ON k-INVARIANTS FOR (∞, n)-CATEGORIES

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ABSTRACT. Every (∞, n)-category can be approximated by its tower of homotopy (m, n)-categories. In this paper, we prove that the successive stages of this tower are classified by k-invariants, analogously to the classical Postnikov system for spaces. Our proof relies on an abstract analysis of Postnikov-type systems equipped with k-invariants, and also yields a construction of k-invariants for algebras over ∞-operads and enriched ∞-categories.

CONTENTS

1. INTRODUCTION

The weak homotopy type of a topological space can be conveniently studied using its Postnikov tower

$$X \longrightarrow \dots \longrightarrow \tau_{\leq a}X \longrightarrow \tau_{\leq a-1}X \longrightarrow \dots \longrightarrow \tau_{\leq 0}X = \pi_0(X).$$

The Postnikov tower allows one (theoretically) to reconstruct X from algebraic and cohomological data. Indeed, the lowest stages of this tower encode the path components of X and its fundamental groupoid. For the higher stages, the passage from $\tau_{\leq a-1}X$ to $\tau_{\leq a}X$ is completely determined by a cohomology class

$$k_a \in \mathrm{H}^{a+1}\big(\tau_{\leq a-1}X, \pi_a(X)\big).$$

Indeed, given a map $f: Y \longrightarrow \tau_{\leq a-1}X$, there exists a lift

$$\begin{tikzcd} & \tau_{\leq a}X \arrow[d] \\ Y \arrow[ur, dotted] \arrow[r, "f"'] & \tau_{\leq a-1}X \end{tikzcd} \tag{1.1}$$

if and only if the cohomology class f^*k_a vanishes on Y. In this case, the i-th homotopy group of the space of lifts (1.1) can be identified (noncanonically) with the $(a-i)$-th cohomology group of Y with coefficients in $f^*\pi_a(X)$. Here it should be noted that the homotopy groups $\pi_a(X)$ typically form a **local system of abelian groups**.

The purpose of this paper is to describe an analogue of the Postnikov tower for (∞, n)-categories. More precisely, every (∞, n)-category $\mathcal{C}$ admits a tower of homotopy (m, n)-categories [Lur09b, Section 3.5] (see Section 6)

$$\mathcal{C} \longrightarrow \dots \longrightarrow \mathrm{ho}_{(m,n)}\,\mathcal{C} \longrightarrow \mathrm{ho}_{(m-1,n)}\,\mathcal{C} \longrightarrow \dots \longrightarrow \mathrm{ho}_{(n,n)}\,\mathcal{C}.$$

Our main result asserts that there are again cohomology classes which control the passage from the homotopy (m, n)-category to the homotopy $(m + 1, n)$-category:

Theorem 1.1 (informal)**.** *For each $a \geq 2$, the extension $\mathrm{ho}_{(n+a,n)}\,\mathcal{C} \longrightarrow \mathrm{ho}_{(n+a-1,n)}\,\mathcal{C}$ is classified by a k-invariant*

$$k_a \in \mathrm{H}^{a+1}\big(\mathrm{ho}_{(n+a-1,n)}\,\mathcal{C}, \pi_a(\mathcal{C})\big),$$

where $\pi_a(\mathcal{C})$ is a ***local system of abelian groups*** *on the (∞, n)-category $\mathrm{ho}_{(n+1,n)}\,\mathcal{C}$.*

In the case of $(\infty, 1)$-categories, these k-invariants have also been constructed explicitly in terms of simplicial categories by Dwyer–Kan–Smith [DKS86]. For $n > 1$, the above result is stated (without proof) and used by Lurie in [Lur09b]. In [HNP19a], we have used this result as part of an obstruction-theoretic proof of the fact that adjunctions in $(\infty, 2)$-categories are uniquely determined at the level of the homotopy 2-category (cf. also [RV16]).

To make Theorem 1.1 more precise, let us recall that for any local system of abelian groups $\mathcal{A}$ on a space X, there exist Eilenberg–Maclane spaces $\mathrm{K}(\mathcal{A}, a) \longrightarrow \tau_{\leq 1}X$, defined in the homotopy category $\mathrm{ho}(\mathcal{S}_{/\tau_{\leq 1}X})$ by the following universal property: for every map $f\colon Y \longrightarrow \tau_{\leq 1}X$, there is a natural bijection

$$\mathrm{H}^a(Y, f^*\mathcal{A}) \cong \pi_0\,\mathrm{Map}_{/\tau_{\leq 1}(X)}\big(Y, \mathrm{K}(\mathcal{A}, a)\big).$$

In fact, the Eilenberg–Maclane spaces $\mathrm{K}(\mathcal{A}, a)$ are related by equivalences

$$\mathrm{K}(\mathcal{A}, a) \xrightarrow{\sim} \Omega_{/\tau_{\leq 1}X}\mathrm{K}(\mathcal{A}, a + 1)$$

where $\Omega_{/\tau_{\leq 1}X}\mathrm{K}(\mathcal{A}, a + 1)$ computes the fiberwise loop space of $\mathrm{K}(\mathcal{A}, a + 1)$ over $\tau_{\leq 1}X$ (at the basepoints given by the canonical section classifying the zero cohomology class). In other words, these Eilenberg–Maclane spaces can be organized into a parametrized spectrum $\mathrm{H}\mathcal{A}$ over $\tau_{\leq 1}X$ such that $\mathrm{K}(\mathcal{A}, a) \simeq \Omega^\infty\big(\Sigma^a\mathrm{H}\mathcal{A}\big)$ [MS06]. From an ∞-categorical perspective, this parametrized spectrum can also be described more precisely as follows [ABG+14]: the local system $\mathcal{A}$ determines a functor of ∞-categories $\mathrm{H}\mathcal{A}\colon \tau_{\leq 1}X \longrightarrow \mathrm{Ab} \longrightarrow \mathrm{Sp}$ sending each $x \in \tau_{\leq 1}X$ to the Eilenberg–Maclane spectrum of the abelian group $\mathcal{A}_x$. By the Grothendieck construction, such an ∞-functor to spectra can equivalently be viewed as a spectrum object in spaces over $\tau_{\leq 1}X$.

In these terms, the k-invariants can be interpreted as maps that fit into commuting squares for $a \geq 2$

$$\begin{tikzcd}
\tau_{\leq a}X \arrow[r] \arrow[d] & \tau_{\leq 1}X \arrow[d, "0"] \\
\tau_{\leq a-1}X \arrow[r, "k_a"'] & \Omega^\infty\big(\Sigma^{a+1}\mathrm{H}\pi_a(X)\big).
\end{tikzcd}$$

Here the right vertical map classifies the zero cohomology class. In fact, this square is homotopy Cartesian, which implies that the space of sections (1.1) is homotopy equivalent to the space of null-homotopies of f^*k_a.

Our more precise version of Theorem 1.1 is then the following:

Theorem 1.2 (Theorem 6.3)**.** *For any (∞, n)-category $\mathcal{C}$ and $a \geq 2$, there is a parametrized spectrum object $\mathrm{H}\pi_a(\mathcal{C})$ internal to (∞, n)-categories, whose base object is $\mathrm{ho}_{(n+1,n)}\,\mathcal{C}$, so that there is a pullback square of (∞, n)-categories*

$$\begin{tikzcd} \mathrm{ho}_{(n+a,n)}\,\mathcal{C} \arrow[r] \arrow[d] & \mathrm{ho}_{(n+1,n)}\,\mathcal{C} \arrow[d, "0"] \\ \mathrm{ho}_{(n+a-1,n)}\,\mathcal{C} \arrow[r, "k_a"'] & \Omega^\infty\big(\Sigma^{a+1}\mathrm{H}\pi_a(\mathcal{C})\big). \end{tikzcd} \tag{1.2}$$

Furthermore, we prove that the parametrized spectrum $\mathrm{H}\pi_a(\mathcal{C})$ can indeed be thought of as an Eilenberg–Maclane spectrum: it is contained in the heart of a certain t-structure on the ∞-category of parametrized spectrum objects over $\mathrm{ho}_{(n+1,n)}\,\mathcal{C}$ (Corollary 6.17). This heart consists of local systems of abelian groups on the (∞, n)-category $\mathrm{ho}_{(n+1,n)}\,\mathcal{C}$, as defined (somewhat informally) by Lurie in [Lur09b] (see Definition 6.13 and Remark 6.15).

To prove Theorem 1.2, the main idea will be to proceed by induction on the categorical dimension n. More precisely, the structure of the Postnikov tower, together with its k-invariants, can be axiomatized in terms of 'Postnikov structures'. We prove that a (functorial) Postnikov structure on a symmetric monoidal ∞-category $\mathcal{V}$ that is compatible with the tensor product gives rise to a Postnikov structure on the ∞-category $\mathrm{Cat}(\mathcal{V})$ of $\mathcal{V}$-enriched ∞-categories (Theorem 5.18). Furthermore, the resulting Postnikov structure on $\mathrm{Cat}(\mathcal{V})$ respects the natural symmetric monoidal structure on $\mathrm{Cat}(\mathcal{V})$ inherited from $\mathcal{V}$. This can be used to proceed inductively.

More generally, this argument can also be used to provide k-invariants for Postnikov towers of algebras over ∞-operads (see Proposition 4.14 and Example 4.24). These k-invariants typically take values in certain André-Quillen cohomology groups, and have also been considered (in specific cases) by Goerss–Hopkins [GH00], Basterra–Mandell [BM05] and Lurie [Lur17].

Outline. Let us now give an outline of this paper: in Section 2, we recall the definition of the tangent bundle of an ∞-category and the related theory of 'square zero extensions'. Furthermore, we discuss the 'square zero' monoidal structure on the tangent bundle of a symmetric monoidal presentable ∞-category $\mathcal{V}$, which is useful to describe tangent bundles to categories of algebras. This square zero monoidal structure is particularly simple when $\mathcal{V}$ is already stable; we discuss this case in a bit more detail in Section 3.

In Section 4, we give an abstract axiomatization of towers of square zero extensions, which we call **Postnikov structures**, as well as multiplicative refinements thereof. In particular, we show how multiplicative Postnikov structures induce (multiplicative) Postnikov structures for algebras over ∞-operads. As the basis of our inductive proof, we show that the Postnikov tower of spaces is part of a multiplicative (functorial) Postnikov structure. Section 5 contains our main result, Theorem 5.18: we show that multiplicative Postnikov structures induce multiplicative Postnikov structures at the level of enriched ∞-categories.

In Section 6, we apply this result inductively to prove that the homotopy (m, n)-categories of an (∞, n)-category are part of a multiplicative Postnikov structure (Theorem 6.3); in particular, this provides the required pullback squares (1.2). Finally, we discuss how the tangent bundle of (∞, n)-categories carries a (family of) t-structures, whose heart consists of the category of local systems of abelian groups on (∞, n)-categories (Definition 6.13). The parametrized spectra $\mathrm{H}\pi_a(\mathcal{C})$ appearing in (1.2) then appear as the Eilenberg–Maclane spectra associated to such local systems.

Conventions. We will make use of the language of ∞-categories, i.e. quasicategories, and ∞-operads, following the standard references [Lur09, Lur17]; we will not distinguish between an ordinary category and its nerve. Furthermore, we will refer to symmetric monoidal ∞-categories as **SM ∞-categories**. Recall that SM ∞-categories and (lax) SM functors form (full) subcategories of the ∞-category of ∞-operads, which we will denote by

$$\mathrm{SMCat} \hookrightarrow \mathrm{SMCat}^{\mathrm{lax}} \hookrightarrow \mathrm{Op}_\infty .$$

A presentable SM ∞-category is a presentable ∞-category equipped with a closed symmetric monoidal structure, i.e. an object in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$.

Given an ∞-operad $\mathcal{O}$, i.e. a map $\mathcal{O}^\otimes \longrightarrow \mathrm{Fin}_*$, and a collection of objects S in the underlying ∞-category $\mathcal{O}_{\langle 1\rangle}$ that is closed under equivalences, the **full suboperad** of $\mathcal{O}$ on S is the full subcategory of $\mathcal{O}^\otimes$ spanned by all objects of the form $x_1 \oplus \cdots \oplus x_n$ with all $x_i \in S$ (cf. [Lur17, Section 2.2.1]).

Let $f\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a SM functor and let W be the class of maps in $\mathcal{C}$ that are sent to equivalences by f. We will say that f is a **monoidal localization** if it defines an initial object in full subcategory of $\mathrm{CAlg}(\mathrm{Cat})_{\mathcal{C}/}$ on those symmetric monoidal functors $g\colon \mathcal{C} \longrightarrow \mathcal{E}$ sending W to equivalences. If $\mathcal{C}$ is a SM ∞-category and $f\colon \mathcal{C} \longrightarrow \mathcal{C}[W^{-1}]$ is a (non-SM) localization such that W is closed under tensor products with objects in $\mathcal{C}$, then f admits a unique lift to a monoidal localization of SM ∞-categories [Lur17, Proposition 4.1.7.4].

If $\mathcal{C}$ and $\mathcal{D}$ are SM ∞-categories, let us define a **reflective monoidal localization** to be an adjoint pair $L\colon \mathcal{C}^\otimes \rightleftarrows \mathcal{D}^\otimes : R$ in the homotopy 2-category of ∞-operads such that $\epsilon\colon LR \to \mathrm{id}_{\mathcal{D}}$ is a natural equivalence. Note that a reflective monoidal localization is determined uniquely by any one of the two maps L and R [RV16]. If (L, R) is a reflective monoidal localization, then the (a priori only *lax* SM) left adjoint L is a monoidal localization in the sense above [Lur17, Corollary 7.3.2.12], [Hau21, Theorem 4.6]. Conversely, if L is a monoidal localization which admits a (fully faithful) right adjoint at the level of the underlying ∞-categories, then it determines a reflective monoidal localization [Lur17, Corollary 7.3.2.7].

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2. Tangent bundles of ∞-categories

The purpose of this section is to recall some elements of the cotangent complex formalism described by Lurie [Lur17, Section 7.3]. In particular, we will recall the definition of the tangent bundle of an ∞-category $\mathcal{C}$ and the notion of a square zero extension. To motivate this terminology, we show in Section 2.2 that the tangent bundle inherits a ‘square zero’ monoidal structure from $\mathcal{V}$. In Section 2.3, we introduce the notion of a ‘t-orientation’ on the tangent bundle, allowing one to make sense of connective (and discrete) objects in its fibers. The tangent bundle of stable (or more generally, additive) ∞-categories has a particularly simple structure, which we discuss in more detail in Section 3.

2.1. **Recollections on tangent bundles and square zero extensions.** Let $\mathcal{V}$ be an ∞-category with finite limits. Following Lurie [Lur17, Definition 7.3.1.9], we define the **tangent bundle** of $\mathcal{V}$ to be the ∞-category

$$\mathcal{T}\mathcal{V} = \mathrm{Exc}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V})$$

of excisive functors $F\colon \mathcal{S}_*^{\mathrm{fin}} \longrightarrow \mathcal{V}$ from the ∞-category of finite pointed spaces, i.e. those functors sending pushout squares to pullback squares. The ∞-category $\mathcal{T}\mathcal{V}$ comes with functors

$$\pi = \mathrm{ev}_*\colon \mathcal{T}\mathcal{V} \longrightarrow \mathcal{V} \qquad\qquad \Omega^\infty = \mathrm{ev}_{S^0}\colon \mathcal{T}\mathcal{V} \longrightarrow \mathcal{V}$$

taking the base, resp. the (parametrized) infinite loop space object underlying such a parametrized spectrum. The functor π is a Cartesian fibration and admits both a left and a right adjoint, both taking the constant excisive functor on an object in $\mathcal{V}$. We refer to the fiber of π at an object $X \in \mathcal{V}$ as the **tangent ∞-category** $\mathcal{T}_X\mathcal{V}$ of $\mathcal{V}$ at X. The diagram

$$\begin{tikzcd}
\mathcal{T}\mathcal{V} \arrow[rr, "E\mapsto {[E(S^0)\to E(*)]}"] \arrow[dr, "\pi"'] & & \mathrm{Fun}\left(\Delta^1, \mathcal{V}\right) \arrow[dl, "\mathrm{codom}"] \\
& \mathcal{V} &
\end{tikzcd}$$

then exhibits each fiber $\mathcal{T}_X\mathcal{V}$ as the stabilization $\mathrm{Sp}(\mathcal{V}_{/X})$ of the over-category $\mathcal{V}_{/X}$ [Lur17, Section 7.3.1]. When $\mathcal{V}$ is presentable, $\mathcal{T}\mathcal{V}$ and each of the fibers $\mathcal{T}_X\mathcal{V}$ are presentable as well and the functor Ω^∞ admits a left adjoint Σ^∞_+.

Definition 2.1. Let $\mathcal{V}$ be a presentable ∞-category. Then the inclusion $\mathcal{T}\mathcal{V} \longrightarrow \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V})$ admits a left adjoint, which we will denote by $X \mapsto X^{\mathrm{exc}}$. We will say that a map $X \longrightarrow Y$ in $\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V})$ is a **$\mathcal{T}\mathcal{V}$-local equivalence** if the map $X^{\mathrm{exc}} \longrightarrow Y^{\mathrm{exc}}$ is an equivalence.

Example 2.2. The tangent bundle $\mathcal{T}\mathcal{S}$ can be thought of as the ∞-category of parametrized spectra (with varying base space). Note that $\mathcal{T}\mathcal{S}$ is in some sense the universal tangent bundle. Indeed, using the tensor product on presentable ∞-categories [Lur17, Section 4.8.1] (with unit $\mathcal{S}$, exhibiting that all presentable ∞-categories are tensored over $\mathcal{S}$), we have that

$$\mathcal{T}\mathcal{V} \simeq \mathcal{T}\mathcal{S} \otimes \mathcal{V}.$$

Indeed, using [Lur17, Proposition 4.8.1.17], the full subcategory of $\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V})$ on the excisive functors coincides under restriction along the Yoneda embedding with the full subcategory of $\mathrm{Fun}^{\mathrm{R}}\left(\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{S})^{\mathrm{op}}, \mathcal{V}\right)$ of right adjoint functors that factor over the localization $(-)^{\mathrm{exc}}\colon \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{S}) \longrightarrow \mathcal{T}\mathcal{S}$.

Remark 2.3. For any $S \in \mathcal{S}_*^{\mathrm{fin}}$ and $C \in \mathcal{V}$, let $h_S \otimes C = \mathrm{Map}(S, -) \otimes C$ be the corresponding corepresentable functor, i.e. the left Kan extension of $C\colon * \longrightarrow \mathcal{V}$ along $S\colon * \longrightarrow \mathcal{S}_*^{\mathrm{fin}}$. Note that $F \in \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V})$ is excisive if and only if it is a local object with respect to the set of maps

$$\left(h_{S_1} \coprod_{h_{S_3}} h_{S_2}\right) \otimes C_\alpha \longrightarrow \left(h_{S_0} \otimes C_\alpha\right) \tag{2.1}$$

for any set of generators $\{C_\alpha\}$ of $\mathcal{V}$ and any pushout square in $\mathcal{S}_*^{\mathrm{fin}}$

$$\begin{tikzcd}
S_0 \arrow[r] \arrow[d] & S_1 \arrow[d] \\
S_2 \arrow[r] & S_3.
\end{tikzcd}$$

In particular, the $\mathcal{T}\mathcal{V}$-local equivalences are strongly generated by this set of maps [Lur09, Proposition 5.5.4.15].

Remark 2.4. For any presentable ∞-category $\mathcal{V}$, the description of the generating $\mathcal{T}\mathcal{V}$-local equivalences from Remark 2.3 shows that evaluation at $* \in \mathcal{S}_*^{\mathrm{fin}}$ sends $\mathcal{T}\mathcal{V}$-local equivalences to equivalences in $\mathcal{V}$. It follows that there is a commuting diagram

$$\begin{tikzcd} \mathcal{T}\mathcal{V} \arrow[r, hook] \arrow[dr] & \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V}) \arrow[r, "(-)^{\mathrm{exc}}"] \arrow[d] & \mathcal{T}\mathcal{V} \arrow[dl] \\ & \mathcal{V}. & \end{tikzcd}$$

The vertical functors are Cartesian (and coCartesian) fibrations, with right adjoint sections taking the constant $\mathcal{S}_*^{\mathrm{fin}}$-diagram. In particular, an arrow in $\mathcal{T}\mathcal{V}$ or $\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V})$ is Cartesian if and only if it is the pullback of a map between constant diagrams. It follows that the (right adjoint) inclusion $\mathcal{T}\mathcal{V} \hookrightarrow \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V})$ preserves Cartesian arrows. When $\mathcal{V}$ is compactly generated, or more generally differentiable [Lur17, Definition 6.1.1.6] (cf. Lemma 6.5), the functor $(-)^{\mathrm{exc}}$ preserves Cartesian arrows by [Lur17, Theorem 6.1.1.10].

