Holomorphic linearization of commuting germs of holomorphic maps

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ABSTRACT. Let $f_1, \ldots, f_h$ be $h \geq 2$ germs of biholomorphisms of $\mathbb{C}^n$ fixing the origin. We investigate the shape a (formal) simultaneous linearization of the given germs can have, and we prove that if $f_1, \ldots, f_h$ commute and their linear parts are almost simultaneously Jordanizable then they are simultaneously formally linearizable. We next introduce a simultaneous Brjuno-type condition and prove that, in case the linear terms of the germs are diagonalizable, if the germs commute and our Brjuno-type condition holds, then they are holomorphically simultaneously linerizable. This answers to a multi-dimensional version of a problem raised by Moser.

1. Introduction

One of the main questions in the study of local holomorphic dynamics (see [A1], [A2], [Bra], or [R3] Chapter 1, for general surveys on this topic) is when a germ of biholomorphism f, fixing the origin, is *holomorphically linearizable*, i.e., when there exists a local holomorphic change of coordinates such that f is conjugated to its linear part Λ.

A way to solve such a problem is to first look for a formal transformation φ solving

$$f \circ \varphi = \varphi \circ \Lambda,$$

i.e., to ask when f is *formally linearizable*, and then to check whether φ is convergent. Moreover, since up to linear changes of the coordinates we can always assume Λ to be in Jordan normal form, i.e.,

$$\Lambda = \begin{pmatrix} \lambda_1 & & & \\ \varepsilon_1 & \lambda_2 & & \\ & \ddots & \ddots & \\ & & \varepsilon_{n-1} & \lambda_n \end{pmatrix},$$

where the eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{C}^*$ are not necessarily distinct, and ε_j can be non-zero only if $\lambda_{j+1} = \lambda_j$, we can reduce ourselves to study such germs, and to search for φ *tangent to the identity*, that is, with linear part equal to the identity.

The answer to this question depends on the set of eigenvalues of the linear part of f, usually called its *spectrum*. In fact, if we denote by $\Lambda = (\lambda_1, \ldots, \lambda_n) \in (\mathbb{C}^*)^n$ the set of

the eigenvalues, then it may happen that there exists a multi-index $Q = (q_1, \ldots, q_n) \in \mathbb{N}^n$, with $|Q| := \sum_{j=1}^n q_j \geq 2$, such that

$$\Lambda^Q - \lambda_j := \lambda_1^{q_1} \cdots \lambda_n^{q_n} - \lambda_j = 0 \tag{1}$$

for some $1 \leq j \leq n$; a relation of this kind is called a *(multiplicative) resonance of f relative to the j-th coordinate*, Q is called a *resonant multi-index relative to the j-th coordinate*, and we put

$$\mathrm{Res}_j(\Lambda) := \{Q \in \mathbb{N}^n \mid |Q| \geq 2, \Lambda^Q = \lambda_j\}. \tag{2}$$

The elements of $\mathrm{Res}(\Lambda) := \bigcup_{j=1}^n \mathrm{Res}_j(\Lambda)$ are simply called *resonant* multi-indices. A *resonant monomial* is a monomial $z^Q := z_1^{q_1} \cdots z_n^{q_n}$ in the j-th coordinate with $Q \in \mathrm{Res}_j(\Lambda)$.

Resonances are the formal obstruction to linearization. Anyway there are formal, and holomorphic, linearization results also in presence of resonances, see for example [R1] and [R3], and references therein.

One generalization of the previous question is to ask when $h \geq 2$ germs of biholomorphisms $f_1, \ldots, f_h$ of $\mathbb{C}^n$ at the same fixed point, which we may place at the origin, are *simultaneously holomorphically linearizable*, i.e., there exists a local holomorphic change of coordinates conjugating f_k to its linear part for each $k = 1, \ldots, h$.

In dimension 1, this problem has been thoroughly studied, also for commuting systems of analytic or smooth circle diffeomorphisms, that are indeed deeply related to commuting systems of germs of holomorphic functions, as explained in [P]. The question about the smoothness of a simultaneous linearization of such a system, raised by Arnold, was brilliantly answered by Herman [H], and extended by Yoccoz [Y1] (see also [Y3]). In [M], Moser raised the problem of smooth linearization of commuting circle diffeomorphisms in connection with the holonomy group of certain foliations of codimension 1, and, using the rapidly convergent Nash-Moser iteration scheme, he proved that if the rotation numbers of the diffeomorphisms satisfy a simultaneous Diophantine condition and if the diffeomorphisms are in some C^∞-neighborhood of the corresponding rotations (the neighborhood being imposed by the constants appearing in the arithmetic condition, as usual in perturbative KAM theorems) then they are C^∞-linearizable, that is, C^∞-conjugated to rotations. We refer to [FK] and references therein for a clear exposition of the one-dimensional problem and for the best results, up to now, in such a context. Furthermore, the problem for commuting germs of holomorphic functions in dimension one has been studied by DeLatte [D], and more recently by Biswas [Bi], under Brjuno-type conditions generalizing Moser's simultaneous Diophantine condition.

In dimension $n \geq 2$ much less is know in the formal and holomorphic settings. Gramchev and Yoshino [GY] have proved a simultaneous holomorphic linearization result for pairwise commuting germs without simultaneous resonances, with diagonalizable linear parts, and under a simultaneous Diophantine condition (further studied by Yoshino in [Yo]) and a few more technical assumptions. In [DG], DeLatte and Gramchev investigated on holomorphic linearization of germs with linear parts having Jordan blocks, leaving as an open problem the study of simultaneous formal and holomorphic linearization of commuting germs with non-diagonalizable linear parts. Recently it has been proved in [R2] that $h \geq 2$ germs $f_1, \ldots, f_h$ of biholomorphisms of $\mathbb{C}^n$, fixing the origin, so that the linear part of f_1 is diagonalizable and f_1 commutes with f_k for any $k = 2, \ldots, h$, under certain arithmetic conditions on the eigenvalues of the linear part of f_1 and some restrictions on their resonances, are simultaneously holomorphically linearizable if and only if there exists a particular complex manifold invariant under $f_1, \ldots, f_h$.

Therefore, there are at least three natural questions arising in this setting:

(Q1) *Is it possible to say anything on the shape a (formal) simultaneous linearization can have?*

(Q2) *Are there any conditions on the eigenvalues of the linear parts of $h \geq 2$ germs of simultaneously formally linearizable biholomorphisms ensuring simultaneous holomorphic linearizability?*

(Q3) *Under which conditions on the eigenvalues of the linear parts of $h \geq 2$ pairwise commuting germs of biholomorphisms can one assert the existence of a simultaneous holomorphic linearization of the given germs? In particular, is there a Brjuno-type condition sufficient for convergence?*

Note that the third question is a natural generalization to dimension $n \geq 2$ of the question raised by Moser [M] in the one-dimensional case (see also the introduction of [FK]).

In this paper we shall give complete answers to these three questions without making any assumption on the resonances. Before stating our answer to the first question, we need the following definition.

Definition 1.1. Let $M_1, \ldots, M_h$ be $h \geq 2$ complex $n \times n$ matrices. We say that $M_1, \ldots, M_h$ are *almost simultaneously Jordanizable*, it there exists a linear change of coordinates A such that $A^{-1}M_1A, \ldots, A^{-1}M_hA$ are *almost in simultaneous Jordan normal form*, i.e., for $k = 1, \ldots, h$ we have

$$A^{-1}M_kA = \begin{pmatrix} \lambda_{k,1} & & & \\ \varepsilon_{k,1} & \lambda_{k,2} & & \\ & \ddots & \ddots & \\ & & \varepsilon_{k,n-1} & \lambda_{k,n} \end{pmatrix}, \quad \varepsilon_{k,j} \neq 0 \Longrightarrow \lambda_{k,j} = \lambda_{k,j+1}. \quad (3)$$

We say that $M_1, \ldots, M_h$ are *simultaneously Jordanizable* if there exists a linear change of coordinates A such that we have (3) with $\varepsilon_{k,j} \in \{0, \varepsilon\}$.

It should be remarked that two commuting matrices are not necessarily almost simultaneously Jordanizable, and that two almost simultaneously Jordanizable matrices do not necessarily commute; see section 2 for details. However, the almost simultaneously Jordanizable hypothesis still is less restrictive than the simultaneously diagonalizable assumption usual in this context.

The following result gives an answer to (Q1).

Theorem 1.1. *Let $f_1, \ldots, f_h$ be $h \geq 2$ formally linearizable germs of biholomorphisms of $\mathbb{C}^n$ fixing the origin and with almost simultaneously Jordanizable linear parts. If $f_1, \ldots, f_h$ are simultaneously formally linearizable, then they are simultaneously formally linearizable via a linearization φ such that $\varphi_{Q,j} = 0$ for each Q and j so that $Q \in \cap_{k=1}^h \mathrm{Res}_j(\Lambda_k)$, and such a linearization is unique.*

We also have a condition ensuring formal simultaneous linearizability.

Theorem 1.2. *Let $f_1, \ldots, f_h$ be $h \geq 2$ formally linearizable germs of biholomorphisms of $\mathbb{C}^n$ fixing the origin and with almost simultaneously Jordanizable linear parts. If $f_1, \ldots, f_h$ all commute pairwise, then they are simultaneously formally linearizable.*

To state our result on simultaneous holomorphic linearizability we need to introduce the following Brjuno-type condition.

