Chapitre 5

A Brief Introduction to Sobolev spaces and applications

5.1 Derivatives in L^2

In this first paragraph we define the Sobolev spaces of L^2-functions whose derivatives in the sense of distributions are also in L^2.

5.1.1 Definition

We begin with the one dimensional case.

Définition 5.1. We denote by $H^1(\mathbb{R})$ the set of functions $u \in L^2(\mathbb{R})$ whose derivative in the sense of distributions is in $L^2(\mathbb{R})$.

We recall that the derivative of $u \in L^2(\mathbb{R})$ in the sense of distributions is said to be in $L^2(\mathbb{R})$ if there exists $v \in L^2(\mathbb{R})$ such that $T'_u = T_v$. In other words,

$$\forall \phi \in C_0^\infty(\mathbb{R}), \quad -\int_{\mathbb{R}} u\phi' \, dx = \int_{\mathbb{R}} v\phi \, dx. \tag{5.1}$$

In this case v is unique and it is denoted by u'.

Remark 5.2. If $f \in C^1(\mathbb{R})$ is compactly supported then it belongs to $H^1(\mathbb{R})$. In general, even if f is of class C^1, f and f' are well defined as functions but they are not necessarily in $L^2(\mathbb{R})$. In this case f is not in $H^1(\mathbb{R})$. On the other hand, a function can be in $H^1(\mathbb{R})$ even if it is not of class C^1.

Example 5.3.

— For $x \in \mathbb{R}$ we set

$$f(x) = \begin{cases} 1 - |x| & \text{if } |x| \leqslant 1, \\ 0 & \text{if } |x| > 1. \end{cases} \tag{5.2}$$

In the sense of distributions we have

$$f'(x) = \begin{cases} 1 & \text{if } x \in]-1, 0[, \\ -1 & \text{if } x \in]0, 1[, \\ 0 & \text{if } |x| > 1. \end{cases}$$

Thus f and f' are in $L^2(\mathbb{R})$, so $f \in H^1(\mathbb{R})$.

— Let H be the Heaviside function defined by (4.11). In the sense of distributions we have $H' = \delta$. But δ is not the distribution given by a L^2 function on $\mathbb{R}$ (see Proposition 4.19), so H is not in $H^1(\mathbb{R})$.

The above definition can be extended to L^2 functions in any dimension $d \in \mathbb{N}^*$ and we can consider any order $k \in \mathbb{N}^*$ of derivatives.

Définition 5.4. For $k \in \mathbb{N}$ we set

$$H^k(\mathbb{R}^d) = \left\{ u \in L^2(\mathbb{R}^d) \,:\, \partial^\alpha u \in L^2(\mathbb{R}^d) \text{ for all } \alpha \in \mathbb{N}^d \text{ such that } |\alpha| \leqslant k \right\},$$

where $\partial^\alpha u$ is the derivative of u in the sense of distributions. In other words, a function $u \in L^2(\mathbb{R}^d)$ belongs to $H^k(\mathbb{R}^d)$ if for all $\alpha \in \mathbb{N}^d$ with $|\alpha| \leqslant k$ there exists $v_\alpha \in L^2(\mathbb{R}^d)$ such that

$$\forall \phi \in C_0^\infty(\mathbb{R}^d), \quad (-1)^{|\alpha|} \int_{\mathbb{R}^d} u \, \partial^\alpha \phi \, dx = \int_{\mathbb{R}^d} v_\alpha \, \phi \, dx.$$

In this case v_α is unique (up to equality almost everywhere) and we set $\partial^\alpha u = v_\alpha$.

Remark 5.5. By the Riesz Theorem and by density of $C_0^\infty(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d)$, a function $u \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ belongs to $H^k(\mathbb{R}^d)$ is and only if for all $\alpha \in \mathbb{N}^d$ with $|\alpha| \leqslant k$ there exists $C_\alpha > 0$ such that

$$\forall \phi \in C_0^\infty(\mathbb{R}^d), \qquad \left| \int_{\mathbb{R}^d} u \partial^\alpha \phi \, dx \right| \leqslant C_\alpha \left\| \phi \right\|_{L^2(\mathbb{R}^d)}.$$

Example 5.6. The function f defined by (5.2) is in $H^1(\mathbb{R})$ but not in $H^2(\mathbb{R})$.

