Torus actions in the normalization problem

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Normalization Problem

Setting

Let $f\colon (\mathbb{C}^n, p) \to (\mathbb{C}^n, p)$ be germ of biholomorphism fixing p.

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Locally, using multi-index notation,

$$f(z) = \Lambda z + \sum_{\substack{Q \in \mathbb{N}^n \\ |Q| \geq 2}} f_Q z^Q,$$

where

$$z^Q := z_1^{q_1} \cdots z_n^{q_n}, \quad f_Q \in \mathbb{C}^n, \quad |Q| := \sum_{j=1}^{n} q_j,$$

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with Λ in Jordan normal form, i.e.,

$$\Lambda = \mathrm{Diag}(\lambda_1, \ldots, \lambda_n) + N$$

$N =$ nilpotent matrix and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}^*$ not necessarily distinct.

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N = nilpotent matrix and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}^*$ not necessarily distinct.

Want to know whether $\exists\varphi\colon (\mathbb{C}^n, O) \to (\mathbb{C}^n, O)$, local holomorphic change of coordinates, $d\varphi_O = \mathrm{Id}$, s.t. $\varphi^{-1}\circ f\circ\varphi$ has a simple form.

Normalization Problem

Linearization

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$$f \circ \varphi = \varphi \circ \Lambda \tag{1}$$

and then to study the convergence of φ.

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- $\varphi_j(z) = z_j + \sum_{|Q|\geq 2} \varphi_{Q,j} z^Q$.
- $\lambda^Q := \lambda_1^{q_1} \cdots \lambda_n^{q_n}$

$$(\lambda^Q - \lambda_j)\,\varphi_{Q,j} = \mathrm{Polynomial}(f_{P,j}, \varphi_{R,k}, \text{ with } P \leq Q, R < Q) \tag{2}$$

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- lexicographic order on $\mathbb{N}^n$

Normalization Problem

Definition

A resonant multi-index for $\lambda \in (\mathbb{C}^*)^n$, rel. to $j \in \{1, \ldots, n\}$ is $Q \in \mathbb{N}^n$, with $|Q| \geq 2$, s.t.

$$\lambda^Q = \lambda_j \, . \qquad (3)$$

$\mathrm{Res}_j(\lambda) := \{Q \in \mathbb{N}^n \mid |Q| \geq 2, \lambda^Q = \lambda_j\}$.

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Resonances $=$ obstruction to *formal* linearization.

Normalization Problem

Poincaré-Dulac normal forms

Theorem (Poincaré-Dulac, 1904)

$\forall f$ as above $\exists \widehat{\varphi}$ formal change of coord., $d\widehat{\varphi}_O = \mathrm{Id}$, s.t. $\widehat{\varphi}^{-1} \circ f \circ \widehat{\varphi} = g \in \mathbb{C}[\![z_1, \ldots, z_n]\!]^n$ where $g(O) = O$, $dg_O = df_O$ and g has only resonant monomials,

$$g_j(z) = \lambda_j z_j + \varepsilon_j z_{j+1} + \sum_{\substack{|Q|\geq 2 \\ \lambda^Q = \lambda_j}} g_{Q,j} z^Q .$$

Moreover, the resonant terms of $\widehat{\varphi}$ can be arbitrarily chosen, and that choice determines uniquely g^{res} and the remaining terms of $\widehat{\varphi}$.

A germ of the form $\Lambda + g^{\mathrm{res}}$, with g^{res} containing only resonant monomials is said in Poincaré-Dulac normal form.

Normalization Problem

Normalization Problem

Given f, $\exists^? \varphi \colon (\mathbb{C}^n, O) \to (\mathbb{C}^n, O)$, holomorphic change of coordinates, $d\varphi_O = \mathrm{Id}$, s.t.

$$\varphi^{-1} \circ f \circ \varphi \text{ is in Poincaré-Dulac normal form?}$$

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Problem

Not uniqueness of the formal change of coordinates $\widehat{\varphi}$ given by Poincaré-Dulac theorem, and not having explicit expression for g^{res}, make very difficult to give estimates for the convergence of $\widehat{\varphi}$.

Torus Actions

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In 2002, N.T. Zung found that to find a Poincaré-Dulac holomorphic normalization for a germ of holomorphic vector field is the same as to find (and linearize) a suitable torus action which preserves the vector field.