Definition 1.2. Let $n \geq 2$ and let $\Lambda_1 = (\lambda_{1,1}, \ldots, \lambda_{1,n}), \ldots, \Lambda_h = (\lambda_{h,1}, \ldots, \lambda_{h,n})$ be $h \geq 2$ n-tuples of complex, not necessarily distinct, non-zero numbers. We say that $\Lambda_1, \ldots, \Lambda_h$ *satisfy the simultaneous Brjuno condition* if there exists a strictly increasing sequence of integers $\{p_\nu\}_{\nu \geq 0}$ with $p_0 = 1$ such that

$$\sum_{\nu \geq 0} \frac{1}{p_\nu} \log \frac{1}{\omega_{\Lambda_1, \ldots, \Lambda_h}(p_{\nu+1})} < +\infty,$$

where for any $m \geq 2$ we set

$$\omega_{\Lambda_1, \ldots, \Lambda_h}(m) = \min_{\substack{2 \leq |Q| \leq m \\ Q \notin \cap_{k=1}^h \cap_{j=1}^n \mathrm{Res}_j(\Lambda_k)}} \varepsilon_Q,$$

with

$$\varepsilon_Q = \min_{1 \leq j \leq n} \max_{1 \leq k \leq h} |\Lambda_k^Q - \lambda_{k,j}|.$$

If $\Lambda_1, \ldots, \Lambda_h$ are the sets of eigenvalues of the linear parts of $f_1, \ldots, f_h$, we shall say that $f_1, \ldots, f_h$ *satisfy the simultaneous Brjuno condition*.

Our holomorphic linearization result answering (Q2) is then the following.

Theorem 1.3. *Let $f_1, \ldots, f_h$ be $h \geq 2$ simultaneously formally linearizable germs of biholomorphism of $\mathbb{C}^n$ fixing the origin and such that their linear parts $\Lambda_1, \ldots, \Lambda_h$ are simultaneously diagonalizable. If $f_1, \ldots, f_h$ satisfy the simultaneous Brjuno condition, then $f_1, \ldots f_h$ are holomorphically simultaneously linearizable.*

Using Theorem 1.3 we are also able to give a positive answer to the generalization (Q3) of Moser's question.

Theorem 1.4. *Let $f_1, \ldots, f_h$ be $h \geq 2$ formally linearizable germs of biholomorphisms of $\mathbb{C}^n$ fixing the origin, with simultaneously diagonalizable linear parts, and satisfying the simultaneous Brjuno condition. Then $f_1, \ldots, f_h$ are simultaneously holomorphically linearizable if and only if they all commute pairwise.*

The structure of the paper is as follows. In the next section we shall discuss properties of simultaneously formally linearizable germs, and we shall give a proof of Theorem 1.1, Theorem 1.2 and other formal results that we can obtain. In section 3 we shall prove Theorem 1.3 and Theorem 1.4 using majorant series. In the appendix we shall discuss the equivalence between various Brjuno-type series.

2. Simultaneously formally linearizable germs

In this section, we shall deal with formal simultaneous linearization. We shall first investigate the properties one can expect from a simultaneous formal linearization, and then we shall provide conditions for simultaneous formal linearizability in presence of resonances.

Definition 2.1. Let $\Lambda \in (\mathbb{C}^*)^n$ and let $j \in \{1, \ldots, n\}$. We say that a multi-index $Q \in \mathbb{N}^n$, with $|Q| \geq 2$, gives a *resonance relation for Λ relative to the j-th coordinate* if

$$\Lambda^Q := \lambda_1^{q_1} \cdots \lambda_n^{q_n} = \lambda_j$$

and we put, as in (2), $\text{Res}_j(\Lambda) = \{Q \in \mathbb{N}^n \mid |Q| \geq 2, \lambda^Q = \lambda_j\}$. The elements of $\text{Res}_j(\Lambda)$ are simply called *resonant* multi-indices with respect to j.

If Λ is a complex $n \times n$ invertible matrix in Jordan normal form, then, with a slight abuse of notation, we shall denote by $\text{Res}_j(\Lambda)$ the resonant multi-indices of the eigenvalues of Λ.

Let us start with the following useful result.

Lemma 2.1. *Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin, and let Λ be an invertible $n \times n$ complex matrix in Jordan normal form, commuting with f. Then the linear part of f commutes with Λ and f contains only monomials that are resonant with respect to the eigenvalues of Λ.*

Proof. We can write f in coordinates, as $f(z) = Mz + \widehat{f}(z) = Mz + \sum_{|Q| \geq 2} f_Q z^Q$. If f commutes with Λ then, comparing terms of the same degree, it is clear that M has to commute with Λ.

If Λ is diagonal, it is obvious that f commutes with Λ if and only if f contains Λ-resonant terms only.

Let us now assume that Λ contains at least a non-trivial Jordan block, that is

$$\Lambda = \begin{pmatrix} \lambda_1 & & & \\ \varepsilon_1 & \lambda_2 & & \\ & \ddots & \ddots & \\ & & \varepsilon_{n-1} & \lambda_n \end{pmatrix}, \quad \varepsilon_j \in \{0, \varepsilon\}, \quad \varepsilon_j \neq 0 \implies \lambda_j = \lambda_{j+1},$$

with at least one non-zero ε_j.

Up to reordering, we may assume $\varepsilon_1 \neq 0$. For each component $j \in \{1, \ldots, n\}$, we have

$$(\Lambda \widehat{f}(z))_j = \lambda_j \widehat{f}_j(z) + \varepsilon_{j-1} \widehat{f}_{j-1}(z),$$

where we set $\varepsilon_{-1} = 0$, and

$$(\widehat{f}(\Lambda z))_j = \sum_{|Q| \geq 2} f_{Q,j} (\Lambda z)^Q$$

$$= \sum_{|Q| \geq 2} f_{Q,j} \lambda^Q z^Q \prod_{k=2}^n \left(1 + \varepsilon_{k-1} \frac{z_{k-1}}{\lambda_k z_k}\right)^{q_k}$$

$$= \sum_{|Q| \geq 2} f_{Q,j} \lambda^Q z^Q \sum_{\substack{0 \leq k_2 \leq q_2 \\ \vdots \\ 0 \leq k_n \leq q_n}} \binom{q_2}{k_2} \cdots \binom{q_n}{k_n} \frac{\varepsilon_1^{k_2} \cdots \varepsilon_{n-1}^{k_n}}{\lambda_2^{k_2} \cdots \lambda_n^{k_n}} z_1^{k_2} z_2^{k_3 - k_2} \cdots z_{n-1}^{k_n - k_{n-1}} z_n^{-k_n},$$

where we are using the convention $0^0 = 1$. Note that $z^P = z^Q \cdot z_1^{k_2} z_2^{k_3 - k_2} \cdots z_{n-1}^{k_n - k_{n-1}} z_n^{-k_n}$ is a monomial with the same degree as z^Q, i.e., $|P| = |Q|$, but subsequent in the lexicographic order, i.e., $P > Q$. Moreover, if Q is a resonant multi-index relative to j, i.e., $\lambda^Q = \lambda_j$, then also P is. In fact, for each Jordan block of order ℓ, we have $\lambda_{i_\ell} = \cdots = \lambda_{i_\ell + \ell}$, so

$$\lambda_{i_\ell}^{q_{i_\ell}} \cdots \lambda_{i_\ell + \ell}^{q_{i_\ell + \ell}} = \lambda_{i_\ell}^{q_{i_\ell} + k_{i_\ell + 1}} \lambda_{i_\ell + 1}^{q_{i_\ell + 1} + k_{i_\ell + 2} - k_{i_\ell + 1}} \cdots \lambda_{i_\ell + \ell}^{q_{i_\ell + \ell} - k_{i_\ell + \ell}}.$$

Now we prove that for each $j \in \{1, \ldots, n\}$ the j-th component of $\widehat{f}$ contains only Λ-resonant monomials. For the first component we have

$$\lambda_1 \widehat{f}_1(z) = \sum_{|Q| \geq 2} f_{Q,1} \lambda^Q z^Q \sum_{\substack{0 \leq k_2 \leq q_2 \\ \vdots \\ 0 \leq k_n \leq q_n}} \binom{q_2}{k_2} \cdots \binom{q_n}{k_n} \frac{\varepsilon_1^{k_2} \cdots \varepsilon_{n-1}^{k_n}}{\lambda_2^{k_2} \cdots \lambda_n^{k_n}} z_1^{k_2} z_2^{k_3 - k_2} \cdots z_{n-1}^{k_n - k_{n-1}} z_n^{-k_n}. \quad (4)$$

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Let $\widetilde{Q}$ be the first, with respect to the lexicographic order, non resonant multi-index so that $f_{\widetilde{Q},1} \neq 0$ and let us compare the coefficients of $z^{\widetilde{Q}}$ in both sides of (4). In the left-hand side we just have $\lambda_1 f_{\widetilde{Q},1}$; in the right-hand side we only have $\lambda^{\widetilde{Q}} f_{\widetilde{Q},1}$, because other contributes could come only by previous multi-indices, but, as observed above, they all give resonances because we are assuming $\widetilde{Q}$ to be the first non resonant multi-index. Hence we have

$$(\lambda^{\widetilde{Q}} - \lambda_1) f_{\widetilde{Q},1} = 0,$$

yielding, since $\lambda^{\widetilde{Q}} \neq \lambda_1$, $f_{\widetilde{Q},1} = 0$, and contradicting the hypothesis. Now we turn to the second component, and since we are assuming $\varepsilon_1 \neq 0$, we have $\lambda_2 = \lambda_1$, so we have

$$\lambda_1 \widehat{f_2}(z) + \varepsilon \widehat{f_1}(z) = \sum_{|Q| \geq 2} f_{Q,2} \lambda^Q z^Q \sum_{\substack{0 \leq k_2 \leq q_2 \\ \vdots \\ 0 \leq k_n \leq q_n}} \binom{q_2}{k_2} \cdots \binom{q_n}{k_n} \frac{\varepsilon_1^{k_2} \cdots \varepsilon_{n-1}^{k_n}}{\lambda_2^{k_2} \cdots \lambda_n^{k_n}} z_1^{k_2} z_2^{k_3 - k_2} \cdots z_{n-1}^{k_n - k_{n-1}} z_n^{-k_n}. \quad (5)$$

Let $\widetilde{Q}$ be the first, with respect to the lexicographic order, non resonant multi-index so that $f_{\widetilde{Q},2} \neq 0$ and let us compare the coefficients of $z^{\widetilde{Q}}$ in both sides of (5). In the left-hand side we just have $\lambda_1 f_{\widetilde{Q},2}$ because we proved above that $\widehat{f_1}$ contains only resonant monomials; in the right-hand side we again have only $\lambda^{\widetilde{Q}} f_{\widetilde{Q},2}$, because other contributes could come only by previous multi-indices, but, as observed above, they all give resonances since $\widetilde{Q}$ is the first non resonant multi-index. Then we repeat the same argument used above and we prove that also $\widehat{f_2}$ contains only resonant monomials. Now we can use the same arguments for the remaining components, and we get the assertion. $\qquad\square$

Remark 2.2. Notice that in the previous result we did not make any hypotheses on the diagonalizability or not of the linear part M of the germ f, because we just wanted to understand what information we can deduce on f assuming its commutation with a matrix in Jordan normal form.