Example 5.7. Let $\alpha \in]-\infty, d-1[$. We have seen in Paragraph 4.3.3 that the derivatives in the sense of distributions of $f : x \mapsto |x|^{-\alpha}$ are given by $\nabla f(x) = -\alpha \, |x|^{-\alpha-2} \, x$. Let $\chi \in C_0^\infty(\mathbb{R}^d)$. Then if $\alpha < \frac{d}{2} - 1$ we have $\chi f \in H^1(\mathbb{R}^d)$ with

$$\nabla(\chi f) = f \nabla \chi + \chi \nabla f \in L^2(\mathbb{R}^d).$$

More generally, we can check that if $\alpha < \frac{d}{2} - k$ for some $k \in \mathbb{N}$ then we have $\chi f \in H^k(\mathbb{R}^d)$.

We can similarly define the Sobolev spaces $H^k(\Omega)$ on any open subset Ω of $\mathbb{R}^d$. We also define the Sobolev spaces $W^{k,p}(\Omega)$ of L^p functions on Ω with all derivatives up to order k in L^p, but we do not discuss these issues in this brief introduction. The discussion of the following paragraph is only valid when $p = 2$ and $\Omega = \mathbb{R}^d$.

5.1.2 Characterisation via the Fourier transform

We can use the Fourier transform to give a simple characterisation of $H^k(\mathbb{R}^d)$. We begin with the following lemma.

Lemma 5.8. Let $k \in \mathbb{N}$. There exist $C_1, C_2 > 0$ such that

$$\forall \xi \in \mathbb{R}^d, \quad C_1 \left(1 + |\xi|^2\right)^k \leqslant \sum_{|\alpha| \leqslant k} \xi^{2\alpha} \leqslant C_2 \left(1 + |\xi|^2\right)^k.$$

Proof. Let $\xi \in \mathbb{R}^d$. For $j \in [\![0, k]\!]$ we have

$$|\xi|^{2j} = (\xi_1^2 + \cdots + \xi_d^2)^j = \sum_{1 \leqslant i_1, \ldots, i_j \leqslant d} \xi_{i_1}^2 \ldots \xi_{i_j}^2 \leqslant j^d \sup_{|\alpha| \leqslant k} \xi^{2\alpha},$$

so

$$(1 + |\xi|^2)^k = \sum_{j=0}^k C_k^j \, |\xi|^{2j} \leqslant \left(\sum_{j=0}^k C_k^j j^d \right) \sum_{|\alpha| \leqslant k} \xi^{2\alpha}.$$

The first inequality follows with some $C_1 > 0$ independant of ξ. Now for $\alpha \in \mathbb{N}^d$ with $|\alpha| \leqslant k$ we have

$$\xi^{2\alpha} \leqslant |\xi|^{2|\alpha|} \leqslant \left(1 + |\xi|^2\right)^k,$$

which gives the second inequality. $\square$

A function u belongs to $H^k(\mathbb{R}^d)$ if its derivatives are in $L^2(\mathbb{R}^d)$. After a Fourier transform, this conditions turns into a condition about $\hat{u}$ multiplied by some polynomial.

Proposition 5.9. *Let $k \in \mathbb{N}^*$ and $u \in L^2(\mathbb{R}^d)$. Then $u \in H^k(\mathbb{R}^d)$ if and only if*

$$\int_{\mathbb{R}^d} \left(1 + |\xi|^2\right)^k |\hat{u}(\xi)|^2 \, d\xi < +\infty. \tag{5.3}$$

Proof. We have $\hat{u} \in L^2(\mathbb{R}^d)$. Let $\alpha \in \mathbb{N}^d$ with $|\alpha| \leqslant k$. If we identify functions with the corresponding distributions we have by Proposition 4.94

$$\mathcal{F}(\partial^\alpha u) = (i\xi)^\alpha \hat{u}.$$

Thus $\partial^\alpha u$ belongs to $L^2(\mathbb{R}^d)$ if and only if the map $\xi \mapsto \xi^\alpha \hat{u}(\xi)$ does. Then $u \in H^k(\mathbb{R}^d)$ if and only if

$$\int_{\mathbb{R}^d} \sum_{|\alpha| \leqslant k} \xi^{2\alpha} \left|\hat{u}(\xi)\right|^2 \, d\xi < +\infty.$$

By Lemma 5.8, this is equivalent to (5.3). □

Remark 5.10. If $u \in L^2(\mathbb{R}^d)$ is such that Δu belongs to $L^2(\mathbb{R}^d)$, then u belongs to $H^2(\mathbb{R}^d)$.

