Brjuno conditions for linearization in presence of resonances

Jasmin Raissy

Abstract. We present a new proof, under a slightly different (and more natural) arithmetic hypothesis, and using direct computations via power series expansions, of a holomorphic linearization result in presence of resonances originally proved by Rüssmann.

1 Introduction

We consider a germ of biholomorphism f of $\mathbb{C}^n$ at a fixed point p, which, up to translation, we may place at the origin O. One of the main questions in the study of local holomorphic dynamics (see [1, 2, 4], or [11, Chapter 1], for general surveys on this topic) is when f is *holomorphically linearizable*, *i.e.*, when there exists a local holomorphic change of coordinates such that f is conjugated to its linear part Λ.

A way to solve such a problem is to first look for a formal transformation φ solving

$$f \circ \varphi = \varphi \circ \Lambda,$$

i.e., to ask when f is *formally linearizable*, and then to check whether φ is convergent. Moreover, since up to linear changes of the coordinates we can always assume Λ to be in Jordan normal form, *i.e.*,

$$\Lambda = \begin{pmatrix} \lambda_1 & & & \\ \varepsilon_2 & \lambda_2 & & \\ & \ddots & \ddots & \\ & & \varepsilon_n & \lambda_n \end{pmatrix},$$

where the eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{C}^*$ are not necessarily distincts, and $\varepsilon_j \in \{0, \varepsilon\}$ can be non-zero only if $\lambda_{j-1} = \lambda_j$, we can reduce ourselves to study such germs, and to search for φ *tangent to the identity*, that is, with linear part equal to the identity.

The answer to this question depends on the set of eigenvalues of $\mathrm{d}f_O$, usually called the *spectrum* of $\mathrm{d}f_O$. In fact, if we denote by $\lambda_1, \ldots, \lambda_n \in \mathbb{C}^*$ the eigenvalues of $\mathrm{d}f_O$, then it may happen that there exists a multi-index $Q = (q_1, \ldots, q_n) \in \mathbb{N}^n$, with $|Q| \geq 2$, such that

$$\lambda^Q - \lambda_j := \lambda_1^{q_1} \cdots \lambda_n^{q_n} - \lambda_j = 0 \tag{1.1}$$

for some $1 \leq j \leq n$; a relation of this kind is called a (*multiplicative*) *resonance* of f *relative to the* j*-th coordinate*, Q is called a *resonant multi-index relative to the* j*-th coordinate*, and we put

$$\mathrm{Res}_j(\lambda) := \{Q \in \mathbb{N}^n \mid |Q| \geq 2, \lambda^Q = \lambda_j\}.$$

The elements of $\mathrm{Res}(\lambda) := \bigcup_{j=1}^n \mathrm{Res}_j(\lambda)$ are simply called *resonant* multi-indices. A *resonant monomial* is a monomial $z^Q := z_1^{q_1} \cdots z_n^{q_n}$ in the j-th coordinate with $Q \in \mathrm{Res}_j(\lambda)$.

Resonances are the formal obstruction to linearization. Indeed, we have the following classical result:

Theorem 1.1 (Poincaré, 1893 [7]; Dulac, 1904 [6]). *Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin O with linear part in Jordan normal form. Then there exists a formal transformation φ of $\mathbb{C}^n$, without constant term and tangent to the identity, conjugating f to a formal power series $g \in \mathbb{C}[[z_1, \ldots, z_n]]^n$ without constant term, with same linear part and containing only resonant monomials. Moreover, the resonant part of the formal change of coordinates φ can be chosen arbitrarily, but once this is done, φ and g are uniquely determined. In particular, if the spectrum of $\mathrm{d}f_O$ has no resonances, f is formally linearizable and the formal linearization is unique.*

A formal transformation g of $\mathbb{C}^n$, without constant term, and with linear part in Jordan normal form with eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{C}^*$, is called *in Poincaré-Dulac normal form* if it contains only resonant monomials with respect to $\lambda_1, \ldots, \lambda_n$.

If f is a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin, a series g in Poincaré-Dulac normal form formally conjugated to f is called a *Poincaré-Dulac* (*formal*) *normal form of* f.

The problem with Poincaré-Dulac normal forms is that, usually, they are not unique. In particular, one may wonder whether it could be possible to have such a normal form including finitely many resonant monomials only. This is indeed the case (see, *e.g.*, Reich [12]) when $\mathrm{d}f_O$ belongs to the so-called *Poincaré domain*, that is when $\mathrm{d}f_O$ is invertible and O is either *attracting*, *i.e.*, all the eigenvalues of $\mathrm{d}f_O$ have modulus less than 1, or *repelling*, *i.e.*, all the eigenvalues of $\mathrm{d}f_O$ have modulus greater than 1 (when $\mathrm{d}f_O$ is still invertible but does not belong to the Poincaré domain, we shall say that it belongs to the *Siegel domain*).

Even without resonances, the holomorphic linearization is not guaranteed. The best positive result is essentially due to Brjuno [5]. To describe such a result, let us introduce the following definitions:

Definition 1.2. For $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ and $m \geq 2$ set

$$\omega_{\lambda_1,\dots,\lambda_n}(m) = \min_{\substack{2\leq|Q|\leq m \\ 1\leq j\leq n}} |\lambda^Q - \lambda_j|. \tag{1.2}$$

If $\lambda_1, \dots, \lambda_n$ are the eigenvalues of $\mathrm{d}f_O$, we shall write $\omega_f(m)$ for $\omega_{\lambda_1,\dots,\lambda_n}(m)$.

It is clear that $\omega_f(m) \neq 0$ for all $m \geq 2$ if and only if there are no resonances. It is also not difficult to prove that if f belongs to the Siegel domain then

$$\lim_{m\to+\infty} \omega_f(m) = 0,$$

which is the reason why, even without resonances, the formal linearization might be divergent.

Definition 1.3. Let $n \geq 2$ and let $\lambda_1, \dots, \lambda_n \in \mathbb{C}^*$ be not necessarily distinct. We say that λ *satisfies the Brjuno condition* if there exists a strictly increasing sequence of integers $\{p_\nu\}_{\nu\geq 0}$ with $p_0 = 1$ such that

$$\sum_{\nu\geq 0} \frac{1}{p_\nu} \log \frac{1}{\omega_{\lambda_1,\dots,\lambda_n}(p_{\nu+1})} < \infty. \tag{1.3}$$

Brjuno's argument for vector fields, when adapted to the case of germs of biholomorphisms, yields the following result (see [8]).

