Holomorphic linearization of commuting germs of holomorphic maps

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Linearization Problem

Given $f\colon (\mathbb{C}^n, p) \to (\mathbb{C}^n, p)$ a germ of biholomorphism, $f(p) = p$, $\exists^? \varphi$ local holomorphic change of coordinates, s.t.

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Classical Idea: first look for a solution of

$$f \circ \varphi = \varphi \circ \Lambda$$

in the setting of formal power series, and then check whether φ is convergent.

Linearization Problem

Given $f\colon (\mathbb{C}^n, O) \to (\mathbb{C}^n, O)$ a germ of biholomorphism, $f(O) = O$, with linear part in Jordan normal form

$$\Lambda = \begin{pmatrix} \lambda_1 & & & \\ \varepsilon_1 & \lambda_2 & & \\ & \ddots & \ddots & \\ & & \varepsilon_{n-1} & \lambda_n \end{pmatrix} \qquad \lambda_1, \ldots, \lambda_n \in \mathbb{C}^*, \quad \varepsilon_j \neq 0 \Rightarrow \lambda_j = \lambda_{j+1},$$

$\exists^? \varphi$ local holomorphic change of coordinates, s.t. $d\varphi_O = \mathrm{Id}$ and

$$\varphi^{-1} \circ f \circ \varphi = \Lambda?$$

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Dimension 1

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  - Yoccoz: Brjuno condition for λ $\iff$ the quadratic polynomial $\lambda z + z^2$ is holormorphically linearizable (and moreover $\lambda z + z^2$ hol. lin. $\Rightarrow f(z) = \lambda z + \cdots$ hol. lin)

Linearization Problem

Dimension $n \geq 2$

Formal Obstruction

A resonant multi-index for $\lambda \in (\mathbb{C}^*)^n$, rel. to $j \in \{1, \ldots, n\}$ is $Q \in \mathbb{N}^n$, with $|Q| = \sum_{h=1}^n q_h \geq 2$, s.t.

$$\Lambda^Q - \lambda_j = 0$$

where $\Lambda^Q := \lambda_1^{q_1} \cdots \lambda_n^{q_n}$.
$\mathrm{Res}_j(\Lambda) := \{Q \in \mathbb{N}^n \mid |Q| \geq 2, \Lambda^Q - \lambda_j = 0\}$.

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But there are formal, and holomorphic, linearization results also in presence of resonances

Theorem (Rüssmann 2002, R. 2010)

f formally linearizable + Brjuno reduced condition $\Rightarrow$ f holomorphically linearizable

Simultaneous Linearization

Simultaneous Linearization Problem

Given $h \geq 2$ germs of biholomorphisms $f_1, \ldots, f_h$ of $\mathbb{C}^n$ at the same fixed point $\exists^? \varphi$ a local holomorphic change of coordinates conjugating f_k to its linear part for each $k = 1, \ldots, h$?

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Moser, 1990: raised the problem of smooth linearization of commuting circle diffeomorphisms in connection with the holonomy group of certain foliations of codimension 1; with the rapidly convergent Nash-Moser iteration scheme, he proved that if the rotation numbers of the diffeomorphisms satisfy a simultaneous Diophantine condition and if the diffeomorphisms are in some C^∞-neighborhood of the corresponding rotations, then they are C^∞-conjugated to rotations.

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Pérez-Marco, 1997: commuting systems of analytic or smooth circle diffeomorphisms are deeply related to commuting systems of germs of holomorphic functions.

Fayad and Khanin, 2009: a finite number of commuting smooth circle diffeomorphisms with simultaneously Diophantine rotation numbers are smoothly conjugated to rotations.

Simultaneous Linearization

Dimension $n \geq 2$

Gramchev and Yoshino, 1999: simultaneous holomorphic linearization for pairwise commuting germs without simultaneous resonances, with diagonalizable linear parts, and under a simultaneous Diophantine condition (further studied by Yoshino, 2004) and a few more technical assumptions.

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–, 2009: $h \geq 2$ germs $f_1, \ldots, f_h$ of biholomorphisms of $\mathbb{C}^n$, fixing the origin, s.t. the linear part of f_1 is diagonalizable and f_1 commutes with f_k for any $k = 2, \ldots, h$, under certain arithmetic conditions on the eigenvalues of the linear part of f_1 and some restrictions on their resonances, are simultaneously holomorphically linearizable if and only if there exists a particular complex manifold invariant under $f_1, \ldots, f_h$.