Exercise 5.11. Let $u \in L^2(\mathbb{R}^d)$ such that $\Delta(\Delta u) + 2\Delta u - u \in L^2(\mathbb{R}^d)$. Prove that $u \in H^4(\mathbb{R}^d)$.

5.1.3 Regularity of functions in Sobolev spaces

It is not clear that being in some Sobolev space is a regularity property for a function u. However, if u has enough weak derivatives in L^2, we recover some regularity in the usual sense.

Proposition 5.12. *Let $k > \frac{d}{2}$ and $u \in H^k(\mathbb{R}^d)$. Then u is continous and goes to 0 at infinity (in the sense that u has a representative which satisfies these properties). In particular it is bounded. More generaly, if $k > n + \frac{d}{2}$ for some $n \in \mathbb{N}$ then u is of class C^n.*

Proof. • By the Cauchy-Schwarz inequality we have

$$\int_{\mathbb{R}^d} \left|\hat{u}(\xi)\right|^2 \, d\xi \leqslant \left(\int_{\mathbb{R}^d} \left(1 + |\xi|^2\right)^{-k} d\xi \right)^{\frac{1}{2}} \left(\left(1 + |\xi|^2\right)^k \left|\hat{u}(\xi)\right|^2 \, d\xi \right)^{\frac{1}{2}} < +\infty,$$

so $\hat{u} \in L^1(\mathbb{R}^d)$. This implies that u is continuous and goes to 0 at infinity. If $k > n + \frac{d}{2}$ then for all $\alpha \in \mathbb{N}^d$ with $|\alpha| \leqslant n$ we have $\partial^\alpha u \in H^{k-n}(\mathbb{R}^d)$, so $\partial^\alpha u$ is a continuous function. This implies that u is of class C^n. □

5.2 Topology on the Sobolev spaces

In this section we define the norms on the Sobolev spaces we have just defined, and we give some properties of these new functional spaces.

5.2.1 Hilbert structure

Theorem 5.1. Let $k \in \mathbb{N}$. For $u, v \in H^k(\mathbb{R}^d)$ we set

$$\langle u, v \rangle_{H^k(\mathbb{R}^d)} = \sum_{|\alpha| \leqslant k} \langle \partial^\alpha u, \partial^\alpha v \rangle_{L^2(\mathbb{R}^d)} . \tag{5.4}$$

This defines an inner product for which $H^k(\mathbb{R}^d)$ is a Hilbert space. Moreover, the correspon-ding norm is equivalent to the norm defined by

$$\|u\|^2 = \int_{\mathbb{R}^d} \left(1 + |\xi|^2\right)^k \left|\hat{u}(\xi)\right|^2 \, d\xi. \tag{5.5}$$

Proof. • The fact that (5.4) defines an inner product on $H^k(\mathbb{R}^d)$ is left as an exercice. We prove that $H^k(\mathbb{R}^d)$ is complete for the corresponding norm, given by

$$\|u\|^2_{H^k(\mathbb{R}^d)} = \sum_{|\alpha| \leqslant k} \|\partial^\alpha u\|^2_{L^2(\mathbb{R}^d)} .$$

Let $(u_n)_{n\in\mathbb{N}}$ be a Cauchy sequence. Then the sequences $(\partial^\alpha u_n)_{n\in\mathbb{N}}$ for $|\alpha| \leqslant k$ are Cauchy sequences in $L^2(\mathbb{R}^d)$. Since $L^2(\mathbb{R}^d)$ is complete, there exist $v_\alpha \in L^2(\mathbb{R}^d)$ for $|\alpha| \leqslant k$ such that $\partial^\alpha u_n$ goes to v_α. For $|\alpha| \leqslant k$ and $\phi \in C_0^\infty(\mathbb{R}^d)$ we have

$$(-1)^{|\alpha|} \int_{\mathbb{R}^d} v_0 \, \partial^\alpha \phi \, dx = (-1)^{|\alpha|} \lim_{n\to+\infty} \int_{\mathbb{R}^d} u_n \, \partial^\alpha \phi \, dx = \lim_{n\to+\infty} \int_{\mathbb{R}^d} \partial^\alpha u_n \, \phi \, dx = \int_{\mathbb{R}^d} v_\alpha \, \phi \, dx.$$