Remark 2.3. Note that Lemma 2.1 does not hold if Λ is just triangular. For example, if we take

$$\Lambda = \begin{pmatrix} \lambda_2 \lambda_3 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ \lambda_3(1 - \lambda_2) & \lambda_3 - \lambda_2 & \lambda_3 \end{pmatrix}$$

with $\lambda_2, \lambda_3 \in \mathbb{C} \setminus \{0, 1\}$ and $\lambda_2 \neq \lambda_3$, and A is any complex 3×3 matrix commuting with Λ (for example Λ itself), then Λ commutes with the germ

$$f(z) = Az + (z_2(z_1 + z_2 + z_3), 0, -z_2(z_1 + z_2 + z_3)),$$

and f clearly contains monomials non resonant with respect to the eigenvalues of Λ.

In the following we shall need the notion of almost simultaneously Jordanizable defined in the introduction in Definition 1.1, that we recall here.

Definition 2.2. Let $M_1, \ldots, M_h$ be $h \geq 2$ complex $n \times n$ matrices. We say that $M_1, \ldots, M_h$ are *almost simultaneously Jordanizable*, it there exists a linear change of coordinates A such that

$A^{-1}M_1A,\ldots,A^{-1}M_hA$ are *almost in simultaneous Jordan normal form*, i.e., for $k = 1,\ldots, h$ we have

$$A^{-1}M_kA = \begin{pmatrix} \lambda_{k,1} & & & \\ \varepsilon_{k,1} & \lambda_{k,2} & & \\ & \ddots & \ddots & \\ & & \varepsilon_{k,n-1} & \lambda_{k,n} \end{pmatrix}, \quad \varepsilon_{k,j} \neq 0 \implies \lambda_{k,j} = \lambda_{k,j+1}. \tag{6}$$

We say that $M_1,\ldots,M_h$ are *simultaneously Jordanizable* if there exists a linear change of coordinates A such that we have (6) with $\varepsilon_{k,j} \in \{0,\varepsilon\}$.

Remark 2.4. Note that the problem of deciding when two $n \times n$ complex matrices are almost simultaneously Jordanizable is not as easy as when the two matrices are diagonalizable. Indeed, whereas $h \geq 2$ diagonalizable matrices are simultaneously diagonalizable if and only if they commute pairwise, and if $h \geq 2$ matrices commute pairwise then they are simultaneously triangularizable (but the converse is clearly false), if two matrices commute then this does not imply that they admit an almost simultaneous Jordan normal form, and it is not true in general that two matrices almost in simultaneous Jordan normal form commute. For example the following two matrices

$$\Lambda = \begin{pmatrix} \lambda & 0 & 0 \\ \varepsilon & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \quad M = \begin{pmatrix} \mu & 0 & 0 \\ \delta & \mu & 0 \\ \beta & 0 & \mu \end{pmatrix} \quad \lambda, \varepsilon, \mu, \delta, \beta \in \mathbb{C}^*$$

commute, but in general it is not possible to almost simultaneously Jordanize them. In fact all the matrices A such that M is almost in simultaneous Jordan normal form with Λ have to be invertible solutions of the following equation

$$AM = \begin{pmatrix} \mu & 0 & 0 \\ \zeta & \mu & 0 \\ 0 & 0 & \mu \end{pmatrix} A$$

(and ζ has to be non-zero because $M \neq \mu I_3$); hence A is of the form

$$A = \begin{pmatrix} \frac{\beta}{\zeta}f + \frac{\delta}{\zeta}e & 0 & 0 \\ d & e & f \\ g & h & -\frac{\delta}{\beta}h \end{pmatrix},$$

which is invertible if and only if $\beta f + \delta e \neq 0$ and $h \neq 0$. But

$$A\Lambda = \begin{pmatrix} \lambda & 0 & 0 \\ \xi & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} A$$

yields $h = 0$, implying that Λ and M are not almost simultaneously Jordanizable (and so they are also not simultaneously Jordanizable).

On the other side, the following two matrices

$$\widetilde{\Lambda} = \begin{pmatrix} \lambda & 0 & 0 \\ \varepsilon & \lambda & 0 \\ 0 & \varepsilon & \lambda \end{pmatrix} \quad \widetilde{M} = \begin{pmatrix} \mu & 0 & 0 \\ \delta & \mu & 0 \\ 0 & 0 & \eta \end{pmatrix} \quad \lambda, \varepsilon, \mu, \delta, \eta \in \mathbb{C}^*$$

are almost in simultaneous Jordan normal form but they do not commute.

In studying the convergence of a formal linearization of a germ of biholomorphism, and hence also in the case of simultaneous formal linearizations, it is useful to be able to use formal linearizations of the special kind we are now going to introduce.

Definition 2.3. Let f be a germ of bihomolomorphism of $\mathbb{C}^n$ fixing the origin and with linear part Λ in Jordan normal form. A tangent to the identity (formal) linearization φ of f is said *non resonant* if for each resonant multi-index relative to the j-th coordinate, $Q \in \mathrm{Res}_j(\Lambda)$, the coefficient $\varphi_{Q,j}$ of z^Q in the power series expansion of the j-th coordinate of φ vanishes, i.e., $\varphi_{Q,j} = 0$.

Definition 2.4. Let $f_1, \ldots, f_h$ be $h \geq 2$ formally linearizable germs of biholomorphisms of $\mathbb{C}^n$ fixing the origin and with linear parts $\Lambda_1, \ldots, \Lambda_h$ almost in simultaneous Jordan normal form. A tangent to the identity (formal) simultaneous linearization φ of $f_1, \ldots, f_h$ is said *non resonant* if for each simultaneous resonant multi-index relative to the j-th coordinate, $Q \in \cap_{k=1}^h \mathrm{Res}_j(\Lambda_k)$, the coefficient $\varphi_{Q,j}$ of z^Q in the power series expansion of the j-th coordinate of φ vanishes, i.e., $\varphi_{Q,j} = 0$.

Let us now investigate the shape a formal simultaneous linearization can have. It can be proven, see [Rü] and [R4], that a formally linearizable germ of biholomorphism is formally linearizable via a non resonant formal linearization, and such a linearization is unique. The same is true also for simultaneously formally linearizable germs, but with a slightly different proof, as shown in the next result, stated in the introduction as Theorem 1.1.

Theorem 2.5. *Let $f_1, \ldots, f_h$ be $h \geq 2$ formally linearizable germs of biholomorphisms of $\mathbb{C}^n$ fixing the origin and with almost simultaneously Jordanizable linear parts. If $f_1, \ldots, f_h$ are simultaneously formally linearizable, then they are simultaneously formally linearizable via a non resonant linearization φ, and such a linearization is unique.*

Proof. We may assume, up to linear changes of the coordinates, that the linear parts $\Lambda_1, \ldots, \Lambda_h$ of $f_1, \ldots, f_h$ are almost in simultaneous Jordan normal form, i.e.,

$$\Lambda_k = \begin{pmatrix} \lambda_{k,1} & & & \\ \varepsilon_{k,1} & \lambda_{k,2} & & \\ & \ddots & \ddots & \\ & & \varepsilon_{k,n-1} & \lambda_{k,n} \end{pmatrix}, \quad \varepsilon_{k,j} \neq 0 \Longrightarrow \lambda_{k,j} = \lambda_{k,j+1},$$

for $k = 1, \ldots, h$.

We know that there exists a formal change of coordinates φ tangent to the identity and such that $\varphi^{-1} \circ f_k \circ \varphi = \Lambda_k$ for all $k = 1, \ldots, h$. If $\varphi_{Q,j} = 0$ for each Q and j so that $Q \in \cap_{k=1}^h \mathrm{Res}_j(\Lambda_k)$, then we are done and we only have to show that such a linearization is unique.