Let $\mathcal{V}$ be an ∞-category with finite limits and $B \in \mathcal{V}$ an object. For every $E \in \mathcal{T}_B\mathcal{V}$, there is a natural map $\Omega^\infty(E) \longrightarrow B$, arising from the map of finite pointed spaces $S^0 \longrightarrow *$. For every map $X \longrightarrow B$, we denote by

$$H^0_{\mathrm{Q}}(X; E) = \pi_0 \operatorname{Map}_{/B}(X, \Omega^\infty(E))$$

the set of homotopy classes of lifts $\eta\colon X \longrightarrow \Omega^\infty(E)$. Since $\Omega^\infty(E)$ is a grouplike $\mathbb{E}_\infty$-monoid over B by Proposition 2.28, this forms an abelian group; its unit is the zero map $0\colon X \longrightarrow B \longrightarrow \Omega^\infty(E)$ induced by the map of finite pointed spaces $* \longrightarrow S^0$. More generally, we will refer to the groups $H^n_{\mathrm{Q}}(X; E) = H^0_{\mathrm{Q}}(X; \Sigma^n E)$ as the n-th **Quillen cohomology** groups of X with coefficients in E. Given a section $\eta\colon X \longrightarrow \Omega^\infty(E)$, we will say that the pullback square

$$\begin{tikzcd} \tilde{X} \arrow[r] \arrow[d] & B \arrow[d, "0"] \\ X \arrow[r, "\eta"'] & \Omega^\infty(E) \end{tikzcd} \tag{2.2}$$

exhibits $\tilde{X}$ as a **square zero extension** of X [Lur17, Definition 7.4.1.6]. When η is homotopic to $0\colon X \longrightarrow B \longrightarrow \Omega^\infty(E)$, we will refer to $\tilde{X} \simeq X \times_B \Omega^{\infty+1}(E)$ as the **trivial square zero extension**.

Remark 2.5. The above definition of a square zero extension looks slightly more general than the one appearing in [Lur17, Definition 7.4.1.6], where it is assumed that $B = X$. However, note that there is a natural map $p\colon X \longrightarrow B$ (induced by the projection $\Omega^\infty(E) \longrightarrow B$); pulling back the parametrized spectrum E along p, one can also realize $\tilde{X}$ as the square zero extension of X classified by the canonical map $\eta'\colon X \longrightarrow \Omega^\infty(p^*E)$.

2.2. **Monoidal structure on the tangent bundle.** Our next goal will be to construct a (closed) symmetric monoidal structure on the tangent bundle $\mathcal{T}\mathcal{V}$ of a presentable SM ∞-category. To this end, let us recall that if $\mathcal{V}$ is a SM ∞-category and $\mathcal{I}$ is an ∞-category, then $\mathrm{Fun}(\mathcal{I}, \mathcal{V})$ can be endowed with a levelwise tensor product, as follows:

Construction 2.6 ([Lur17, Remark 2.1.3.4])**.** Let $\mathcal{V}$ be a SM ∞-category, encoded by a coCartesian fibration of ∞-operads $\mathcal{V}^{\otimes} \longrightarrow \mathrm{Fin}_*$. If $\mathcal{I}$ is another ∞-category, let us consider the map

$$\mathrm{Fun}(\mathcal{I}, \mathcal{V})^{\otimes_{\mathrm{lev}}} := \mathrm{Fun}(\mathcal{I}, \mathcal{V}^{\otimes}) \times_{\mathrm{Fun}(\mathcal{I}, \mathrm{Fin}_*)} \mathrm{Fin}_* \longrightarrow \mathrm{Fin}_* .$$

This is again a coCartesian fibration of ∞-operads [Lur17, Remark 2.1.3.4], which endows the functor category $\mathrm{Fun}(\mathcal{I}, \mathcal{V})$ with a symmetric monoidal structure that we will refer to as the **levelwise tensor product**. For every $f\colon \mathcal{I} \longrightarrow \mathcal{J}$, the restriction functor $f^*\colon \mathrm{Fun}(\mathcal{J}, \mathcal{V}) \longrightarrow \mathrm{Fun}(\mathcal{I}, \mathcal{V})$ has the natural structure of a symmetric monoidal functor because the induced map $f^*\colon \mathrm{Fun}(\mathcal{J}, \mathcal{V})^{\otimes_{\mathrm{lev}}} \longrightarrow \mathrm{Fun}(\mathcal{I}, \mathcal{V})^{\otimes_{\mathrm{lev}}}$ preserves coCartesian arrows over Fin_*. On the other hand, every SM functor $\mathcal{V} \longrightarrow \mathcal{W}$ induces a SM functor $\mathrm{Fun}(\mathcal{I}, \mathcal{V})^{\otimes_{\mathrm{lev}}} \longrightarrow \mathrm{Fun}(\mathcal{I}, \mathcal{W})^{\otimes_{\mathrm{lev}}}$ by postcomposition.

For future reference, let us mention two alternative descriptions of the levelwise tensor product:

Remark 2.7. The levelwise tensor product is adjoint to the **Boardman–Vogt tensor product**. Indeed, we can view $\mathcal{I}$ as an ∞-operad via the functor $\mathcal{I} \longrightarrow * \longrightarrow \mathrm{Fin}_*$ where the second functor is the inclusion of the object $\langle 1 \rangle$. For any ∞-operad $\mathcal{O}$, recall that the ∞-category of ∞-operad maps $\mathcal{O}^{\otimes} \otimes_{\mathrm{BV}} \mathcal{I} \longrightarrow \mathcal{V}^{\otimes}$ is then equivalent to the ∞-category $\mathrm{BiFunc}(\mathcal{O}^{\otimes}, \mathcal{I}; \mathcal{V}^{\otimes})$ of (dotted) bifunctors of ∞-operads [Lur17, Definition 2.2.5.3]

$$\begin{tikzcd}
\mathcal{O}^{\otimes} \times \mathcal{I} \arrow[rr, dotted] \arrow[d] & & \mathcal{V}^{\otimes} \arrow[d] \\
\mathrm{Fin}_* \times * \arrow[r, "\mathrm{id} \times \{\langle 1 \rangle\}"] & \mathrm{Fin}_* \times \mathrm{Fin}_* \arrow[r, "\wedge"] & \mathrm{Fin}_* .
\end{tikzcd}$$

Since the bottom horizontal composite can simply be identified with the identity functor on Fin_*, the ∞-category $\mathrm{BiFunc}(\mathcal{O}^{\otimes}, \mathcal{I}; \mathcal{V}^{\otimes})$ is equivalent to the ∞-category of functors $f\colon \mathcal{O}^{\otimes} \times \mathcal{I} \longrightarrow \mathcal{V}^{\otimes}$ relative to Fin_* with the following equivalent properties:

(a) for each inert map $\alpha\colon x \to y$ in $\mathcal{O}^{\otimes}$ and each equivalence $\beta\colon i \to j$ in $\mathcal{I}$, $f(\alpha, \beta)$ is an inert map in $\mathcal{V}^{\otimes}$.
(b) for each inert map $\alpha\colon x \to y$ in $\mathcal{O}^{\otimes}$ and each object $i \in \mathcal{I}$, $f(\alpha, \mathrm{id}_i)$ is an inert map in $\mathcal{V}^{\otimes}$.

Note that these conditions are indeed equivalent since $f(\alpha, \beta) \simeq f(\mathrm{id}_y, \beta) \circ f(\alpha, \mathrm{id}_i)$, where $f(\mathrm{id}_y, \beta)$ is an equivalence. The ∞-category of functors f satisfying condition (b) is in turn equivalent to the ∞-category of ∞-operad maps $\mathcal{O}^{\otimes} \longrightarrow \mathrm{Fun}(\mathcal{I}, \mathcal{V})^{\otimes_{\mathrm{lev}}}$. Consequently, we have natural equivalences

$$\mathrm{Alg}_{\mathcal{O} \otimes_{\mathrm{BV}} \mathcal{I}}(\mathcal{V}^{\otimes}) \simeq \mathrm{BiFunc}\big(\mathcal{O}^{\otimes}, \mathcal{I}; \mathcal{V}^{\otimes}\big) \simeq \mathrm{Alg}_{\mathcal{O}}\left(\mathrm{Fun}(\mathcal{I}, \mathcal{V})^{\otimes_{\mathrm{lev}}}\right).$$

Let us point out that by symmetry of the Boardman–Vogt tensor product, we also have that $\mathrm{Alg}_{\mathcal{O}}\left(\mathrm{Fun}(\mathcal{I}, \mathcal{V})^{\otimes_{\mathrm{lev}}}\right) \simeq \mathrm{Alg}_{\mathcal{O} \otimes_{\mathrm{BV}} \mathcal{I}}(\mathcal{V}^{\otimes}) \simeq \mathrm{Fun}\big(\mathcal{I}, \mathrm{Alg}_{\mathcal{O}}(\mathcal{V}^{\otimes})\big)$.

Remark 2.8. If $\mathcal{I}$ has coproducts, then the levelwise tensor product can be identified with the **Day convolution product**. Indeed, let $\mathcal{I}^{\amalg}$ be the corresponding coCartesian ∞-operad [Lur17, Definition 2.4.3.7] and let us write $\mathrm{Fun}(\mathcal{I}, \mathcal{V})^{\otimes_{\mathrm{Day}}} \longrightarrow \mathrm{Fin}_*$ for the ∞-operad obtained from $\mathcal{I}^{\amalg}$ and $\mathcal{V}^{\otimes}$ by Day convolution [Lur17, Definition 2.2.6.1]. By [Lur17, Proposition 2.2.6.16], this is a coCartesian fibration of ∞-operads that endows $\mathrm{Fun}(\mathcal{I}, \mathcal{V})$ with a (closed) SM structure. For any ∞-operad $\mathcal{O}$, we then have equivalences of ∞-categories of maps of ∞-operads (i.e. algebras)

$$\begin{aligned}\mathrm{Alg}_{\mathcal{O}}\left(\mathrm{Fun}(\mathcal{I}, \mathcal{V})^{\otimes_{\mathrm{Day}}}\right) &\simeq \mathrm{Alg}_{\mathcal{O}\times\mathcal{I}^{\amalg}}(\mathcal{V}^{\otimes}) \simeq \mathrm{Fun}(\mathcal{I}, \mathrm{Alg}_{\mathcal{O}}(\mathcal{V}^{\otimes}))\\ &\simeq \mathrm{Alg}_{\mathcal{O}\otimes_{\mathrm{BV}}\mathcal{I}}(\mathcal{V}^{\otimes}) \simeq \mathrm{Alg}_{\mathcal{O}}\left(\mathrm{Fun}(\mathcal{I}, \mathcal{V})^{\otimes_{\mathrm{lev}}}\right).\end{aligned}$$

Here $\mathcal{O} \times \mathcal{I}^{\amalg}$ is the product of ∞-operads, given explicitly by $\mathcal{O}^{\otimes} \times_{\mathrm{Fin}_*} \mathcal{I}^{\amalg} \longrightarrow \mathrm{Fin}_*$. The first equivalence then follows from the universal property of the Day convolution [Lur17, Definition 2.2.6.1], the second from [Lur17, Theorem 2.4.3.18] and the last two equivalences follow from the relation between the levelwise tensor product and the Boardman–Vogt tensor product (which is symmetric).

Lemma 2.9. *Let $\mathcal{V}$ be a presentable SM ∞-category and $f \colon \mathcal{I} \longrightarrow \mathcal{J}$ a functor between ∞-categories with finite coproducts that preserves finite coproducts. Then the SM functor $f^* \colon \mathrm{Fun}(\mathcal{J}, \mathcal{V})^{\otimes_{\mathrm{lev}}} \longrightarrow \mathrm{Fun}(\mathcal{I}, \mathcal{V})^{\otimes_{\mathrm{lev}}}$ admits a symmetric monoidal left adjoint $f_!$.*

Proof. Remark 2.8 identifies the lax SM functor $f^* \colon \mathrm{Fun}(\mathcal{J}, \mathcal{V})^{\otimes_{\mathrm{lev}}} \longrightarrow \mathrm{Fun}(\mathcal{I}, \mathcal{V})^{\otimes_{\mathrm{lev}}}$ (which happens to be strong SM) with the lax SM functor $f^* \colon \mathrm{Fun}(\mathcal{J}, \mathcal{V})^{\otimes_{\mathrm{Day}}} \longrightarrow \mathrm{Fun}(\mathcal{I}, \mathcal{V})^{\otimes_{\mathrm{Day}}}$ arising from naturality of the Day convolution product. The latter admits a SM left adjoint $f_!$ (given by left Kan extension) by [LNP22, Remark 3.31]. □

Proposition 2.10. *Let $\mathcal{V}$ be a presentable SM ∞-category and endow $\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V})$ with the levelwise tensor product $\otimes_{\mathrm{lev}}$. Then the localization of $\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V})$ at the $\mathcal{TV}$-local equivalences is monoidal. In particular:*

- *The localization functor $(-)^{\mathrm{exc}} \colon \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V}) \longrightarrow \mathcal{TV}$ has a unique lift to a SM functor between SM ∞-categories with domain given by $\left(\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V}), \otimes_{\mathrm{lev}}\right)$.*
- *The closed SM structure on $\mathcal{TV}$ is given by $X \otimes Y = \left(X \otimes_{\mathrm{lev}} Y\right)^{\mathrm{exc}}$.*

Proof. By [Lur17, Proposition 4.1.7.4], it suffices to verify that $X \otimes_{\mathrm{lev}} Y \longrightarrow X \otimes_{\mathrm{lev}} Y'$ is a $\mathcal{TV}$-local equivalence for every $X : \mathcal{S}_*^{\mathrm{fin}} \longrightarrow \mathcal{V}$ and every $\mathcal{TV}$-local equivalence $Y \longrightarrow Y'$. Since the $\mathcal{TV}$-local equivalences are closed under colimits and $\otimes_{\mathrm{lev}}$ preserves colimits in each variable, we may assume that $Y \longrightarrow Y'$ is a generating local equivalence of the form (2.1) and that $X = h_T \otimes D$. Since the tensoring $\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{S}) \times \mathcal{V} \longrightarrow \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V})$ is monoidal (for the levelwise tensor product), there are equivalences

$$X \otimes_{\mathrm{lev}} Y' := \left(h_T \otimes D\right) \otimes_{\mathrm{lev}} \left(h_{S_0} \otimes C\right) \simeq \left(h_T \times h_{S_0}\right) \otimes \left(C \otimes D\right) \simeq \left(h_{T\vee S_0}\right) \otimes \left(C \otimes D\right).$$

The last equivalence uses that the copresheaf $h_T \times h_{S_0} = \mathrm{Map}(T, -) \times \mathrm{Map}(S_0, -)$ (valued in spaces) is corepresentable by the coproduct $T \vee S_0$ in $\mathcal{S}_*^{\mathrm{fin}}$. Similarly, we have that

$$\begin{aligned} X \otimes_{\mathrm{lev}} Y &:= \left(h_T \otimes D\right) \otimes_{\mathrm{lev}} \left(\left(h_{S_1} \coprod_{h_{S_3}} h_{S_2}\right) \otimes C\right) \\ &\simeq \left(h_T \times \left(h_{S_1} \coprod_{h_{S_3}} h_{S_2}\right)\right) \otimes (D \otimes C) \simeq \left(h_{T\vee S_1} \coprod_{h_{T\vee S_3}} h_{T\vee S_2}\right) \otimes (D \otimes C)\end{aligned}$$

where the last equivalence uses that $\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{S})$ is Cartesian closed and that $h_T \times h_{S_i} = h_{T\vee S_i}$. It therefore suffices to show that the map

$$\left(h_{T\vee S_1} \coprod_{h_{T\vee S_3}} h_{T\vee S_2}\right) \otimes (D \otimes C) \longrightarrow h_{T\vee S_0} \otimes (D \otimes C)$$

is a $\mathcal{TV}$-local equivalence. This is obvious since

$$\begin{tikzcd} T \vee S_0 \arrow[r] \arrow[d] & T \vee S_1 \arrow[d] \\ T \vee S_2 \arrow[r] & T \vee S_3 \end{tikzcd}$$

is a pushout square in $\mathcal{S}_*^{\mathrm{fin}}$. □

Lemma 2.11. *Let $\mathcal{V}$ be a presentable SM ∞-category and endow $\mathcal{TV}$ with the closed symmetric monoidal structure from Proposition 2.10. Then:*

(1) The functor $\pi\colon \mathcal{TV} \longrightarrow \mathcal{V}$ admits a natural symmetric monoidal structure.

(2) The induced oplax symmetric monoidal structure on the left adjoint to π [Lur17, Corollary 7.3.2.7] *is strong monoidal. Consequently, $\mathcal{TV}$ is tensored over $\mathcal{V}$ via the formula*

$$C \otimes X = (C \otimes_{\mathrm{lev}} X(-))^{\mathrm{exc}}.$$

(3) $\Omega^\infty : \mathcal{TV} \longrightarrow \mathcal{V}$ has a natural lax symmetric monoidal structure.

Remark 2.12. The lax monoidal structure on Ω^∞ induces an oplax symmetric monoidal structure on $\Sigma^\infty_+\colon \mathcal{V} \longrightarrow \mathcal{TV}$ [HHLN23]. This does not make Σ^∞_+ a strong monoidal functor. For example, taking $\mathcal{V} = \mathcal{S}$, we have that $\Sigma^\infty_+(X) \in \mathrm{Sp}(\mathcal{S}_{/X})$ corresponds to the constant parametrized spectrum over X with fiber given by the sphere spectrum $\mathbb{S}$. Unraveling the definitions (e.g. using equivalence (2.3)), one then sees that $\Sigma^\infty_+(X) \otimes \Sigma^\infty_+(Y)$ corresponds to the constant parametrized spectrum over $X \times Y$ with fiber $\mathbb{S} \vee \mathbb{S}$, while $\Sigma^\infty_+(X \times Y)$ has fiber $\mathbb{S}$.

Proof. Let $t\colon * \longrightarrow \mathcal{S}_*^{\mathrm{fin}}$ be the inclusion of the initial (and also terminal) object. By Construction 2.6 and Lemma 2.9, restriction and left Kan extension along t yield an adjoint pair of SM functors $\mathrm{cst} = t_!\colon \mathcal{V} \rightleftarrows \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V}) : t^* = \mathrm{ev}_*$, where the left adjoint takes the constant diagram and the right adjoint evaluates at $*$.

For (1), we then note that ev_* is itself a left adjoint and sends $\mathcal{TV}$-local equivalences to equivalences in $\mathcal{V}$, since the domain and codomain of the generating $\mathcal{TV}$-local equivalences (2.1) are both sent to C_α. It follows that $\pi\colon \mathcal{TV} \longrightarrow \mathcal{V}$ is symmetric monoidal for $\otimes$ as well. For (2), one simply notes that the SM functor $\mathrm{cst}\colon \mathcal{V} \longrightarrow \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V})$ already takes values in $\mathcal{TV} \subseteq \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V})$. For (3), note that Ω^∞ is the composite of the lax symmetric monoidal inclusion $\mathcal{TV} \longrightarrow \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V})$ and the symmetric monoidal functor $\mathrm{ev}_{S^0}\colon \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V}) \longrightarrow \mathcal{V}$ (for the levelwise tensor product on the domain). □

For any functor $X\colon \mathcal{S}_*^{\mathrm{fin}} \longrightarrow \mathcal{V}$, there is a canonical (counit) map $X(*) \longrightarrow X$, where we consider $X(*) \in \mathcal{V}$ as a constant diagram.

Lemma 2.13. *Let $X, Y : \mathcal{S}_*^{\mathrm{fin}} \longrightarrow \mathcal{V}$ be two functors. Then the pushout-product map*

$$\psi(X,Y)\colon X(*) \otimes_{\mathrm{lev}} Y \coprod_{X(*) \otimes_{\mathrm{lev}} Y(*)} X \otimes_{\mathrm{lev}} Y(*) \longrightarrow X \otimes_{\mathrm{lev}} Y$$

is a $\mathcal{TV}$-local equivalence.

Proof. Suppose that $X = \operatorname{colim} X_i$ for some diagram of functors X_i. Since evaluation and taking the constant diagram preserve colimits, we can identify the pushout-product map $\psi(X,Y)$ with the colimit $\operatorname{colim}_i \psi(X_i, Y)$ in the arrow category of $\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V})$. As $\mathcal{TV}$-local equivalences are stable under colimits, we can therefore reduce to the case where $X = h_S \otimes C$ and $Y = h_T \otimes D$ are corepresentables.

Using that the constant diagram on $X(*)$ is given by $h_* \otimes X(*)$, the pushout-product map can then be identified with

$$h_T \otimes (C \otimes D) \coprod_{h_* \otimes (C \otimes D)} h_S \otimes (C \otimes D) \longrightarrow (h_S \otimes C) \otimes_{\mathrm{lev}} (h_T \otimes D).$$

As in the proof of Proposition 2.10, the codomain can be identified with $h_{S \vee T} \otimes (C \otimes D)$. The above map is then a $\mathcal{T}\mathcal{V}$-local equivalence because

$$\begin{tikzcd} S \vee T \arrow[r] \arrow[d] & S \vee * \arrow[d] \\ * \vee T \arrow[r] & * \vee * \end{tikzcd}$$

is a coCartesian square (see Remark 2.3). □

The above lemma can be described somewhat informally as follows: we can identify an object of $\mathcal{T}\mathcal{V}$ with a tuple consisting of $C \in \mathcal{V}$ and $E \in \mathrm{Sp}(\mathcal{V}_{/C})$. Using the tensoring of $\mathcal{T}\mathcal{V}$ over $\mathcal{V}$ from Lemma 2.11, we then have an equivalence

$$(C, E) \otimes (D, F) \simeq \big(C \otimes D, (C \otimes F) \oplus (E \otimes D)\big) \tag{2.3}$$

where the direct sum is taken in the fiber $\mathcal{T}_{C \otimes D}\mathcal{V}$. This justifies the following terminology:

Definition 2.14. Let $\mathcal{V}$ be a presentable SM ∞-category. The **square zero tensor product** on $\mathcal{T}\mathcal{V}$ is the symmetric monoidal structure provided by Proposition 2.10.