Torus Actions

Germs commuting with a torus action

Theorem (–, 2009)

f commutes with a holom. effective $\mathbb{T}^r$-action on $(\mathbb{C}^n, O)$, $1 \leq r \leq n$, with weight matrix $\Theta \in M_{n\times r}(\mathbb{Z})$

$$\Updownarrow$$

$\exists\varphi$ local holom. change of coord. s.t. $\varphi^{-1} \circ f \circ \varphi$ contains only Θ-resonant monomials.

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f commutes with $A\colon \mathbb{T}^r \times (\mathbb{C}^n, O) \to (\mathbb{C}^n, O)$, $A(x, O) = O$, means

$$f(A(x,z)) = A(x, f(z)).$$

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A^{lin} is semi-simple and $\mathrm{Sp}(A^{\mathrm{lin}}(x,\cdot)) = \{\exp(2\pi i \sum_{k=1}^r x_k \theta_j^k)\}_{j=1,\dots,n}$ where $\Theta = (\theta_j^k) \in M_{n\times r}(\mathbb{Z})$ is the weight matrix of A.

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$z^Q e_j$, with $|Q| \ge 1$, is Θ-resonant if

$$\langle Q, \theta^k\rangle := \sum_{h=1}^n q_h \theta_h^k = \theta_j^k \quad \forall\, k = 1, \dots, r.$$

Torus Actions

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An additive resonant multi-index for $\theta \in \mathbb{C}^n$, rel. to $j \in \{1, \ldots, n\}$ is $Q \in \mathbb{N}^n$, with $|Q| \geq 2$, s.t.

$$\langle Q, \theta \rangle = \theta_j \,. \tag{4}$$

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Corollary (–, 2009)

f is holomorphically linearizable

$$\Updownarrow$$

it commutes with a $\mathbb{T}^r$-action, $1 \leq r \leq n$, with Θ having no resonances of degree $|Q| \geq 2$.

Strategy

Torus Actions   Holomorphic Normalization

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$\bigcap_{k=1}^{r} \mathrm{Res}_j^+(\theta^k)$	$\mathrm{Res}_j(\lambda)$

Toric Degree

Definition

The toric degree of $\lambda \in (\mathbb{C}^*)^n$ is the min $r \in \mathbb{N}$ s.t. $\exists \alpha_1, \ldots, \alpha_r \in \mathbb{C}^*$ and $\exists \theta^{(1)}, \ldots, \theta^{(r)} \in \mathbb{Z}^n$ s.t.

$$[\varphi] = \left[\sum_{k=1}^{r} \alpha_k \theta^{(k)} \right] \in (\mathbb{C}/\mathbb{Z})^n,$$

where $[\varphi]$ is the unique in $(\mathbb{C}/\mathbb{Z})^n$ s.t. $\lambda = e^{2\pi i [\varphi]}$.
$\theta^{(1)}, \ldots, \theta^{(r)}$ are a r-tuple of toric vectors associated to λ, with toric coefficients $\alpha_1, \ldots, \alpha_r$.

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- $\alpha_1, \ldots, \alpha_r$ are $\mathbb{Z}$-independent,
- $\theta^{(1)}, \ldots, \theta^{(r)}$ are $\mathbb{Q}$-linearly independent.

Main Result

Theorem (–, 2009)

Take f as above. Then, in all but one case, f is holomorphically normalizable

$$\Updownarrow$$

f commutes with a holom. effective torus action on $(\mathbb{C}^n, O)$ of dim depending on $\mathrm{tordeg}(\lambda)$ with columns of Θ related to toric vectors associated to λ.

First distinction

Proposition (–, 2009)

$\exists$ *toric* r*-tuple assoc. to* λ *with coeff.* $\mathbb{Z}$*-independent with* 1

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- Torsion case: $1, \alpha_1, \dots, \alpha_r$ $\mathbb{Z}$-dependent

In the torsion case we can always consider reduced toric r-tuples, i.e., $\eta^{(1)}, \dots, \eta^{(r)}$ with coeff. $\beta_1, \dots, \beta_r$ s.t. $\beta_1 = 1/m$ with $m \in \mathbb{N} \setminus \{0, 1\}$ and $m, \eta_1^{(1)}, \dots, \eta_n^{(1)}$ coprime; $\eta^{(2)}, \dots, \eta^{(r)}$ are called reduced torsion-free toric vectors assoc. to λ.