If there is at least one multi-index $Q \in \cap_{k=1}^h \mathrm{Res}_j(\Lambda_k)$ with $j \in \{1, \ldots, n\}$ and such that $\varphi_{Q,j} \neq 0$, then we can construct another formal simultaneous linearization ψ which is non-resonant. Since we can write $f_k = \varphi \circ \Lambda_k \circ \varphi^{-1}$ for each $k = 1, \ldots, h$, ψ has to satisfy

$$\varphi^{-1} \circ \psi \circ \Lambda_k = \Lambda_k \circ \varphi^{-1} \circ \psi,$$

i.e., we need to construct ψ not containing monomials simultaneously resonant for $\Lambda_1, \ldots, \Lambda_h$ and such that (by Lemma 2.1) $\varphi^{-1} \circ \psi$ contains only monomials simultaneously resonant for

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$\Lambda_1, \ldots, \Lambda_h$. Writing

$$\varphi_j^{-1}(z) = z_j \left(1 + \sum_{\substack{Q \in N_j \\ \Lambda_k^Q = 1, k=1,\ldots,h}} \widetilde{\varphi}_{Q,j} z^Q + \sum_{\substack{Q \in N_j \\ \Lambda_k^Q \neq 1, \text{ for some } k=1,\ldots,h}} \widetilde{\varphi}_{Q,j} z^Q \right), \quad j = 1, \ldots, n,$$

where

$$N_j := \{ Q \in \mathbb{Z}^n \mid |Q| \geq 1, q_j \geq -1, q_h \geq 0 \text{ for all } h \neq j \},$$

we want to find ψ of the form

$$\psi_j(z) = z_j \left(1 + \sum_{\substack{Q \in N_j \\ \Lambda_k^Q \neq 1, \text{ for some } k=1,\ldots,h}} \psi_{Q,j} z^Q \right), \quad j = 1, \ldots, n,$$

such that

$$(\varphi^{-1} \circ \psi)_j(z) = z_j \left(1 + \sum_{\substack{Q \in N_j \\ \Lambda_k^Q = 1, k=1,\ldots,h}} g_{Q,j} z^Q \right), \quad j = 1, \ldots, n.$$

Using the same argument of the proof of Poincaré-Dulac Theorem (see [Ar] pp. 192–193 or [R3] pp. 40–41), in the j-th coordinate of $\varphi^{-1} \circ \psi$, the coefficient of z^Q with Q so that $\Lambda_k^Q \neq 1$ for at least one k is equal to

$$\psi_{Q,j} + \text{Polynomial}(\varphi^{-1}, \text{previous } \psi_{P,l}),$$

where the polynomial in the previous formula is "universal" in the sense that it depends only on the fact that we are composing two power series and it does not depend on the coefficients of φ^{-1} and on the previous $\psi_{P,l}$, that are in fact just arguments of this universal polynomial. Hence it suffices to put

$$\psi_{Q,j} = -\text{Polynomial}(\varphi^{-1}, \text{previous } \psi_{P,l}).$$

Note that for the first non-resonant multi-indices, due to degree considerations, we just have to put $\psi_{Q,j} = -\widetilde{\varphi}_{Q,j}$.

We proved that there exists a formal non-resonant tangent to the identity simultaneous linearization φ of the given germs, containing only monomials that are not simultaneous resonant for the eigenvalues of $\Lambda_1, \ldots, \Lambda_h$. Let us assume by contradiction that there exists another such a linearization ψ and $\psi \not\equiv \varphi$. Writing $\varphi(z) = z + \sum_{|Q| \geq 2} \varphi_Q z^Q$ and $\psi(z) = z + \sum_{|Q| \geq 2} \psi_Q z^Q$, let $\widetilde{Q}$ be the first multi-index, with respect to the lexicographic order, so that $\varphi_{\widetilde{Q}} \neq \psi_{\widetilde{Q}}$ and let $\ell \in \{1, \ldots, n\}$ be the minimal index such that $\varphi_{\widetilde{Q},\ell} \neq \psi_{\widetilde{Q},\ell}$. Since for each $Q \in \cap_{k=1}^h \text{Res}_\ell(\Lambda_k)$ we know that $\varphi_{Q,\ell} = \psi_{Q,\ell} = 0$, there is at least one germ f_k such that $\Lambda_k^{\widetilde{Q}} \neq \lambda_{k,\ell}$. We know, again by the proof of Poincaré-Dulac Theorem, that

$$\left(\Lambda_k^{\widetilde{Q}} - \lambda_{k,\ell} \right) \varphi_{\widetilde{Q},\ell} = \text{Polynomial}(f_k, \text{previous } \varphi_{P,j}),$$

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and, similarly,

$$\left(\Lambda_k^{\widetilde{Q}} - \lambda_{k,\ell}\right)\psi_{\widetilde{Q},\ell} = \mathrm{Polynomial}(f_k, \text{previous } \psi_{P,j}),$$

where, again, the polynomial in the previous formulas is "universal", because again it depends only on the fact that we are composing power series and it does not depend on the coefficients of f_k and on the previous $\varphi_{P,j}$ or $\psi_{P,j}$, that are in fact just arguments of this universal polynomial. Hence, since we are assuming that $\varphi_{P,j} = \psi_{P,j}$ for all the multi-indices $P < \widetilde{Q}$ and, for $P = \widetilde{Q}$, for all $j < \ell$, we have that

$$\mathrm{Polynomial}(f_k, \text{previous } \varphi_{P,j}) = \mathrm{Polynomial}(f_k, \text{previous } \psi_{P,j})$$

implying that $\varphi_{\widetilde{Q},\ell} = \psi_{\widetilde{Q},\ell}$, and contradicting the hypothesis. $\qquad\square$

Remark 2.6. The universal polynomials we dealt with in the last proof can be interpreted and computed using the mould formalism introduced by Écalle (see [É]), and in this sense the latter proof is a mouldian proof.

As announced in the introduction as Theorem 1.2, we shall now give a condition ensuring formal simultaneous linearizability.

Theorem 2.7. *Let $f_1, \ldots, f_h$ be $h \geq 2$ formally linearizable germs of biholomorphisms of $\mathbb{C}^n$ fixing the origin and with almost simultaneously Jordanizable linear parts. If $f_1, \ldots, f_h$ all commute pairwise, i.e., $f_p \circ f_q = f_q \circ f_p$ for any p and q in $\{1, \ldots, h\}$, then they are simultaneously formally linearizable.*

Proof. We may assume without loss of generality that the linear parts $\Lambda_1, \ldots, \Lambda_h$ of the germs are all almost in simultaneous Jordan normal form. Since f_1 is formally linearizable, it is possibly to linearize it with a non-resonant formal linearization φ_1 (see [Rü] and [R4]). Then, thanks to the commutation hypothesis, by Lemma 2.1, $\widetilde{f}_2 = \varphi_1^{-1} \circ f_2 \circ \varphi_1, \ldots, \widetilde{f}_h = \varphi_1^{-1} \circ f_h \circ \varphi_1$ contain only Λ_1-resonant terms. Now we claim that it is possible to find a formal change of coordinates φ_2 fixing the origin, tangent to the identity, containing only Λ_1-resonant terms that are not Λ_2-resonant, and conjugating $\widetilde{f}_2$ to a germ g_2 with same linear part and in Poincaré-Dulac normal form. In fact, we have to solve

$$\left(\Lambda_2 + \widetilde{f}_2^{\mathrm{Res}(\Lambda_1)}\right) \circ \left(I + \varphi_2^{\mathrm{Res}(\Lambda_1)}\right) = \left(I + \varphi_2^{\mathrm{Res}(\Lambda_1)}\right) \circ \left(\Lambda_2 + g_2^{\mathrm{Res}(\Lambda_1)\cap\mathrm{Res}(\Lambda_2)}\right),$$

where $\widetilde{f}^{\mathrm{Res}(\Lambda_1)}$ contains only monomial resonant with respect to the eigenvalues of Λ_1, and so on, that is

$$\Lambda_2 \varphi_2^{\mathrm{Res}(\Lambda_1)} + \widetilde{f}_2^{\mathrm{Res}(\Lambda_1)} \circ \left(I + \varphi_2^{\mathrm{Res}(\Lambda_1)}\right) = g_2^{\mathrm{Res}(\Lambda_1)\cap\mathrm{Res}(\Lambda_2)} + \varphi_2^{\mathrm{Res}(\Lambda_1)} \circ \left(\Lambda_2 + g_2^{\mathrm{Res}(\Lambda_1)\cap\mathrm{Res}(\Lambda_2)}\right),$$

which is solvable by the usual Poincaré-Dulac procedure and setting $\varphi_{2,Q,j}^{\mathrm{Res}(\Lambda_1)} = 0$ whenever $\Lambda_2^Q = \lambda_{2,j}$. But since f_2 is formally linearizable, the linear form is its unique Poincaré-Dulac normal form (see [R4] Theorem 2.3), so $g_2 \equiv \Lambda_2$. Hence, since the given germs commute pairwise, $\varphi_2^{-1} \circ \varphi_1^{-1} \circ f_3 \circ \varphi_1 \circ \varphi_2, \ldots, \varphi_2^{-1} \circ \varphi_1^{-1} \circ f_h \circ \varphi_1 \circ \varphi_2$ contain only monomials that are simultaneously Λ_1 and Λ_2 resonant, and we can iterate the procedure finding a formal linearization φ_3 of $\varphi_2^{-1} \circ \varphi_1^{-1} \circ f_3 \circ \varphi_1 \circ \varphi_2$ containing only monomials that are Λ_1 and Λ_2 resonant but not Λ_3-resonant. We can then iteratively perform the same procedure getting $\varphi_4, \ldots, \varphi_h$ formal transformations such that $\varphi_1 \circ \cdots \circ \varphi_h$ is a simultaneous formal linearization of $f_1, \ldots, f_h$. $\qquad\square$

Remark 2.8. The simultaneous formal linearization obtained in the last proof is non-resonant. Moreover, we can perform the same procedure with a different permutation of the indices, i.e., starting with $f_{\sigma(1)}$ and then continuing with $f_{\sigma(2)}$ and so on, where σ is any permutation of $\{1, \ldots, n\}$, and, by Theorem 2.5, we always get the same linearization.

Remark 2.9. The hypothesis on the pairwise commutation is indeed necessary. In fact, if Λ_1 and Λ_2 are two commuting matrices almost in simultaneous Jordan normal form such that $\mathrm{Res}(\Lambda_1) \neq \varnothing$ and $\mathrm{Res}(\Lambda_2) \neq \varnothing$, but $\mathrm{Res}(\Lambda_1) \cap \mathrm{Res}(\Lambda_2) = \varnothing$, the unique formal transformation tangent to the identity and commuting with both Λ_1 and Λ_2 is the identity, so any non-linear germ f_3 with linear part in Jordan normal form and commuting with Λ_1 (that is, containing only Λ_1-resonant terms) but not with Λ_2 cannot be simultaneously linearizable with Λ_1 and Λ_2.

Note that, in the proofs of the previous results, we needed to assume the almost simultaneous Jordanizability of the linear parts of the given germs, and indeed we cannot perform those proofs just assuming that those linear parts are just simultaneously triangularizable, because, as already remarked, Lemma 2.1 does not hold for general triangular matrices. Anyway, this is more than what was usually known in the previous literature, where in general linearization results are proved only for germs with diagonalizable linear part. Moreover, in the particular case of diagonalizable linear parts, recalling that $h \geq 2$ diagonalizable commuting complex $n \times n$ matrices are simultaneously diagonalizable, we have the following equivalence.