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Are there any conditions on the eigenvalues of the linear parts of $h \geq 2$ germs of simultaneously formally linearizable biholomorphisms ensuring simultaneous holomorphic linearizability?

Q3: generalization of Moser's question

Under which conditions on the eigenvalues of the linear parts of $h \geq 2$ pairwise commuting germs of biholomorphisms can one assert the existence of a simultaneous holomorphic linearization of the given germs? In particular, is there a Brjuno-type condition sufficient for convergence?

Q1: shape of simultaneous linearization

Proposition (–, 2010)

Let $f_1, \ldots, f_h$ be $h \geq 2$ formally linearizable germs of biholomorphisms of $(\mathbb{C}^n, O)$, with almost simultaneously Jordanizable linear parts.
$f_1, \ldots, f_h$ simultaneously formally linearizable $\Longrightarrow \exists! \varphi$ formal simultaneous linearization s.t. $\varphi_{Q,j} = 0 \ \forall Q, j : Q \in \cap_{k=1}^{h} \mathrm{Res}_j(\Lambda_k)$.

Q1: shape of simultaneous linearization

Definition

$M_1, \ldots, M_h$, $h \geq 2$ complex $n \times n$ matrices are almost simultaneously Jordanizable, if $\exists$ a linear change of coordinates A s.t. $A^{-1}M_1A, \ldots, A^{-1}M_hA$ are almost in simultaneous Jordan normal form, i.e., for $k = 1, \ldots, h$ we have

$$A^{-1}M_kA = \begin{pmatrix} \lambda_{k,1} & & & \\ \varepsilon_{k,1} & \lambda_{k,2} & & \\ & \ddots & \ddots & \\ & & \varepsilon_{k,n-1} & \lambda_{k,n} \end{pmatrix}, \quad \varepsilon_{k,j} \neq 0 \Rightarrow \lambda_{k,j} = \lambda_{k,j+1}. \quad (1)$$

$M_1, \ldots, M_h$ are simultaneously Jordanizable if $\exists$ a linear change of coordinates A s.t. we have (1) with $\varepsilon_{k,j} \in \{0, \varepsilon\}$.

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The hypothesis on the pairwise commutation is indeed necessary: If Λ_1 and Λ_2 are two commuting matrices almost in simultaneous Jordan n.f. s.t. $\mathrm{Res}(\Lambda_1) \neq \emptyset$ and $\mathrm{Res}(\Lambda_2) \neq \emptyset$, but $\mathrm{Res}(\Lambda_1) \cap \mathrm{Res}(\Lambda_2) = \emptyset$, the unique formal transformation tangent to the identity and commuting with both Λ_1 and Λ_2 is the identity, so any non-linear germ f_3 with linear part in Jordan normal form and commuting with Λ_1 (i.e., containing only Λ_1-resonant terms) but not with Λ_2 cannot be simultaneously linearizable with Λ_1 and Λ_2.

Q2: conditions on the eigenvalues

Theorem (–, 2010)

Let $f_1, \ldots, f_h$ be $h \geq 2$ simultaneously formally linearizable germs of biholomorphism of $(\mathbb{C}^n, O)$ s.t. their linear parts $\Lambda_1, \ldots, \Lambda_h$ are simultaneously diagonalizable. If $f_1, \ldots, f_h$ satisfy the simultaneous Brjuno condition, then $f_1, \ldots f_h$ are holomorphically simultaneously linearizable.

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Definition

$\Lambda_1 = (\lambda_{1,1}, \dots, \lambda_{1,n}), \dots, \Lambda_h = (\lambda_{h,1}, \dots, \lambda_{h,n})$, $h \geq 2$ n-tuples of complex, not nec. distinct, non-zero numbers, satisfy the simultaneous Brjuno condition if

$$\sum_{\nu \geq 0} \frac{1}{2^\nu} \log \frac{1}{\omega_{\Lambda_1,\dots,\Lambda_h}(2^{\nu+1})} < +\infty,$$

where $\forall m \geq 2$

$$\omega_{\Lambda_1,\dots,\Lambda_h}(m) := \min_{\substack{2 \leq |Q| \leq m \\ Q \notin \cap_{k=1}^h \cap_{j=1}^n \mathrm{Res}_j(\Lambda_k)}} \varepsilon_Q,$$

with

$$\varepsilon_Q = \max_{1 \leq k \leq h} \min_{\substack{1 \leq j \leq n \\ Q \notin \cap_{k=1}^h \cap_{j=1}^n \mathrm{Res}_j(\Lambda_k)}} |\Lambda_k^Q - \lambda_{k,j}|.$$

If $\Lambda_1, \dots, \Lambda_h$ are the sets of eigenvalues of the linear parts of $f_1, \dots, f_h$, we say that $f_1, \dots, f_h$ satisfy the simultaneous Brjuno condition.

