Notice that if $H^1(\Omega) \neq H_0^1(\Omega)$ then $H^{-1}(\Omega)$ *is not* the dual space of $H^1(\Omega)$.

We recall that by the Riesz Theorem, we can identify a Hilbert space with its dual. However, in this kind of context we usually already identify $L^2(\Omega)$ with its dual. With this identification we have

$$H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega),$$

with continuous injections. The first inclusion is clear by definition of the Sovolev space $H_0^1(\Omega)$. Now a function $u \in L^2(\Omega)$ is identified with the continuous linear form on $L^2(\Omega)$ defined by

$$v \mapsto \langle u, v\rangle_{L^2(\Omega)} . \tag{2.33}$$

By restriction, this also defines a continuous linear form on $H_0^1(\Omega)$. In this sense, we can say that u belongs to $H^{-1}(\Omega)$. However, all the elements of $H^{-1}(\Omega)$ cannot be identified with a function in $L^2(\Omega)$. For instance, on $\mathbb{R}$, the Dirac distribution

$$\delta : v \mapsto v(0) \tag{2.34}$$

defines a continuous linear for on $H^1(\mathbb{R}) = H_0^1(\mathbb{R})$, and it is not of the form (2.33) (notice that this example is specific to the dimension 1, a Dirac distribution is not in $H^{-1}(\Omega)$ in dimension $d \geqslant 2$, however with the trace Theorem we can generalize this example in higher dimension, see Exercise 16).

Let $f \in L^2(\Omega)$ and $F \in L^2(\Omega, \mathbb{R}^d)$. Then $\varphi = f - \operatorname{div} F$, where the derivatives are understood in the sense of distributions, also defines a continuous linear form on $H_0^1(\Omega)$ (which is not necessarily in $L^2(\Omega)$). For $v \in H_0^1(\Omega)$ it is given by

$$\varphi(v) = \langle f, v\rangle + \sum_{j=1}^{d} \langle F_j, \partial_j u\rangle .$$

In particular we have

$$\|\varphi\|_{H^{-1}(\Omega)} \leqslant \|f\|_{L^2(\Omega)} + \sum_{j=1}^{d} \|F_j\|_{L^2(\Omega)} . \tag{2.35}$$

In fact, using the Riesz Theorem in $H_0^1(\Omega)$ we see that any $\varphi \in H^{-1}(\Omega)$ can be written in this form with $u \in H_0^1(\Omega)$ and $F = \nabla u$. Moreover, in this case we have an equality in (2.35). See Theorem 5.9.1 in [Evans] (see Exercise 17 for the particular case of the Dirac distribution (2.34)).

***Exercise* 16.** Let $f \in L^2(\mathbb{R})$. Prove that the map

$$v \in C_0^\infty(\mathbb{R}^2) \mapsto \int_{\mathbb{R}} f(x)v(x,0)\, dx$$

extends to a continuous linear form on $H^1(\mathbb{R}^2)$.

***Exercise* 17.** Find $u \in H^1(\mathbb{R})$ such that

$$\forall v \in H^1(\mathbb{R}), \quad v(0) = \int_{\mathbb{R}} uv + \int_{\mathbb{R}} u'v'.$$

2.8 Exercises

***Exercise* 18.** Show that there is no continuous linear map $\gamma : L^2(\mathbb{R}_+^*) \to \mathbb{R}$ such that $\gamma(u) = 0$ for all $u \in C^0([0,+\infty[) \cap L^2(\mathbb{R}_+^*)$.

***Exercise* 19.** For which values of $k \in \mathbb{N}$, $p \in [1,+\infty]$ and $q \in [1,+\infty]$ do we have $W^{k,p}(\mathbb{R}^d) \subset L^q(\mathbb{R}^d)$?

***Exercise* 20.** **1.** Let $u \in H_0^1(\mathbb{R}_+^2)$. For $(x_1,x_2) \in \mathbb{R}^2$ we set

$$\tilde{u}(x_1,x_2) = \begin{cases} u(x_1,x_2) & \text{if } x_1 > 0, \\ 0 & \text{if } x_1 \leqslant 0. \end{cases}$$

Prove that $\tilde{u} \in H^1(\mathbb{R}^2)$ and give an expression for the derivatives of $\tilde{u}$. In particular, what can we say about $\|\tilde{u}\|_{H^1(\mathbb{R}^2)}$?
2. Let Ω be an open subset of $\mathbb{R}^2$. Let $u \in H_0^1(\Omega)$. Prove that the extension of u by 0 on $\mathbb{R}^d$ belongs to $H^1(\mathbb{R}^2)$.