Proposition 2.10. *Let $f_1, \ldots, f_h$ be $h \geq 2$ formally linearizable germs of biholomorphisms of $\mathbb{C}^n$ fixing the origin and with simultaneously diagonalizable linear parts. Then $f_1, \ldots, f_h$ are simultaneously formally linearizable if and only if they all commute pairwise.*

Proof. If $f_1, \ldots, f_h$ are simultaneously formally linearizable, since their linear parts are simultaneously diagonalizable, then they all commute and we are done.

The converse follows from Theorem 2.7. $\qquad\square$

Remark 2.11. Note that having simultaneously diagonalizable linear parts is equivalent to having diagonalizable pairwise commuting linear parts, but being simultaneously formally linearizable does not imply that the linear parts are pairwise commuting, even when the linear parts are diagonalizable.

It is possible to find cases of formal simultaneous linearization even without assuming that the linear parts are almost simultaneously Jordanizable or that the germs commute pairwise, as shown in the following results. However, in those cases one has to assume other conditions, for example on the nature of resonances. We refer to [R2] and [R3] for the definitions of *only level s resonances* and *simultaneous osculating manifold*.

Proposition 2.12. *Let $f_1, \ldots, f_h$ be $h \geq 2$ germs of biholomorphism of $\mathbb{C}^n$ fixing the origin. Assume that the spectrum of the linear part of f_1 has only level s resonances and that f_1 commutes with f_k for $k = 2, \ldots, h$. Then $f_1, \ldots, f_h$ are simultaneously formally linearizable if and only if there exists a germ of formal complex manifold M at O of codimension s, invariant under f_h for each $h = 1, \ldots, m$, which is a simultaneous osculating manifold for $f_1, \ldots, f_m$ and such that $f_1|_M, \ldots, f_m|_M$ are simultaneously formally linearizable.*

Proof. It is clear from the proof of Theorem 2.5 of [R2] that it is possible to perform it in the formal category, where to formally linearize f_1, under these hypotheses, it suffices to have the only level s hypothesis. $\qquad\square$

Proposition 2.13. *Let $f_1, \ldots, f_h$ be $h \geq 2$ germs of biholomorphisms of $\mathbb{C}^n$ fixing the origin and such that the eigenvalues of the linear part of f_1 have no resonances. If f_1 commutes with*

f_k for $k = 2, \ldots, h$, then $f_1, \ldots, f_h$ are simultaneously formally linearizable via a unique formal linearization.

Proof. Up to linear conjugacy, we may assume that the linear part Λ_1 of f_1 is in Jordan normal form. Since the eigenvalues of Λ_1 are non-resonant, f_1 is formally linearizable via a unique linearization φ with no constant term and with identity linear part. Then, by Lemma 2.1, thanks to the commutation hypothesis, $\varphi^{-1} \circ f_k \circ \varphi$ contains only monomials that are resonant with respect to the eigenvalues of Λ_1, and we get the assertion. $\qquad\square$

In particular, if the eigenvalues of the linear part of all f_k have no resonances, the unique formal linearization φ_k of f_k is indeed the unique simultaneous formal linearization of the germs $f_1, \ldots, f_h$.

Another example of the results we can obtain using this kind of arguments is the following.

Corollary 2.14. *Let $f_1, \ldots, f_h$ be $h \geq 2$ germs of biholomorphisms of $\mathbb{C}^n$ fixing the origin and such that the linear parts Λ_1 and Λ_2 of f_1 and f_2 are almost simultaneously Jordanizable and have no common resonances (i.e., $\mathrm{Res}_j(\Lambda_1) \cap \mathrm{Res}_j(\Lambda_2) = \varnothing$ for $j = 1, \ldots, n$). If f_1 and f_2 commute with f_k for $k = 1, \ldots, h$, then $f_1, \ldots, f_h$ are simultaneously formally linearizable.*

3. Convergence under the simultaneous Brjuno condition

In this section we shall prove a holomorphic simultaneous linearization result in presence of resonances. Note that, even if in the previous section we proved formal simultaneous linearization results just assuming almost simultaneous Jordanizability of the linear parts, to prove convergence in "generic cases" we need to assume simultaneous diagonalizability of the linear parts because of Yoccoz's counter example (see [Y2] pp. 83–85) to holomorphic linearization in case of linear part in (non diagonal) Jordan form.

We shall now recall the arithmetical Brjuno-type condition, stated in the introduction in Definition 1.2, that we shall need to prove our holomorphic linearization result.

Definition 3.1. Let $n \geq 2$ and let $\Lambda_1 = (\lambda_{1,1}, \ldots, \lambda_{1,n}), \ldots, \Lambda_h = (\lambda_{h,1}, \ldots, \lambda_{h,n})$ be $h \geq 2$ n-tuples of complex, not necessarily distinct, non-zero numbers. For any $m \geq 2$ we set

$$\omega_{\Lambda_1, \ldots, \Lambda_h}(m) = \min_{\substack{2 \leq |Q| \leq m \\ Q \notin \cap_{k=1}^h \cap_{j=1}^n \mathrm{Res}_j(\Lambda_k)}} \varepsilon_Q,$$

with

$$\varepsilon_Q = \min_{1 \leq j \leq n} \max_{1 \leq k \leq h} |\Lambda_k^Q - \lambda_{k,j}|.$$

Note that, since $Q \notin \cap_{k=1}^h \cap_{j=1}^n \mathrm{Res}_j(\Lambda_k)$, we have ε_Q always non-zero, and hence also $\omega(m)$ is always non-zero. If $\Lambda_1, \ldots, \Lambda_h$ are the sets of eigenvalues of the linear parts of $f_1, \ldots, f_h$, we shall write $\omega_{f_1, \ldots, f_h}(m)$ for $\omega_{\Lambda_1, \ldots, \Lambda_h}(m)$.

Definition 3.2. Let $n \geq 2$ and let $\Lambda_1 = (\lambda_{1,1}, \ldots, \lambda_{1,n}), \ldots, \Lambda_h = (\lambda_{h,1}, \ldots, \lambda_{h,n})$ be $h \geq 2$ n-tuples of complex, not necessarily distinct, non-zero numbers. We say that $\Lambda_1, \ldots, \Lambda_h$ *satisfy the simultaneous Brjuno condition* if there exists a strictly increasing sequence of integers $\{p_\nu\}_{\nu \geq 0}$ with $p_0 = 1$ such that

$$\sum_{\nu \geq 0} \frac{1}{p_\nu} \log \frac{1}{\omega_{\Lambda_1, \ldots, \Lambda_h}(p_{\nu+1})} < +\infty. \tag{7}$$

If $\Lambda_1, \ldots, \Lambda_h$ are the sets of eigenvalues of the linear parts of $f_1, \ldots, f_h$, we shall say that $f_1, \ldots, f_h$ *satisfy the simultaneous Brjuno condition.*

Recall that (7) is equivalent to

$$\sum_{\nu \geq 0} \frac{1}{2^\nu} \log \frac{1}{\omega_{\Lambda_1, \ldots, \Lambda_h}(2^{\nu+1})} < +\infty,$$

and a proof of this equivalence can be found in [Brj] and [R3].

We shall give in the appendix other conditions equivalent to the simultaneous Brjuno condition.

Remark 3.1. Note that if $\Lambda_1, \ldots, \Lambda_h$ do not satisfy the simultaneous Brjuno condition, then each of them does not satisfy the reduced Brjuno condition, i.e.,

$$\sum_{\nu \geq 0} \frac{1}{2^\nu} \log \frac{1}{\omega_{\Lambda_k}(2^{\nu+1})} = +\infty,$$

for $k = 1, \ldots, h$, where

$$\omega_{\Lambda_k}(m) := \min_{\substack{2 \leq |Q| \leq m \\ 1 \leq j \leq n \\ Q \notin \mathrm{Res}_j(\Lambda_k)}} |\Lambda_k^Q - \lambda_{k,j}|.$$

In particular, if $\Lambda_1, \ldots, \Lambda_h$ are simultaneously Cremer, i.e.,

$$\limsup_{m \to +\infty} \frac{1}{m} \log \frac{1}{\omega_{\Lambda_1, \ldots, \Lambda_h}(m)} = +\infty,$$

and hence they do not satisfy the simultaneous Brjuno condition, then at least one of them has to be Cremer, i.e.,

$$\limsup_{m \to +\infty} \frac{1}{m} \log \frac{1}{\omega_{\Lambda_k}(m)} = +\infty, \tag{8}$$

and the other ones do not satisfy the reduced Brjuno condition.

Furthermore it is possible to find $\Lambda_1, \ldots, \Lambda_h$ satisfying the simultaneous Brjuno condition, with Λ_k not satisfying the reduced Brjuno condition for any $k = 1, \ldots, h$, as shown in the next result, descending from Theorem 5.1 of [Yo] and Theorem 2.1 of [M].

Proposition 3.2. *Let $h > n \geq 2$. Then there exists a set of h-tuples of linearly independent vectors $\Lambda_1, \ldots, \Lambda_h \in (\mathbb{C}^*)^n$, with the density of continuum, satisfying the simultaneous Brjuno condition, whereas for any $p_1, \ldots, p_h \in \mathbb{Z} \setminus \{0\}$ the vector $\sum_{k=1}^h p_k \Lambda_k$ does not satisfy the reduced Brjuno condition.*

Proof. The result descends easily from the proof of Theorem 5.1 of [Yo]. In fact, since

$$\min_{\substack{1 \leq j \leq n \\ Q \notin \cap_{k=1}^h \mathrm{Res}_j(\Lambda_k)}} \sum_{k=1}^h |\Lambda_k^Q - \lambda_{k,j}| = \sum_{k=1}^h \min_{\substack{1 \leq j \leq n \\ Q \notin \cap_{k=1}^h \mathrm{Res}_j(\Lambda_k)}} |\Lambda_k^Q - \lambda_{k,j}| \leq h \cdot \min_{\substack{1 \leq j \leq n \\ Q \notin \cap_{k=1}^h \mathrm{Res}_j(\Lambda_k)}} \max_{1 \leq k \leq h} |\Lambda_k^Q - \lambda_{k,j}|,$$

it is clear that the simultaneous Diophantine condition introduced by Yoshino in Section 5 of [Yo] implies our simultaneous Brjuno condition.