For any SM left adjoint $f\colon \mathcal{V} \longrightarrow \mathcal{W}$, postcomposition with f defines a SM left adjoint $\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V}) \longrightarrow \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{W})$ that descends to a natural SM left adjoint $\mathcal{T}(f)\colon \mathcal{T}\mathcal{V} \longrightarrow \mathcal{T}\mathcal{W}$ between localizations.

Remark 2.15. Let $\emptyset$ be the initial object of $\mathcal{V}$. Since $\{\emptyset\} \hookrightarrow \mathcal{V}$ is stable under the binary tensor product of $\mathcal{V}$ and $\pi\colon \mathcal{T}\mathcal{V} \longrightarrow \mathcal{V}$ is symmetric monoidal, the full subcategory $\mathcal{T}_\emptyset\mathcal{V} = \mathcal{T}\mathcal{V} \times_{\mathcal{V}} \{\emptyset\} \hookrightarrow \mathcal{T}\mathcal{V}$ inherits a nonunital SM structure from $\mathcal{T}\mathcal{V}$. Lemma 2.13 shows that for all $E, F \in \mathcal{T}_\emptyset\mathcal{V}$, the tensor product $E \otimes F$ is the zero object in $\mathcal{T}_\emptyset\mathcal{V}$.

Example 2.16. Let $\mathcal{V}$ be a cartesian closed presentable ∞-category. In this case, the levelwise monoidal structure on $\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V})$ induced by the cartesian product on $\mathcal{V}$ is simply the cartesian monoidal structure. Since the (reflective) full subcategory $\mathcal{T}\mathcal{V} \hookrightarrow \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V})$ is closed under the cartesian product, the induced square zero monoidal structure on $\mathcal{T}\mathcal{V}$ is simply the cartesian product as well.

Proposition 2.17. *Let $\mathcal{V}$ be a presentable SM ∞-category and let $\mathcal{O}$ be an ∞-operad. Then there is an equivalence of ∞-categories*

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{T}\mathcal{V}) \simeq \mathcal{T}(\mathrm{Alg}_{\mathcal{O}}(\mathcal{V}))$$

where $\mathcal{T}\mathcal{V}$ is endowed with the square zero monoidal structure.

Proof. The fully faithful functor $\mathcal{T}\mathcal{V} \longrightarrow \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V})$ is lax symmetric monoidal and hence realizes $\mathcal{T}\mathcal{V}^{\otimes}$ as a full suboperad of the ∞-operad $\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V})^{\otimes}$. The ∞-category of $\mathcal{O}$-algebras in $\mathcal{T}\mathcal{V}$ then embeds as the full subcategory of $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V}))$ whose underlying functors are excisive. Using Remark 2.7 together with the commutativity of the Boardman–Vogt tensor product [Lur17, Proposition 2.2.5.13], we obtain an equivalence

$$\begin{tikzcd} \mathrm{Alg}_{\mathcal{O}}(\mathcal{T}\mathcal{V}) \arrow[d, phantom, "\cap" rotate=90, "|"] & & \mathcal{T}(\mathrm{Alg}_{\mathcal{O}}(\mathcal{V})) \arrow[d, phantom, "\cap"] \\ \mathrm{Alg}_{\mathcal{O}}\big(\mathrm{Fun}(\mathcal{S}^{\mathrm{fin}}_*, \mathcal{V}), \otimes_{\mathrm{lev}}\big) \arrow[rr, "\sim"] \arrow[dr] & & \mathrm{Fun}(\mathcal{S}^{\mathrm{fin}}_*, \mathrm{Alg}_{\mathcal{O}}(\mathcal{V})) \arrow[dl] \\ & \mathrm{Fun}(\mathcal{S}^{\mathrm{fin}}_*, \mathcal{V}) & \end{tikzcd}$$

of ∞-categories over $\mathrm{Fun}(\mathcal{S}^{\mathrm{fin}}_*, \mathcal{V})$, where the diagonal functors are induced by forgetting algebra structures. In particular, this equivalence identifies the full subcategory $\mathrm{Alg}_{\mathcal{O}}(\mathcal{T}\mathcal{V})$ on the left hand side with the full subcategory on the right spanned by diagrams of $\mathcal{O}$-algebras in $\mathcal{V}$ whose underlying diagrams are excisive. But this is the same as diagrams $\mathcal{S}^{\mathrm{fin}}_* \longrightarrow \mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$ that are themselves excisive, because the forgetful functor from $\mathcal{O}$-algebras to $\mathcal{V}$ detects limits [Lur17, Corollary 3.2.2.4]. We conclude that the horizontal equivalence above identifies $\mathrm{Alg}_{\mathcal{O}}(\mathcal{T}\mathcal{V})$ with $\mathcal{T}(\mathrm{Alg}_{\mathcal{O}}(\mathcal{V}))$, so the desired result follows. $\square$

The following result provides a symmetric monoidal refinement of Example 2.2:

Proposition 2.18. *Let* $\mathcal{V} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ *and consider the commuting square in* $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$

$$\begin{tikzcd} \mathcal{S} \arrow[r, "\eta"] \arrow[d, "\mathrm{cst}"'] & \mathcal{V} \arrow[d, "\mathrm{cst}"] \\ \mathcal{T}\mathcal{S} \arrow[r, "\mathcal{T}(\eta)"] & \mathcal{T}\mathcal{V}. \end{tikzcd}$$

where the vertical functors are the SM left adjoints to the projection functors and the horizontal functors are induced by the map η *from the initial presentable SM* ∞*-category. This is a pushout square in* $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$.

The proof requires some results about the tensor product of presentable ∞-categories [Lur17, Section 4.8.1]. Let us recall that there is a sub-∞-operad $\mathrm{Pr}^{\mathrm{L},\otimes} \subseteq \mathrm{Cat}^{\mathrm{big},\times}$ of the cartesian operad of big ∞-categories, whose objects are presentable ∞-categories and $\mathrm{Map}_{\mathrm{Pr}^{\mathrm{L},\otimes}}\big(\mathcal{C}_1, \dots, \mathcal{C}_n; \mathcal{D}\big)$ is the union of path components of $\mathrm{Map}_{\mathrm{Cat}^{\mathrm{big}}}\big(\mathcal{C}_1 \times \cdots \times \mathcal{C}_n, \mathcal{D}\big)$ spanned by the functors preserving colimits in each variable. Then the ∞-operad $\mathrm{Pr}^{\mathrm{L},\otimes}$ describes a (closed) symmetric monoidal structure on Pr^{L} [Lur17, Proposition 4.8.1.15].

In the proof below, let us refer to functors $g\colon \mathcal{C}_1 \times \mathcal{C}_2 \longrightarrow \mathcal{D}$ preserving colimits in each variable simply as **bifunctors** and let us say that such a bifunctor g is initial if it defines an initial object in the ∞-category of presentable ∞-categories (with left adjoints between them) equipped with a bifunctor from $\mathcal{C}_1 \times \mathcal{C}_2$. An initial bifunctor $g\colon \mathcal{C}_1 \times \mathcal{C}_2 \longrightarrow \mathcal{D}$ exhibits $\mathcal{D}$ as the tensor product of $\mathcal{C}_1$ and $\mathcal{C}_2$ in Pr^{L}.

Lemma 2.19. *Let* $\mathcal{C}_1, \mathcal{C}_2$ *and* $\mathcal{D}$ *be presentable* ∞*-categories,* $g\colon \mathcal{C}_1 \times \mathcal{C}_2 \longrightarrow \mathcal{D}$ *a bifunctor and consider the functor*

$$\Psi(g)\colon \mathcal{D} \xrightarrow{h} \mathcal{P}(\mathcal{D}) \xrightarrow{g^*} \mathcal{P}(\mathcal{C}_1 \times \mathcal{C}_2).$$

Then $\Psi(g)$ *takes values in the full subcategory of right adjoint functors*

$$\mathrm{Fun}^{\mathrm{R}}(\mathcal{C}_1^{\mathrm{op}}, \mathcal{C}_2) \subseteq \mathrm{Fun}(\mathcal{C}_1^{\mathrm{op}}, C_2) \subseteq \mathrm{Fun}(\mathcal{C}_1^{\mathrm{op}}, \mathcal{P}(\mathcal{C}_2)) \simeq \mathcal{P}(\mathcal{C}_1 \times \mathcal{C}_2)$$

and g *is an initial bifunctor if and only if* $\Psi(g)\colon \mathcal{D} \longrightarrow \mathrm{Fun}^{\mathrm{R}}(\mathcal{C}_1^{\mathrm{op}}, \mathcal{C}_2)$ *is an equivalence.*

Proof. This follows from the proof of [Lur17, Proposition 4.8.1.17]. Indeed, the argument in loc. cit. shows that $\Psi(g)$ is in fact a right adjoint functor with values in the full subcategory $\mathrm{Fun}^{\mathrm{R}}(\mathcal{C}_1^{\mathrm{op}}, \mathcal{C}_2)$ and that the assignment $g \mapsto \Psi(g)$ determines a natural equivalence of spaces

$$\mathrm{Map}_{\mathrm{Pr}^{\mathrm{L},\otimes}}\big(\mathcal{C}_1, C_2; \mathcal{D}\big) \simeq \mathrm{Map}_{\mathrm{Pr}^{\mathrm{R}}}\big(\mathcal{D}, \mathrm{Fun}^{\mathrm{R}}(\mathcal{C}_1^{\mathrm{op}}, \mathcal{C}_2)\big) \simeq \mathrm{Map}_{\mathrm{Pr}^{\mathrm{L}}}\big(\mathrm{Fun}^{\mathrm{R}}(\mathcal{C}_1^{\mathrm{op}}, \mathcal{C}_2), \mathcal{D}\big).$$

In particular (as concluded in loc. cit.), it follows that the presentable ∞-category $\mathrm{Fun}^{\mathrm{R}}(\mathcal{C}_1^{\mathrm{op}}, \mathcal{C}_2)$ corepresents bifunctors, i.e. $\mathcal{C}_1 \otimes \mathcal{C}_2 \simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{C}_1^{\mathrm{op}}, \mathcal{C}_2)$. This immediately implies that g is an initial bifunctor if and only if $\Psi(g) \colon \mathcal{D} \longrightarrow \mathrm{Fun}^{\mathrm{R}}(\mathcal{C}_1^{\mathrm{op}}, \mathcal{C}_2)$ is an equivalence. □

Proof of Proposition 2.18. Since $\mathcal{S}$ is the initial object in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ and coproducts of $\mathbb{E}_\infty$-algebras are given by the tensor product in the underlying ∞-category [Lur17, Proposition 3.2.4.7], it will suffice to verify that the SM left adjoint functor $F \colon \mathcal{T}\mathcal{S} \otimes \mathcal{V} \longrightarrow \mathcal{T}\mathcal{V}$ induced by the commuting square is an equivalence. To verify this, we need to show that the underlying functor (forgetting SM structures) is an equivalence.

To this end, note that the proof of [Lur17, Proposition 3.2.4.7] implies that F can be identified with the composite functor

$$F \colon \mathcal{T}\mathcal{S} \otimes \mathcal{V} \xrightarrow{\mathcal{T}(\eta)\otimes \mathrm{cst}} \mathcal{T}\mathcal{V} \otimes \mathcal{T}\mathcal{V} \xrightarrow{\otimes} \mathcal{T}\mathcal{V}.$$

The corresponding bifunctor is therefore given by

$$f \colon \mathcal{T}\mathcal{S} \times \mathcal{V} \xrightarrow{\mathcal{T}(\eta)\times \mathrm{cst}} \mathcal{T}\mathcal{V} \times \mathcal{T}\mathcal{V} \xrightarrow{\otimes} \mathcal{T}\mathcal{V}.$$

To see that F is an equivalence, we need to show that the bifunctor f satisfies the condition of Lemma 2.19, i.e. that $\Psi(f) \colon \mathcal{T}\mathcal{V} \longrightarrow \mathrm{Fun}^{\mathrm{R}}(\mathcal{T}\mathcal{S}^{\mathrm{op}}, \mathcal{V})$ is an equivalence. To identify the codomain of $\Psi(f)$, consider the functor $h^{\mathrm{exc}} \colon \mathcal{S}_*^{\mathrm{fin,op}} \hookrightarrow \mathcal{P}(\mathcal{S}_*^{\mathrm{fin,op}}) \longrightarrow \mathcal{T}\mathcal{S}$ given by the Yoneda embedding followed by the localization from Remark 2.3. The universal properties of the Yoneda embedding and this localization imply that restriction along h^{exc} induces an equivalence

$$(h^{\mathrm{exc}})^* \colon \mathrm{Fun}^{\mathrm{R}}(\mathcal{T}\mathcal{S}^{\mathrm{op}}, \mathcal{V}) \xrightarrow{\sim} \mathrm{Exc}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V}) = \mathcal{T}\mathcal{V}.$$

Using this equivalence, $\Psi(f)$ can be identified with the functor $\Psi(f) \colon \mathcal{T}\mathcal{V} \longrightarrow \mathrm{Exc}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V})$ sending $X \in \mathcal{T}\mathcal{V}$ to the functor $\mathcal{S}_*^{\mathrm{fin}} \longrightarrow \mathcal{V}$ classifying the correspondence

$$\mathcal{S}_*^{\mathrm{fin}} \times \mathcal{V} \longrightarrow \mathcal{S}; \quad (S, v) \longmapsto \mathrm{Map}_{\mathcal{T}\mathcal{V}}\big((h_S \otimes 1_{\mathcal{V}})^{\mathrm{exc}} \otimes \mathrm{cst}(v), X\big).$$

Here we used that $T(\eta) \colon \mathcal{T}\mathcal{S} \longrightarrow \mathcal{T}\mathcal{V}$ sends h_S^{exc} to the excisive approximation of $(h_S \otimes 1_{\mathcal{V}})$. By Proposition 2.10, the tensor product $(h_S \otimes 1_{\mathcal{V}})^{\mathrm{exc}} \otimes \mathrm{cst}(v)$ in $\mathcal{T}\mathcal{V}$ is naturally equivalent to $(h_S \otimes v)^{\mathrm{exc}} \in \mathcal{T}\mathcal{V}$. This object has the universal property that

$$\mathrm{Map}_{\mathcal{T}\mathcal{V}}\big((h_S \otimes v)^{\mathrm{exc}}, X\big) \simeq \mathrm{Map}_{\mathcal{V}}(v, X(S)).$$

It follows that $\Psi(f)$ can simply be identified with the identity on $\mathcal{T}\mathcal{V}$. In particular, it is an equivalence, so that Lemma 2.19 shows that f is an initial bifunctor and F is an equivalence, as desired. □

2.3. ***t*-orientations on tangent categories.** In later sections, we will consider various examples of tangent bundles whose fibers are stable categories with a natural "connective part". Let us axiomatize this situation as follows:

Definition 2.20. Let $p\colon \mathcal{E} \longrightarrow \mathcal{B}$ be a **stable Cartesian fibration**, i.e. a Cartesian fibration such that each fiber $\mathcal{E}_X$ is stable and each arrow $f\colon X \longrightarrow X'$ in $\mathcal{B}$ induces an exact functor $f^*\colon \mathcal{E}_Y \longrightarrow \mathcal{E}_X$. A ***t*-orientation** on $p\colon \mathcal{E} \longrightarrow \mathcal{B}$ is a tuple of full subcategories $\left(\mathcal{E}^{\geq 0}, \mathcal{E}^{\leq 0}\right)$ of $\mathcal{E}$ such that:

(1) For each p-Cartesian arrow $E \longrightarrow F$ in $\mathcal{E}$ with $F \in \mathcal{E}^{\leq 0}$, we have that $E \in \mathcal{E}^{\leq 0}$.
(2) For every $X \in \mathcal{B}$, the tuple
$$\left(\mathcal{E}^{\geq 0} \cap \mathcal{E}_X,\ \mathcal{E}^{\leq 0} \cap \mathcal{E}_X\right)$$
defines a t-structure on the stable ∞-category $\mathcal{E}_X$.

In this case, we will refer to $\mathcal{E}^{\heartsuit} = \mathcal{E}^{\geq 0} \cap \mathcal{E}^{\leq 0}$ as the *heart* of the t-orientation.

Example 2.21. Let $\pi\colon \mathcal{T}\mathcal{V} \longrightarrow \mathcal{V}$ be the tangent bundle of a presentable ∞-category. Then each $\mathcal{T}_X\mathcal{V}$ carries a t-structure such that $\mathcal{T}_X^{\leq -1}\mathcal{V}$ is the full subcategory of $E \in \mathcal{T}_X\mathcal{V}$ such that $\Omega^\infty(E) \simeq X$ is the terminal object in $\mathcal{V}_{/X}$ [Lur17, Proposition 1.4.3.4]. Since such objects are stable under base change along a map $X' \longrightarrow X$ in the base, it follows that $\mathcal{T}\mathcal{V}$ comes with a **canonical *t*-orientiation** in which $\mathcal{T}^{\leq -1}\mathcal{V}$ consists of those E such that $\Omega^\infty(E) \simeq \pi(E)$.

Note that Condition (1) of Definition 2.20 is equivalent to $\mathcal{E}^{\leq 0} \longrightarrow \mathcal{B}$ being a Cartesian fibration and the inclusion $\mathcal{E}^{\leq 0} \hookrightarrow \mathcal{E}$ preserving Cartesian edges.

Lemma 2.22. *Let $p\colon \mathcal{E} \longrightarrow \mathcal{B}$ be a stable Cartesian fibration with a t-orientation $(\mathcal{E}^{\geq 0}, \mathcal{E}^{\leq 0})$. Then:*

(1) the restriction of the projection p to each of the three subcategories $\mathcal{E}^{\geq 0}, \mathcal{E}^{\leq 0}$ and $\mathcal{E}^{\heartsuit}$ is a Cartesian fibration.
(2) there exists a commuting square of adjunctions over $\mathcal{B}$, i.e. in $\mathrm{Cat}_\infty/\mathcal{B}$, of the form
$$\begin{tikzcd}
\mathcal{E}^{\heartsuit} \arrow[r, hook, shift left] \arrow[r, phantom, "\perp"] \arrow[d, hook, shift left] \arrow[d, phantom, "\dashv"] & \mathcal{E}^{\leq 0} \arrow[l, shift left, "\tau_{\geq 0}"] \arrow[d, hook, shift left] \arrow[d, phantom, "\dashv"] \\
\mathcal{E}^{\geq 0} \arrow[u, shift left, "\tau_{\leq 0}"] \arrow[r, hook, shift left] \arrow[r, phantom, "\perp"] & \mathcal{E}. \arrow[l, shift left, "\tau_{\geq 0}"] \arrow[u, shift left, "\tau_{\leq 0}"]
\end{tikzcd}$$
Furthermore, all right adjoint functors preserve Cartesian edges.

In particular, $\mathcal{E}^{\heartsuit} \longrightarrow \mathcal{B}$ is a Cartesian fibration whose fibers are (ordinary) abelian categories.

Proof. For each $X \in \mathcal{B}$, the fiber $\mathcal{E}_X$ comes equipped with a t-structure. In particular, for each X there are coreflective localizations [Lur17, Proposition 1.2.1.5]
$$\begin{tikzcd}
\mathcal{E}_X^{\geq 0} \arrow[r, hook, shift left] \arrow[r, phantom, "\perp"] & \mathcal{E}_X \arrow[l, shift left, "\tau_{\geq 0}"]
\end{tikzcd} \qquad\qquad \begin{tikzcd}
\mathcal{E}_X^{\heartsuit} \arrow[r, hook, shift left] \arrow[r, phantom, "\perp"] & \mathcal{E}_X^{\leq 0}. \arrow[l, shift left, "\tau_{\geq 0}"]
\end{tikzcd} \tag{2.4}$$
The functors $\tau_{\geq 0}$ realize their codomain as the localization of the domain at the (-1)-coconnective morphisms, i.e. those morphisms whose cofiber in $\mathcal{E}_X$ is contained in $\mathcal{E}_X^{\leq 0}$. By Condition (1) from Definition 2.20, each morphism $f\colon X \longrightarrow Y$ in $\mathcal{E}$ induces a left t-exact functor $f^*\colon \mathcal{E}_Y \longrightarrow \mathcal{E}_X$

between the fibers. It follows that the (-1)-coconnective morphisms in (each fiber of) $\mathcal{E}$ and $\mathcal{E}^{\leq 0}$ are stable under the functors f^*. Let us pass to a universe $\mathcal{U}$ such that $\mathcal{E}$ and $\mathcal{B}$ are $\mathcal{U}$-small and write $\mathcal{X}^{\geq 0}$ (resp. $\mathcal{X}^{\heartsuit}$) for the $\mathcal{U}$-small ∞-category obtained from $\mathcal{E}$ (resp. $\mathcal{E}^{\leq 0}$) by localizing at the (-1)-coconnective arrows in each fiber. We can then apply [Hin16, Proposition 2.1.4] in the ($\mathcal{U}$-small) setting where the marked arrows in $\mathcal{B}$ are just the equivalences to obtain maps of Cartesian fibrations (preserving Cartesian arrows)

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "\tau_{\geq 0}"] \arrow[dr, "\pi"'] & & \mathcal{X}^{\geq 0} \arrow[dl] \\ & \mathcal{B} & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{E}^{\leq 0} \arrow[rr, "\tau_{\geq 0}"] \arrow[dr, "\pi"'] & & \mathcal{X}^{\heartsuit} \arrow[dl] \\ & \mathcal{B}. & \end{tikzcd} \tag{2.5}$$

By loc. cit., on the fiber over an object $X \in \mathcal{B}$ these maps can be identified with the localization functors from (2.4). In particular, it follows from [Lur17, Proposition 7.3.2.6] that the localizations from (2.5) both admit a left adjoint over $\mathcal{B}$. These left adjoints are (fiberwise) fully faithful and identify $\mathcal{X}^{\geq 0}$ and $\mathcal{X}^{\heartsuit}$ with the full subcategories $\mathcal{E}^{\geq 0}$ and $\mathcal{E}^{\heartsuit}$, respectively. In particular, this shows that the projections from $\mathcal{E}^{\geq 0}$ and $\mathcal{E}^{\heartsuit}$ to $\mathcal{B}$ are Cartesian fibrations, proving (1). Furthermore, the functors from (2.5) provide the horizontal right adjoints (relative to $\mathcal{B}$) in (2). Finally, the inclusions $\mathcal{E}^{\heartsuit} \longrightarrow \mathcal{E}^{\geq 0}$ and $\mathcal{E}^{\leq 0} \longrightarrow \mathcal{E}$ admit left adjoints over $\mathcal{B}$ by [Lur17, Proposition 7.3.2.6]. □

Let us now specialize to the case of the tangent bundle.