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Theorem (–, 2010)

Let $f_1, \ldots, f_h$ be $h \geq 2$ formally linearizable germs of biholomorphisms of $(\mathbb{C}^n, O)$ with simultaneously diagonalizable linear parts, and satisfying the simultaneous Brjuno condition. Then $f_1, \ldots, f_h$ are sim. hol. linearizable $\iff$ they all commute pairwise.

Remark

If $\Lambda_1, \ldots, \Lambda_h$ do not satisfy the sim. Brjuno cond., then each of them does not satisfy the reduced Brjuno cond., i.e.,

$$\sum_{\nu \geq 0} \frac{1}{2^\nu} \log \frac{1}{\omega_{\Lambda_k}(2^{\nu+1})} = +\infty, \quad k = 1, \ldots, h,$$

$$\omega_{\Lambda_k}(m) := \min_{\substack{2 \leq |Q| \leq m \\ 1 \leq j \leq n \\ Q \notin \mathrm{Res}_j(\Lambda_k)}} |\Lambda_k^Q - \lambda_{k,j}|.$$

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In particular, if $\Lambda_1, \ldots, \Lambda_h$ are simultaneously Cremer, i.e.,

$$\limsup_{m \to +\infty} \frac{1}{m} \log \frac{1}{\omega_{\Lambda_1, \ldots, \Lambda_h}(m)} = +\infty,$$

and hence they do not satisfy the sim. Brjuno cond., then at least one of them has to be Cremer, i.e.,

$$\limsup_{m \to +\infty} \frac{1}{m} \log \frac{1}{\omega_{\Lambda_k}(m)} = +\infty,$$

and the other ones do not satisfy the reduced Brjuno condition.

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Furthermore (following Moser and Yoshino) it is possible to find $\Lambda_1, \ldots, \Lambda_h$ satisfying the simultaneous Brjuno condition, with Λ_k not satisfying the reduced Brjuno condition for any $k = 1, \ldots, h$.

Thanks!

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- if $h \geq 2$ matrices commute pairwise $\Rightarrow$ they are simultaneously triangularizable (but the converse is clearly false).
- if two matrices commute then this does not imply that they admit an almost simultaneous Jordan normal form, and it is not true in general that two matrices almost in simultaneous Jordan normal form commute

Examples

The two matrices

$$\Lambda = \begin{pmatrix} \lambda & 0 & 0 \\ \varepsilon & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \quad M = \begin{pmatrix} \mu & 0 & 0 \\ \delta & \mu & 0 \\ \beta & 0 & \mu \end{pmatrix} \quad \lambda, \varepsilon, \mu, \delta, \beta \in \mathbb{C}^*$$

commute, but they are not almost simultaneously Jordanizable. In fact $\forall A$ s.t. AMA^{-1} is almost in sim. Jordan n.f. with Λ we have

$$AM = \begin{pmatrix} \mu & 0 & 0 \\ \zeta & \mu & 0 \\ 0 & 0 & \mu \end{pmatrix} A \qquad (M \neq \mu I_3 \Rightarrow \zeta \neq 0)$$

$$\Rightarrow A = \begin{pmatrix} \frac{\beta}{\zeta} f + \frac{\delta}{\zeta} e & 0 & 0 \\ d & e & f \\ g & h & -\frac{\delta}{\beta} h \end{pmatrix}, \text{ and } \det A \neq 0 \iff \beta f + \delta e \neq 0, h \neq 0.$$

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$$A\Lambda = \begin{pmatrix} \lambda & 0 & 0 \\ \xi & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} A \Longrightarrow h = 0 \Longrightarrow \Lambda, M \text{ not almost sim Jordanizable}$$

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$$\widetilde{\Lambda} = \begin{pmatrix} \lambda & 0 & 0 \\ \varepsilon & \lambda & 0 \\ 0 & \varepsilon & \lambda \end{pmatrix} \quad \widetilde{M} = \begin{pmatrix} \mu & 0 & 0 \\ \delta & \mu & 0 \\ 0 & 0 & \eta \end{pmatrix} \quad \lambda, \varepsilon, \mu, \delta, \eta \in \mathbb{C}^*$$

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