This proves that in the sense of distributions we have $\partial^\alpha v_0 = v_\alpha$. Then $v_0 \in H^k(\mathbb{R}^d)$ and

$$\|u_n - v_0\|_{H^k(\mathbb{R}^d)} \xrightarrow[n\to+\infty]{} 0.$$

Thus the sequence $(u_n)_{n\in\mathbb{N}}$ has a limit $H^k(\mathbb{R}^d)$. This proves that $H^k(\mathbb{R}^d)$ is complete.

- By the Parseval identity we have for $u \in H^k(\mathbb{R}^d)$

$$\|u\|^2_{H^k(\mathbb{R}^d)} = \sum_{|\alpha|\leqslant k} \|\partial^\alpha u\|^2_{L^2(\mathbb{R}^d)} = \sum_{|\alpha|\leqslant k} \|\xi^\alpha \hat{u}\|^2_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \sum_{|\alpha|\leqslant k} \xi^{2\alpha} |\hat{u}(\xi)|^2 \, d\xi.$$

As in the proof of Proposition 5.9, we conclude that this norm is equivalent to (5.5) with Lemma 5.8. □

Remark 5.13. If $u \in H^k(\mathbb{R}^d)$ for all $k \in \mathbb{N}$ then we have $u \in C^\infty(\mathbb{R}^d)$.

5.2.2 Density of smooth functions

In this paragraph we prove the density of smooth functions in the Sobolev spaces.

Theorem 5.2. Let $k \in \mathbb{N}$. Then $C_0^\infty(\mathbb{R}^d)$ is dense in $H^k(\mathbb{R}^d)$.

Proof. Let $u \in H^k(\mathbb{R}^d)$ and $\varepsilon > 0$. Let $\chi \in C_0^\infty(\mathbb{R}^d)$ be supported in $B(0,2)$ and equal to 1 on $B(0,1)$. For $m \in \mathbb{N}^*$ and $x \in \mathbb{R}^d$ we set $\chi_m(x) = \chi(\frac{x}{m})$. Let $\alpha \in \mathbb{N}^d$ with $|\alpha| \leqslant k$. Let $m \in \mathbb{N}^*$. By the Leibniz rule we have $\chi_m u \in H^k(\mathbb{R}^d)$ and

$$\partial^\alpha(\chi_m u) - \chi_m \partial^\alpha u = \sum_{\substack{0\leqslant\beta\leqslant\alpha \\ \beta\neq\alpha}} \binom{\alpha}{\beta} \partial^{\alpha-\beta}\chi_m \partial^\beta u.$$

Since $(1-\chi_m)$ and $\partial^{\alpha-\beta}\chi_m$ vanish on $B(0,m)$ for all β, we have by the dominated convergence theorem

$$\begin{aligned}
&\|\partial^\alpha(\chi_m u) - \partial^\alpha u\|_{L^2(\mathbb{R}^d)} \\
&\quad = (\chi_m - 1)\partial^\alpha u + \big(\partial^\alpha(\chi_m u) - \chi_m \partial^\alpha u\big) \\
&\quad \leqslant \|\chi_m - 1\|_{L^\infty(\mathbb{R}^d)} \int_{|x|\geqslant m} |\partial^\alpha u(x)|^2 \, dx + \sum_{\substack{0\leqslant\beta\leqslant\alpha \\ \beta\neq\alpha}} \binom{\alpha}{\beta} \|\partial^{\alpha-\beta}\chi_m\|_{L^\infty(\mathbb{R}^d)} \int_{|x|\geqslant m} \left|\partial^\beta u(x)\right|^2 dx \\
&\quad \xrightarrow[m\to+\infty]{} 0.
\end{aligned}$$