Theorem 1.4 (Brjuno, 1971 [5]). *Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin, such that $\mathrm{d}f_O$ is diagonalizable. Assume moreover that the spectrum of $\mathrm{d}f_O$ has no resonances and satisfies the Brjuno condition. Then f is holomorphically linearizable.*

In the resonant case, one can still find formally linearizable germs, (see for example [9] and [10]), so two natural questions arise.

(Q1) *How many Poincaré-Dulac formal normal forms does a formally linearizable germ have*?

(Q2) *Is it possible to find arithmetic conditions on the eigenvalues of the spectrum of $\mathrm{d}f_O$ ensuring holomorphic linearizability of formally linearizable germs*?

Rüssmann gave answers to both questions in [13], an I.H.E.S. preprint which is no longer available, and that was finally published in [14]. The answer to the first question is the following (the statement is slightly different from the original one presented in [14] but perfectly equivalent):

Theorem 1.5 (Rüssmann, 2002 [14]). *Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin. If f is formally linearizable, then the linear form is its unique Poincaré-Dulac normal form.*

To answer to the second question, Rüssmann introduced the following condition, that we shall call Rüssmann condition.

Definition 1.6. Let $n \geq 2$ and let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}^*$ be not necessarily distinct. We say that $\lambda = (\lambda_1, \ldots, \lambda_n)$ *satisfies the Rüssmann condition* if there exists a function $\Omega\colon \mathbb{N} \to \mathbb{R}$ such that:

(i) $k \leq \Omega(k) \leq \Omega(k+1)$ for all $k \in \mathbb{N}$;
(ii) $\sum_{k\geq 1} \frac{1}{k^2} \log \Omega(k) < +\infty$, and
(iii) $|\lambda^Q - \lambda_j| \geq \frac{1}{\Omega(|Q|)}$ for all $j = 1, \ldots n$ and for each multi-index $Q \in \mathbb{N}$ with $|Q| \geq 2$ not giving a resonance relative to j.

Rüssmann proved the following generalization of Brjuno's Theorem 1.4 (the statement is slightly different from the original one presented in [14] but perfectly equivalent).

Theorem 1.7 (Rüssmann, 2002 [14]). *Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin and such that $\mathrm{d}f_O$ is diagonalizable. If f is formally linearizable and the spectrum of $\mathrm{d}f_O$ satisfies the Rüssmann condition, then f is holomorphically linearizable.*

We refer to [14] for the original proof and we limit ourselves to briefly recall here the main ideas. To prove these results, Rüssmann first studies the process of Poincaré-Dulac formal normalization using a functional iterative approach, without assuming anything on the diagonalizability of $\mathrm{d}f_O$. With this functional technique he proves Theorem 1.5; then he constructs a formal iteration process converging to a zero of the operator $\mathcal{F}(\varphi) := f \circ \varphi - \varphi \circ \Lambda$ (where Λ is the linear part of f), and, assuming Λ diagonal, he gives estimates for each iteration step, proving that, under what we called the Rüssmann condition, the process converges to a holomorphic linearization.

In this paper, we shall first present a direct proof of Theorem 1.5 using power series expansions. Then we shall give a direct proof, using explicit computations with power series expansions and then proving convergence via majorant series, of an analogue of Theorem 1.7 under the following slightly different assumption, which is the natural generalization to the resonant case of the condition introduced by Brjuno.

Definition 1.8. Let $n \geq 2$ and let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}^*$ be not necessarily distinct. For $m \geq 2$ set

$$\widetilde{\omega}_{\lambda_1,\ldots,\lambda_n}(m) = \min_{\substack{2\leq|Q|\leq m \\ Q\notin \mathrm{Res}_j(\lambda)}} \min_{1\leq j\leq n} |\lambda^Q - \lambda_j|,$$

where $\mathrm{Res}_j(\lambda)$ is the set of multi-indices $Q \in \mathbb{N}^n$, with $|Q| \geq 2$, giving a resonance relation for $\lambda = (\lambda_1, \ldots, \lambda_n)$ relative to $1 \leq j \leq n$, *i.e.*, $\lambda^Q - \lambda_j = 0$. If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $\mathrm{d}f_O$, we shall write $\widetilde{\omega}_f(m)$ for $\widetilde{\omega}_{\lambda_1,\ldots,\lambda_n}(m)$.

Definition 1.9. Let $n \geq 2$ and let $\lambda = (\lambda_1, \ldots, \lambda_n) \in (\mathbb{C}^*)^n$. We say that λ *satisfies the reduced Brjuno condition* if there exists a strictly increasing sequence of integers $\{p_\nu\}_{\nu\geq 0}$ with $p_0 = 1$ such that

$$\sum_{\nu\geq 0} \frac{1}{p_\nu} \log \frac{1}{\widetilde{\omega}_{\lambda_1,\ldots,\lambda_n}(p_{\nu+1})} < \infty.$$

We shall then prove:

Theorem 1.10. *Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin and such that $\mathrm{d}f_O$ is diagonalizable. If f is formally linearizable and the spectrum of $\mathrm{d}f_O$ satisfies the reduced Brjuno condition, then f is holomorphically linearizable.*

We shall also show that Rüssmann condition implies the reduced Brjuno condition and so our result implies Theorem 1.7. The converse is known to be true in dimension 1, as proved by Rüssmann in [14], but is not known in higher dimension.

The structure of this paper is as follows. In the next section we shall discuss properties of formally linearizable germs, and we shall give our direct proof of Theorem 1.5. In Section 3 we shall prove Theorem 1.10 using majorant series. In the last section we shall discuss relations between Rüssmann condition and the reduced Brjuno condition.

ACKNOWLEDGEMENTS. I would like to thank Marco Abate for helpful comments on a draft of this work.

2 Formally linearizable germs

In general, a germ f can have several Poincaré-Dulac formal normal forms; however, we can say something on the shape of the formal conjugations between them. We have in fact the following result.