Definition 2.23. Let $\mathcal{V}$ be an SM ∞-category with finite limits. A t-orientation on $\mathcal{T}\mathcal{V}$ is **monoidal** if $\mathcal{T}^{\geq 0}\mathcal{V}$ is closed under the square-zero tensor product and contains the unit.

Example 2.24. Consider the full subcategories of excisive functors $F\colon \mathcal{S}_*^{\mathrm{fin}} \longrightarrow \mathcal{S}$

$$\mathcal{T}^{\geq 0}\mathcal{S} \subseteq \mathcal{T}\mathcal{S} \qquad\qquad \mathcal{T}^{\leq 0}\mathcal{S} \subseteq \mathcal{T}\mathcal{S}$$

such that for every n, the map $F(S^n) \longrightarrow F(*)$ has n-connected, resp. n-truncated fibers. This defines a t-orientation on $\mathcal{T}\mathcal{S}$, whose restriction to each fiber $\mathcal{T}_X\mathcal{S} \simeq \mathrm{Fun}(X, \mathrm{Sp})$ consists of diagrams of connective, resp. coconnective spectra. Furthermore, this t-orientation is monoidal (the square zero monoidal structure simply being the Cartesian product by Example 2.16). In particular, the heart $\mathcal{T}^{\heartsuit}\mathcal{S}$ can be identified with the ∞-category of **local systems of abelian groups**. The inclusion $\mathcal{T}^{\heartsuit}\mathcal{S} \subseteq \mathcal{T}\mathcal{S}$ sends a local system of abelian groups $\mathcal{A}$ to the corresponding parametrized Eilenberg–Maclane spectrum $\mathrm{H}\mathcal{A}$.

Let $\mathcal{V}$ be a SM ∞-category with finite limits and suppose that $\mathcal{T}\mathcal{V}$ carries a monoidal t-orientation. If $\mathcal{O}$ is an ∞-operad, we can use Proposition 2.17 to identify the Cartesian fibration $\pi\colon \mathcal{T}\,\mathrm{Alg}_{\mathcal{O}}(\mathcal{V}) \longrightarrow \mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$ with $\mathrm{Alg}_{\mathcal{O}}(\mathcal{T}\mathcal{V}) \longrightarrow \mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$. Using this identification, consider the two full subcategories

$$\mathcal{T}^{\geq 0}\,\mathrm{Alg}_{\mathcal{O}}(\mathcal{V}) = \mathrm{Alg}_{\mathcal{O}}(\mathcal{T}^{\geq 0}\mathcal{V}) \qquad\qquad \mathcal{T}^{\leq 0}\,\mathrm{Alg}_{\mathcal{O}}(\mathcal{V}) = \mathrm{Alg}_{\mathcal{O}}(\mathcal{T}^{\leq 0}\mathcal{V})$$

where we view $\mathcal{T}^{\geq 0}\mathcal{V}$ and $\mathcal{T}^{\leq 0}\mathcal{V}$ as full suboperads of $\mathcal{T}\mathcal{V}$. In other words, these are the full subcategories of $\mathcal{O}$-algebras in $\mathcal{T}\mathcal{V}$ whose underlying objects (for every colour $x \in \mathcal{O}$) are 0-connective, resp. 0-coconnective in $\mathcal{T}\mathcal{V}$.

Proposition 2.25. *These two full subcategories $\mathcal{T}^{\geq 0}\,\mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$ and $\mathcal{T}^{\leq 0}\,\mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$ define a monoidal t-orientiation on $\mathcal{T}\,\mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$. For every colour $x \in \mathcal{O}$, the forgetful functor $x^*\colon \mathcal{T}\,\mathrm{Alg}_{\mathcal{O}}(\mathcal{V}) \longrightarrow \mathcal{T}\mathcal{V}$ is t-exact, i.e. it preserves both* 0*-connective and* 0*-coconnective objects.*

Proof. First, to see that the t-orientation is monoidal, note that the full subcategory $\mathcal{T}^{\geq 0}\operatorname{Alg}_{\mathcal{O}}(\mathcal{V}) \simeq \operatorname{Alg}_{\mathcal{O}}(\mathcal{T}^{\geq 0}\mathcal{V}) \subseteq \operatorname{Alg}_{\mathcal{O}}(\mathcal{T}\mathcal{V})$ is closed under tensor products, since evaluation on the set of colors detects tensor products of algebras.

To verify condition (1) of Definition 2.20, notice that a morphism in $\operatorname{Alg}_{\mathcal{O}}(\mathcal{T}\mathcal{V})$ is π-Cartesian if and only if for every colour $x \in \mathcal{O}$, its image under $x^*\colon \operatorname{Alg}_{\mathcal{O}}(\mathcal{T}\mathcal{V}) \longrightarrow \mathcal{T}\mathcal{V}$ is a Cartesian arrow [Lur17, Corollary 3.2.2.3]. This immediately implies that for every Cartesian arrow in $\operatorname{Alg}_{\mathcal{O}}(\mathcal{T}\mathcal{V})$ whose codomain is contained in $\mathcal{T}^{\leq 0}\operatorname{Alg}_{\mathcal{O}}(\mathcal{V})$, the domain is contained in $\mathcal{T}^{\leq 0}\operatorname{Alg}_{\mathcal{O}}(\mathcal{T}\mathcal{V})$ as well.

For condition (2), consider the adjoint pair $\mathcal{T}^{\geq 0}\mathcal{V} \rightleftarrows \mathcal{T}\mathcal{V}$ from Lemma 2.22. Since the inclusion $\mathcal{T}^{\geq 0}\mathcal{V} \longrightarrow \mathcal{T}\mathcal{V}$ is symmetric monoidal, its right adjoint $\tau_{\geq 0}$ inherits a lax symmetric monoidal structure [Lur17, Corollary 7.3.2.7]. We therefore obtain an adjoint pair at the level of $\mathcal{O}$-algebras which is natural with respect to restriction along maps of ∞-operads $\mathcal{O} \to \mathcal{O}'$ (cf. [Lur17, Remark 7.3.2.13]). In particular, both adjoints commute with the forgetful functor for each colour $x \in \mathcal{O}$

$$\begin{tikzcd}
\mathcal{T}^{\geq 0}\operatorname{Alg}_{\mathcal{O}}(\mathcal{V}) \simeq \operatorname{Alg}_{\mathcal{O}}(\mathcal{T}^{\geq 0}\mathcal{V}) \arrow[r, hook, shift left] \arrow[r, phantom, "\perp"] \arrow[d, "x^*"'] & \operatorname{Alg}_{\mathcal{O}}(\mathcal{T}\mathcal{V}) \arrow[l, shift left] \arrow[d, "x^*"] \\
\mathcal{T}^{\geq 0}\mathcal{V} \arrow[r, hook, shift left] \arrow[r, phantom, "\perp"] & \mathcal{T}\mathcal{V}. \arrow[l, shift left]
\end{tikzcd}$$

Since the unit of the adjoint pair $\mathcal{T}^{\geq 0}\mathcal{V} \rightleftarrows \mathcal{T}\mathcal{V}$ is an equivalence and its counit maps to an equivalence in $\mathcal{V}$ by Lemma 2.22, the induced adjunction on $\mathcal{O}$-algebras restricts to an adjunction between the fibers over an $\mathcal{O}$-algebra A

$$\begin{tikzcd}
\mathcal{T}_A^{\geq 0}\operatorname{Alg}_{\mathcal{O}}(\mathcal{V}) \arrow[r, hook, shift left] \arrow[r, phantom, "\perp"] \arrow[d, "x^*"'] & \mathcal{T}_A\operatorname{Alg}_{\mathcal{O}}(\mathcal{V}) \arrow[l, shift left, "\tau_{\geq 0}"] \arrow[d, "x^*"] \\
\mathcal{T}_{x^*A}^{\geq 0}\mathcal{V} \arrow[r, hook, shift left] \arrow[r, phantom, "\perp"] & \mathcal{T}_{x^*A}\mathcal{V}. \arrow[l, shift left, "\tau_{\geq 0}"]
\end{tikzcd}$$

The left and right adjoint both commute with the forgetful functors and the unit of the adjunction is an equivalence. In particular, it follows that an object $E \in \mathcal{T}_A\operatorname{Alg}_{\mathcal{O}}(\mathcal{V})$ is:

(a) contained in $\mathcal{T}_A^{\geq 0}\operatorname{Alg}_{\mathcal{O}}(\mathcal{V})$ if and only if $\tau_{\geq 0}(E) \simeq E$.
(b) contained in $\mathcal{T}_A^{\leq -1}\operatorname{Alg}_{\mathcal{O}}(\mathcal{V})$ if and only if for every colour $x \in \mathcal{O}$, $x^*E \in \mathcal{T}_{x^*A}\mathcal{V}$ is (-1)-coconnective, i.e. $\tau_{\geq 0}(x^*E) \simeq 0$. In turn, this is equivalent to $\tau_{\geq 0}(E) \simeq 0$ in $\mathcal{T}_A\operatorname{Alg}_{\mathcal{O}}(\mathcal{V})$.

By [Lur17, Proposition 1.2.1.16], the subcategories $\mathcal{T}_A^{\geq 0}\operatorname{Alg}_{\mathcal{O}}(\mathcal{V})$ and $\mathcal{T}_A^{\leq 0}\operatorname{Alg}_{\mathcal{O}}(\mathcal{V})$ then determine a t-structure on $\mathcal{T}_A\operatorname{Alg}_{\mathcal{O}}(\mathcal{V})$ if and only if the essential image of

$$\tau_{\geq 0}\colon \mathcal{T}_A\operatorname{Alg}_{\mathcal{O}}(\mathcal{V}) \longrightarrow \mathcal{T}_A\operatorname{Alg}_{\mathcal{O}}(\mathcal{V})$$

is closed under extensions. Since this functor is idempotent, (a) identifies its essential image with $\mathcal{T}_A^{\geq 0}\operatorname{Alg}_{\mathcal{O}}(\mathcal{V})$, which is closed under extensions because the forgetful functors x^* (which detect connectivity) preserve extensions and each $\mathcal{T}_{x^*A}^{\geq 0}\mathcal{V}$ is closed under extensions. $\square$

In the remainder of this section, we will show that for a large class of presentable ∞-categories $\mathcal{V}$, the connective objects for the canonical t-orientation on $\mathcal{T}\mathcal{V}$ (Example 2.21) admit a simpler combinatorial description than that of an excisive functor. To this end, let us start by recalling that every $E \in \mathcal{T}_X\mathcal{V}$ defines a reduced excisive functor $E\colon \mathcal{S}_*^{\mathrm{fin}} \longrightarrow \mathcal{V}_{/X}$. Restricting E to the full subcategory of finite pointed sets, we obtain a very special Γ-space object in $\mathcal{V}_{/X}$ in the sense of Segal, whose underlying object is $\Omega^\infty(E)$. Indeed, for any two finite pointed sets S, T, the pushout square of finite pointed spaces (in fact, sets)

$$\begin{tikzcd} S \vee T \arrow[r] \arrow[d] & * \vee T \arrow[d] \\ S \vee * \arrow[r] & * \end{tikzcd}$$

induces an equivalence $E(S \vee T) \longrightarrow E(S) \times E(T)$, from which the grouplike Segal conditions follow. In other words, $\Omega^\infty(E)$ has the structure of a grouplike $\mathbb{E}_\infty$-monoid in the sense of [GGN15].

Conversely, in the presence of loop space machinery, every grouplike $\mathbb{E}_\infty$-monoid arises from a spectrum. For later purposes, let us make this slightly more precise: suppose that $\mathcal{V}$ is a presentable ∞-category and let

$$\mathrm{Grp}(\mathcal{V}) \subseteq \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{V}) \qquad\qquad \mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{V}) \subseteq \mathrm{Fun}(\mathrm{Fin}_*, \mathcal{V})$$

denote the ∞-categories of grouplike monoids, resp. grouplike $\mathbb{E}_\infty$-monoids in $\mathcal{V}$. Note that both arise as full (reflective) subcategories of diagrams satisfying the grouplike Segal conditions [Lur09, Definition 7.2.2.1], [Lur17, Section 2.4.2, Definition 5.2.6.2], [GGN15]. In addition, there is an adjoint pair

$$B\colon \mathrm{Grp}(\mathcal{V}) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{V}_*\colon \Omega \tag{2.6}$$

where the left adjoint sends a grouplike monoid to its bar construction and the right adjoint sends a pointed object in $\mathcal{V}$ to its loop space (endowed with the group structure coming from the usual cogroup structure $S^1 \longrightarrow S^1 \vee S^1$).

Definition 2.26. Let $\mathcal{V}$ be a presentable ∞-category. We will say that $\mathcal{V}$ **has loop space machinery** if it satisfies the following conditions:

(1) the Cartesian product $\mathcal{V} \times \mathcal{V} \xrightarrow{\times} \mathcal{V}$ preserves geometric realizations.
(2) the unit of the adjunction (2.6) is an equivalence.

We will say that $\mathcal{V}$ has **parametrized loop space machinery** if each slice ∞-category $\mathcal{V}_{/X}$ has loop space machinery.

Example 2.27. Note that $\mathcal{V}$ has loop space machinery if and only if $\mathcal{V}_{*/}$ has loop space machinery. Using this, one readily sees that all ∞-toposes and stable presentable ∞-categories have parametrized loop space machinery. More generally, a prestable presentable ∞-category (i.e. the connective part of a t-structure on a stable ∞-category [Lur18, Section C.1]) has parametrized loop space machinery. If $\mathcal{V}$ has (parametrized) loop space machinery and $U\colon \mathcal{W} \longrightarrow \mathcal{V}$ is a right adjoint functor preserving sifted colimits and detecting equivalences (in particular, it is monadic), then $\mathcal{W}$ has (parametrized) loop space machinery.

Recall that a simplicial object $\Delta^{\mathrm{op}} \to \mathcal{D}$ in some ∞-category $\mathcal{D}$ is said to be n**-skeletal** if it is left Kan extended from $\Delta^{\mathrm{op}}_{\leq n} \subseteq \Delta^{\mathrm{op}}$.

Proposition 2.28. *Let $\mathcal{V}$ be a presentable ∞-category with loop space machinery and consider the adjoint pair*

$$\mathbf{B}\colon \mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{V}) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{Sp}(\mathcal{V}) = \mathrm{Exc}_{\mathrm{red}}(\mathcal{S}^{\mathrm{fin}}_*, \mathcal{V})\colon \Omega^\infty$$

whose right adjoint restricts a reduced excisive functor along the inclusion $i\colon \mathrm{Fin}_* \longrightarrow \mathcal{S}^{\mathrm{fin}}_*$. *Then the left adjoint* $\mathbf{B}$ *is fully faithful and a functor* $F\colon \mathcal{S}^{\mathrm{fin}}_* \longrightarrow \mathcal{V}$ *lies in its essential image (in particular, it will be reduced excisive) if and only if it satisfies the following two conditions:*

(1) Its restriction to Fin_* *satisfies the grouplike Segal conditions.*
*(2) It preserves all **finite geometric realizations**, i.e. colimits of simplicial diagrams that are* n*-skeletal for some* n*.*

Before turning to the proof of Proposition 2.28, let us mention some consequences:

Definition 2.29. For a presentable ∞-category $\mathcal{V}$, let us say that a functor $A\colon \mathrm{Fin}_* \longrightarrow \mathcal{V}$ is a **Segal $\mathbb{E}_\infty$-groupoid** if for any two finite pointed sets $S, T \in \mathrm{Fin}_*$, the square

$$\begin{tikzcd} A(S\vee T) \arrow[r] \arrow[d] & A(*\vee T) \arrow[d] \\ A(S\vee *) \arrow[r] & A(*) \end{tikzcd}$$

is cartesian. We will write $\mathrm{Gpd}_{\mathbb{E}_\infty}(\mathcal{V}) \subseteq \mathrm{Fun}(\mathrm{Fin}_*, \mathcal{V})$ for the full subcategory on the Segal $\mathbb{E}_\infty$-groupoids.

Note that a Segal $\mathbb{E}_\infty$-groupoid in $\mathcal{V}$ with $A(*) = X$ is equivalent to a grouplike $\mathbb{E}_\infty$-monoid in $\mathcal{V}_{/X}$.

Corollary 2.30. *Let* $\mathcal{V}$ *be a presentable* ∞*-category with parametrized loop space machinery. Then the following hold:*

(1) There is a relative adjoint pair

$$\begin{tikzcd} \mathrm{Gpd}_{\mathbb{E}_\infty}(\mathcal{V}) \arrow[rr, shift left, "\mathbf{B}"] \arrow[rr, phantom, "\perp"] \arrow[dr, "\mathrm{ev}_*"'] & & \mathcal{T}\mathcal{V} = \mathrm{Exc}(\mathcal{S}^{\mathrm{fin}}_*, \mathcal{V}) \arrow[ll, shift left, "\Omega^\infty"] \arrow[dl, "\pi"] \\ & \mathcal{V} & \end{tikzcd}$$

whose right adjoint restricts an excisive functor along the inclusion $i\colon \mathrm{Fin}_* \longrightarrow \mathcal{S}^{\mathrm{fin}}_*$.

(2) The left adjoint $\mathbf{B}$ *is fully faithful and a functor* $F\colon \mathcal{S}^{\mathrm{fin}}_* \longrightarrow \mathcal{V}$ *lies in its essential image (in particular, it will be excisive) if and only if it preserves finite geometric realizations and* i^*F *is a Segal* $\mathbb{E}_\infty$*-groupoid.*

(3) The connective part $\mathcal{T}^{\geq 0}\mathcal{V}$ *of the canonical* t*-orientation (Example 2.21) on* $\mathcal{T}\mathcal{V}$ *coincides with the essential image of* $\mathbf{B}$*.*

Proof. Note that each excisive functor $E\colon \mathcal{S}^{\mathrm{fin}}_* \longrightarrow \mathcal{V}$ can also be considered as a reduced excisive functor with values in $\mathcal{V}_{/E(*)}$. The restriction to Fin_* then defines a grouplike $\mathbb{E}_\infty$-monoid in $\mathcal{V}_{/E(*)}$, or equivalently, an $\mathbb{E}_\infty$-groupoid in $\mathcal{V}$. It follows that there is a well-defined functor $\Omega^\infty\colon \mathcal{T}\mathcal{V} \longrightarrow \mathrm{Gpd}_{\mathbb{E}_\infty}(\mathcal{V})$ compatible with the projections to $\mathcal{V}$. For each $X \in \mathcal{V}$, the induced functor between fibers admits a fully faithful left adjoint by Proposition 2.28 (applied to $\mathcal{V}_{/X}$).

For (1), we now note that the projections ev_* and π are both Cartesian fibrations, so that Ω^∞ admits a relative left adjoint $\mathbf{B}$ [Lur17, Proposition 7.3.2.6]. For (2), note that $\mathbf{B}$ is given fiberwise by the fully faithful left adjoint from Proposition 2.28. Since π and ev_* are also coCartesian fibrations, this implies that $\mathbf{B}$ is fully faithful (by [Lur09, Proposition 2.4.4.2]) and that its essential image is as asserted in (2).

For (3), the proof of [Lur17, Proposition 1.4.3.4] shows that it suffices to verify that the essential image of $\mathbf{B}\colon \mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{V}_{/X}) \hookrightarrow \mathcal{T}_X\mathcal{V}$ is closed under extensions. For this, we just need to verify that the additive presentable ∞-category $\mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{V}_{/X})$ satisfies the following condition [Lur18, Proposition C.1.2.2]: for each map $Y \longrightarrow \Sigma Z$ in $\mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{V}_{/X})$ to a suspension with fiber $F \longrightarrow Y$, the natural map $0 \amalg_F Y \longrightarrow \Sigma Z$ from the cofiber is an equivalence. To see this, consider the following diagram in $\mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{V}_{/X})$:

$$\begin{tikzcd}
\dots\ F \oplus Z \oplus Z \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d] & F \oplus Z \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & F \arrow[r] \arrow[d] & Y \arrow[d] \\
\dots\ Z \oplus Z \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & Z \arrow[r, shift left] \arrow[r, shift right] & 0 \arrow[r] & \Sigma Z
\end{tikzcd}$$

Here the bottom row is the standard augmented simplicial object that computes ΣZ as a geometric realization of coproducts (by restricting along the cofinal functor $\big(\Delta_{/\Lambda^2_0}\big)^{\mathrm{op}} \longrightarrow \Lambda^2_0$ and taking the left Kan extension along the left fibration $\big(\Delta_{/\Lambda^2_0}\big)^{\mathrm{op}} \longrightarrow \Delta^{\mathrm{op}}$). The top row is obtained from the bottom row by base change along $Y \longrightarrow \Sigma Z$ and each of the left vertical maps can be identified with the evident projection onto a summand. However, note that the simplicial structure of the top row is not just the direct sum of the bottom row and the constant diagram on F.