***Exercise* 21.** We recall that for $s \in [0,1]$ and $u \in L^2(\mathbb{R}^d)$ we have set

$$\|u\|_{H^s(\mathbb{R}^d)}^2 = \int_{\mathbb{R}^d} \left(1 + |\xi|^2\right)^s |\hat{u}(\xi)|^2 \, d\xi.$$

Then $H^s(\mathbb{R}^d)$ is the set of $u \in L^2(\mathbb{R}^d)$ such that $\|u\|_{H^s(\mathbb{R}^d)} < +\infty$.
1. Check that $H^0(\mathbb{R}^d) = L^2(\mathbb{R}^d)$ and that $H^1(\mathbb{R}^d)$ coincides with the space already defined, with equivalent norm.
2. Let $s \in]0,1[$. Prove that there exists $C > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}^d)$ we have

$$\int_{y\in\mathbb{R}^d} \int_{x\in\mathbb{R}^d} \frac{|u(x) - u(y)|^2}{|x-y|^{d+2s}} \, dx \, dy = C \int_{\xi\in\mathbb{R}^d} |\xi|^{2s} |\hat{u}(\xi)|^2 \, d\xi.$$

3. Deduce that the quantity

$$\left(\|u\|_{L^2(\mathbb{R}^d)}^2 + \int_{y\in\mathbb{R}^d} \int_{x\in\mathbb{R}^d} \frac{|u(x) - u(y)|^2}{|x-y|^{d+2s}} \, dx \, dy \right)^{\frac{1}{2}}$$

defines a norm on $H^s(\mathbb{R}^d)$, equivalent to the norm defined above.

***Exercise* 22.** In this exercise we prove that for $u \in H^1(\mathbb{R}^d)$ (real valued) we have $|u| \in H^1(\mathbb{R}^d)$, $\nabla u = 0$ almost everywhere on $u^{-1}(\{0\})$ and $\nabla |u| = \mathsf{sign}(u) \nabla u$ on $u^{-1}(\mathbb{R}^d \setminus \{0\})$.
1. Let $G : \mathbb{R} \to \mathbb{R}$ be of class C^1, globally Lipschitz and such that $G(0) = 0$.
a. Show that G' is bounded on $\mathbb{R}$.
b. Prove that $G \circ u \in H^1(\mathbb{R}^d)$ with $\nabla(G \circ u) = (G' \circ u)\nabla u$.
2. For $t \in \mathbb{R}$ we set

$$H_-(t) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x \leqslant 0, \end{cases} \quad \text{and} \quad H_+(t) = \begin{cases} 1 & \text{if } x \geqslant 0, \\ 0 & \text{if } x < 0. \end{cases}$$

For $n \in \mathbb{N}^*$ we set

$$H_n(t) = \begin{cases} 1 & \text{if } t \geqslant \frac{1}{n}, \\ nt & \text{if } 0 \leqslant t \leqslant \frac{1}{n}, \\ 0 & \text{if } t \leqslant 0. \end{cases}$$

Then we set $V_n(t) = \int_{-\infty}^t H_n(s) \, ds$.
a. Prove that $(V_n \circ u) \in H^1(\mathbb{R}^d)$ with $\nabla(V_n \circ u) = (H_n \circ u)\nabla u$.
b. For $t \in \mathbb{R}$ we set

$$g(t) = \begin{cases} t & \text{if } t > 0, \\ 0 & \text{if } t < 0. \end{cases}$$

Prove that $(g \circ u) \in H^1(\mathbb{R}^d)$ with $\nabla(g \circ u) = (H_- \circ u)\nabla u$.
c. Prove that $\nabla(g \circ u) = (H_+ \circ u)\nabla u$.
d. Deduce that $\nabla u = 0$ almost everywhere on $u^{-1}(\{0\})$.
3. Conclude.