Proposition 2.1. *Let f and g be two germs of biholomorphism of $\mathbb{C}^n$ fixing the origin, with the same linear part Λ and in Poincaré-Dulac normal form. If there exists a formal transformation φ of $\mathbb{C}^n$, with no constant term and tangent to the identity, conjugating f and g, then φ contains only monomials that are resonant with respect to the eigenvalues of Λ.*

Proof. Since f and g are in Poincaré-Dulac normal form, Λ is in Jordan normal form. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of Λ. We shall prove that a formal solution $\varphi = I + \widehat{\varphi}$ of

$$f \circ \varphi = \varphi \circ g \tag{2.1}$$

contains only monomials that are resonant with respect to $\lambda_1, \dots, \lambda_n$. Using the standard multi-index notation, for each $j \in \{1, \dots, n\}$ we can write

$$f_j(z) = \lambda_j z_j + \varepsilon_j z_{j-1} + z_j f_j^{\mathrm{res}}(z) = \lambda_j z_j + \varepsilon_j z_{j-1} + z_j \sum_{\substack{Q \in N_j \\ \lambda^Q = 1}} f_{Q,j} z^Q,$$

$$g_j(z) = \lambda_j z_j + \varepsilon_j z_{j-1} + z_j g_j^{\mathrm{res}}(z) = \lambda_j z_j + \varepsilon_j z_{j-1} + z_j \sum_{\substack{Q \in N_j \\ \lambda^Q = 1}} g_{Q,j} z^Q,$$

and

$$\varphi_j(z) = z_j \left(1 + \varphi_j^{\mathrm{res}}(z) + \varphi_j^{\neq\mathrm{res}}(z)\right) = z_j + z_j \sum_{\substack{Q \in N_j \\ \lambda^Q = 1}} \varphi_{Q,j} z^Q + z_j \sum_{\substack{Q \in N_j \\ \lambda^Q \neq 1}} \varphi_{Q,j} z^Q,$$

where

$$N_j := \{Q \in \mathbb{Z}^n \mid |Q| \geq 1, q_j \geq -1, q_h \geq 0 \text{ for all } h \neq j\},$$

and $\varepsilon_j \in \{0, 1\}$ can be non-zero only if $\lambda_j = \lambda_{j-1}$. With these notations, the left-hand side of the j-th coordinate of (2.1) becomes

$$\begin{aligned}
&(f \circ \varphi)_j(z) \\
&= \lambda_j \varphi_j(z) + \varepsilon_j \varphi_{j-1}(z) + \varphi_j(z) \sum_{\substack{Q \in N_j \\ \lambda^Q = 1}} f_{Q,j} \prod_{k=1}^n \varphi_k(z)^{q_k} \\
&= \lambda_j z_j \left(1 + \varphi_j^{\mathrm{res}}(z) + \varphi_j^{\neq\mathrm{res}}(z)\right) + \varepsilon_j z_{j-1} \left(1 + \varphi_{j-1}^{\mathrm{res}}(z) + \varphi_{j-1}^{\neq\mathrm{res}}(z)\right) \\
&\quad + z_j \left(1 + \varphi_j^{\mathrm{res}}(z) + \varphi_j^{\neq\mathrm{res}}(z)\right) \sum_{\substack{Q \in N_j \\ \lambda^Q = \lambda_j}} f_{Q,j} z^Q \prod_{k=1}^n \left(1 + \varphi_k^{\mathrm{res}}(z) + \varphi_k^{\neq\mathrm{res}}(z)\right)^{q_k},
\end{aligned} \tag{2.2}$$

while the j-th coordinate of the right-hand side of (2.1) becomes

$$
\begin{aligned}
&(\varphi\circ g)_j(z)\\
&=g_j(z)+g_j(z)\sum_{\substack{Q\in N_j\\ \lambda^Q=1}}\varphi_{Q,j}\prod_{k=1}^n g_k(z)^{q_k}+g_j(z)\sum_{\substack{Q\in N_j\\ \lambda^Q\neq 1}}\varphi_{Q,j}\prod_{k=1}^n g_k(z)^{q_k}\\
&=\lambda_j z_j+\varepsilon_j z_{j-1}+z_j g_j^{\mathrm{res}}(z)\\
&\quad+\left(\lambda_j z_j+\varepsilon_j z_{j-1}+z_j g_j^{\mathrm{res}}(z)\right)\sum_{\substack{Q\in N_j\\ \lambda^Q=1}}\varphi_{Q,j}z^Q\prod_{k=1}^n\left(\lambda_k+\varepsilon_k\frac{z_{k-1}}{z_k}+g_k^{\mathrm{res}}(z)\right)^{q_k}\\
&\quad+\left(\lambda_j z_j+\varepsilon_j z_{j-1}+z_j g_j^{\mathrm{res}}(z)\right)\sum_{\substack{Q\in N_j\\ \lambda^Q\neq 1}}\varphi_{Q,j}z^Q\prod_{k=1}^n\left(\lambda_k+\varepsilon_k\frac{z_{k-1}}{z_k}+g_k^{\mathrm{res}}(z)\right)^{q_k}.
\end{aligned}
\tag{2.3}
$$

Furthermore, notice that if P and Q are two multi-indices such that $\lambda^P=\lambda^Q=1$, then we have $\lambda^{\alpha P+\beta Q}=1$ for every $\alpha,\beta\in\mathbb{Z}$.

We want to prove that $\varphi_{Q,j}=0$ for each multi-index $Q\in N_j$ such that $\lambda^Q\neq 1$. Let us assume by contradiction that this is not true, and let $\widetilde{Q}$ be the first (with respect to the lexicographic order) multi-index in $N:=\bigcup_{j=1}^n N_j$ so that $\lambda^{\widetilde{Q}}\neq 1$ and $\varphi_{\widetilde{Q},j}\neq 0$. Let j be the minimal in $\{1,\dots,n\}$ such that $\widetilde{Q}\in N_j$, and let us compute the coefficient of the monomial $z^{\widetilde{Q}+e_j}$ in (2.2) and (2.3). In (2.2) we only have $\lambda_j\varphi_{\widetilde{Q},j}$ because, since $f-\Lambda$ is of second order and resonant, other contributions could come only from coefficients $\varphi_{P,k}$ with $|P|<|\widetilde{Q}|$ and $\lambda^P\neq 1$, but there are no such coefficients thanks to the minimality of $\widetilde{Q}$ and j. In (2.3) we can argue analogously, but we have also to take care of the monomials divisible by $\varepsilon_k^h(z_{k-1}/z_k)^h z^P$, with $\lambda^P=1$; in this last case, if $\varepsilon_k\neq 0$, we obtain a multi-index $P-he_k+he_{k-1}$, and again $\lambda^{P-he_k+he_{k-1}}=1$ because $\lambda_k=\lambda_{k-1}$. Then in (2.3) we only have $\lambda^{\widetilde{Q}+e_j}\varphi_{\widetilde{Q},j}$. Hence, we have

$$(\lambda^{\widetilde{Q}+e_j}-\lambda_j)\varphi_{\widetilde{Q},j}=0,$$

yielding

$$\varphi_{\widetilde{Q},j}=0,$$

because $\lambda^{\widetilde{Q}}\neq 1$ and $\lambda_j\neq 0$, and contradicting the hypothesis. $\square$

Remark 2.2. It is clear from the proof that Proposition 2.1 holds also in the formal category, *i.e.*, for $f,g\in\mathbb{C}_O[\![z_1,\dots,z_n]\!]$ formal power series without constant terms in Poincaré-Dulac normal form.