Torsion-free Case

Main Theorem in the torsion-free case

$\forall$ r-tuple of toric vectors $\theta^{(1)}, \ldots, \theta^{(r)}$ associated to λ

$$\mathrm{Res}_j(\lambda) = \bigcap_{k=1}^{r} \mathrm{Res}_j^+(\theta^{(k)}) \quad \forall\, j = 1, \ldots, n$$

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Main Theorem

f in the torsion-free case is holomorphically normalizable

$$\Updownarrow$$

f commutes with a holom. effective $\mathbb{T}^r$-action on $(\mathbb{C}^n, O)$, $r = \mathrm{tordeg}(\lambda)$, with columns of Θ that are a r-tuple of toric vectors associated to λ.

Torsion Case

$\forall$ reduced toric r-tuple $\eta^{(1)}, \ldots, \eta^{(r)}$ associated to λ

$$\bigcap_{k=2}^{r} \mathrm{Res}_j^{+}(\eta^{(k)}) \supseteq \mathrm{Res}_j(\lambda) \supseteq \bigcap_{k=1}^{r} \mathrm{Res}_j^{+}(\eta^{(k)}). \tag{5}$$

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We have the following sub-cases:

- Impure torsion case: for a reduced toric r-tuple ($\Rightarrow \forall$)

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- Pure torsion case: the first inclusion is always strict, and *either*
  - λ can be simplified: $\exists$ a reduced toric r-tuple, said simple, s.t. $\mathrm{Res}_j(\lambda) = \bigcap_{k=1}^{r} \mathrm{Res}_j^+(\eta^{(k)})\ \forall j$,

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- Pure torsion case: the first inclusion is always strict, and *either*
  - λ can be simplified: $\exists$ a reduced toric r-tuple, said simple, s.t. $\mathrm{Res}_j(\lambda) = \bigcap_{k=1}^{r} \mathrm{Res}_j^+(\eta^{(k)})$ $\forall j$, *or*
  - λ cannot be simplified: $\forall$ reduced toric r-tuple, $\exists j$ s.t. the inclusions in (5) are strict.

Main Theorem

f in the impure torsion case is holomorphically normalizable

$$\Updownarrow$$

f commutes with a holom. effective $\mathbb{T}^{r-1}$-action on $(\mathbb{C}^n, O)$, $r = \mathrm{tordeg}(\lambda)$, with columns of Θ that are reduced torsion-free toric vectors associated to λ.

Pure Torsion Case

λ can be simplified

Main Theorem

f in the pure torsion case s.t. λ can be simplified is holomorphically normalizable

$$\Updownarrow$$

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- *df_O diagonalizable: f commutes with a holom. effective $\mathbb{T}^r$-action on $(\mathbb{C}^n, O)$, $r = \operatorname{tordeg}(\lambda)$, with columns of Θ that are a reduced simple r-tuple of toric vectors associated to λ;*

Pure Torsion Case

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Main Theorem

f in the pure torsion case s.t. λ can be simplified is holomorphically normalizable

$$\Updownarrow$$

- *df_O diagonalizable: f commutes with a holom. effective $\mathbb{T}^r$-action on $(\mathbb{C}^n, O)$, $r = \mathrm{tordeg}(\lambda)$, with columns of Θ that are a reduced simple r-tuple of toric vectors associated to λ;*
- *df_O not diagonalizable: if $\lambda_j = \lambda_h \Rightarrow \eta_j^{(k)} = \eta_h^{(k)}\ \forall\, k = 1, \ldots, r$, then same statement as above.*

Pure Torsion Case

λ cannot be simplified

Proposition (–, 2009)

f in the pure torsion case s.t. λ cannot be simplified. If f commutes with a holom. effective $\mathbb{T}^r$-action on $(\mathbb{C}^n, O)$, $r = \mathrm{tordeg}(\lambda)$, with columns of Θ that are reduced toric vectors associated to λ, then f is holomorphically normalizable.

Remark: Torsion

Écalle, in 1992, introduced the following notion of torsion.