Since the forgetful functor $\mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{V}_{/X}) \longrightarrow \mathcal{V}_{/X}$ detects geometric realizations and $\mathcal{V}$ has parametrized loop space machinery (so that the fiber product $\times_{\Sigma Z}$ preserves geometric realizations), the top row is then a colimit diagram as well. The canonical map $0 \amalg_F Y \longrightarrow \Sigma Z$ is then an equivalence, since it can be identified with the geometric realization of the natural equivalence of simplicial objects $0 \amalg_F \big(F \oplus Z^{\oplus \bullet - 1}\big) \longrightarrow Z^{\oplus \bullet - 1}$. $\square$

Corollary 2.31. *Let $\mathcal{V}$ be a presentable SM ∞-category with parametrized loop space machinery. Then there is a commuting square of presentable SM ∞-categories and symmetric monoidal left adjoint functors*

$$\begin{tikzcd}
\mathrm{Fun}(\mathrm{Fin}_*, \mathcal{V}) \arrow[r, "i_!"] \arrow[d] & \mathrm{Fun}(\mathcal{S}^{\mathrm{fin}}_*, \mathcal{V}) \arrow[d, "(-)^{\mathrm{exc}}"] \\
\mathrm{Gpd}_{\mathbb{E}_\infty}(\mathcal{V}) \arrow[r, "\mathbf{B}"] & \mathcal{T}\mathcal{V}
\end{tikzcd}$$

where the top ∞-categories come equipped with the levelwise monoidal structure and the vertical functors are monoidal localizations. In particular, the canonical t-orientation (Example 2.21) is monoidal.

Proof. Corollary 2.30 already provides the desired square of presentable ∞-categories and left adjoints without monoidal structures. Here the functors $\mathrm{Fun}(\mathrm{Fin}_*, \mathcal{V}) \longrightarrow \mathrm{Gpd}_{\mathbb{E}_\infty}(\mathcal{V})$ and $\mathrm{Fun}(\mathcal{S}^{\mathrm{fin}}_*, \mathcal{V}) \longrightarrow \mathcal{T}\mathcal{V}$ are the localizations whose right adjoints are the evident inclusions of the full subcategories of Segal $\mathbb{E}_\infty$-groupoids and excisive functors. Since $\mathbf{B}$ is a fully faithful functor, the localization $\mathrm{Fun}(\mathrm{Fin}_*, \mathcal{V}) \longrightarrow \mathrm{Gpd}_{\mathbb{E}_\infty}(\mathcal{V})$ precisely inverts the class W of maps that are sent to equivalences by $(-)^{\mathrm{exc}} \circ i$.

To refine this commuting square to a commuting square of SM functors, observe that Fin_* and $\mathcal{S}^{\mathrm{fin}}_*$ both admit finite coproducts (given by wedge sums) and that the inclusion $i\colon \mathrm{Fin}_* \hookrightarrow \mathcal{S}^{\mathrm{fin}}_*$ preserves coproducts. Lemma 2.9 now implies that $i_!\colon \mathrm{Fun}(\mathrm{Fin}_*, \mathcal{V})^{\otimes_{\mathrm{lev}}} \longrightarrow \mathrm{Fun}(\mathcal{S}^{\mathrm{fin}}_*, \mathcal{V})^{\otimes_{\mathrm{lev}}}$ admits a natural SM structure (adjoint to the SM structure on i^*). The functor $(-)^{\mathrm{exc}}$ is a SM localization by Proposition 2.10.

Since $(-)^{\mathrm{exc}} \circ i$ is monoidal, the class W of arrows in $\mathrm{Fun}(\mathrm{Fin}_*, \mathcal{V})$ is closed under the tensor product with an object. It follows that the localization $\mathrm{Fun}(\mathrm{Fin}_*, \mathcal{V}) \longrightarrow \mathrm{Gpd}_{\mathbb{E}_\infty}(\mathcal{V})$ is a symmetric monoidal localization [Lur17, Proposition 4.1.7.4]; the functor $\mathbf{B}\colon \mathrm{Gpd}_{\mathbb{E}_\infty}(\mathcal{V}) \longrightarrow \mathcal{T}\mathcal{V}$ then has a unique SM structure making the square commute.

For the conclusion about the canonical t-orientation being monoidal, note that $\mathrm{Gpd}_{\mathbb{E}_\infty}(\mathcal{V}) \hookrightarrow \mathcal{T}\mathcal{V}$ is a fully faithful symmetric monoidal functor whose essential image coincides with $\mathcal{T}^{\geq 0}\mathcal{V}$ by Corollary 2.30. This implies that $\mathcal{T}^{\geq 0}\mathcal{V}$ contains the monoidal unit and is closed under the tensor product, as desired. ☐

Let us now turn to the proof of Proposition 2.28, which requires some preliminaries.

Lemma 2.32. *Let $i\colon \mathrm{Fin}_* \longrightarrow \mathcal{S}_*^{\mathrm{fin}}$ be the natural fully faithful inclusion. Then restriction and left Kan extension define an adjoint pair*

$$i_!\colon \mathrm{Fun}(\mathrm{Fin}_*, \mathcal{V}) \underset{\longleftarrow}{\overset{\hookrightarrow}{\perp}} \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V})\colon i^*$$

whose left adjoint is fully faithful. The essential image of $i_!$ consists exactly of those functors $F\colon \mathcal{S}_*^{\mathrm{fin}} \longrightarrow \mathcal{V}$ that preserve finite geometric realizations.

Proof. Note that $i_!$ is fully faithful because i is. To identify the essential image, let us factor the Yoneda embedding as

$$\mathrm{Fin}_* \xrightarrow{\ i\ } \mathcal{S}_*^{\mathrm{fin}} \xrightarrow{\ j\ } \mathcal{P}(\mathrm{Fin}_*)$$

where j sends $T \in \mathcal{S}_*^{\mathrm{fin}}$ to $\mathrm{Map}_{\mathcal{S}_*^{\mathrm{fin}}}(i(-), T)$. Note that for each finite pointed set $S \in \mathrm{Fin}_*$, the functor $\mathrm{Map}_{\mathcal{S}_*^{\mathrm{fin}}}(i(S), -)$ preserves all finite geometric realizations in $\mathcal{S}_*^{\mathrm{fin}}$, since it sends $T \mapsto T^{\times |S|-1}$. Consequently, j preserves finite geometric realizations as well. Since every finite pointed space is the geometric realization of some n-skeletal simplicial diagram in Fin_* and the Yoneda embedding is fully faithful on Fin_*, it follows that j is fully faithful.

We then have a sequence of adjunctions given by restriction and left Kan extension

$$\mathrm{Fun}(\mathrm{Fin}_*, \mathcal{V}) \underset{i^*}{\overset{i_!}{\underset{\longleftarrow}{\overset{\hookrightarrow}{\perp}}}} \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V}) \underset{j^*}{\overset{j_!}{\underset{\longleftarrow}{\overset{\hookrightarrow}{\perp}}}} \mathrm{Fun}\left(\mathcal{P}(\mathrm{Fin}_*), \mathcal{V}\right)$$

where the left adjoints are fully faithful. By [Lur09, Lemma 5.1.5.5], the essential image of $j_!i_!$ coincides with those functors $\mathcal{P}(\mathrm{Fin}_*) \longrightarrow \mathcal{V}$ preserving all colimits. Consequently, the essential image of $i_!$ consists of those functors whose left Kan extension along j defines a colimit-preserving functor $\mathcal{P}(\mathrm{Fin}_*) \longrightarrow \mathcal{V}$.

Since j preserves finite geometric realizations, it follows that any functor in the image of $i_!$ preserves finite geometric realizations. Conversely, given $F\colon \mathcal{S}_*^{\mathrm{fin}} \longrightarrow \mathcal{V}$ preserving finite geometric realizations, we have to verify that the counit map

$$i_!i^*F(T) \longrightarrow F(T)$$

is a natural equivalence for $T \in \mathcal{S}_*^{\mathrm{fin}}$. Note that the domain and codomain both preserve finite geometric realizations in T. Since each T is the realization of a finite simplicial diagram in Fin_*, we can reduce to the case where $T \in \mathrm{Fin}_*$. But F and $i_!i^*F$ agree on finite pointed sets by construction. ☐

Recall that S^1 arises as the geometric realization of the 1-skeletal (finite) pointed simplicial set $N_\bullet(\Delta^1/\partial\Delta^1)\colon \Delta^{\mathrm{op}} \longrightarrow \mathrm{Fin}_*$, given explicitly in simplicial degree n by the finite pointed set $\langle n\rangle$ with $(n+1)$-elements [Seg74, p.295]. For every $S \in \mathcal{S}_*^{\mathrm{fin}}$, the levelwise smash product $N_\bullet(\Delta^1/\partial\Delta^1) \wedge S$ then determines a simplicial diagram in $\mathcal{S}_*^{\mathrm{fin}}$, given in degree n by the n-fold wedge sum $\langle n\rangle \wedge S = S^{\vee n}$.

Lemma 2.33. *Suppose that $\mathcal{V}$ is a presentable ∞-category with loop space machinery and that $A\colon \mathrm{Fin}_* \longrightarrow \mathcal{V}$ satisfies the grouplike Segal conditions. Let $F = i_!A\colon \mathcal{S}_*^{\mathrm{fin}} \longrightarrow \mathcal{V}$ be its image under the left adjoint from Lemma 2.32. For each $S \in \mathcal{S}_*^{\mathrm{fin}}$, the simplicial diagram*

$$F\big(N_\bullet(\Delta^1/\partial\Delta^1) \wedge S\big)\colon \Delta^{\mathrm{op}} \longrightarrow \mathcal{V}$$

endows $F(S)$ with the structure of a grouplike monoid in the sense of [Lur09, Definition 7.2.2.1].

Proof. Consider the functor $Q\colon \mathcal{S}_*^{\mathrm{fin}} \longrightarrow \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{V})$ sending S to $F\big(N_\bullet(\Delta^1/\partial\Delta^1) \wedge S\big)$. We have to show that Q takes values in the full subcategory $\mathrm{Grp}(\mathcal{V}) \subseteq \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{V})$ of simplicial objects satisfying the grouplike Segal conditions.

Observe that the full subcategory $\mathrm{Grp}(\mathcal{V}) \subseteq \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{V})$ of simplicial objects X satisfying the grouplike Segal conditions (i.e. the grouplike Segal maps $X(n) \longrightarrow X(1)^{\times n}$ are equivalences) is stable under geometric realizations: for every simplicial diagram $X_\bullet$ in $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{V})$, the grouplike Segal maps $|X_\bullet(n)| \longrightarrow |X_\bullet(1)|^{\times n}$ are equivalent to the geometric realizations of simplicial diagram of Segal maps $X_\bullet(n) \longrightarrow X_\bullet(1)^{\times n}$. On the other hand, the functor Q preserves finite geometric realizations since F preserves finite geometric realizations (Lemma 2.32). Since every object in $\mathcal{S}_*^{\mathrm{fin}}$ is the geometric realization of a k-skeletal simplicial diagram of finite pointed sets, it thus suffices to show that $F\big(N_\bullet(\Delta^1/\partial\Delta^1) \wedge S\big)$ is a grouplike monoid when S is a finite pointed set.

When $S = \langle m\rangle$, the simplicial object $F\big(N_\bullet(\Delta^1/\partial\Delta^1) \wedge \langle m\rangle\big)$ can be identified explicitly as follows: it is obtained from $A((-) \wedge \langle m\rangle)\colon \mathrm{Fin}_* \longrightarrow \mathcal{V}$ by restricting along the functor $N_\bullet(\Delta^1/\partial\Delta^1)\colon \Delta^{\mathrm{op}} \longrightarrow \mathrm{Fin}_*$ from [Seg74, p.295]. Since A satisfies the grouplike Segal conditions, $A((-) \wedge \langle m\rangle) \simeq A(-)^{\times m}$ satisfies the grouplike Segal conditions as well. The simplicial object obtained by restriction then satisfies the grouplike Segal conditions as well (as asserted somewhat implicitly in loc. cit., see in particular Proposition 1.5). □

Proof of Proposition 2.28. Consider the adjoint pair $(i_!, i^*)$ from Lemma 2.32. We claim that for every $A\colon \mathrm{Fin}_* \longrightarrow \mathcal{V}$ satisfying the grouplike Segal conditions, the functor $F := i_!(A)\colon \mathcal{S}_*^{\mathrm{fin}} \longrightarrow \mathcal{V}$ is reduced excisive. Assuming this, the adjoint pair $(i_!, i^*)$ simply restricts to an adjoint pair between spectra and grouplike $\mathbb{E}_\infty$-monoids, i.e. $\mathbf{B} = i_!$ and $\Omega^\infty = i^*$. The characterization of the essential image of $\mathbf{B}$ then follows from Lemma 2.32.

To verify the claim, note that $F(*) \simeq A(*) \simeq *$, so F is reduced. Since $* \in \mathcal{S}_*^{\mathrm{fin}}$ is the initial object, there is a canonical lift $\tilde{F}\colon \mathcal{S}_*^{\mathrm{fin}} \longrightarrow \mathcal{V}_*$ such that postcomposing with the forgetful functor $\mathcal{V}_* \longrightarrow \mathcal{V}$ yields F: indeed, $\tilde{F}$ is simply given by the functor sending S to the pointed object $* \simeq F(*) \longrightarrow F(S)$ in $\mathcal{V}$. Since the forgetful functor $\mathcal{V}_* \longrightarrow \mathcal{V}$ preserves limits, the functor F is excisive if and only if $\tilde{F}$ is excisive.

To see that $\tilde{F}$ is excisive, it suffices to verify that for every $S \in \mathcal{S}_*^{\mathrm{fin}}$, the natural map

$$\tilde{F}(S) \longrightarrow \Omega\tilde{F}(\Sigma S) \tag{2.7}$$

is an equivalence [Lur17, Proposition 1.4.2.13]. Using that $\Sigma S = S^1 \wedge S$ is the geometric realization of the 1-skeletal simplicial diagram $\mathrm{N}_\bullet(\Delta^1/\partial\Delta^1) \wedge S$ and that F (and hence $\tilde{F}$) preserves finite geometric realizations, we have that $\tilde{F}(\Sigma S)$ is the bar construction of the group object from Lemma 2.33. The map (2.7) can then be identified with the map underlying the canonical map of grouplike monoids $F(S) \longrightarrow \Omega B(F(S))$, which is an equivalence because $\mathcal{V}$ has loop space machinery. $\square$

3. Tangent bundles of stable ∞-categories

The purpose of this section is to spell out the various definitions from Section 2 in the case where $\mathcal{V}$ is a stable or additive presentable ∞-category, for which the tangent bundle has a much simpler description.

3.1. **Trivializing the tangent bundle.** Let $\mathcal{V}$ be a pointed ∞-category with finite limits and consider the full subcategory $\mathrm{Ret} \subseteq \mathcal{S}_*^{\mathrm{fin}}$ on $*$ and S^0. Then Ret is equivalent to the retract category [Lur09, Definition 4.4.5.2] and there are functors

$$\begin{tikzcd} \mathcal{T}\mathcal{V} \arrow[r, "G"] \arrow[dr, "\mathrm{ev}_*"'] & \mathrm{Fun}(\mathrm{Ret}, \mathcal{V}) \arrow[r, "\mathrm{fib}"] \arrow[d, "\mathrm{ev}_*"] & \mathcal{V} \times \mathcal{V} \arrow[dl, "\pi_1"] \\ & \mathcal{V} & \end{tikzcd} \tag{3.1}$$

Here the first horizontal functor is given by restriction and fib sends a retract diagram $X \to Y \to X$ to the tuple $\big(X, Y \times_X *\big)$. The first functor exhibits $\mathcal{T}\mathcal{V}$ as the fiberwise stabilization of $\mathrm{Fun}(\mathrm{Ret}, \mathcal{V})$: for every $X \in \mathcal{V}$ the induced functor $\mathcal{T}_X\mathcal{V} \longrightarrow \mathrm{Fun}(\mathrm{Ret}, \mathcal{V})_X \simeq (\mathcal{V}_{/X})_*$ on fibers over X exhibits its domain as the stabilization of its target.

Now suppose that $\mathcal{V}$ is an additive ∞-category. Then the functor fib is an equivalence, with inverse sending (X, Y) to $X \longrightarrow X \oplus Y \longrightarrow X$ (see e.g. [CDH+20, Lemma 1.5.12]). In this case, we therefore obtain an equivalence

$$\begin{tikzcd} \mathcal{T}\mathcal{V} \arrow[rr, "\simeq"] \arrow[dr, "\pi"'] & & \mathcal{V} \times \mathrm{Sp}(\mathcal{V}) \arrow[dl, "\pi_1"] \\ & \mathcal{V} & \end{tikzcd}$$

between $\mathcal{T}\mathcal{V}$ and the fiberwise stabilization of $\mathcal{V} \times \mathcal{V}$ over $\mathcal{V}$. For stable $\mathcal{V}$, the situation is even simpler:

Lemma 3.1. *If $\mathcal{V}$ is a stable ∞-category, then both G and* fib *are equivalences, so that there is an equivalence $\mathcal{T}\mathcal{V} \simeq \mathcal{V} \times \mathcal{V}$ such that $\pi(X, Y) \simeq X$ and $\Omega^\infty(X, Y) \simeq X \oplus Y$.*

Proof. The functor fib is an equivalence since $\mathcal{V}$ is additive, so that the fibers of $\mathrm{Fun}(\mathrm{Ret}, \mathcal{V})$ are equivalent to $\mathcal{V}$ and hence already stable, which in turn implies that the functor G exhibiting the fiberwise stabilization is an equivalence (cf. [HNP19b, Corollary 2.2.5] for a similar argument). $\square$

Lemma 3.2. *Let $\mathcal{V}$ be an additive presentable ∞-category and let $\Sigma^\infty \colon \mathcal{V} \longrightarrow \mathrm{Sp}(\mathcal{V})$ be the left adjoint functor exhibiting $\mathrm{Sp}(\mathcal{V})$ as the stabilization of $\mathcal{V}$. Then the following induced square of tangent categories is Cartesian:*

$$\begin{tikzcd} \mathcal{T}\mathcal{V} \arrow[r, "\mathcal{T}(\Sigma^\infty)"] \arrow[d, "\pi"'] & \mathcal{T}(\mathrm{Sp}(\mathcal{V})) \arrow[d, "\pi"] \\ \mathcal{V} \arrow[r, "\Sigma^\infty"] & \mathrm{Sp}(\mathcal{V}). \end{tikzcd}$$

Proof. The left adjoint functor $\Sigma^\infty\colon \mathcal{V} \longrightarrow \mathrm{Sp}(\mathcal{V})$ commutes with the functor fib because one can identify $Y \times_X 0 \simeq Y \amalg_X 0$ for a retract diagram $X \to Y \to X$. This implies that the functor $\mathcal{T}(\Sigma^\infty)$ is obtained from the functor $\Sigma^\infty \times \Sigma^\infty_+\colon \mathcal{V} \times \mathcal{V} \longrightarrow \mathrm{Sp}(\mathcal{V}) \times \mathrm{Sp}(\mathcal{V})$ by stabilizing the second factor, which readily implies the result. $\square$

3.2. **Square zero monoidal structure.** If $\mathcal{V}$ is a stable presentable SM ∞-category, then the square zero monoidal structure on $\mathcal{T}\mathcal{V} \simeq \mathcal{V} \times \mathcal{V}$ (Definition 2.14) can be made more explicit using the following:

Definition 3.3. Let $\mathcal{D}$ be a presentable SM ∞-category. We will say that an object $D \in \mathcal{D}$ is **square zero** if the canonical map $\emptyset \longrightarrow D \otimes D$ from the initial object is an equivalence and denote by $\mathrm{SqZ}(\mathcal{D}) \subseteq \mathcal{D}$ the full subcategory on the square zero objects. Note that every SM left adjoint $F\colon \mathcal{D} \longrightarrow \mathcal{D}'$ restricts to a natural map $F\colon \mathrm{SqZ}(\mathcal{D}) \longrightarrow \mathrm{SqZ}(\mathcal{D}')$.

Recall that the ∞-category of **$\mathcal{V}$-linear SM ∞-categories** is given by the ∞-category $\mathrm{CAlg}_{\mathcal{V}}(\mathrm{Pr}^{\mathrm{L}}) \simeq \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})_{\mathcal{V}/}$ of presentable SM ∞-categories $\mathcal{D}$ equipped with a symmetric monoidal left adjoint functor $\mathcal{V} \longrightarrow \mathcal{D}$.