We can now give a direct proof of Theorem 1.5, *i.e.*, that when a germ is formally linearizable, then the linear form is its unique Poincaré-Dulac normal form.

Theorem 2.3. *Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin. If f is formally linearizable, then the linear form is its unique Poincaré-Dulac normal form.*

Proof. Let Λ be the linear part of f. Up to linear conjugacy, we may assume that Λ is in Jordan normal form. If the eigenvalues $\lambda_1, \dots, \lambda_n$ of Λ have no resonances, then there is nothing to prove. Let us then assume that we have resonances, and let us assume by contradiction that there is another Poincaré-Dulac formal normal form $g \not\equiv \Lambda$ associated to f. Since f is formally linearizable and it is formally conjugated to g, also g is formally linearizable. Thanks to Proposition 2.1, any formal linearization ψ of g tangent to the identity contains only monomials resonant with respect to $\lambda_1, \dots, \lambda_n$; hence, writing $g = \Lambda + g^{\mathrm{res}}$ and $\psi = I + \psi^{\mathrm{res}}$, the conjugacy equation $g \circ \psi = \psi \circ \Lambda$ becomes

$$\begin{aligned}\Lambda + \Lambda\psi^{\mathrm{res}} + g^{\mathrm{res}} \circ (I + \psi^{\mathrm{res}}) &= (\Lambda + g^{\mathrm{res}}) \circ (I + \psi^{\mathrm{res}})\\ &= (I + \psi^{\mathrm{res}}) \circ \Lambda\\ &= \Lambda + \psi^{\mathrm{res}} \circ \Lambda\\ &= \Lambda + \Lambda\psi^{\mathrm{res}},\end{aligned}$$

because $\psi^{\mathrm{res}} \circ \Lambda = \Lambda\psi^{\mathrm{res}}$. Hence there must be

$$g^{\mathrm{res}} \circ \psi \equiv 0,$$

and composing on the right with ψ^{-1} we get $g^{\mathrm{res}} \equiv 0$. $\square$

Remark 2.4. As a consequence of the previous result, we get that *any formal normalization given by the Poincaré-Dulac procedure applied to a formally linerizable germ f is indeed a formal linearization of the germ.* In particular, we have *uniqueness of the Poincaré-Dulac normal form (which is linear and hence holomorphic), but not of the formal linearizations.* Hence *a formally linearizable germ f is formally linearizable via a formal transformation $\varphi = \mathrm{Id} + \widehat{\varphi}$ containing only non-resonant monomials.* In fact, thanks to the standard proof of Poincaré-Dulac Theorem (see [11, Theorem 1.3.25]), we can consider the formal normalization obtained with the Poincaré-Dulac procedure and imposing $\varphi_{Q,j} = 0$ for all Q and j such that $\lambda^Q = \lambda_j$; and this formal transformation φ, by Theorem 2.3, conjugates f to its linear part.

3 Convergence under the reduced Brjuno condition

Now we have all the ingredients needed to prove Theorem 1.10.

Theorem 3.1. *Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin and such that $\mathrm{d}f_O$ is diagonalizable. If f is formally linearizable and the spectrum of $\mathrm{d}f_O$ satisfies the reduced Brjuno condition, then f is holomorphically linearizable.*

Proof. Up to linear changes of the coordinates, we may assume that the linear part Λ of f is diagonal, *i.e.*, $\Lambda = \mathrm{Diag}(\lambda_1, \dots, \lambda_n)$. From the conjugacy equation

$$f \circ \varphi = \varphi \circ \Lambda, \tag{3.1}$$

writing $f(z) = \Lambda z + \sum_{|L|\geq 2} f_L z^L$, and $\varphi(w) = w + \sum_{|Q|\geq 2} \varphi_Q w^Q$, where f_L and φ_Q belong to $\mathbb{C}^n$, we have that coefficients of φ have to verify

$$\sum_{|Q|\geq 2} A_Q \varphi_Q w^Q = \sum_{|L|\geq 2} f_L \left(\sum_{|M|\geq 1} \varphi_M w^M \right)^L, \tag{3.2}$$

where

$$A_Q = \lambda^Q I_n - \Lambda.$$

The matrices A_Q are not invertible only when $Q \in \bigcup_{j=1}^n \mathrm{Res}_j(\lambda)$, but, thanks Remark 2.4, we can set $\varphi_{Q,j} = 0$ for all $Q \in \mathrm{Res}_j(\lambda)$; hence we just have to consider $Q \notin \bigcap_{j=1}^n \mathrm{Res}_j(\lambda)$, and, to prove the convergence of the formal conjugation φ in a neighbourhood of the origin, it suffices to show that

$$\sup_Q \frac{1}{|Q|} \log \|\varphi_Q\| < \infty, \tag{3.3}$$

for $|Q| \geq 2$ and $Q \notin \cap_{j=1}^n \mathrm{Res}_j(\lambda)$.