Hence we can use exactly the same construction of Yoshino in the proof of Theorem 5.1 of [Yo] observing that, since the set of Cremer points, i.e., of the numbers $\theta \in \mathbb{R}$ satisfying

$$\limsup_{m \to +\infty} \frac{1}{m} \log \frac{1}{\min_{2 \leq k \leq m} |e^{2\pi i k \theta} - 1|} = +\infty,$$

is residual, i.e., countable intersection of open dense sets, we can perform the same argument substituting, at the end of the proof of Theorem 5.1 of [Yo], Liouville with Cremer, and we are done. $\qquad \square$

Remark 3.3. Other possible definitions for a simultaneous Brjuno condition could had been given using the following functions:

$$\overline{\omega}_{\Lambda_1,\ldots,\Lambda_h}(m) := \min_{\substack{2 \leq |Q| \leq m \\ Q \notin \cap_{k=1}^h \cap_{j=1}^n \mathrm{Res}_j(\Lambda_k)}} \max_{1 \leq k \leq h} \min_{1 \leq j \leq n} |\Lambda_k^Q - \lambda_{k,j}|,$$

or

$$\widetilde{\omega}_{\Lambda_1,\ldots,\Lambda_h}(m) := \max_{1 \leq k \leq h} \min_{\substack{2 \leq |Q| \leq m \\ Q \notin \cap_{k=1}^h \cap_{j=1}^n \mathrm{Res}_j(\Lambda_k)}} \min_{1 \leq j \leq n} |\Lambda_k^Q - \lambda_{k,j}|,$$

and asking for

$$\sum_{\nu \geq 0} \frac{1}{2^\nu} \log \overline{\omega}_{\Lambda_1,\ldots,\Lambda_h}(2^{\nu+1}) < +\infty,$$

or

$$\sum_{\nu \geq 0} \frac{1}{2^\nu} \log \widetilde{\omega}_{\Lambda_1,\ldots,\Lambda_h}(2^{\nu+1}) < +\infty.$$

However, this definitions are clearly more restrictive than the one we chose, since we have:

$$\widetilde{\omega}_{\Lambda_1,\ldots,\Lambda_h}(m) \leq \overline{\omega}_{\Lambda_1,\ldots,\Lambda_h}(m) \leq \omega_{\Lambda_1,\ldots,\Lambda_h}(m),$$

and hence

$$\sum_{\nu \geq 0} \frac{1}{2^\nu} \log \frac{1}{\omega_{\Lambda_1,\ldots,\Lambda_h}(2^{\nu+1})} \leq \sum_{\nu \geq 0} \frac{1}{2^\nu} \log \frac{1}{\overline{\omega}_{\Lambda_1,\ldots,\Lambda_h}(2^{\nu+1})} \leq \sum_{\nu \geq 0} \frac{1}{2^\nu} \log \frac{1}{\widetilde{\omega}_{\Lambda_1,\ldots,\Lambda_h}(2^{\nu+1})},$$

but the inequalities are generically strict.

Now we can state and prove our holomorphic simultaneous linearization result, Theorem 1.3, whose proof will be an adaptation to our case of the existing methods introduced by Brjuno [Brj], and Siegel [S1, S2] (see also Pöschel [Pö]). We recall here the statement of Theorem 1.3.

Theorem 3.4. *Let $f_1,\ldots,f_h$ be $h \geq 2$ simultaneously formally linearizable germs of biholomorphism of $\mathbb{C}^n$ fixing the origin and such that their linear parts $\Lambda_1,\ldots,\Lambda_h$ are simultaneously diagonalizable. If $f_1,\ldots,f_h$ satisfy the simultaneous Brjuno condition, then $f_1,\ldots f_h$ are holomorphically simultaneously linearizable.*

Proof. Without loss of generality we may assume that $\Lambda_1,\ldots\Lambda_h$ are diagonal, that is

$$\Lambda_k = \mathrm{Diag}(\lambda_{k,1},\ldots,\lambda_{k,n}).$$

Since each f_k is holomorphic in a neighbourhood of the origin, there exists a positive number ρ such that $\|f_L^{(k)}\| \leq \rho^{|L|}$ for $|L| \geq 2$. The functional equation

$$f_k \circ \varphi = \varphi \circ \Lambda_k,$$

remains valid under the linear change of coordinates $f_k(z) \mapsto \sigma f_k(z/\sigma)$, $\varphi(w) \mapsto \sigma \varphi(w/\sigma)$ with $\sigma = \max\{1, \rho^2\}$. Therefore we may assume that

$$\forall |L| \geq 2 \qquad\qquad\qquad \|f_L^{(k)}\| \leq 1,$$

for $k = 1,\ldots,h$.

Thanks to Theorem 2.5, we may assume that $\varphi_{Q,j} = 0$ for all $Q \in \cap_{k=1}^h \mathrm{Res}_j(\Lambda_k)$, hence for any multi-index $Q \in \mathbb{N}^n \setminus \cap_{k=1}^h \cap_{j=1}^n \mathrm{Res}_j(\Lambda_k)$ with $|Q| \geq 2$ we have

$$\|\varphi_Q\| \leq \varepsilon_Q^{-1} \sum_{\substack{Q_1 + \cdots + Q_\nu = Q \\ \nu \geq 2}} \|\varphi_{Q_1}\| \cdots \|\varphi_{Q_\nu}\|, \tag{9}$$

where $\varepsilon_Q = \min_{1 \leq j \leq n} \max_{1 \leq k \leq h} |\Lambda_k^Q - \lambda_{k,j}|$.

Now, the proof follows closely the proof of the main Theorem in [Pö]. For the benefit of the reader, we shall report it here.

Following Pöschel [Pö], we inductively define:

$$\begin{cases} \alpha_m = \displaystyle\sum_{\substack{m_1 + \cdots + m_\nu = m \\ \nu \geq 2}} \alpha_{m_1} \cdots \alpha_{m_\nu}, & \text{for } m \geq 2 \\ \alpha_1 = 1, \end{cases}$$

and

$$\begin{cases} \delta_Q = \varepsilon_Q^{-1} \displaystyle\max_{\substack{Q_1 + \cdots + Q_\nu = Q \\ \nu \geq 2}} \delta_{Q_1} \cdots \delta_{Q_\nu}, & \text{for } Q \in \mathbb{N}^n \setminus \cap_{k=1}^h \cap_{j=1}^n \mathrm{Res}_j(\Lambda_k) \text{ with } |Q| \geq 2, \\ \delta_E = 1, & \text{for any } E \in \mathbb{N}^n \text{ with } |E| = 1. \end{cases}$$

Then, by induction, we have

$$\|\varphi_Q\| \leq \alpha_{|Q|} \delta_Q,$$

for every $Q \in \mathbb{N}^n \setminus \cap_{k=1}^h \cap_{j=1}^n \mathrm{Res}_j(\Lambda_k)$ with $|Q| \geq 2$. Therefore, to establish

$$\sup_Q \frac{1}{|Q|} \log \|\varphi_Q\| < +\infty, \tag{10}$$

it suffices to prove analogous estimates for α_m and δ_Q.

It is easy to estimate α_m, and we refer to [Pö] (see also [R4]) for a detailed proof of

$$\sup_m \frac{1}{m} \log \alpha_m < +\infty.$$

To estimate δ_Q we have to take care of small divisors. First of all, for each multi-index $Q \notin \cap_{k=1}^h \cap_{j=1}^n \mathrm{Res}_j(\Lambda_k)$ with $|Q| \geq 2$ we can associate to δ_Q a decomposition of the form

$$\delta_Q = \varepsilon_{L_0}^{-1} \varepsilon_{L_1}^{-1} \cdots \varepsilon_{L_p}^{-1}, \tag{11}$$

where $L_0 = Q$, $|Q| > |L_1| \geq \cdots \geq |L_p| \geq 2$ and $L_a \notin \cap_{k=1}^h \cap_{j=1}^n \mathrm{Res}_j(\Lambda_k)$ for all $a = 1, \ldots, p$ and $p \geq 1$. In fact, we choose a decomposition $Q = Q_1 + \cdots + Q_\nu$ such that the maximum in the expression of δ_Q is achieved; obviously, Q_a does not belong to $\cap_{k=1}^h \cap_{j=1}^n \mathrm{Res}_j(\Lambda_k)$ for all $a = 1, \ldots, \nu$. We can then express δ_Q in terms of $\varepsilon_{Q_j}^{-1}$ and $\delta_{Q'_j}$ with $|Q'_j| < |Q_j|$. Carrying on this process, we eventually arrive at a decomposition of the form (11). Furthermore, for each multi-index $Q \notin \cap_{k=1}^h \cap_{j=1}^n \mathrm{Res}_j(\Lambda_k)$ with $|Q| \geq 2$, we can choose an index k_Q and an index i_Q so that

$$\varepsilon_Q = |\Lambda_{k_Q}^Q - \lambda_{k_Q, i_Q}|.$$

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For $m \geq 2$ and $1 \leq j \leq n$, we can define

$$N_m^j(Q)$$

to be the number of factors ε_L^{-1} in the expression (11) of δ_Q, satisfying

$$\varepsilon_L < \theta \, \omega_{f_1,\ldots,f_h}(m), \quad \text{and} \quad i_L = j,$$

where $\omega_{f_1,\ldots,f_h}(m)$ is defined as

$$\omega_{f_1,\ldots,f_h}(m) = \min_{\substack{2 \leq |Q| \leq m \\ Q \notin \cap_{k=1}^h \cap_{j=1}^n \mathrm{Res}_j(\Lambda_k)}} \varepsilon_Q,$$

and θ is the positive real number satisfying

$$4\theta = \min_{\substack{1 \leq p \leq n \\ 1 \leq k \leq h}} |\lambda_{k,p}| \leq 1.$$

The last inequality can always be satisfied by replacing f_k by f_k^{-1} if necessary. Moreover we also have $\omega_{f_1,\ldots,f_h}(m) \leq 2$. Notice that $\omega_{f_1,\ldots,f_h}(m)$ is non-increasing with respect to m and under our assumptions $\omega_{f_1,\ldots,f_h}(m)$ tends to zero as m goes to infinity.