Therefore there exists $m \in \mathbb{N}^*$ such that

$$\|u - \chi_m u\|_{H^k(\mathbb{R}^d)} \leqslant \frac{\varepsilon}{2}.$$

We set $v = \chi_m u$. Now let $(\rho_n)_{n\in\mathbb{N}}$ be an approximation of the identity with $\rho_n \in C_0^\infty(\mathbb{R}^d)$ for all $n \in \mathbb{N}$. For $n \in \mathbb{N}$ the function $u_n = (\rho_n * v)$ is smooth because ρ_n is, and it is compactly supported because v and ρ_n are. Let $\alpha \in \mathbb{N}^d$ with $|\alpha| \leqslant k$. We have $\partial^\alpha u_n = (\rho_n * \partial^\alpha v)$ so

$$\|\partial^\alpha u_n - \partial^\alpha v\|_{L^2(\mathbb{R}^d)} \xrightarrow[n\to+\infty]{} 0.$$

Thus there exists $n \in \mathbb{N}$ such that

$$\|u_n - v\|_{H^k(\mathbb{R}^d)} \leqslant \frac{\varepsilon}{2}.$$

The conclusion follows. □

With this density result we can extend to $H^k(\mathbb{R}^d)$ many results known for functions of class C^k. We give for instance the Green formula in $\mathbb{R}^d$.

Proposition 5.14 (Green Formula on $\mathbb{R}^d$)**.** *Let $u, v \in H^1(\mathbb{R}^d)$ and $j \in [\![1,d]\!]$. We have*

$$\int_{\mathbb{R}^d} \partial_j u \, v \, dx = -\int_{\mathbb{R}^d} u \, \partial_j v \, dx.$$

Proof. Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be sequences in $C_0^\infty(\mathbb{R}^d)$ which go to u and v in $H^1(\mathbb{R}^d)$. An integration by parts gives, for all $n \in \mathbb{N}$,

$$\int_{\mathbb{R}^d} \partial_j u_n \, v_n \, dx = - \int_{\mathbb{R}^d} u_n \, \partial_j v_n \, dx.$$

Taking the limit $n \to +\infty$ gives the result. □

5.3 Examples of applications for partial differential equations

5.3.1 The Helmholtz equation

In section 2.6 we have discussed on $\mathbb{R}^d$ the equation

$$-\Delta u + u = f,$$

where $f \in L^2(\mathbb{R}^d)$. Using the Fourier transform we saw that for any $f \in \mathcal{S}(\mathbb{R}^d)$ this problem has a unique solution $u \in \mathcal{S}(\mathbb{R}^d)$ and that the map $R : f \mapsto u$ extends to a continuous map on $L^2(\mathbb{R}^d)$.

The Sobolev spaces are the relevent context to discuss this kind of equation. We first observe that the operator $(-\Delta + \mathrm{Id})$ defines a continuous map from $H^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$. In fact, it defines a bijection with continuous inverse.

Proposition 5.15. *Let $f \in L^2(\mathbb{R}^d)$. Then there exists a unique $u \in H^2(\mathbb{R}^d)$ such that*

$$-\Delta u + u = f$$

in the sense of distributions, and there exists $C > 0$ independant of f such that

$$\|u\|_{H^2(\mathbb{R}^d)} \leqslant C \|f\|_{L^2(\mathbb{R}^d)} .$$

If moreover $f \in H^k(\mathbb{R}^d)$ for some $k \in \mathbb{N}$ then $u \in H^{k+2}(\mathbb{R}^d)$.

Proof. If we consider on $H^2(\mathbb{R}^d)$ the norm given by (5.5) we see that for all $u \in H^2(\mathbb{R}^d)$ we have

$$\|(-\Delta + \mathrm{Id})u\|_{L^2(\mathbb{R}^d)} = \|u\|_{H^2(\mathbb{R}^d)} ,$$

so $(-\Delta + \mathrm{Id})$ defines a continuous and injective map from $H^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$. Moreover, its inverse, defined on $\mathsf{Ran}(-\Delta + \mathrm{Id})$, is continuous. It remains to prove that $(-\Delta + \mathrm{Id})$ is surjective (or equivalently that $\mathsf{Ran}(-\Delta + \mathrm{Id}) = L^2(\mathbb{R}^d)$). For this we use the Fourier transform as in Section 2.6. Given $f \in L^2(\mathbb{R}^d)$ we consider the function $u \in L^2(\mathbb{R}^d)$ such that

$$\hat{u}(\xi) = \frac{\hat{f}(\xi)}{\xi^2 + 1}.$$

Then $u \in H^2(\mathbb{R}^d)$ by Proposition 5.9, and it belongs to $H^{k+2}(\mathbb{R}^d)$ if $f \in H^k(\mathbb{R}^d)$. Taking the inverse Fourier transform in the equality $(\xi^2+1)\hat{u}(\xi) = \hat{f}(\xi)$ we obtain $(-\Delta+\mathrm{Id})u = f$. This completes the proof. □