Since f is holomorphic in a neighbourhood of the origin, there exists a positive number ρ such that $\|f_L\| \leq \rho^{|L|}$ for $|L| \geq 2$. The functional equation (3.1) remains valid under the linear change of coordinates $f(z) \mapsto \sigma f(z/\sigma)$, $\varphi(w) \mapsto \sigma\varphi(w/\sigma)$ with $\sigma = \max\{1, \rho^2\}$. Therefore we may assume that

$$\forall |L| \geq 2 \qquad \|f_L\| \leq 1.$$

It follows from (3.2) that for any multi-index $Q \in \mathbb{N}^n \setminus \bigcap_{j=1}^n \mathrm{Res}_j(\lambda)$ with $|Q| \geq 2$ we have

$$\|\varphi_Q\| \leq \varepsilon_Q^{-1} \sum_{\substack{Q_1+\cdots+Q_\nu=Q \\ \nu\geq 2}} \|\varphi_{Q_1}\| \cdots \|\varphi_{Q_\nu}\|, \tag{3.4}$$

where

$$\varepsilon_Q = \min_{\substack{1\le j\le n \\ Q\notin \mathrm{Res}_j(\lambda)}} |\lambda^Q - \lambda_j|.$$

We can define, inductively, for $m \geq 2$

$$\alpha_m = \sum_{\substack{m_1+\cdots+m_\nu=j \\ \nu\ge 2}} \alpha_{m_1}\cdots\alpha_{m_\nu},$$

and

$$\delta_Q = \varepsilon_Q^{-1} \max_{\substack{Q_1+\cdots+Q_\nu=Q \\ \nu\ge 2}} \delta_{Q_1}\cdots\delta_{Q_\nu},$$

for $Q \in \mathbb{N}^n \setminus \bigcap_{j=1}^n \mathrm{Res}_j(\lambda)$ with $|Q| \geq 2$, with $\alpha_1 = 1$ and $\delta_E = 1$, where E is any integer vector with $|E| = 1$. Then, by induction, we have that

$$\|\varphi_Q\| \leq \alpha_{|Q|}\delta_Q,$$

for every $Q \in \mathbb{N}^n \setminus \bigcap_{j=1}^n \mathrm{Res}_j(\lambda)$ with $|Q| \geq 2$. Therefore, to establish (3.3) it suffices to prove analogous estimates for α_m and δ_Q.

It is easy to estimate α_m. Let $\alpha = \sum_{m\geq 1} \alpha_m t^m$. We have

$$\alpha - t = \sum_{m\ge 2} \alpha_m t^m = \sum_{m\ge 2}\left(\sum_{h\ge 1}\alpha_h t^h\right)^m = \frac{\alpha^2}{1-\alpha}.$$

This equation has a unique holomorphic solution vanishing at zero

$$\alpha = \frac{t+1}{4}\left(1 - \sqrt{1 - \frac{8t}{(1+t)^2}}\right),$$

defined for $|t|$ small enough. Hence,

$$\sup_m \frac{1}{m}\log\alpha_m < \infty,$$

as we want.

To estimate δ_Q we have to take care of small divisors. First of all, for each multi-index $Q \notin \bigcap_{j=1}^n \mathrm{Res}_j(\lambda)$ with $|Q| \geq 2$ we can associate to δ_Q a decomposition of the form

$$\delta_Q = \varepsilon_{L_0}^{-1}\varepsilon_{L_1}^{-1}\cdots\varepsilon_{L_p}^{-1}, \tag{3.5}$$

where $L_0 = Q$, $|Q| > |L_1| \geq \cdots \geq |L_p| \geq 2$ and $L_j \notin \bigcap_{j=1}^n \mathrm{Res}_j(\lambda)$ for all $j = 1, \ldots, p$ and $p \geq 1$. In fact, we choose a decomposition $Q =$

$Q_1+\cdots+Q_\nu$ such that the maximum in the expression of δ_Q is achieved; obviously, Q_j does not belong to $\bigcap_{j=1}^n \operatorname{Res}_j(\lambda)$ for all $j = 1, \ldots, \nu$. We can then express δ_Q in terms of $\varepsilon_{Q_j}^{-1}$ and $\delta_{Q'_j}$ with $|Q'_j| < |Q_j|$. Carrying on this process, we eventually arrive at a decomposition of the form (3.5). Furthermore, for each multi-index $Q \notin \bigcap_{j=1}^n \operatorname{Res}_j(\lambda)$ with $|Q| \geq 2$, we can choose an index i_Q so that

$$\varepsilon_Q = |\lambda^Q - \lambda_{i_Q}|.$$

The rest of the proof follows closely in [9, proof of Theorem 5.1]. For the benefit of the reader, we report it here.

For $m \geq 2$ and $1 \leq j \leq n$, we can define

$$N_m^j(Q)$$

to be the number of factors ε_L^{-1} in the expression (3.5) of δ_Q, satisfying

$$\varepsilon_L < \theta\, \widetilde{\omega}_f(m), \quad \text{and} \quad i_L = j,$$

where $\widetilde{\omega}_f(m)$ is defined in Definition 1.8, and in this notation can be expressed as

$$\widetilde{\omega}_f(m) = \min_{\substack{2\leq|Q|\leq m \\ Q\notin\bigcap_{j=1}^n \operatorname{Res}_j(\lambda)}} \varepsilon_Q,$$

and θ is the positive real number satisfying

$$4\theta = \min_{1\leq h\leq n} |\lambda_h| \leq 1.$$

The last inequality can always be satisfied by replacing f by f^{-1} if necessary. Moreover we also have $\widetilde{\omega}_f(m) \leq 2$.

Notice that $\widetilde{\omega}_f(m)$ is non-increasing with respect to m and under our assumptions $\widetilde{\omega}_f(m)$ tends to zero as m goes to infinity. The following is the key estimate.

Lemma 3.2. *For $m \geq 2$, $1 \leq j \leq n$ and $Q \notin \bigcap_{j=1}^n \operatorname{Res}_j(\lambda)$, we have*

$$N_m^j(Q) \leq \begin{cases} 0, & \text{if } |Q| \leq m, \\ \dfrac{2|Q|}{m} - 1, & \text{if } |Q| > m. \end{cases}$$

Proof. The proof is done by induction on $|Q|$. Since we fix m and j throughout the proof, we write N instead of N_m^j.

For $|Q| \leq m$,

$$\varepsilon_Q \geq \widetilde{\omega}_f(|Q|) \geq \widetilde{\omega}_f(m) > \theta\, \widetilde{\omega}_f(m),$$

hence $N(Q) = 0$.

Assume now that $|Q| > m$. Then $2|Q|/m - 1 \geq 1$. Write

$$\delta_Q = \varepsilon_Q^{-1} \delta_{Q_1} \cdots \delta_{Q_\nu}, \quad Q = Q_1 + \cdots + Q_\nu, \quad \nu \geq 2,$$

with $|Q| > |Q_1| \geq \cdots \geq |Q_\nu|$; note that $Q - Q_1$ does not belong to $\bigcap_{j=1}^n \mathrm{Res}_j(\lambda)$, otherwise the other Q_h's would be in $\bigcap_{j=1}^n \mathrm{Res}_j(\lambda)$. We have to consider the following different cases.