Definition

The *torsion* of $\lambda \in (\mathbb{C}^*)^n$ is the natural integer τ such that

$$\frac{1}{\tau}\mathbb{Z} = \mathbb{Q} \cap \left(\mathbb{Z} \bigoplus_{1 \leq j \leq n} \left(\frac{\log(\lambda_j)}{2\pi i} \mathbb{Z} \right) \right) .$$

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Lemma (–, 2009)

$\lambda \in (\mathbb{C}^*)^n$ *is torsion-free* $\iff$ *its torsion is* 1

Final Remarks

- algorithmic way to compute resonances

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- algorithmic way to compute resonances
- examples for each case $\Longrightarrow$ consistent classification
- example of techniques to construct torus actions
- torsion is not enough to measure the difference between germs of holomorphic vector fields and germs of biholomorphisms

Thanks

Examples

Example (Torsion-free case)

$$\left[\sqrt{2}\begin{pmatrix}3\\2\\-1\end{pmatrix}+2i\begin{pmatrix}2\\3\\1\end{pmatrix}\right]\in(\mathbb{C}/\mathbb{Z})^3$$

Example (Torsion case)

$$\left[\frac{1}{7}\begin{pmatrix}3\\2\\-1\end{pmatrix}+2i\begin{pmatrix}2\\3\\1\end{pmatrix}\right]\in(\mathbb{C}/\mathbb{Z})^3$$

Example (Impure torsion case)

$$\left[\frac{1}{3}\begin{pmatrix}0\\0\\1\\1\end{pmatrix}+\sqrt{2}\begin{pmatrix}-12\\0\\0\\1\end{pmatrix}+\sqrt{3}\begin{pmatrix}0\\5\\2\\0\end{pmatrix}\right]\in(\mathbb{C}/\mathbb{Z})^4$$

has toric degree 3.

Example (Impure torsion case)

$$\mathrm{Res}_1^+(\eta^{(2)}) = \{(q_1, q_2, q_3, 12(q_1 - 1)) \in \mathbb{N}^4 \mid 13q_1 + q_2 + q_3 \geq 14\}$$
$$\mathrm{Res}_2^+(\eta^{(2)}) = \{(q_1, q_2, q_3, 12q_1) \in \mathbb{N}^4 \mid 13q_1 + q_2 + q_3 \geq 2\}$$
$$\mathrm{Res}_3^+(\eta^{(2)}) = \mathrm{Res}_2^+(\eta^{(2)})$$
$$\mathrm{Res}_4^+(\eta^{(2)}) = \{(q_1, q_2, q_3, 12q_1 + 1) \in \mathbb{N}^4 \mid 13q_1 + q_2 + q_3 \geq 1\},$$

and

$$\mathrm{Res}_1^+(\eta^{(3)}) = \{(q_1, 0, 0, q_4) \in \mathbb{N}^4 \mid q_1 + q_4 \geq 2\}$$
$$\mathrm{Res}_2^+(\eta^{(3)}) = \{(q_1, 1, 0, q_4) \in \mathbb{N}^4 \mid q_1 + q_4 \geq 1\}$$
$$\mathrm{Res}_3^+(\eta^{(3)}) = \{(q_1, 0, 1, q_4) \in \mathbb{N}^4 \mid q_1 + q_4 \geq 1\}$$
$$\mathrm{Res}_4^+(\eta^{(3)}) = \mathrm{Res}_1^+(\eta^{(3)}).$$

$$\forall P = (p, 0, 0, 12p) \text{ with } p \geq 1 \quad \langle P, \eta^{(1)} \rangle = 12p \in 3\,\mathbb{Z}.$$

Then it is easy to verify that for $j = 1, \ldots, 4$

$$\mathrm{Res}_j([\varphi]) = \mathrm{Res}_j^+(\eta^{(2)}) \cap \mathrm{Res}_j^+(\eta^{(3)}).$$

Examples

Example (Pure torsion case that can be simplified)

$$[\varphi] = \left[\frac{1}{3}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} + \sqrt{2}\begin{pmatrix}1\\6\\0\\0\end{pmatrix} + \sqrt{3}\begin{pmatrix}0\\0\\-1\\5\end{pmatrix} \right] \in (\mathbb{C}/\mathbb{Z})^4,$$

has toric degree 3. We have $\mathrm{Res}_j^+(\eta^{(1)}) = \emptyset$, for $j = 1, \dots, 4$, and it is not difficult to verify that

$$\mathrm{Res}_2(\lambda) = \{(0, 1, 5q, q) \mid q \in \mathbb{N}^*\} \neq \emptyset$$

$$\mathrm{Res}_j(\lambda) = \mathrm{Res}_j^+(\eta^{(2)}) \cap \mathrm{Res}_j^+(\eta^{(3)}) \;\; j = 1, 3, 4.$$