5.3.2 The Heat equation

Given $u_0 \in L^2(\mathbb{R}^d)$ we consider the heat equation

$$\forall t > 0, \quad \frac{du(t)}{dt} = \Delta u(t) \tag{5.6}$$

with the initial condition

$$u(0) = u_0. \tag{5.7}$$

It was already discussed in Paragraph 4.5.4 with the help of the convolution product. Here we use the Fourier transform to give a solution of the heat equation. We begin with a result of uniqueness.

Proposition 5.16. *Let $u \in C^0(\mathbb{R}_+, H^2(\mathbb{R}^d)) \cap C^1(\mathbb{R}_+^*, L^2(\mathbb{R}^d))$ be a solution of (5.6)-(5.7) with $u_0 = 0$. Then $u(t) = 0$ for all $t \geqslant 0$.*

Proof. The map $t \mapsto \|u(t)\|_{L^2(\mathbb{R}^d)}^2$ takes non-negative values, it is continuous on $[0, +\infty[$ and it is differentiable on $]0, +\infty[$. For $t > 0$ we have by the Green formula

$$\frac{d}{dt} \|u(t)\|_{L^2(\mathbb{R}^d)}^2 = 2 \operatorname{Re} \langle u(t), \Delta u(t) \rangle_{L^2(\mathbb{R}^d)} = -2 \|\nabla u(t)\|_{L^2(\mathbb{R}^d)}^2 \leqslant 0.$$

Since $\|u(0)\|_{L^2(\mathbb{R}^d)}^2 = 0$, we deduce that $\|u(t)\|_{L^2(\mathbb{R}^d)}^2 = 0$ for all $t > 0$. □

Proposition 5.17. *Let $k \in \mathbb{N}$ and $u_0 \in H^k(\mathbb{R}^d)$. For $t \geqslant 0$ we consider the unique $u(t) \in L^2(\mathbb{R}^d)$ such that for all $t \geqslant 0$ and $\xi \in \mathbb{R}$ we have*

$$\hat{u}(t, \xi) = e^{-t\xi^2} \widehat{u_0}(\xi).$$

Then u is continuous from $[0, +\infty[$ to $H^k(\mathbb{R}^d)$ and differentiable from $]0, +\infty[$ to $H^N(\mathbb{R}^d)$ for any $N \in \mathbb{N}$, and it solves the heat equation (5.6)-(5.7).

Proof. For $t_0 \geqslant 0$ and $t \geqslant 0$ we have

$$\|u(t) - u(t_0)\|_{H^k(\mathbb{R}^d)}^2 = \int_{\mathbb{R}^d} \left| e^{-t|\xi|^2} - e^{-t_0|\xi|^2} \right|^2 \left(1 + |\xi|^2\right)^k |\widehat{u_0}(\xi)|^2 \, d\xi.$$

For all $t \geqslant 0$ and $\xi \in \mathbb{R}^d$ we have

$$\left| e^{-t|\xi|^2} - e^{-t_0|\xi|^2} \right|^2 \left(1 + |\xi|^2\right)^k |\widehat{u_0}(\xi)|^2 \leqslant \left(1 + |\xi|^2\right)^k |\widehat{u_0}(\xi)|^2,$$

and the right-hand side defines an integrable function on $\mathbb{R}^d$, so by the dominated convergence theorem we have

$$\|u(t) - u(t_0)\|_{H^k(\mathbb{R}^d)}^2 \xrightarrow[t \to t_0]{} 0.$$

Now let $N \in \mathbb{N}$ and $t_0 > 0$. For $t > 0$, $t \neq t_0$, we have

$$\left\| \frac{u(t) - u(t_0)}{t - t_0} - \Delta u(t_0) \right\|_{H^N(\mathbb{R}^d)}^2 = \int_{\mathbb{R}^d} \left(1 + |\xi|^2\right)^N \left| \frac{e^{-t|\xi|^2} - e^{-t_0|\xi|^2}}{t - t_0} + |\xi|^2 e^{-t_0|\xi|^2} \right|^2 |\hat{u}(\xi)|^2 \, d\xi.$$