Case 1: $\varepsilon_Q \geq \theta\, \widetilde{\omega}_f(m)$ and i_Q arbitrary, or $\varepsilon_Q < \theta\, \widetilde{\omega}_f(m)$ and $i_Q \neq j$. Then

$$N(Q) = N(Q_1) + \cdots + N(Q_\nu),$$

and applying the induction hypotheses to each term we get $N(Q) \leq (2|Q|/m) - 1$.

Case 2: $\varepsilon_Q < \theta\, \widetilde{\omega}_f(m)$ and $i_Q = j$. Then

$$N(Q) = 1 + N(Q_1) + \cdots + N(Q_\nu),$$

and there are three different subcases.

Case 2.1: $|Q_1| \leq m$. Then

$$N(Q) = 1 < \frac{2|Q|}{m} - 1,$$

as we want.

Case 2.2: $|Q_1| \geq |Q_2| > m$. Then there is ν' such that $2 \leq \nu' \leq \nu$ and $|Q_{\nu'}| > m \geq |Q_{\nu'+1}|$, and we have

$$N(Q) = 1 + N(Q_1) + \cdots + N(Q_{\nu'}) \leq 1 + \frac{2|Q|}{m} - \nu' \leq \frac{2|Q|}{m} - 1.$$

Case 2.3: $|Q_1| > m \geq |Q_2|$. Then

$$N(Q) = 1 + N(Q_1),$$

and there are again three different subcases.

Case 2.3.1: $i_{Q_1} \neq j$. Then $N(Q_1) = 0$ and we are done.

Case 2.3.2: $|Q_1| \leq |Q| - m$ and $i_{Q_1} = j$. Then

$$N(Q) \leq 1 + 2\,\frac{|Q| - m}{m} - 1 < \frac{2|Q|}{m} - 1.$$

Case 2.3.3: $|Q_1| > |Q| - m$ and $i_{Q_1} = j$. The crucial remark is that $\varepsilon_{Q_1}^{-1}$ gives no contribute to $N(Q_1)$, as shown in the next lemma.

Lemma 3.3. *If $Q > Q_1$ with respect to the lexicographic order, Q, Q_1 and $Q - Q_1$ are not in $\bigcap_{j=1}^{n} \mathrm{Res}_j(\lambda)$, $i_Q = i_{Q_1} = j$ and*

$$\varepsilon_Q < \theta\, \widetilde{\omega}_f(m) \quad \text{and} \quad \varepsilon_{Q_1} < \theta\, \widetilde{\omega}_f(m),$$

then $|Q - Q_1| = |Q| - |Q_1| \geq m$.

Proof. Before we proceed with the proof, notice that the equality $|Q - Q_1| = |Q| - |Q_1|$ is obvious since $Q > Q_1$.

Since we are supposing $\varepsilon_{Q_1} = |\lambda^{Q_1} - \lambda_j| < \theta\, \widetilde{\omega}_f(m)$, we have

$$|\lambda^{Q_1}| > |\lambda_j| - \theta\, \widetilde{\omega}_f(m) \geq 4\theta - 2\theta = 2\theta.$$

Let us suppose by contradiction $|Q - Q_1| = |Q| - |Q_1| < m$. By assumption, it follows that

$$\begin{aligned} 2\theta\, \widetilde{\omega}_f(m) &> \varepsilon_Q + \varepsilon_{Q_1} \\ &= |\lambda^Q - \lambda_j| + |\lambda^{Q_1} - \lambda_j| \\ &\geq |\lambda^Q - \lambda^{Q_1}| \\ &\geq |\lambda^{Q_1}|\, |\lambda^{Q - Q_1} - 1| \\ &\geq 2\theta\, \widetilde{\omega}_f(|Q - Q_1| + 1) \\ &\geq 2\theta\, \widetilde{\omega}_f(m), \end{aligned}$$

which is impossible. □

Using Lemma 3.3, Case 1 applies to δ_{Q_1} and we have

$$N(Q) = 1 + N(Q_{1_1}) + \cdots + N(Q_{1_{\nu_1}}),$$

where $|Q| > |Q_1| > |Q_{1_1}| \geq \cdots \geq |Q_{1_{\nu_1}}|$ and $Q_1 = Q_{1_1} + \cdots + Q_{1_{\nu_1}}$. We can do the analysis of Case 2 again for this decomposition, and we finish unless we run into Case 2.3.2 again. However, this loop cannot happen more than $m + 1$ times and we have to finally run into a different case. This completes the induction and the proof of Lemma 3.2. □

Since the spectrum of $\mathrm{d}f_O$ satisfies the reduced Brjuno condition, there exists a strictly increasing sequence $\{p_\nu\}_{\nu \geq 0}$ of integers with $p_0 = 1$ and such that

$$\sum_{\nu \geq 0} \frac{1}{p_\nu} \log \frac{1}{\widetilde{\omega}_f(p_{\nu+1})} < \infty. \tag{3.6}$$

We have to estimate

$$\frac{1}{|Q|} \log \delta_Q = \sum_{j=0}^{p} \frac{1}{|Q|} \log \varepsilon_{L_j}^{-1}, \quad Q \notin \bigcap_{j=1}^{n} \mathrm{Res}_j(\lambda).$$

By Lemma 3.2,

$$\begin{aligned}\operatorname{card}\left\{0\le j\le p : \theta\,\widetilde{\omega}_f(p_{\nu+1})\le\varepsilon_{L_j}<\theta\,\widetilde{\omega}_f(p_\nu)\right\}&\le N^1_{p_\nu}(Q)+\cdots N^n_{p_\nu}(Q)\\&\le\frac{2n|Q|}{p_\nu}\end{aligned}$$

for $\nu \geq 1$. It is also easy to see from the definition of δ_Q that the number of factors $\varepsilon_{L_j}^{-1}$ is bounded by $2|Q|-1$. In particular,

$$\operatorname{card}\left\{0 \le j \le p : \theta\,\widetilde{\omega}_f(p_1) \le \varepsilon_{L_j}\right\} \le 2n|Q| = \frac{2n|Q|}{p_0}.$$