Examples

Example (Pure torsion case that can be simplified)

However, we can write

$$[\varphi] = \left[\frac{1}{3} \begin{pmatrix} 1 \\ -2 \\ 1 \\ -5 \end{pmatrix} + \sqrt{2} \begin{pmatrix} 1 \\ 6 \\ 0 \\ 0 \end{pmatrix} + \sqrt{3} \begin{pmatrix} 0 \\ 0 \\ -1 \\ 5 \end{pmatrix} \right],$$

and it is not difficult to verify that, in this representation, we have, for $j = 1, \ldots, 4$

$$\mathrm{Res}_j(\lambda) = \bigcap_{k=1}^{3} \mathrm{Res}_j^+(\xi^{(k)}).$$

Examples

Example (Pure torsion case that cannot be simplified)

$$\left[\frac{1}{7}\begin{pmatrix}1\\3\end{pmatrix} + \sqrt{2}\begin{pmatrix}1\\-6\end{pmatrix}\right] \in (\mathbb{C}/\mathbb{Z})^2,$$

has toric degree 2 and torsion 7. We have

$$\mathrm{Res}_1^+(\eta^{(2)}) = \{(6h+1, h) \mid h \geq 1\}, \quad \mathrm{Res}_2^+(\eta^{(2)}) = \{(6h, h+1) \mid h \geq 1\},$$

$$\mathrm{Res}_1^+(\eta^{(1)}) \cap \mathrm{Res}_1^+(\eta^{(2)}) = \mathrm{Res}_2^+(\eta^{(1)}) \cap \mathrm{Res}_2^+(\eta^{(2)}) = \emptyset,$$

then

$$\mathrm{Res}_1^+(\eta^{(2)}) \supset \mathrm{Res}_1(\lambda) = \{(42h+1, 7h) \mid h \geq 1\} \supset \mathrm{Res}_1^+(\eta^{(1)}) \cap \mathrm{Res}_1^+(\eta^{(2)})$$

$$\mathrm{Res}_2^+(\eta^{(2)}) \supset \mathrm{Res}_2(\lambda) = \{(42h, 7h+1) \mid h \geq 1\} \supset \mathrm{Res}_2^+(\eta^{(1)}) \cap \mathrm{Res}_2^+(\eta^{(2)}).$$

Furthermore, it is easy to check that λ cannot be simplified.

Construction of Torus Action

Theorem (–, 2009)

Let f be as above and commute with a set of integrable holomorphic vector fields $X_1, \ldots, X_m$, $1 \leq m \leq n$. Then f commutes with a holom. effective $\mathbb{T}^r$-action on $(\mathbb{C}^n, O)$, $r = \mathrm{tordeg}(X_1)$, with columns of the weight matrix that are a r-tuple of toric vectors associated to X_1.

Where, if $1 \leq m \leq n$, f commute with a set of integrable holomorphic vector fields if $\exists X_1, \ldots, X_m$ s.t.

$$df(X_j) = X_j \circ f$$

$\forall j = 1, \ldots, m$ that are integrable, i.e.,

- $X_1, \ldots, X_m$ germs of holom. v.f. of $(\mathbb{C}^n, O)$, $X_j(O) = 0$, $\mathrm{order}(X_j) = 1$, $[X_j, X_k] = 0\ \forall j, k$, and $X_1 \wedge \cdots \wedge X_m \not\equiv 0$;
- (ii) $\exists g_1, \ldots, g_{n-m}$ germs of holom. functions of $(\mathbb{C}^n, O)$ s.t. $X_j(g_k) = 0\ \forall j, k$, and $dg_1 \wedge \cdots \wedge dg_{n-m} \not\equiv 0$.

Construction of Torus Action

Toric degree of a vector field

Writing $X = \sum_{j=1}^{n} \varphi_j z_j \frac{\partial}{\partial z_j} + \cdots$, the toric degree of X is the min $r \in \mathbb{N}$ s.t. $\exists \alpha_1, \ldots, \alpha_r \in \mathbb{C}^*$ and $\exists \theta^{(1)}, \ldots, \theta^{(r)} \in \mathbb{Z}^n$ s.t.

$$\varphi = \sum_{k=1}^{r} \alpha_k \theta^{(k)}.$$