The following is the key estimate, and it descends from Brjuno.

Lemma 3.5. *For $m \geq 2$, $1 \leq j \leq n$, and $Q \notin \cap_{k=1}^h \cap_{j=1}^n \mathrm{Res}_j(\Lambda_k)$, we have*

$$N_m^j(Q) \leq \begin{cases} 0, & \text{if } |Q| \leq m, \\ \dfrac{2|Q|}{m} - 1, & \text{if } |Q| > m. \end{cases}$$

The proof of Lemma 3.5 can be obtained adapting the proof of Brjuno's lemma contained in the addendum of [Pö], and we report it here for the sake of completeness.

Proof. The proof is done by induction on $|Q|$. Since we fix m and j throughout the proof, we shall write N instead of N_m^j.

For $|Q| \leq m$,

$$\varepsilon_Q \geq \omega_{f_1,\ldots,f_h}(|Q|) \geq \omega_{f_1,\ldots,f_h}(m) > \theta \, \omega_{f_1,\ldots,f_h}(m),$$

hence $N(Q) = 0$.

Assume now that $|Q| > m$. Then $2|Q|/m - 1 \geq 1$. Write

$$\delta_Q = \varepsilon_Q^{-1} \delta_{Q_1} \cdots \delta_{Q_\nu},$$

with

$$Q = Q_1 + \cdots + Q_\nu, \quad \nu \geq 2, \quad \text{and} \quad |Q| > |Q_1| \geq \cdots \geq |Q_\nu|;$$

notice that $Q - Q_1$ does not belong to $\cap_{k=1}^h \cap_{j=1}^n \mathrm{Res}_j(\Lambda_k)$, otherwise the others Q_h's would be in $\cap_{k=1}^h \cap_{j=1}^n \mathrm{Res}_j(\Lambda_k)$.

In this decomposition, only $|Q_1|$ can be greater than $M := \max(|Q| - m, m)$. If this is the case, we can decompose δ_{Q_1} in the same way, and repeating this step at most $m - 1$ times, we obtain the decomposition (where $P_1 = Q_1$)

$$\delta_Q = \varepsilon_Q^{-1} \varepsilon_{P_1}^{-1} \cdots \varepsilon_{P_\mu}^{-1} \delta_{L_1} \cdots \delta_{L_\nu},$$

where $\mu \geq 1$, $\nu \geq 2$ and

$$Q > P_1 > \cdots > Q_\mu,$$
$$L_1 + \cdots + L_\nu = Q,$$
$$|P_\mu| > M \geq |L_1| \geq \cdots \geq |L_\nu|.$$

Here, $Q > L$ means, as in [Pö], that $Q - L \in \mathbb{N}^n$ is not identically zero. The crucial point is that at most *one* of the ε_K's can contribute to $N(Q)$, which is the content of the following lemma descending from Siegel.

Lemma 3.6. *If $Q > L$, the multi-indices Q, L and $Q - L$ are not in $\cap_{k=1}^{h} \cap_{j=1}^{n} \operatorname{Res}_j(\Lambda_k)$, and*

$$\varepsilon_Q < \theta\, \omega_{f_1,\ldots,f_h}(m), \quad \varepsilon_L < \theta\, \omega_{f_1,\ldots,f_h}(m), \quad i_Q = i_L,$$

then $|Q - L| \geq m$.

The proof of Lemma 3.6 can be obtained adapting the proof of Siegel's lemma contained in the addendum of [Pö], and we report it here for the sake of completeness.

Proof. Thanks to the definition, for all $k \in \{1, \ldots, h\}$ we have

$$|\Lambda_k^Q - \lambda_{k,i_Q}| \leq |\Lambda_{k_Q}^Q - \lambda_{k_Q,i_Q}| \quad \text{and} \quad |\Lambda_k^L - \lambda_{k,i_L}| \leq |\Lambda_{k_L}^L - \lambda_{k_L,i_L}|;$$

moreover, since we are supposing $\varepsilon_L < \theta\, \omega_{f_1,\ldots,f_h}(m)$, we have

$$|\Lambda_k^L| > |\lambda_{k,i_L}| - \theta\, \omega_{f_1,\ldots,f_h}(m)$$
$$\geq 4\theta - 2\theta = 2\theta.$$

It follows that

$$\begin{aligned}
2\theta\, \omega_{f_1,\ldots,f_h}(m) &> \varepsilon_Q + \varepsilon_L \\
&\geq |\Lambda_k^Q - \lambda_{k,i_Q}| + |\Lambda_k^L - \lambda_{k,i_L}| \\
&\geq |\Lambda_k^Q - \Lambda_k^L| \\
&\geq |\Lambda_k^L| |\Lambda_k^{Q-L} - 1| \\
&\geq 2\theta\, \omega_{f_1,\ldots,f_h}(|Q - L| + 1),
\end{aligned}$$

and therefore $\omega_{f_1,\ldots,f_h}(|Q - L| + 1) < \omega_{f_1,\ldots,f_h}(m)$, which implies $|Q - L| \geq m$ by the monotonicity of $\omega_{f_1,\ldots,f_h}$. $\qquad\square$

Thanks to the previous result, it follows from the decomposition of δ_Q that

$$N(Q) \leq 1 + N(L_1) + \cdots + N(L_\nu).$$

Choose $0 \leq \rho \leq \nu$ such that $|L_\rho| > m \geq |L_{\rho+1}|$. By the induction hypothesis, all terms with $|L| \leq m$ vanish, and we obtain

$$\begin{aligned}
N(Q) &\leq 1 + N(L_1) + \cdots + N(L_\rho) \\
&\leq 1 + 2\frac{|L_1 + \cdots + L_\rho|}{m} - \rho \\
&\leq \begin{cases} 1, & \rho = 0 \\ 2\frac{|Q| - m}{m}, & \rho = 1 \\ 2\frac{|L_1 + \cdots + L_\rho|}{m} - 1, & \rho \geq 2, \end{cases} \\
&\leq 2\frac{|Q|}{m} - 1,
\end{aligned}$$

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concluding the proof. $\qquad\square$

Since the $f_1, \ldots, f_h$ satisfy the simultaneous Brjuno condition, there exists a strictly increasing sequence $\{p_\nu\}_{\nu \geq 0}$ of integers with $p_0 = 1$, and such that

$$\sum_{\nu \geq 0} \frac{1}{p_\nu} \log \frac{1}{\omega_{f_1, \ldots, f_h}(p_{\nu+1})} < +\infty. \tag{12}$$

We have to estimate

$$\frac{1}{|Q|} \log \delta_Q = \sum_{j=0}^{p} \frac{1}{|Q|} \log \varepsilon_{L_j}^{-1}, \quad Q \notin \cap_{k=1}^{h} \cap_{j=1}^{n} \mathrm{Res}_j(\Lambda_k).$$

By Lemma 3.5,

$$\mathrm{card}\left\{0 \leq j \leq p : \theta\, \omega_{f_1, \ldots, f_h}(p_{\nu+1}) \leq \varepsilon_{L_j} < \theta\, \omega_{f_1, \ldots, f_h}(p_\nu)\right\} \leq \frac{2n|Q|}{p_\nu}$$

for $\nu \geq 1$. It is also easy to see from the definition of δ_Q that the number of factors $\varepsilon_{L_j}^{-1}$ is bounded by $2|Q| - 1$, and so the previous inequality holds also for $\nu = 0$ when the upper bound is dropped. Therefore,

$$\begin{aligned}
\frac{1}{|Q|} \log \delta_Q &\leq 2n \sum_{\nu \geq 0} \frac{1}{p_\nu} \log \frac{1}{\theta\, \omega_{f_1, \ldots, f_h}(p_{\nu+1})} \\
&= 2n \left(\sum_{\nu \geq 0} \frac{1}{p_\nu} \log \frac{1}{\omega_{f_1, \ldots, f_h}(p_{\nu+1})} + \log \frac{1}{\theta} \sum_{\nu \geq 0} \frac{1}{p_\nu} \right).
\end{aligned} \tag{13}$$

Since $\omega_{f_1, \ldots, f_h}(m)$ tends to zero monotonically as m goes to infinity, we can choose some $\overline{m}$ such that $1 > \omega_{f_1, \ldots, f_h}(m)$ for all $m > \overline{m}$, and we obtain

$$\sum_{\nu \geq \nu_0} \frac{1}{p_\nu} \leq \frac{1}{\log(1/\omega_{f_1, \ldots, f_h}(\overline{m}))} \sum_{\nu \geq \nu_0} \frac{1}{p_\nu} \log \frac{1}{\omega_{f_1, \ldots, f_h}(p_{\nu+1})},$$

where ν_0 verifies the inequalities $p_{\nu_0 - 1} \leq \overline{m} < p_{\nu_0}$. Thus both series in parentheses in (13) converge thanks to (12). Therefore

$$\sup_Q \frac{1}{|Q|} \log \delta_Q < +\infty$$

and this concludes the proof. $\qquad\square$

Combining Proposition 2.10 and Theorem 3.4 we obtain the following equivalence, presented as Theorem 1.4 in the introduction.