Then,

$$\begin{aligned}\frac{1}{|Q|}\log\delta_Q &\le 2n\sum_{\nu\ge 0}\frac{1}{p_\nu}\log\frac{1}{\theta\,\widetilde{\omega}_f(p_{\nu+1})}\\&= 2n\left(\sum_{\nu\ge 0}\frac{1}{p_\nu}\log\frac{1}{\widetilde{\omega}_f(p_{\nu+1})}+\log\frac{1}{\theta}\sum_{\nu\ge 0}\frac{1}{p_\nu}\right).\end{aligned}\tag{3.7}$$

Since $\widetilde{\omega}_f(m)$ tends to zero monotonically as m goes to infinity, we can choose some $\overline{m}$ such that $1 > \widetilde{\omega}_f(m)$ for all $m > \overline{m}$, and we get

$$\sum_{\nu\ge\nu_0}\frac{1}{p_\nu} \le \frac{1}{\log(1/\widetilde{\omega}_f(\overline{m}))}\sum_{\nu\ge\nu_0}\frac{1}{p_\nu}\log\frac{1}{\widetilde{\omega}_f(p_{\nu+1})},$$

where ν_0 verifies the inequalities $p_{\nu_0-1} \le \overline{m} < p_{\nu_0}$. Thus both series in parentheses in (3.7) converge thanks to (3.6). Therefore

$$\sup_Q \frac{1}{|Q|}\log\delta_Q < \infty$$

and this concludes the proof. □

When there are no resonances, we obtain Brjuno's Theorem 1.4.

Remark 3.4. If the reduced Brjuno condition is not satisfied, then there are formally linearizable germs that are not holomorphically linearizable. A first example is the following: let us consider the following germ of biholomorphism f of $(\mathbb{C}^2, O)$:

$$\begin{aligned}f_1(z,w) &= \lambda z + z^2,\\ f_2(z,w) &= w,\end{aligned}\tag{3.8}$$

with $\lambda = e^{2\pi i\theta}$, $\theta \in \mathbb{R} \setminus \mathbb{Q}$, not a Brjuno number. We are in presence of resonances because $\mathrm{Res}_1(\lambda, 1) = \{P \in \mathbb{N}^2 \mid P = (1, p), p \geq 1\}$ and $\mathrm{Res}_2(\lambda, 1) = \{P \in \mathbb{N}^2 \mid P = (0, p), p \geq 2\}$. It is easy to prove that f is formally linearizable, but not holomorphically linearizable, because otherwise the holomorphic function $\lambda z + z^2$ would be holomorphically linearizable contradicting Yoccoz's result [15].

A more general example is the following:

Example 3.5. Let $n \geq 2$, and let $\lambda_1, \ldots, \lambda_s \in \mathbb{C}^*$, be $1 \leq s < n$ complex non-resonant numbers such that

$$\limsup_{m\to+\infty} \frac{1}{m} \log \frac{1}{\omega_{\lambda_1,\ldots,\lambda_s}(m)} = +\infty. \tag{3.9}$$

Then it is possible to find (see *e.g.* [11, Theorem 1.5.1]) a germ f of biholomorphism of $\mathbb{C}^s$ fixing the origin, with $\mathrm{d}f_O = \mathrm{Diag}(\lambda_1, \ldots, \lambda_s)$, formally linearizable (since there are no resonances) but not holomorphically linearizable. It is also possible to find $\mu_1, \ldots, \mu_r \in \mathbb{C}^*$, with $r = n - s$, such that the n-tuple $\lambda = (\lambda_1, \ldots, \lambda_s, \mu_1, \ldots, \mu_r) \in (\mathbb{C}^*)^n$ *has only level s resonances* (see [9], where this definition was first introduced, for details), *i.e.*, for $1 \leq j \leq s$ we have

$$\mathrm{Res}_j(\lambda) = \{P \in \mathbb{N}^n \,\big|\, |P| \geq 2,\ p_l = \delta_{jl} \text{ for } l = 1, \ldots, s, \text{ and } \mu_1^{p_{s+1}} \cdots \mu_r^{p_n} = 1\},$$

where δ_{jl} is the Kroenecker's delta, and for $s + 1 \leq h \leq n$ we have

$$\mathrm{Res}_h(\lambda) = \{P \in \mathbb{N}^n \,\big|\, |P| \geq 2,\ p_1 = \cdots = p_s = 0,\ \mu_1^{p_{s+1}} \cdots \mu_r^{p_n} = \mu_{h-s}\}.$$

Then any germ of biholomorphism F of $\mathbb{C}^n$ fixing the origin of the form

$$\begin{aligned} F_j(z, w) &= f_j(z) && \text{for } j = 1, \ldots, s, \\ F_h(z, w) &= \mu_{h-s} w_{h-s} + \widetilde{F}_h(z, w) && \text{for } h = s + 1, \ldots, n, \end{aligned}$$

with

$$\mathrm{ord}_z(\widetilde{F}_h) \geq 1,$$

for $h = s + 1, \ldots, n$, where $(z, w) = (z_1, \ldots, z_s, w_1, \ldots w_r)$ are local coordinates of $\mathbb{C}^n$ at the origin, is formally linearizable (see [9, Theorem 4.1]), but $\lambda = (\lambda_1, \ldots, \lambda_s, \mu_1, \ldots, \mu_r)$ does not satisfy the reduced Brjuno condition (because of (3.9)) and F is not holomorphically linearizable. In fact, if F were holomorphically linearizable via a linearization Φ, tangent to the identity, then $F \circ \Phi = \Phi \circ \mathrm{Diag}(\lambda_1, \ldots, \lambda_s, \mu_1, \ldots, \mu_r)$. Hence, for each $1 \leq j \leq s$, we would have

$$\begin{aligned} (F \circ \Phi)_j(z, w) &= \lambda_j \Phi_j(z, w) + \widetilde{f}_j(\Phi_1(z, w), \ldots, \Phi_s(z, w)) \\ &= (\Phi \circ \mathrm{Diag}(\lambda_1, \ldots, \lambda_s, \mu_1, \ldots, \mu_r))_j(z, w) \\ &= \Phi_j(\lambda_1 z_1, \ldots, \lambda_s z_s, \mu_1 w_1, \ldots, \mu_r w_r), \end{aligned}$$

yielding

$$(F \circ \Phi)_j(z, 0) = \Phi_j(\lambda_1 z_1, \dots, \lambda_s z_s, 0, \dots, 0),$$

and thus the holomorphic germ φ of $\mathbb{C}^s$ fixing the origin defined by $\varphi_j(z) = \Phi_j(z, 0)$ for $j = 1, \dots, s$, would coincide with the unique formal linearization of f, that would then be convergent contradicting the hypotheses.