Definition 3.4. Let $\mathcal{W}$ be a $\mathcal{V}$-linear SM ∞-category together with a square zero object $M \in \mathcal{W}$. We say that this exhibits $\mathcal{W}$ as the **free $\mathcal{V}$-algebra on a square zero object** if for each $\mathcal{D} \in \mathrm{CAlg}_{\mathcal{V}}(\mathrm{Pr}^{\mathrm{L}})$, evaluation at M defines a natural equivalence

$$\mathrm{ev}_M\colon \mathrm{Fun}^{\otimes}_{\mathcal{V}}(\mathcal{W}, \mathcal{D}) \longrightarrow \mathrm{SqZ}(\mathcal{D}).$$

Remark 3.5. Consider a pushout square in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$

$$\begin{tikzcd} \mathcal{V}_1 \arrow[r] \arrow[d] & \mathcal{V}_2 \arrow[d] \\ \mathcal{W}_1 \arrow[r, "f"'] & \mathcal{W}_2. \end{tikzcd}$$

If $M \in \mathcal{W}_1$ exhibits $\mathcal{W}_1$ as the free $\mathcal{V}_1$-algebra on a square zero object, then $f(M)$ exhibits $\mathcal{W}_2 \simeq \mathcal{V}_2 \otimes_{\mathcal{V}_1} \mathcal{W}_1$ as the free $\mathcal{V}_2$-algebra on a square zero object: indeed, the evaluation at $f(M)$ factors as two equivalences

$$\mathrm{ev}_{f(M)}\colon \mathrm{Fun}^{\otimes}_{\mathcal{V}_2}(\mathcal{V}_2 \otimes_{\mathcal{V}_1} \mathcal{W}_1, \mathcal{D}) \xrightarrow[\sim]{f^*} \mathrm{Fun}^{\otimes}_{\mathcal{V}_1}(\mathcal{W}_1, \mathcal{D}) \xrightarrow[\sim]{\mathrm{ev}_M} \mathrm{SqZ}(\mathcal{D}).$$

Proposition 3.6. *There exists a free $\mathcal{S}$-algebra $\mathcal{S}[\epsilon]$ on a square zero object. Furthermore, the functor*

$$\{A, M\} \longrightarrow \mathcal{S}[\epsilon]$$

that sends A to the monoidal unit and M to the (universal) square zero object, exhibits $\mathcal{S}[\epsilon]$ as the free presentable ∞-category on the two-element set $\{A, M\}$.

In particular, the tensor product functor $\otimes\colon \mathcal{S}[\epsilon] \times \mathcal{S}[\epsilon] \longrightarrow \mathcal{S}[\epsilon]$ is the unique functor preserving colimits in each variable given on generating objects by $A \otimes A = A$, $A \otimes M = M \otimes A = M$ and $M \otimes M = \emptyset$ is the initial object.

Proof. First, let $\mathrm{Fin}^{\mathrm{bij}}$ be the category of finite sets and bijections, with monoidal structure given by disjoint union. By [Lur17, Proposition 2.2.4.9], the inclusion of the 1-element set $\{\underline{1}\}\colon * \longrightarrow \mathrm{Fin}^{\mathrm{bij}}$ exhibits $\mathrm{Fin}^{\mathrm{bij}}$ as the free symmetric monoidal ∞-category on $*$. By [Lur17, Corollary 4.8.1.12] (and the fact that $\mathrm{Fin}^{\mathrm{bij}} \simeq \mathrm{Fin}^{\mathrm{bij,op}}$), the ∞-category $\mathrm{Fun}(\mathrm{Fin}^{\mathrm{bij}}, \mathcal{S})$ of symmetric sequences admits a unique closed symmetric monoidal structure such that the Yoneda embedding

$$\mathrm{Fin}^{\mathrm{bij}} \xrightarrow{\quad h \quad} \mathrm{Fun}(\mathrm{Fin}^{\mathrm{bij}}, \mathcal{S})$$

admits a symmetric monoidal structure. In particular, the (co)representable h_0 on the empty set is the monoidal unit and the universal property of $\big(\mathrm{Fin}^{\mathrm{bij}}, \sqcup\big)$ and [Lur17, Proposition 4.8.1.10] imply that the map $\{h_1\}\colon * \longrightarrow \mathrm{Fun}(\mathrm{Fin}^{\mathrm{bij}}, \mathcal{S})$ exhibits $\mathrm{Fun}(\mathrm{Fin}^{\mathrm{bij}}, \mathcal{S})$ as the free presentable SM ∞-category on $*$. Finally, [Lur17, Remark 4.8.1.13] asserts that the resulting symmetric monoidal structure on $\mathrm{Fun}(\mathrm{Fin}^{\mathrm{bij}}, \mathcal{S})$ is in fact given by Day convolution.

Let us now denote by $\mathcal{S}[\epsilon]$ the (reflective) localization of symmetric sequences at the set of maps $\emptyset \longrightarrow h_n$ from the initial object, for all $n \geq 2$. Then $\mathcal{S}[\epsilon] \subseteq \mathrm{Fun}(\mathrm{Fin}^{\mathrm{bij}}, \mathcal{S})$ is the full subcategory of symmetric sequences X such that $X(n) \simeq *$ for all $n \geq 2$. In particular, the functor $\{A, M\} \longrightarrow \mathcal{S}[\epsilon]$ sending $A \mapsto h_0$ and $M \mapsto h_1$ exhibits $\mathcal{S}[\epsilon]$ as the free presentable ∞-category on $\{A, M\}$.

Note that for any $m \geq 0$ and $n \geq 2$, the map $\emptyset \otimes h_m \longrightarrow h_n \otimes h_m$ is equivalent to the map $\emptyset \longrightarrow h_{n+m}$, so (by the same argument as in Proposition 2.10) this exhibits $\mathcal{S}[\epsilon]$ as a symmetric monoidal localization of $\mathrm{Fun}(\mathrm{Fin}^{\mathrm{bij}}, \mathcal{S})$. By the universal property of symmetric monoidal localizations, the square zero object $h_1 \in \mathcal{S}[\epsilon]$ then realizes $\mathcal{S}[\epsilon]$ as the free presentable SM ∞-category on a square zero object. □

Corollary 3.7. *For every presentable SM ∞-category $\mathcal{V}$, there exists a free $\mathcal{V}$-algebra $\mathcal{V}[\epsilon]$ on a square zero object.*

Proof. Proposition 3.6 provides the existence of the free $\mathcal{S}$-algebra on a square zero object $\mathcal{S}[\epsilon]$. By Remark 3.5, $\mathcal{V} \otimes_{\mathcal{S}} \mathcal{S}[\epsilon]$ then provides the free $\mathcal{V}$-algebra on a square zero object. □

Remark 3.8. Let $\mathcal{V}$ be a presentable SM ∞-category. Then the free $\mathcal{V}$-algebra $\mathcal{V}[\epsilon]$ on a square zero object can also be described in terms of a variant of the Day convolution product applicable to *promonoidal* ∞-categories, as developed in recent work of Nardin–Shah [NS20]. More precisely, one can check that the 2-coloured operad $\mathcal{M}\mathrm{Com}$ for commutative algebras and modules is such a promonoidal (∞-)category. Since the underlying category of $\mathcal{M}\mathrm{Com}$ is simply the set $\{A, M\}$, this endows $\mathrm{Fun}(\{A, M\}, \mathcal{V})$ with a Day convolution product which has the property that

$$\begin{aligned}
(h_A \otimes C) \otimes (h_A \otimes D) &= h_A \otimes (C \otimes D)\\
(h_A \otimes C) \otimes (h_M \otimes D) &= h_M \otimes (C \otimes D)\\
(h_M \otimes C) \otimes (h_M \otimes D) &= \emptyset.
\end{aligned}$$

In particular, the square zero object $h_M \otimes 1_{\mathcal{V}}$ induces a symmetric monoidal functor from $\mathcal{V}[\epsilon]$ to this Day convolution, which is easily seen to be an equivalence. The universal property of the Day convolution therefore implies that for any ∞-operad $\mathcal{O}$, there is a natural equivalence

$$\mathrm{Alg}_{\mathcal{O}}\big(\mathcal{V}[\epsilon]\big) \simeq \mathrm{Alg}_{\mathcal{O}\times\mathcal{M}\mathrm{Com}}(\mathcal{V}).$$

The ∞-operad $\mathcal{M}\mathcal{O} = \mathcal{O} \times \mathcal{M}\mathrm{Com}$ is the ∞-operad for $\mathcal{O}$-algebras and (operadic) modules over them [Hin15, HNP19b]. Combining this with Proposition 2.17 and Proposition 3.10 below, one finds that $\mathcal{T}\,\mathrm{Alg}_{\mathcal{O}}(\mathcal{V}) \simeq \mathrm{Alg}_{\mathcal{M}\mathcal{O}}(\mathcal{V})$ for every stable presentable SM ∞-category $\mathcal{V}$ (see also [Sch97, BM05, Lur17]).

We will now relate the free $\mathcal{V}$-algebra on a square zero object to $\mathcal{T}\mathcal{V}$:

Construction 3.9. Let $\mathcal{V}$ be a stable presentable SM ∞-category and consider the following cofiber sequence of excisive functors

$$h_* \otimes 1_{\mathcal{V}} \xrightarrow{i\otimes \mathrm{id}} h_{S^0} \otimes 1_{\mathcal{V}} \longrightarrow M_{\mathcal{V}}$$

where $i \otimes \mathrm{id}$ is the canonical map of corepresentables induced by $* \longrightarrow S^0$. Lemma 2.13 shows that the pushout-product of $h_* \otimes 1_{\mathcal{V}} \longrightarrow h_{S^0} \otimes 1_{\mathcal{V}}$ with itself in $\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V})$ is a $\mathcal{T}\mathcal{V}$-local equivalence. Consequently, the pushout-product map in the monoidal localization $\mathcal{T}\mathcal{V}$ becomes an equivalence. Since the cofiber of a pushout-product map is the tensor product of the cofibers (see e.g. [GPS14, Theorem 6.2] for a proof at the level of stable monoidal derivators), it follows that $M_{\mathcal{V}} \otimes M_{\mathcal{V}} \simeq 0$ in $\mathcal{T}\mathcal{V}$.

Proposition 3.10. *Let $\mathcal{V}$ be a stable presentable SM ∞-category and consider $\mathcal{T}\mathcal{V}$ as a $\mathcal{V}$-linear SM ∞-category via the left adjoint $\mathcal{V} \longrightarrow \mathcal{T}\mathcal{V}$ to the projection. Then the square zero object $M_{\mathcal{V}} \in \mathcal{T}\mathcal{V}$ exhibits $\mathcal{T}\mathcal{V}$ as the free $\mathcal{V}$-algebra on a square zero object.*

Proof. Since $\mathcal{V}$ is a stable presentable SM ∞-category, the canonical SM left adjoint $\mathcal{S} \longrightarrow \mathcal{V}$ factors canonically over spectra [Lur17, Corollary 4.8.2.19]. This gives rise to the following diagram in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$:

$$\begin{tikzcd}
\mathcal{S} \arrow[r] \arrow[dd] & \mathrm{Sp} \arrow[r] \arrow[d] & \mathcal{V} \arrow[d] \\
& \mathrm{Sp}[\epsilon] \arrow[r] \arrow[d, "\phi"'] & \mathcal{V}[\epsilon] \arrow[d, "\phi_{\mathcal{V}}"] \\
\mathcal{T}\mathcal{S} \arrow[r] & \mathcal{T}\,\mathrm{Sp} \arrow[r] & \mathcal{T}\mathcal{V}.
\end{tikzcd}$$

Here each composite vertical functor is the left adjoint to the projection (i.e. taking constant $\mathcal{S}_*^{\mathrm{fin}}$-diagrams). For $\mathcal{V}$ and the ∞-category of spectra, this left adjoint factors over the free algebra on a square zero object: ϕ is the functor classifying the the square zero object $M_{\mathrm{Sp}} \in \mathcal{T}\,\mathrm{Sp}$ and $\phi_{\mathcal{V}}$ classifies $M_{\mathcal{V}}$. Since the functor $\mathcal{T}\,\mathrm{Sp} \longrightarrow \mathcal{T}\mathcal{V}$ is a monoidal left adjoint, it sends M_{Sp} to $M_{\mathcal{V}}$ to that the diagram commutes.

Now notice that by Proposition 2.18, the total square and the left rectangle are both pushout squares in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$. On the other hand, Remark 3.5 shows that the top right square is coCartesian. It therefore follows that the bottom right square is coCartesian as well. Consequently, $\phi_{\mathcal{V}}$ is an equivalence as soon as ϕ is an equivalence, so we can reduce to the case $\mathcal{V} = \mathrm{Sp}$. In this case, let us consider the composite functor

$$\{A, M\} \longrightarrow \mathcal{S}[\epsilon] \longrightarrow \mathrm{Sp}[\epsilon]$$

sending A to the monoidal unit and M to the universal square zero object. Proposition 3.6 asserts that the first functor exhibits $\mathcal{S}[\epsilon]$ as the free presentable ∞-category generated by $\{A, M\}$ and Remark 3.5 and [Lur17, Proposition 4.8.2.18] imply that the second functor exhibits $\mathrm{Sp}[\epsilon]$ as the stabilization of $\mathcal{S}[\epsilon]$. The composite therefore exhibits $\mathrm{Sp}[\epsilon]$ as the free stable presentable ∞-category generated by $\{A, M\}$.

Now observe that by construction the monoidal functor

$$\phi\colon\ \mathrm{Fun}(\{A, M\}, \mathrm{Sp}) \simeq \mathrm{Sp}[\epsilon] \longrightarrow \mathcal{T}\,\mathrm{Sp}$$

is given on generators by $\phi(h_A) = 1_{\mathcal{T}\,\mathrm{Sp}} = h_* \otimes 1_{\mathrm{Sp}}$ and $\phi(h_M) = M_{\mathrm{Sp}} = \mathrm{cof}\big(h_* \otimes 1_{\mathrm{Sp}} \longrightarrow h_{S^0} \otimes 1_{\mathrm{Sp}}\big)$. It follows that the right adjoint to ϕ is is given by the composite functor $\mathcal{T}\,\mathrm{Sp} \longrightarrow$

$\mathrm{Fun}(\mathrm{Ret}, \mathrm{Sp}) \longrightarrow \mathrm{Sp} \times \mathrm{Sp}$ appearing (3.1), which is an equivalence since Sp is stable. We conclude that ϕ is an equivalence, as desired. □

Remark 3.11. If $\mathcal{V}$ is an additive presentable SM ∞-category, its stabilization $\mathrm{Sp}(\mathcal{V})$ carries an induced symmetric monoidal structure and $\Sigma^\infty\colon \mathcal{V} \longrightarrow \mathrm{Sp}(\mathcal{V})$ is a symmetric monoidal functor [GGN15, Theorem 5.1]. The pullback square of Lemma 3.2 then becomes a pullback square of SM ∞-categories. Proposition 3.10 then provides an explicit description of the square zero monoidal structure on $\mathcal{T}\mathcal{V} \simeq \mathcal{V} \times \mathrm{Sp}(\mathcal{V})$, given informally by the formula

$$(C, E) \otimes_{\mathcal{T}\mathcal{V}} (D, F) \simeq \big(C \otimes_{\mathcal{V}} D, (\Sigma^\infty C \otimes_{\mathrm{Sp}(\mathcal{V})} F) \oplus (E \otimes_{\mathrm{Sp}(\mathcal{V})} \Sigma^\infty D)\big). \tag{3.2}$$

Example 3.12. Let $\mathcal{V}$ be a stable presentable SM ∞-category and suppose that $\mathcal{O}$ is a monochromatic ∞-operad in arity ≥ 1, i.e. $\mathcal{O}^\otimes_{\langle 0\rangle} = \emptyset$ and $\mathcal{O}^\otimes_{\langle 1\rangle}$ is a category with (up to equivalence) one object. This implies that $\mathrm{Alg}_\mathcal{O}(\mathcal{V})$ is pointed, i.e. the terminal algebra 0 is also the initial algebra (by [Lur17, Proposition 3.1.3.13]).

For any $A \in \mathrm{Alg}_\mathcal{O}(\mathcal{V})$, $\mathcal{T}_A \mathrm{Alg}_\mathcal{O}(\mathcal{V})$ can be identified with the ∞-category of operadic A-modules (see [HNP19b, Corollary 1.0.5] or [Lur17, Theorem 7.3.4.13]). Alternatively, Remark 3.8 identifies $\mathcal{T}\mathrm{Alg}_\mathcal{O}(\mathcal{V}) \simeq \mathrm{Alg}_{\mathcal{M}\mathcal{O}}(\mathcal{V})$ with the ∞-category of $\mathcal{O}$-algebras and modules over them.

Now, given such an A-module E, the $\mathcal{O}$-algebra $\Omega^\infty(E)$ can be identified with the split square zero extension $A \oplus E$. For any section $\eta\colon A \longrightarrow A \oplus E$, we then obtain pullback squares of the form

$$\begin{tikzcd} \Omega^\infty(0, E[-1]) \arrow[r] \arrow[d] & A_\eta \arrow[r] \arrow[d] & \Omega^\infty(A, E) \arrow[d] \\ \Omega^\infty(0,0) = 0 \arrow[r] & A \arrow[r, "\eta"] & \Omega^\infty(A, E). \end{tikzcd}$$

Here the map $0 \longrightarrow A$ is the initial map of $\mathcal{O}$-algebras and total pullback arises as the image under Ω^∞ of the pullback in $\mathcal{T}\mathrm{Alg}_\mathcal{O}(\mathcal{V}) \simeq \mathrm{Alg}_{\mathcal{M}\mathcal{O}}(\mathcal{V})$ of the map $(A, 0) \longrightarrow (A, E)$ along the initial map $(0, 0) \longrightarrow (A, E)$. In particular, $\Omega^\infty(0, E[-1])$ is the image of an $\mathcal{O}$-algebra $(0, E[-1])$ under the nonunital lax symmetric monoidal functor

$$\Omega^\infty\colon \mathcal{T}_0\mathcal{V} = \mathcal{T}\mathcal{V} \times_\mathcal{V} \{0\} \hookrightarrow \mathcal{T}\mathcal{V} \longrightarrow \mathcal{V}$$

where the first functor is the inclusion of the nonunital SM sub-∞-category from Remark 2.15. We have seen there that the tensor product on $\mathcal{T}_0\mathcal{V}$ is null-homotopic, so that each operation in $\mathcal{O}$ of arity ≥ 2 acts on $(0, E[-1])$ by a null-homotopic map. Consequently, the resulting map $A_\eta \longrightarrow A$ indeed behaves like a square zero extension in the sense of algebra: its fiber $\Omega^\infty(0, E[-1])$ is an $\mathcal{O}$-algebra on which all operations in $\mathcal{O}$ of arity ≥ 2 act by null-homotopic maps (cf. [Lur17, Proposition 7.4.1.14]).

3.3. ***t*-orientations.** Let us conclude with some remarks about t-orientations on tangent bundles of additive and monoidal ∞-categories.

Example 3.13. Let $\mathcal{V}$ be an additive presentable ∞-category, so that $\mathcal{T}\mathcal{V} \simeq \mathcal{V} \times \mathrm{Sp}(\mathcal{V})$ (Lemma 3.2). Then any t-structure on $\mathrm{Sp}(\mathcal{V})$ determines a t-orientation on $\mathcal{T}\mathcal{V}$. Now suppose that $\mathcal{V}$ is furthermore symmetric monoidal and recall that the square zero tensor product on $\mathcal{T}\mathcal{V}$ can be identified with the tensor product on $\mathcal{V} \times \mathrm{Sp}(\mathcal{V})$ given by Remark 3.11. From this description, one sees that a t-structure on $\mathrm{Sp}(\mathcal{V})$ determines a *monoidal* t-orientation on $\mathcal{T}\mathcal{V}$ if and only if $\mathrm{Sp}(\mathcal{V})^{\geq 0}$ is closed under taking the tensor product in $\mathrm{Sp}(\mathcal{V})$ with objects of the form $\Sigma^\infty(X)$, for $X \in \mathcal{V}$.

Example 3.14. Suppose that $\mathcal{V}$ is an additive presentable ∞-category and consider the canonical t-orientation on $\mathcal{T}\mathcal{V} \simeq \mathcal{V} \times \mathrm{Sp}(\mathcal{V})$ (Example 2.21). A tuple (C, E) is then contained in $\mathcal{T}^{\leq -1}\mathcal{V}$ if and only if $\Omega^\infty(E) = 0$ in $\mathcal{V}$. The proof of [Lur17, Proposition 1.4.3.4] shows that $(C, E) \in \mathcal{T}^{\geq 0}\mathcal{V}$ if and only if E is contained in the smallest subcategory of $\mathrm{Sp}(\mathcal{V})$ which is closed under colimits and extensions and contains all $\Sigma^\infty(X)$ for $X \in \mathcal{V}$. If $\mathcal{V}$ is furthermore closed SM, then Example 3.13 shows that the canonical t-orientation is monoidal.

When $\mathcal{V}$ is stable, the canonical t-orientation simply produces the trivial t-structure ($\mathcal{T}^{\geq 0}\mathcal{V} = \mathcal{T}\mathcal{V}$). If $\mathcal{V}$ is prestable [Lur18, Definition C.1.2.1], the canonical t-orientation has $\mathcal{T}^{\geq 0}\mathcal{V} \simeq \mathcal{V} \times \mathcal{V}$ under the equivalence $\mathcal{T}\mathcal{V} \simeq \mathcal{V} \times \mathrm{Sp}(\mathcal{V})$ [Lur18, Proposition C.1.2.2].

Example 3.15. Suppose that $\mathcal{V}$ is a prestable SM ∞-category and $\mathcal{O}$ an ∞-operad. Endowing $\mathcal{T}\mathcal{V} \simeq \mathcal{V} \times \mathrm{Sp}(\mathcal{V})$ with its canonical monoidal t-orientation and applying Proposition 2.25, we obtain a t-orientation on $\mathcal{T}\,\mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$, and hence a t-structure on $\mathcal{T}_A\,\mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$ for any $\mathcal{O}$-algebra A. Under the identification $\mathcal{T}_A\,\mathrm{Alg}_{\mathcal{O}}(\mathcal{V}) \simeq \mathrm{Mod}_A(\mathrm{Sp}(\mathcal{V}))$ from Example 3.12, this is simply the t-structure whose connective part is given by $\mathrm{Mod}_A(\mathcal{V}) \subseteq \mathrm{Mod}_A(\mathrm{Sp}(\mathcal{V}))$.