If $|t - t_0| \leqslant \frac{t_0}{2}$ and $\xi \in \mathbb{R}$ we have

$$(1 + \xi^2)^N \left| \frac{e^{-t|\xi|^2} - e^{-t_0|\xi|^2}}{t - t_0} + |\xi|^2 e^{-t_0|\xi|^2} \right|^2 |\hat{u}(\xi)|^2 \leqslant \frac{t_0}{4} \left(1 + |\xi|^2\right)^N |\xi|^4 e^{-\frac{t_0|\xi|^2}{2}} |\hat{u}(\xi)|^2.$$

Then we can apply the dominated convergence theorem, and we obtain that the map $t \mapsto u(t) \in H^N(\mathbb{R}^d)$ is differentiable at t_0 with

$$u'(t_0) = \Delta u(t_0).$$

The conclusion follows. □

5.3.3 The Wave equation

Let $c > 0$. We consider on $\mathbb{R}^d$ the wave equation

$$\partial_t^2 u - c^2 \Delta u = 0, \tag{5.8}$$

with initial conditions

$$u(0) = u_0, \quad \partial_t u(0) = u_1, \tag{5.9}$$

for some $u_0 \in H^2(\mathbb{R}^d)$ and $u_1 \in H^1(\mathbb{R}^d)$.

We recall that for $u_0 \in C^2(\mathbb{R})$ and $u_1 \in C^1(\mathbb{R})$ the problem (5.8)-(5.9) has a unique solution $u \in C^2(\mathbb{R}^2)$, given by

$$u(t, x) = \frac{u_0(x + ct) + u_0(x - ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} u_1(s) \, ds.$$

Theorem 5.3. Let $u_0 \in H^2(\mathbb{R}^d)$ and $u_1 \in H^1(\mathbb{R}^d)$. Then the problem (5.8)-(5.9) has a unique solution $u \in C^0(\mathbb{R}, H^2(\mathbb{R}^d)) \cap C^1(\mathbb{R}, H^1(\mathbb{R}^d)) \cap C^2(\mathbb{R}, L^2(\mathbb{R}^d))$.

Assume that u is a solution. Taking the Fourier transform $\hat{u}$ of u with respect to x we obtain for $t \in \mathbb{R}$ and $\xi \in \mathbb{R}^d$

$$\frac{\partial^2}{\partial t^2}\hat{u}(t,\xi) + c^2\xi^2\hat{u}(t,\xi) = 0.$$

Moreover $\hat{u}(0,\xi) = \widehat{u_0}(\xi)$ and $\partial_t \hat{u}(0,\xi) = \widehat{u_1}(\xi)$. For each $\xi \in \mathbb{R}^d$ we solve this second order equation with respect to t. This gives

$$\hat{u}(t,\xi) = \cos(ct\,|\xi|)\widehat{u_0}(\xi) + t\,\mathrm{sinc}(ct\,|\xi|)\widehat{u_1}(\xi), \tag{5.10}$$

where, for $\theta \in \mathbb{R}$,

$$\mathrm{sinc}(\theta) = \begin{cases} \frac{\sin(\theta)}{\theta} & \text{if } \theta \neq 0, \\ 1 & \text{if } \theta = 0. \end{cases}$$

Conversely, for all $t \in \mathbb{R}$ we define $u(t)$ as the inverse Fourier transform of (5.10) with respect to ξ. Then we check that u is indeed a solution of (5.8)-(5.9).

The uniqueness is given by the linearity of the problem and the following result about the conservation of the energy.

Proposition 5.18. *Let $u \in C^0(\mathbb{R}, H^2(\mathbb{R}^d)) \cap C^1(\mathbb{R}, H^1(\mathbb{R}^d)) \cap C^2(\mathbb{R}, L^2(\mathbb{R}^d))$ be a solution of* (5.8)*. For $t \in \mathbb{R}$ we set*

$$E(t) = \|\partial_t u(t)\|^2_{L^2(\mathbb{R}^d)} + \|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)}.$$

Then for all $t \in \mathbb{R}$ we have $E(t) = E(0)$.