4 Rüssmann condition vs. reduced Brjuno condition

Rüssmann proves that, in dimension 1, his condition is equivalent to Brjuno condition (see [14, Lemma 8.2]), and he also proves the following result.

Lemma 4.1 (Rüssmann, 2002 [14]). *Let $\Omega\colon \mathbb{N} \to (0, +\infty)$ be a monotone non decreasing function, and let $\{s_\nu\}$ be defined by $s_\nu := 2^{q+\nu}$, with $q \in \mathbb{N}$. Then*

$$\sum_{\nu \geq 0} \frac{1}{s_\nu} \log \Omega(s_{\nu+1}) \leq \sum_{k \geq 2^{q+1}} \frac{1}{k^2} \log \Omega(k).$$

We have the following relation between the Rüssmann and the reduced Brjuno condition.

Lemma 4.2. *Let $n \geq 2$ and let $\lambda = (\lambda_1, \dots, \lambda_n) \in (\mathbb{C}^*)^n$. If λ satisfies Rüssmann condition, then it also satisfies the reduced Brjuno condition.*

Proof. The function $\widetilde{\omega}_{\lambda_1,\dots\lambda_n}(m)$ defined in Definition 1.8 satisfies

$$\widetilde{\omega}_{\lambda_1,\dots\lambda_n}(m)^{-1} \leq \widetilde{\omega}_{\lambda_1,\dots\lambda_n}(m+1)^{-1}$$

for all $m \in \mathbb{N}$, and

$$|\lambda^Q - \lambda_j| \geq \widetilde{\omega}_{\lambda_1,\dots\lambda_n}(|Q|)$$

for each $j = 1, \dots, n$ and each multi-index $Q \in \mathbb{N}$ with $|Q| \geq 2$ not giving a resonance relative to j. Furthermore, by its definition, it is clear that any other function $\Omega\colon \mathbb{N} \to \mathbb{R}$ such that $k \leq \Omega(k) \leq \Omega(k+1)$ for all $k \in \mathbb{N}$, and satisfying, for any $j = 1, \dots n$,

$$|\lambda^Q - \lambda_j| \geq \frac{1}{\Omega(|Q|)}$$

for each multi-index $Q \in \mathbb{N}$ with $|Q| \geq 2$ not giving a resonance relative to j, is such that

$$\frac{1}{\widetilde{\omega}_{\lambda_1,\dots\lambda_n}(m)} \leq \Omega(m)$$

for all $m \in \mathbb{N}$. Hence

$$\sum_{\nu\geq 0} \frac{1}{p_\nu} \log \frac{1}{\widetilde{\omega}_{\lambda_1,\dots,\lambda_n}(p_{\nu+1})} < \sum_{\nu\geq 0} \frac{1}{p_\nu} \log \Omega(p_{\nu+1})$$

for any strictly increasing sequence of integers $\{p_\nu\}_{\nu\geq 0}$ with $p_0 = 1$. Since λ satisfies Rüssmann condition, thanks to Lemma 4.1, there exists a function Ω as above such that

$$\sum_{\nu\geq 0} \frac{1}{s_\nu} \log \Omega(s_{\nu+1}) < +\infty,$$

with $\{s_\nu\}$ be defined by $s_\nu := 2^{q+\nu}$, with $q \in \mathbb{N}$, and we are done. $\square$

We do not know whether the Rüssmann condition is equivalent to the reduced Brjuno condition in the multi-dimensional case. As we said, Rüssmann is able to prove that this is true in dimension one, but to do so he strongly uses the one-dimensional characterization of these conditions via continued fraction.

Added in proofs. In: J. Raissy, *Holomorphic linearization of commuting germs of holomorphic maps*, arXiv:1005.3434v1, it is proved that the Rüssmann condition and the reduced Brjuno condition are indeed equivalent.

References

[1] M. Abate, *Discrete local holomorphic dynamics*, In: "Proceedings of 13th Seminar of Analysis and its Applications, Isfahan, 2003", S. Azam *et al.* (eds.), University of Isfahan, Iran, 2005, 1–32.

[2] M. Abate, *Discrete holomorphic local dynamical systems*, In: "Holomorphic Dynamical Systems", G. Gentili, J. Guenot, G. Patrizio (eds.), Lectures notes in Math. 1998, Springer Verlag, Berlin, 2010, 1–55.

[3] V. I. Arnold "Geometrical Methods in the Theory of Ordinary Differential Equations", Springer-Verlag, Berlin, 1988.

[4] F. Bracci, *Local dynamics of holomorphic diffeomorphisms*, Boll. UMI **7–B** (8) (2004), 609–636.

[5] A. D. Brjuno *Analytic form of differential equations*, Trans. Moscow Math. Soc. **25** (1971), 131–288; **26** (1972), 199–239.

[6] H. Dulac, "Recherches sur les Points Singuliers des Équationes Différentielles, J. École polytechnique II série cahier IX, 1904, 1–125.

[7] H. Poincaré, "Œuvres, Tome I", Gauthier-Villars, Paris, 1928, XXXVI–CXXIX.

[8] J. PÖSCHEL, *On invariant manifolds of complex analytic mappings near fixed points*, Exp. Math. **4** (1986),97–109.
[9] J. RAISSY, *Linearization of holomorphic germs with quasi-Brjuno fixed points*, Mathematische Zeitschrift **264** (4) (2010), 881–900.
[10] J. RAISSY, *Simultaneous linearization of holomorphic germs in presence of resonances*, Conform. Geom. Dyn. **13** (2009), 217–224.
[11] J. RAISSY, *Geometrical methods in the normalization of germs of biholomorphisms*, Ph.D Thesis, Università di Pisa, 2009.
[12] L. REICH, *Das Typenproblem bei formal-biholomorphien Abbildungen mit anziehendem Fixpunkt*, Math. Ann. **179** (1969), 227–250.
[13] H. RÜSSMANN, *On the convergence of power series transformations of analytic mappings near a fixed point into a normal form*, Preprint I.H.E.S. **M/77/178** (1977).
[14] H. RÜSSMANN, *Stability of elliptic fixed points of analytic area-preserving mappings under the Brjuno condition*, Ergodic Theory Dynam. Systems **22** (2002), 1551–1573.
[15] J.-C. YOCCOZ, Théorème de Siegel, nombres de Bruno et polynômes quadratiques, Astérisque **231** (1995), 3–88.