4. Postnikov structures

The goal of this section is to give an axiomatic description of a decomposition of an object in a nice ∞-category, together with the data of 'k-invariants', analogous to the Postnikov tower of a space.

Definition 4.1. Let $\mathcal{V}$ be an ∞-category with finite limits. A **Postnikov structure** on an object X in $\mathcal{V}$ consists of the following data:

(1) An infinite tower
$$X \longrightarrow \dots \longrightarrow X_a \longrightarrow \dots \longrightarrow X_1$$
of objects $X_a \in \mathcal{V}$ under X for $a \geq 1$, exhibiting X as the limit of $\{X_a\}_{a\geq 1}$.

(2) For each $a \geq 2$, an object $K_a \colon \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{V}$ in $\mathcal{T}\mathcal{V}$ together with a Cartesian square
$$\begin{tikzcd} X_a \arrow[r] \arrow[d] & \pi(K_a) \arrow[d, "0"] \\ X_{a-1} \arrow[r, "k_a"'] & \Omega^\infty K_a \end{tikzcd}$$
exhibiting $X_a \longrightarrow X_{a-1}$ as a square zero extension (see (2.2)).

Note that the convention to start at $a = 1$ is rather arbitrary, and in various cases it can be more natural to start at $a = 0$.

Warning 4.2. The notion of a Postnikov structure on an object X is a priori unrelated to the tower of truncations of X, i.e. its underlying tower need not be given by the Postnikov tower of X in the sense of [Lur09, Definition 5.5.6.23]. For example, Theorem 6.3 yields a Postnikov structure on an (∞, n)-category $\mathcal{C}$ whose underlying tower consists of the homotopy categories $\mathrm{ho}_{(n+a,n)}(\mathcal{C})$ and not on its truncations $\tau_{\leq n+a}(\mathcal{C})$ in $\mathrm{Cat}_{(\infty,n)}$.

Warning 4.2 notwithstanding, we will see that a good source of Postnikov structures is given by the usual Postnikov tower together with its k-invariants:

Example 4.3. The motivating example of a Postnikov structure is the usual Postnikov tower of a space X, together with the data of its k-invariants. In this case, $X_a = \tau_{\leq a} X$ and the K_a are given by the (suspended) parametrized Eilenberg–Maclane spectra $K_a = \Sigma^{a+1}\mathrm{H}\pi_a(X)$ over $\tau_{\leq 1} X$. We will come back to this in Example/Proposition 4.15.

To study naturality of Postnikov structures, it will be convenient to organize the data of an object X equipped with a Postnikov structure into a single diagram $T\colon \mathcal{E} \longrightarrow \mathcal{V}$. To this end, let us start by recalling the following definition:

Definition 4.4. Let $\phi\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a functor of ∞-categories. We will denote by $M(\phi)$ the domain of the coCartesian fibration classified by $\phi\colon \Delta^1 \longrightarrow \mathrm{Cat}_\infty$. By [Lur09, Lemma 3.2.3.3], $M(\phi)$ can be identified with the **mapping simplex** [Lur09, Section 3.2.2], i.e. it can be identified with the pushout of ∞-categories

$$M(\phi) := \mathcal{D} \coprod_{\{1\}\times\mathcal{C}} \Delta^1 \times \mathcal{C}.$$

Using the coCartesian fibration $M(\phi) \longrightarrow \Delta^1$, one can understand $M(\phi)$ as follows: an object of $M(\phi)$ is either an object of $\mathcal{C}$ or an object of $\mathcal{D}$, and for each $c, c' \in \mathcal{C}$ and $d, d' \in \mathcal{D}$ we have

$$\begin{aligned} \mathrm{Map}_{M(\phi)}\big(c, c'\big) &= \mathrm{Map}_{\mathcal{C}}(c, c') & \qquad \mathrm{Map}_{M(\phi)}\big(c, d\big) &= \mathrm{Map}_{\mathcal{D}}(\phi(c), d) \\ \mathrm{Map}_{M(\phi)}(d, d') &= \mathrm{Map}_{\mathcal{D}}(d, d') & \qquad \mathrm{Map}_{M(\phi)}(d, c) &= \emptyset \end{aligned}$$

with the evident composition. Let us write $\phi_*\colon \Delta^1 \times \mathcal{C} \longrightarrow M(\phi)$ for the natural map into the pushout, sending $(0, c)$ to c and $(1, c)$ to $\phi(c)$.

Construction 4.5. For any integer a, let $\kappa_a\colon \{a \to (a-1)\} \longrightarrow \mathcal{S}_*^{\mathrm{fin}}$ be the functor sending the walking arrow $a \to (a-1)$ to $* \to S^0$ and let $\mathcal{E}_a = M(\kappa_\alpha)$ be its mapping simplex. As in Definition 4.4, we will identify the objects of $\mathcal{E}_a$ with the objects of $\mathcal{S}_*^{\mathrm{fin}}$, together with two additional objects a, $a-1$. The functor $\sigma_{a*}\colon \Delta^1 \times \{a \to a-1\} \longrightarrow \mathcal{E}_a$ is then given explicitly by

$$\sigma_a(0, a) = a, \qquad \sigma_a(0, a-1) = a-1, \qquad \sigma_a(1, a) = * \qquad \sigma_a(1, a-1) = S^0.$$

For any integer m, let us then define $\mathcal{E}_{\geq m}$ to as the pushout of ∞-categories:

$$\begin{tikzcd} \mathbb{Z}^{\mathrm{op}}_{\geq m} \arrow[r] \arrow[d] & \Big[\mathcal{E}_{m+1} \coprod_{\{m+1\}} \mathcal{E}_{m+2} \coprod_{\{m+2\}} \mathcal{E}_{m+3} \coprod \dots\Big] \arrow[d] \\ \big(\mathbb{Z}^{\mathrm{op}}_{\geq m}\big)^{\triangleleft} \arrow[r] & \mathcal{E}_{\geq m} \end{tikzcd}$$

where the left vertical functor is the usual inclusion into the cone and the top horizontal functor sends each map $a \to (a-1)$ in $\mathbb{Z}^{\mathrm{op}}_{\geq m}$ to the nondegenerate arrow corresponding arrow in $\mathcal{E}_a$. Given $T\colon \mathcal{E}_{\geq m} \longrightarrow \mathcal{V}$, we then observe that:

- the restriction of T to $\mathcal{S}_*^{\mathrm{fin}} \subseteq \mathcal{E}_a$ corresponds to K_a.
- the restriction of T along $\sigma_{a*}\colon \Delta^1 \times \{a \to a-1\} \subseteq \mathcal{E}_a$ corresponds to the square

$$\begin{tikzcd} X_a \arrow[r] \arrow[d] & K_a(*) = \pi(K_a) \arrow[d] \\ X_{a-1} \arrow[r, "k_a"'] & K_a(S^0) = \Omega^\infty(K_a). \end{tikzcd}$$

- the restriction of T to $(\mathbb{Z}^{\mathrm{op}}_{\geq m})^{\triangleleft} \subseteq \mathcal{E}$ encodes the tower $X \to \ldots \to\to X_{m+1} \to X_m$.

By default, we will take $\mathcal{E} = \mathcal{E}_{\geq 1}$.

Definition 4.6. We define the **∞-category of objects equipped with a Postnikov structure** to be the full subcategory

$$\mathrm{PoStr}(\mathcal{V}) \subseteq \mathrm{Fun}(\mathcal{E}, \mathcal{V})$$

of diagrams T for which (a) the restriction to each $\mathcal{S}^{\mathrm{fin}}_* \subseteq \mathcal{E}_a$ is excisive, (b) the restriction along each σ_{a*} is a Cartesian square and (c) the restriction to $(\mathbb{Z}^{\mathrm{op}}_{\geq 1})^{\triangleleft}$ is a limit cone.

Remark 4.7. The conditions determining $\mathrm{PoStr}(\mathcal{V})$ inside $\mathrm{Fun}(\mathcal{E}, \mathcal{V})$ assert that certain designated cone diagrams $\mathcal{I}^{\triangleleft}_{\alpha} \longrightarrow \mathcal{E}$, with $\mathcal{I}_\alpha$ contractible (either a span or $\mathbb{Z}^{\mathrm{op}}_{\geq 1}$), are sent to limit cones. In particular, $\mathrm{PoStr}(\mathcal{V}) \subseteq \mathrm{Fun}(\mathcal{E}, \mathcal{V})$ is closed under limits.

Evaluating at the cone point of the tower $\infty \in (\mathbb{Z}^{\mathrm{op}}_{\geq 1})^{\triangleleft} \subseteq \mathcal{E}$ determines a limit-preserving functor $\mathrm{ev}_\infty \colon \mathrm{PoStr}(\mathcal{V}) \longrightarrow \mathcal{V}$.

Definition 4.8. A **Postnikov structure** on an ∞-category $\mathcal{V}$ is defined to be a section $\Phi \colon \mathcal{V} \longrightarrow \mathrm{PoStr}(\mathcal{V})$ of the functor $\mathrm{ev}_\infty : \mathrm{PoStr}(\mathcal{V}) \longrightarrow \mathcal{V}$.

Warning 4.9. Note the distinction between a Postnikov structure on an *object in* an ∞-category $\mathcal{V}$ (Definition 4.1) and a Postnikov structure *on* an ∞-category $\mathcal{V}$: the former is a single diagram in $\mathcal{V}$, while the latter is a family of diagrams depending functorially on $X \in \mathcal{V}$. This should not cause any confusion, since it is always clear from the context if we are dealing with a functor on $\mathcal{V}$.

Definition 4.10. Let $\mathcal{V}$ be a SM ∞-category and endow $\mathrm{Fun}(\mathcal{E}, \mathcal{V})$ with the levelwise tensor product. We define the **∞-operad of objects equipped with a Postnikov structure** to be the full suboperad

$$\mathrm{PoStr}(\mathcal{V})^{\otimes} \subseteq \mathrm{Fun}(\mathcal{E}, \mathcal{V})^{\otimes_{\mathrm{lev}}}$$

spanned by the objects from Definition 4.8. A **multiplicative Postnikov structure** on $\mathcal{V}$ is a section of the map $\mathrm{ev}_\infty \colon \mathrm{PoStr}(\mathcal{V})^{\otimes} \longrightarrow \mathcal{V}^{\otimes}$ in the ∞-category of ∞-operads.

Using that $\mathrm{PoStr}(\mathcal{V})^{\otimes}$ is a full suboperad of $\mathrm{Fun}(\mathcal{E}, \mathcal{V})^{\otimes}$, Definition 4.10 can be rephrased as follows: a multiplicative Postnikov structure on $\mathcal{V}$ is a lax symmetric monoidal section $\Phi \colon \mathcal{V} \longrightarrow \mathrm{Fun}(\mathcal{E}, \mathcal{V})$ of $\mathrm{ev}_\infty \colon \mathrm{Fun}(\mathcal{E}, \mathcal{V}) \longrightarrow \mathcal{V}$ with the property that the underlying functor of Φ is a Postnikov structure (Definition 4.8).

Remark 4.11. In general, the ∞-operad $\mathrm{PoStr}(\mathcal{V})^{\otimes}$ need not be a SM ∞-category.

Remark 4.12. Suppose that $\Phi \colon \mathcal{V} \longrightarrow \mathrm{Fun}(\mathcal{E}, \mathcal{V})$ is a multiplicative Postnikov structure. Restricting to the copy of $\mathcal{S}^{\mathrm{fin}}_* \subseteq \mathcal{E}_m$ in level m, one obtains a lax monoidal functor $K_m(\Phi) \colon \mathcal{V} \longrightarrow \mathrm{Fun}(\mathcal{S}^{\mathrm{fin}}_*, \mathcal{V})$ taking values in $\mathcal{T}\mathcal{V} \subseteq \mathrm{Fun}(\mathcal{S}^{\mathrm{fin}}_*, \mathcal{V})$. Since $\mathcal{T}\mathcal{V}$ is a monoidal localization of $\mathrm{Fun}(\mathcal{S}^{\mathrm{fin}}_*, \mathcal{V})$, each $K_m(\Phi)$ defines a lax monoidal functor $\mathcal{V} \longrightarrow \mathcal{T}\mathcal{V}$ to the tangent bundle, equipped with the square zero monoidal structure (Definition 2.14).

Example 4.13. Suppose that the monoidal structure on $\mathcal{V}$ is given by the Cartesian product. Then the levelwise monoidal structure on $\mathrm{Fun}(\mathcal{E}, \mathcal{V})$ is the Cartesian monoidal structure as well. Consequently (cf. [Lur17, Section 2.4.1]), strong symmetric monoidal functors $\mathcal{V} \to \mathrm{Fun}(\mathcal{E}, \mathcal{V})$ simply correspond to product preserving functors $\mathcal{V} \to \mathrm{Fun}(\mathcal{E}, \mathcal{V})$, i.e., to functors $\mathcal{V} \to \mathrm{Fun}(\mathcal{E}, \mathcal{V})$ each of whose components $\mathcal{V} \to \mathcal{V}$ are product preserving. In particular, any Postnikov structure

$\Phi\colon \mathcal{V} \longrightarrow \mathrm{Fun}(\mathcal{E},\mathcal{V})$ on $\mathcal{V}$ whose individual components $\Phi_i\colon \mathcal{V}\to\mathcal{V}$ are product preserving canonically refines to a multiplicative Postnikov structure.

The main point of multiplicative Postnikov structures is that they induce such structures on categories of algebras:

Proposition 4.14. *Let $\mathcal{O}$ be an ∞-operad and let $\mathcal{V}$ be a symmetric monoidal ∞-category equipped with a multiplicative Postnikov structure $\Phi\colon \mathcal{V}\longrightarrow \mathrm{Fun}(\mathcal{E},\mathcal{V})$. Then the induced map*

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{V}) \xrightarrow{\Phi_*} \mathrm{Alg}_{\mathcal{O}}\big(\mathrm{Fun}(\mathcal{E},\mathcal{V})\big) \simeq \mathrm{Fun}\big(\mathcal{E}, \mathrm{Alg}_{\mathcal{O}}(\mathcal{V})\big)$$

is also a multiplicative Postnikov structure.

Proof. First, note that we can view $\mathcal{E}$ as an ∞-operad (with only unary operations), so that $\mathrm{Fun}(\mathcal{E},\mathcal{V})\simeq \mathrm{Alg}_{\mathcal{E}}(\mathcal{V})$. By Remark 2.7, the symmetry of the Boardman-Vogt tensor product of ∞-operads [Lur17, Proposition 2.2.5.13] then induces a commuting diagram of symmetric monoidal ∞-categories

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}\otimes_{\mathrm{BV}}\mathcal{E}}(V) \simeq \mathrm{Alg}_{\mathcal{O}}\big(\mathrm{Fun}(\mathcal{E},\mathcal{V})\big) \arrow[r, "\simeq"] \arrow[d, "\mathrm{Alg}_{\mathcal{O}}(\mathrm{ev}_\infty)"'] & \mathrm{Fun}\big(\mathcal{E},\mathrm{Alg}_{\mathcal{O}}(\mathcal{V})\big) \simeq \mathrm{Alg}_{\mathcal{E}\otimes_{\mathrm{BV}}\mathcal{O}}(V) \arrow[d, "\mathrm{ev}_\infty"] \\
\mathrm{Alg}_{\mathcal{O}}(\mathcal{V}) \arrow[r, "="'] & \mathrm{Alg}_{\mathcal{O}}(\mathcal{V})
\end{tikzcd}$$

in which the horizontal arrows are equivalences. It follows that $\Phi_*=\mathrm{Alg}_{\mathcal{O}}(\Phi)$ defines a lax symmetric monoidal section of ev_∞. To see that Φ_* takes values in the full sub-∞-category $\mathrm{PoStr}(\mathrm{Alg}_{\mathcal{O}}(\mathcal{V}))\subseteq \mathrm{Fun}(\mathcal{E},\mathrm{Alg}_{\mathcal{O}}(\mathcal{V}))$, consider the commuting diagram

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}}(\mathcal{V}) \arrow[r, "\Phi_*"] \arrow[d] & \mathrm{Alg}_{\mathcal{O}}(\mathrm{Fun}(\mathcal{E},\mathcal{V})) \arrow[r, "\simeq"] \arrow[d] & \mathrm{Fun}(\mathcal{E},\mathrm{Alg}_{\mathcal{O}}(\mathcal{V})) \arrow[d] \\
\mathrm{Fun}(\mathcal{O}_{\langle 1\rangle},\mathcal{V}) \arrow[r, "\Phi_*"'] & \mathrm{Fun}(\mathcal{O}_{\langle 1\rangle},\mathrm{Fun}(\mathcal{E},\mathcal{V})) \arrow[r, "\simeq"'] & \mathrm{Fun}(\mathcal{E},\mathrm{Fun}(\mathcal{O}_{\langle 1\rangle},\mathcal{V}))
\end{tikzcd}$$

where $\mathcal{O}_{\langle 1\rangle}$ is the underlying ∞-category of $\mathcal{O}$ [Lur17, Remark 2.1.1.25]. Since the vertical functors preserve limits and detect equivalences, the top horizontal composite defines a Postnikov structure if and only if the bottom horizontal composite does (since an $\mathcal{E}$-diagram is a Postnikov structure if it sends certain sub-diagrams to limit diagrams). But for the bottom horizontal composite this is clear, since limits are computed pointwise. ☐

4.1. **Examples.** Together with Proposition 4.14, the main sources of examples of multiplicative Postnikov structures are the following:

Example/Proposition 4.15. *Let $\mathcal{S}$ be the ∞-category of spaces. Then the Postnikov tower $X\to\ldots\to\tau_{\leq 2}(X)\to\tau_{\leq 1}(X)$, together with its k-invariants, gives rise to a multiplicative Postnikov structure on $(\mathcal{S},\times)$.*

Proof. Since we consider $\mathcal{S}$ with the Cartesian monoidal structure, Example 4.13 shows that it suffices to construct the Postnikov structure without its lax monoidal structure, and only check at the end that the individual components are product preserving. Now the underlying Postnikov structure can be produced at the level of simplicial sets (and is classical, cf. [DK84, GJ09]). Indeed, for every Kan complex X, let us make the following definitions:

(a) Let $P_a(X) = \mathrm{cosk}_{a+1}(X)$ be the $(a+1)$-coskeleton and note that there is a canonical weak equivalence $P_1(X) \longrightarrow \mathrm{N}(\Pi_1(X))$ to the nerve of the fundamental groupoid.

(b) For every $a \geq 2$, there is a functor $\pi_a(X)\colon \Pi_1(X) \longrightarrow \mathrm{Ab}$ sending a vertex x of X to the corresponding homotopy group. Let us recall that this homotopy group can be presented as quotient of the set of maps of pointed simplicial sets $\big(\mathrm{sk}_a\Delta^{a+1}, \{0\}\big) \longrightarrow (X, x)$ by pointed homotopy.

(c) For every pointed simplicial set S, taking the free reduced $\pi_a(X)$-module on its simplices yields a simplicial $\Pi_1(X)$-set $\pi_a(X) \otimes S\colon \Pi_1(X) \longrightarrow \mathrm{sSet}$. Taking $S = \Delta^n/\mathrm{sk}_{n-1}\Delta^n$, this gives the functor sending each vertex x of X to the classical (minimal) model for the Eilenberg–Maclane space $K(\pi_a(X, x), n)$. Recall that the latter is characterized up to *isomorphism* by the following universal property: there is a natural bijection between the set of maps of simplicial sets $T \longrightarrow K(\pi_a(X, x), n)$ and the set of n-cocycles in the normalized cochain complex of T with coefficients in $\pi_a(X, x)$.

(d) Recall that there is a classifying space functor $(-)_{h\Pi_1(X)}$ from $\mathrm{Fun}(\Pi_1(X), \mathrm{sSet})$ to simplicial sets, given by the following explicit point-set model for the homotopy colimit: $Y_{h\Pi_1(X)}$ has n-simplices given by tuples of $x_0 \to \ldots \to x_n$ in $\Pi_1(X)$ and an n-simplex of $Y(x_0)$. In particular, $(*)_{h\Pi_1(X)} = \mathrm{N}(\Pi_1(X))$ is the nerve of the fundamental groupoid.

(e) Let $\mathrm{sSet}_*^{\mathrm{fin}}$ denote the full subcategory of pointed simplicial sets whose image in the ∞-category $\mathcal{S}_*$ of pointed spaces is finite. We then define $K_{X,a}\colon \mathrm{sSet}_*^{\mathrm{fin}} \longrightarrow \mathrm{sSet}$ by

$$K_{X,a}(T) = \big[\pi_a(X) \otimes (T \wedge S^{a+1})\big]_{h\Pi_1(X)}$$

where $S^{a+1} = \Delta^{a+1}/\mathrm{sk}_a\Delta^{a+1}$.