Theorem 3.7. *Let $f_1, \ldots, f_h$ be $h \geq 2$ formally linearizable germs of biholomorphisms of $\mathbb{C}^n$ fixing the origin, with simultaneously diagonalizable linear parts, and satisfying the simultaneous Brjuno condition. Then $f_1, \ldots, f_h$ are simultaneously holomorphically linearizable if and only if they all commute pairwise.*

Proof. If $f_1, \ldots, f_h$ are simultaneously holomorphically linearizable, since their linear parts are simultaneously diagonalizable, then they all commute and we are done.

On the other hand, if $f_1, \ldots, f_h$ commute pairwise then they are simultaneously formally linearizable, by Proposition 2.10, hence the assertion follows from Theorem 3.4. $\qquad\square$

We can also deduce the following result.

Corollary 3.8. *Let $f_1, \ldots, f_h$ be $h \geq 2$ germs of biholomorphism of $\mathbb{C}^n$ fixing the origin and such that the linear part of f_1 is diagonalizable, and its eigenvalues have no resonances. If the linear parts of $f_1, \ldots, f_h$ are simultaneously diagonalizable, f_1 commutes with f_k for $k = 2, \ldots, h$ and $f_1, \ldots, f_h$ satisfy the simultaneous Brjuno condition, then $f_1, \ldots, f_h$ are holomorphically simultaneously linearizable.*

Proof. It follows from Proposition 2.13 that $f_1, \ldots, f_h$ are simultaneously formally linearizable, and then the assertion follows from Theorem 3.4. $\qquad\square$

We also obtain the following generalization of Theorem 2.5 of [R1], to which we refer for the definitions of *only level s resonances* and *simultaneous osculating manifold*.

Proposition 3.9. *Let $f_1, \ldots, f_h$ be $h \geq 2$ germs of biholomorphism of $\mathbb{C}^n$ fixing the origin with simultaneously diagonalizable linear parts, and satisfying the simultaneous Brjuno condition. Assume that the spectrum of the linear part of f_1 has only level s resonances and that f_1 commutes with f_k for $k = 2, \ldots, h$. Then $f_1, \ldots, f_h$ are simultaneously holomorphically linearizable if and only if there exists a germ of complex manifold M at O of codimension s, invariant under f_h for each $h = 1, \ldots, m$, which is a simultaneous osculating manifold for $f_1, \ldots, f_m$ and such that $f_1|_M, \ldots, f_m|_M$ are simultaneously holomorphically linearizable.*

Proof. Proposition 2.12 implies that $f_1, \ldots, f_h$ are simultaneously formally linearizable, hence they are in the hypotheses of Theorem 3.4 and this concludes the proof. $\qquad\square$

Remark 3.10. It is possible to prove a simultaneous holomorphic linearization result using the same functional technique of Rüssmann [Rü], yielding the following statement.

Given $f_1, \ldots, f_h$, $h \geq 2$ pairwise commuting germs of biholomorphism of $\mathbb{C}^n$ fixing the origin and such that their linear parts $\Lambda_1, \ldots, \Lambda_h$ are non-resonant and diagonalizable, if

$$\sum_{\nu \geq 0} \frac{1}{2^\nu} \log \frac{1}{\widetilde{\omega}_{\Lambda_1, \ldots, \Lambda_h}(2^{\nu+1})} < +\infty, \tag{14}$$

where

$$\widetilde{\omega}_{\Lambda_1, \ldots, \Lambda_h}(m) = \max_{1 \leq k \leq h} \min_{\substack{2 \leq |Q| \leq m \\ 1 \leq j \leq n}} |\Lambda_k^Q - \lambda_{k,j}|,$$

then $f_1, \ldots f_h$ are holomorphically simultaneously linearizable.

In fact, by the commutation hypothesis, $\Lambda_1, \ldots, \Lambda_h$ are simultaneously diagonalizable, so we may assume them to be diagonal. Moreover, since $\Lambda_1, \ldots, \Lambda_h$ are non-resonant, each f_h admits a unique formal linearization which is their unique formal simultaneous linearization φ, thanks to the commutation hypothesis.

We can then perform the argument of Section 6 of [Rü] for each f_k, getting estimates for the convergence of φ substituting his function Ω with the function $\omega_{\Lambda_k}(m) := \min_{\substack{2 \leq |Q| \leq m \\ 1 \leq j \leq n}} |\Lambda_k^Q - \lambda_{k,j}|$ for each $k = 1, \ldots, h$, and we get the assertion because we can estimate the convergence of φ using the maximum of these $\omega_{\Lambda_1}(m), \ldots, \omega_{\Lambda_h}(m)$, that is using (14).

Note that in this case we use one of the generalizations to a simultaneous Brjuno condition we introduced in Remark 3.3, which is, as we remarked, stronger than the simultaneous Brjuno condition we introduced. Moreover, to use Theorem 2.1 of [Rü] we have to assume $\Lambda_1, \ldots, \Lambda_h$ to be non-resonant to have a unique formal linearization, and to be able to apply Rüssmann's procedure to each germ separately, which is, again, a bit stronger condition than just asking for

our germs to be simultaneously formally linearizable. However, at least in principle, it should be possible to prove Theorem 3.4 with functional arguments.

Appendix: Equivalence of various Brjuno-type series

In this appendix we prove the equivalence of three Brjuno-type condition/series. In fact in the literature studying of the linearization problem one can find at least the following three series:

$$B(\omega) = \sum_{\nu \geq 0} \frac{1}{2^\nu} \log \frac{1}{\omega(2^{\nu+1})},$$

$$R(\omega) = \sum_{k \geq 1} \frac{1}{k^2} \log \frac{1}{\omega(k)},$$

and

$$\Gamma(\omega) = \sum_{k \geq 1} \frac{1}{k(k+1)} \log \frac{1}{\omega(k)},$$

where $\omega \colon \mathbb{N} \to \mathbb{R}$ is a non-increasing monotone function, usually containing the information on the small divisors one has to estimate. The first one was introduced by Brjuno in [Brj] while the other two can be found for example in [D], [Rü] and [GM].

Rüssmann proved that, in dimension 1, the convergence of $R(\omega)$ is equivalent to the convergence of $B(\omega)$ (see Lemma 8.2 of [Rü]), and he also proved the following result.

Lemma 4.1. (Rüssmann, 2002 [Rü]) *Let* $\Omega \colon \mathbb{N} \to (0, +\infty)$ *be a monotone non decreasing function, and let* $\{s_\nu\}$ *be defined by* $s_\nu := 2^{q+\nu}$*, with* $q \in \mathbb{N}$*. Then*

$$\sum_{\nu \geq 0} \frac{1}{s_\nu} \log \Omega(s_{\nu+1}) \leq \sum_{k \geq 2^{q+1}} \frac{1}{k^2} \log \Omega(k).$$

We proved (see Lemma 4.2 of [R4]) that in fact if $R(\omega) < +\infty$ then also $B(\omega) < +\infty$. In fact, we can prove the following equivalence.

Theorem 4.2. *Let* $\omega \colon \mathbb{N} \to (0,1)$ *be a non-increasing monotone function and consider the following series*

$$B(\omega) = \sum_{\nu \geq 0} \frac{1}{2^\nu} \log \frac{1}{\omega(2^{\nu+1})}, \quad R(\omega) = \sum_{k \geq 1} \frac{1}{k^2} \log \frac{1}{\omega(k)}, \ and \ \Gamma(\omega) = \sum_{k \geq 1} \frac{1}{k(k+1)} \log \frac{1}{\omega(k)}.$$

Then we have the following inequalities:

$$\Gamma(\omega) \leq R(\omega) \leq 2\,\Gamma(\omega), \tag{15}$$

and

$$\Gamma(\omega) \leq \frac{1}{2} B(\omega) \leq 2\,\Gamma(\omega) - \log \frac{1}{\omega(1)}. \tag{16}$$

Proof. The first inequalities are immediate to recover, because for any $k \geq 1$ we have

$$\frac{1}{k(k+1)} \leq \frac{1}{k^2} \leq \frac{2}{k(k+1)}.$$

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To prove (16), recall that

$$\sum_{k=a}^{b-1} \frac{1}{k(k+1)} = \frac{1}{a} - \frac{1}{b}.$$

Then, for the first inequality in (16) we have that

$$\frac{1}{\omega(2^\nu)} \leq \cdots \leq \frac{1}{\omega(2^{\nu+1}-1)} \leq \frac{1}{\omega(2^{\nu+1})},$$

hence

$$\Gamma(\omega) = \sum_{\nu \geq 0} \sum_{k=2^\nu}^{2^{\nu+1}-1} \frac{1}{k(k+1)} \log \frac{1}{\omega(k)}$$

$$\leq \sum_{\nu \geq 0} \log \frac{1}{\omega(2^{\nu+1})} \sum_{k=2^\nu}^{2^{\nu+1}-1} \frac{1}{k(k+1)}$$

$$= \frac{1}{2} B(\omega).$$

For the other inequality, we use

$$\frac{1}{\omega(2^{\nu+1})} \leq \cdots \leq \frac{1}{\omega(2^{\nu+2}-1)} \leq \frac{1}{\omega(2^{\nu+2})};$$

hence

$$\frac{1}{2} B(\omega) = 2 \sum_{\nu \geq 0} \log \frac{1}{\omega(2^{\nu+1})} \sum_{k=2^{\nu+1}}^{2^{\nu+2}-1} \frac{1}{k(k+1)}$$

$$\leq 2 \sum_{\nu \geq 0} \sum_{k=2^{\nu+1}}^{2^{\nu+2}-1} \frac{1}{k(k+1)} \log \frac{1}{\omega(k)}$$

$$= 2\,\Gamma(\omega) - \log \frac{1}{\omega(1)},$$

and we are done. $\qquad\qquad\square$

As a corollary we obtain the following equivalence.

Corollary 4.3. *With the above definitions, we have*

$$B(\omega) < +\infty \iff \Gamma(\omega) < +\infty \iff R(\omega) < +\infty.$$

In particular, the simultaneous Brjuno condition, or any Brjuno condition, can be expressed through the convergence of any of the three series B, R, and Γ.

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