(f) By [DK84, 1.2(vi)], there is a natural map of simplicial sets for each $a \geq 2$

$$k_a\colon P_{a-1}(X) \longrightarrow K_{X,a}(S^0) = \big[K(\pi_a(X), a+1)\big]_{h\Pi_1(X)}.$$

Explicitly, this map is given as follows. The simplicial set $K_{X,a}(S^0)$ is $(a+1)$-coskeletal and the map $K_{X,a}(S^0) \longrightarrow \mathrm{N}(\Pi_1(X))$ induces an isomorphism on a-skeleta. The map k_a then coincides with $P_{a-1}(X) \longrightarrow P_1(X) \longrightarrow \mathrm{N}(\Pi_1(X))$ on the a-skeleton, and sends an $(a+1)$-simplex of $P_{a-1}(X)$, i.e. a map $\sigma\colon \mathrm{sk}_a\Delta^{a+1} \longrightarrow X$, to the associated element in $\pi_a\big(X, \sigma(0)\big)$ (see point (b)).

By construction, the map k_a is trivial on all $(a+1)$ simplices in $P_{a-1}(X)$ that arise as the image of an $(a+1)$-simplex in $P_a(X)$, so that there is a commuting square

$$\begin{tikzcd} P_a(X) \arrow[r] \arrow[d] & K_{X,a}(*) \arrow[d, "0"] \\ P_{a-1}(X) \arrow[r, "k_a"'] & K_{X,a}(S^0). \end{tikzcd} \tag{4.1}$$

For any Kan complex X, the functor $K_{X,a}$ preserves weak equivalences of simplicial sets and hence determines a functor of ∞-categories $K_{X,a}\colon \mathcal{S}_*^{\mathrm{fin}} \longrightarrow \mathcal{S}$. It is straightforward to verify the conditions of Proposition 2.28, which imply that $K_{X,a}$ is excisive because $\mathcal{S}$ admits loopspace machinery (Example 2.27). Furthermore, the square (4.1) defines a pullback square in the ∞-category $\mathcal{S}$ by [DK84, Lemma 2.3] and the sequence $X \to \ldots \to P_a(X) \to P_{a-1}(X) \to \ldots$ is a homotopy limit sequence. It follows that the above construction defines, for every Kan complex X, a simplicial model for a Postnikov structure on the underlying object in the ∞-category $\mathcal{S}$.

All of the above data is strictly functorial in maps of Kan complexes and sends weak equivalences to pointwise weak equivalences of simplicial sets. It therefore defines a section

$$\begin{tikzcd} \mathcal{S} \arrow[r, dotted] & \mathrm{PoStr}(\mathcal{S}) \arrow[l, bend right, "\mathrm{ev}_\infty"'] \end{tikzcd}$$

on ∞-categorical localizations, as desired. To verify that the individual components of this tower are product preserving we note that:

(1) For each $a \geq 1$ the Postnikov piece functor $P_a(X)$ is product preserving. Indeed on the level of Kan complexes it is given by $\mathrm{cosk}_a(-)$, which is product preserving on the nose.

(2) For each $a \geq 1$ and $T \in \mathcal{S}_*^{\mathrm{fin}}$, the functor

$$X \longmapsto K_{X,a}(T) = \Big(\pi_a(X) \otimes (T \wedge S^{a+1})\Big)_{h\Pi_1(X)}$$

is product preserving. Indeed, this follows from the fact that :
- $\Pi_1(-)$ is product preserving;
- taking a'th homotopy groups is product preserving when considered as a functor from pointed spaces to abelian groups. In other words, the map $\pi_a(X \times Y, (x, y)) \to \pi_a(X, x) \times \pi_a(Y, y)$ is an isomorphism.
- for a fixed finite set I the functor $A \mapsto A \otimes I = A^I$ from abelian groups to sets is product preserving;
- products in spaces commute with homotopy quotients in each variable separately. Indeed, for two diagrams of simplicial sets $X \colon \mathcal{G} \longrightarrow \mathrm{sSet}$ and $Y \colon \mathcal{H} \longrightarrow \mathrm{sSet}$ indexed by groupoids, the map $(X \times Y)_{h(\mathcal{G}\times\mathcal{H})} \longrightarrow X_{h\mathcal{G}} \times Y_{h\mathcal{H}}$ is an isomorphism by the explicit formula from (d).

It follows that the Postnikov structure is multiplicative. □

Example 4.16. As mentioned in its construction, the multiplicative Postnikov structure of Example 4.15 is not just lax symmetric monoidal, but strongly symmetric monoidal: it is a product preserving functor $\Phi \colon \mathcal{S} \longrightarrow \mathrm{Fun}(\mathcal{E}, \mathcal{S})$. It follows that for any small ∞-category with finite products $\mathcal{T}$, the ∞-category $\mathrm{Fun}^\times(\mathcal{T}, \mathcal{S})$ of product preserving functors $\mathcal{T} \to \mathcal{S}$ comes with a Postnikov structure

$$\mathrm{Fun}^\times(\mathcal{T}, \mathcal{S}) \xrightarrow{\Phi_*} \mathrm{Fun}^\times(\mathcal{T}, \mathrm{Fun}(\mathcal{E}, \mathcal{S})) \simeq \mathrm{Fun}(\mathcal{E}, \mathrm{Fun}^\times(\mathcal{T}, \mathcal{S})).$$

For every $A \in \mathrm{Fun}^\times(\mathcal{T}, \mathcal{S})$, this provides a refinement of the tower $A \to \dots \to \tau_{\leq 2}A \to \tau_{\leq 1}A$ of truncations of A. In particular, when $\mathcal{T}$ is an algebraic theory, this shows that $\mathcal{T}$-algebras over in $\mathcal{S}$ have Postnikov towers equipped with k-invariants (cf. [GH00] for algebras over simplicial operads).

Example 4.17. Let $\mathcal{X}$ be an ∞-topos in which Postnikov towers converge [Lur09, Definition 5.5.6.23], i.e. $\mathcal{X} \longrightarrow \lim_n \tau_{\leq n}\mathcal{X}$ is the limit of its full subcategories of truncated objects (this implies that $\mathcal{X}$ is hypercomplete). In this case, there exists a reflective localization $L \colon \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \rightleftarrows \mathcal{X} \colon i$ such that L is left exact and preserves (limits of) Postnikov towers. We then obtain a Postnikov structure on $\mathcal{X}$

$$\mathcal{X} \xrightarrow{i} \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \xrightarrow{\Phi_*} \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Fun}(\mathcal{E}, \mathcal{S})) \simeq \mathrm{Fun}(\mathcal{E}, \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})) \xrightarrow{L_*} \mathrm{Fun}(\mathcal{E}, \mathcal{X}).$$

Indeed, this sends every object $X \in \mathcal{X}$ to the Postnikov structure of the presheaf $i(X)$ (applying Example 4.15 pointwise in $\mathcal{C}$), and then applies L to the resulting diagram of presheaves. Since L is left exact and preserves Postnikov towers, the resulting $\mathcal{E}$-diagram in $\mathcal{X}$ is indeed a Postnikov structure.

Observation 4.18. The proof of Example 4.15 admits the following modification: let $\mathcal{S}^{\pi-\mathrm{ab}} \subseteq \mathcal{S}$ be the full subcategory consisting of those spaces X such that each homotopy group $\pi_1(X, x)$ is abelian and acts trivially on the higher $\pi_n(X, x)$. Then there exists a multiplicative Postnikov structure

$$\mathcal{S}^{\pi-\mathrm{ab}} \longrightarrow \mathrm{PoStr}(\mathcal{S}^{\pi-\mathrm{ab}}) \subseteq \mathrm{PoStr}(\mathcal{S})$$

whose value on a space X is the Postnikov structure $X \to \ldots \to \tau_{\leq 1}X \to \pi_0(X)$ including the zeroth stage. Furthermore, the k-invariants are given by maps

$$k_a\colon \tau_{\leq a-1}X \longrightarrow \Omega^\infty(K_a(X))$$

where $K_a(X)$ is the parametrized spectrum over $\pi_0(X)$ whose fiber over $x \in \pi_0(X)$ denotes the suspended Eilenberg–Maclane spectrum $H(\pi_a(X, x))[a+1]$. Indeed, this follows from the fact that the category of simplicial sets with homotopy type in $\mathcal{S}^{\pi-\mathrm{ab}}$ is closed under coskeleta and products, together with the fact that the local system of homotopy groups from (b) arises as the pullback of a local system along the map $\Pi_1(X) \longrightarrow \pi_0(X)$.

Example 4.19. Let $\mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{S}^{\pi-\mathrm{ab}})$ be the ∞-category of $\mathbb{E}_\infty$-spaces whose underlying space has trivial actions of π_1. Proposition 4.14 and Observation 4.18 imply that the Postnikov tower $A \to \ldots \to \tau_{\leq 1}A \to \tau_{\leq 0}A$ is part of a multiplicative Postnikov structure Φ^{ab} on $\big(\mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{S}^{\pi-\mathrm{ab}}), \times\big)$.

Let A be a grouplike $\mathbb{E}_\infty$-space. Then A is in particular contained in $\mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{S}^{\pi-\mathrm{ab}})$. The corresponding Postnikov structure $\Phi^{\mathrm{ab}}_A\colon \mathcal{E} \longrightarrow \mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{S}^{\pi-\mathrm{ab}})$ has the property that $\pi_0(\Phi^{\mathrm{ab}}_A)$ is the constant diagram with value $\pi_0(A)$. In particular, Φ^{ab}_A takes values in the full subcategory of grouplike $\mathbb{E}_\infty$-monoids. It follows that the multiplicative Postnikov structure Φ^{ab} restricts to a multiplicative Postnikov structure on grouplike $\mathbb{E}_\infty$-monoids, which fits into a commuting square

$$\begin{tikzcd}
\mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{S}) \arrow[r, "\Phi^{\mathrm{ab}}"] \arrow[d, "\mathrm{forget}"'] & \mathrm{PoStr}\big(\mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{S})\big) \arrow[d, "\mathrm{forget}"] \\
\mathcal{S}^{\pi-\mathrm{ab}} \arrow[r, "4.18"] & \mathrm{PoStr}(\mathcal{S}).
\end{tikzcd}$$

The Postnikov structure on the category of grouplike $\mathbb{E}_\infty$-spaces (or equivalently, connective spectra) from Example 4.19 admits a generalization to more general **complete Grothendieck prestable ∞-categories** [Lur18, Definition C.1.2.12, Definition C.1.4.2].

Remark 4.20. Recall that a presentable ∞-category $\mathcal{V}$ is a complete Grothendieck prestable ∞-category if and only if the left adjoint $\mathcal{V} \longrightarrow \mathrm{Sp}(\mathcal{V})$ to its stabilization is fully faithful and exhibits $\mathcal{V} \simeq \mathrm{Sp}(\mathcal{V})^{\geq 0}$ as the connective part of a left complete t-structure on $\mathrm{Sp}(\mathcal{V})$ with the property that the coconnective part $\mathrm{Sp}(\mathcal{V})^{\leq 0} \subseteq \mathrm{Sp}(\mathcal{V})$ is closed under filtered colimits (see [Lur18, Proposition C.1.4.1] and its proof). In particular, this implies that $\mathcal{V}$ is an additive ∞-category and that the full subcategory of 0-truncated objects $\tau_{\leq 0}\mathcal{V}$ is an abelian category, equivalent to the heart $\mathrm{Sp}(\mathcal{V})^{\heartsuit}$. As usual, we will write $\pi_a X \in \mathrm{Sp}(\mathcal{V})^{\heartsuit}$ for the homotopy groups with respect to the t-structure. Finally, we will say that a map $f\colon A \longrightarrow B$ in $\mathcal{V}$ is **a-connective** if its cofiber $\mathrm{cof}(f) \in \mathcal{V} \subseteq \mathrm{Sp}(\mathcal{V})$ is $(a+1)$-connective (in other words, the $(a+1)$-fold suspension of an object in $\mathcal{V}$).

For later purposes, let us record the following properties of prestable ∞-categories:

Remark 4.21. Consider a square $F\colon \Delta^1 \times \Delta^1 \longrightarrow \mathcal{V}$ in a prestable ∞-category in which all maps induce isomorphisms on $\tau_{\leq 0}$. Then the square is Cartesian if and only if it is coCartesian in $\mathcal{V}$. Indeed, the condition that all maps induce isomorphisms in $\tau_{\leq 0}\mathcal{V} \simeq \mathrm{Sp}(\mathcal{V})^{\heartsuit}$ implies that the square is Cartesian in $\mathcal{V} \simeq \mathrm{Sp}(\mathcal{V})^{\geq 0}$ if and only if it is Cartesian in $\mathrm{Sp}(\mathcal{V})$, and likewise for being coCartesian. Since pullback and pushout squares in $\mathrm{Sp}(\mathcal{V})$ coincide, the result follows.

Lemma 4.22. *Let $\mathcal{V}$ be a SM prestable ∞-category such that the tensor product preserves finite colimits in each variable and let $n \geq 0$ and $a \geq 1$. For each $1 \leq i \leq n$, suppose we have an a-connective map $f_i\colon A_i \longrightarrow B_i$ and a 0-connective map $g_i\colon A_i \to A'_i$, and let $B'_i = A'_i \amalg_{A_i} B_i$ be their pushout. For the induced square*

$$\begin{tikzcd}
\bigotimes_{i=1}^n A_i \arrow[r] \arrow[d] & \bigotimes_{i=1}^n A'_i \arrow[d] \\
\bigotimes_{i=1}^n B_i \arrow[r] & \bigotimes_{i=1}^n B'_i
\end{tikzcd}$$

the natural map $Q \longrightarrow \bigotimes_{i=1}^n B'_i$ from the pushout is $(a+1)$-connective.

Proof. Observe that there are pushout squares

$$\begin{tikzcd}
\bigotimes_{i=1}^n A'_i \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & \bigotimes_{i=1}^n B'_i \arrow[d] \\
0 \arrow[r] & \mathrm{cof}\big(\bigotimes_{i=1}^n f_i\big) \arrow[r, "\theta"] & \mathrm{cof}\big(\bigotimes_{i=1}^n f'_i\big)
\end{tikzcd}$$

where $f'_i\colon A'_i \longrightarrow B'_i$ is the pushout of f_i. It suffices to that θ is $(a+1)$-connective. Using (a suspension of) [Lur17, Lemma 7.4.1.30] and the fact that f_i and f'_i have equivalent cofibers, θ fits into a commuting square

$$\begin{tikzcd}
\mathrm{cof}\big(\bigotimes_{i=1}^n f_i\big) \arrow[r, "\theta"] \arrow[d] & \mathrm{cof}\big(\bigotimes_{i=1}^n f'_i\big) \arrow[d] \\
\bigoplus_{i=1}^n \big(B_1 \otimes \dots \otimes \mathrm{cof}(f_i) \otimes \dots B_n\big) \arrow[r, "\theta'"] & \bigoplus_{i=1}^n \big(B'_1 \otimes \dots \otimes \mathrm{cof}(f_i) \otimes \dots B'_n\big)
\end{tikzcd}$$

where the vertical maps are $(2a)$-connective (hence in particular $(a+1)$-connective). It then remains to verify that θ' is $(a+1)$-connective, which follows from the fact that each $B_i \longrightarrow B'_i$ is 0-connective and $\mathrm{cof}(f_i)$ is $(a+1)$-connective. $\square$

Example/Proposition 4.23. *Let $\mathcal{V}$ be a complete Grothendieck prestable ∞-category and let us write $\mathrm{PoStr}^{\mathrm{cn}}(\mathcal{V}) \subseteq \mathrm{PoStr}(\mathcal{V})$ for the full sub-∞-category of objects equipped with a Postnikov structure (indexed over all $a \geq 0$) with the following properties:*

(a) *For each $a \geq 0$, the map $X \longrightarrow X_a$ exhibits $X_a \simeq \tau_{\leq a}X$ as the a-truncation of X.*

(b) For each $a \geq 1$, $\pi(K_a)$ is 0-truncated and all maps in the pullback square

$$\begin{tikzcd} X_a \arrow[r] \arrow[d] & \pi(K_a) \arrow[d, "0"] \\ X_{a-1} \arrow[r, "k_a"] & \Omega^\infty K_a \end{tikzcd} \tag{4.2}$$

induce isomorphisms on 0-truncations.

(c) For each $a \geq 1$, the object $K_a \in \mathcal{T}\mathcal{V}$ is contained in the connective part for the canonical t-orientation (Example 3.14).

Then the map $\mathrm{ev}_\infty\colon \mathrm{PoStr}^{\mathrm{cn}}(\mathcal{V}) \longrightarrow \mathcal{V}$ is an equivalence. If $\mathcal{V}$ is furthermore closed SM, then ev_∞ refines to an equivalence $\mathrm{PoStr}^{\mathrm{cn}}(\mathcal{V})^\otimes \longrightarrow \mathcal{V}^\otimes$ between $\mathcal{V}^\otimes$ and the full suboperad of $\mathrm{PoStr}(\mathcal{V})^\otimes$ spanned by the objects equipped with Postnikov structures satisfying the above properties.

In particular, each object $A \in \mathcal{V}$ comes with a unique Postnikov structure satisfying the above three conditions, and the resulting Postnikov structure on $\mathcal{V}$ carries a unique multiplicative structure if $\mathcal{V}$ is symmetric monoidal. Let us point out that by Remark 4.21, the square (4.2) is also a pushout square. Together with condition (a), this implies that for each $a \geq 1$, the square (4.2) can be identified with the (co)Cartesian square

$$\begin{tikzcd} \tau_{\leq a} A \arrow[r] \arrow[d] & \pi_0(A) \arrow[d] \\ \tau_{\leq a-1} A \arrow[r, "k_a"] & \pi_0(A) \oplus \Sigma^{a+1}\pi_a(A) \end{tikzcd} \tag{4.3}$$

since the cofiber of the left (and hence right) vertical map is $\pi_a(X)[a+1]$ and the right vertical map is the inclusion of a summand (since it admits a retraction). Taking algebras, we then obtain the following:

Example 4.24. Let $\mathcal{V}$ be a complete SM Grothendieck prestable ∞-category and let $\mathcal{O}$ be an ∞-operad. For example, one can take $\mathcal{V} = \mathrm{Sp}^{\geq 0}$ to be the ∞-category of connective spectra with the smash product. Combining Proposition 4.14 and Example/Proposition 4.23, we find that the Postnikov tower $A \longrightarrow \dots \longrightarrow \tau_{\leq 1} A \longrightarrow \tau_{\leq 0} A$ of an $\mathcal{O}$-algebra in $\mathcal{V}$ is part of a (multiplicative) Postnikov structure on $\mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$.

By Example 3.12, this means that each stage of the Postnikov tower fits into a pullback square of $\mathcal{O}$-algebras (4.3) where $\pi_0(A) \oplus \Sigma^{a+1}\pi_a(A)$ is the trivial square zero extension of $\pi_0(A)$ by the operadic module $\Sigma^{a+1}\pi_a(A)$. By specializing to $\mathcal{O} = \mathbb{E}_n$, this recovers [Lur17, Corollary 7.4.1.28].

The remainder of this section is devoted to a proof of Example/Proposition 4.23. To avoid repetition, let us prove the claim for a symmetric monoidal $\mathcal{V}$; the much simpler non-monoidal case can be proven in the same way, removing all references to the monoidal structure. Our proof will proceed by induction, where the inductive step relies on an analysis of the ∞-operad of pullback squares (4.2). To this end, let us introduce some auxiliary categories:

Construction 4.25. For each $a \geq 1$, let us denote by

$$\mathcal{E}_a := M(\kappa_a), \qquad \mathcal{E}_a^{\mathrm{cn}} := M(\kappa'_a)$$

the mapping simplices (Definition 4.4) of the functors $\kappa_a\colon \{a \to (a-1)\} \longrightarrow \mathcal{S}_*^{\mathrm{fin}}$, as in Construction 4.5, and $\kappa'_a\colon \{a \to (a-1)\} \longrightarrow \mathrm{Fin}_*$ sending $a \mapsto *$ and $(a-1) \mapsto S^0$. Note that

$\mathcal{E}_a^{\mathrm{cn}}$ is an ordinary category, since it is the unstraightening of a diagram of ordinary categories. In particular, Definition 4.4 provides a full description of $\mathcal{E}_a^{\mathrm{cn}}$, without need of specifying further homotopy coherences.

Now consider the chain of functors

$$a\colon * \xrightarrow{\ 0\ } \Delta^2 \xhookrightarrow{\ j\ } \mathcal{E}_a^{\mathrm{cn}} \xrightarrow{\ \tilde{i}\ } \mathcal{E}_a \tag{4.4}$$

where j is the inclusion of the full subcategory $\big\{a \to (a-1) \to *\big\}$ in $\mathcal{E}_a^{\mathrm{cn}}$ (using the description from Definition 4.4) and $\tilde{i}$ is the cobase change of the inclusion $i\colon \mathrm{Fin}_* \hookrightarrow \mathcal{S}_*^{\mathrm{fin}}$.

Definition 4.26. Let us denote by

$$\mathrm{Ext}_a^{\otimes} \hookrightarrow \mathrm{Fun}(\mathcal{E}_a, \mathcal{V})^{\otimes_{\mathrm{lev}}}, \qquad \mathrm{Ext}_a^{\mathrm{cn},\otimes} \hookrightarrow \mathrm{Fun}(\mathcal{E}_a^{\mathrm{cn}}, \mathcal{V})^{\otimes_{\mathrm{lev}}} \qquad \mathrm{Trun}_a^{\otimes} \hookrightarrow \mathrm{Fun}(\Delta^2, \mathcal{V})^{\otimes_{\mathrm{lev}}}$$

the three full sub-∞-operads defined as follows:

(1) $\mathrm{Trun}_a^{\otimes}$ is spanned by sequences $T_a \to T_{a-1} \to T_*$ with $T_a \in \tau_{\leq a}\mathcal{V}$ and exhibiting $T_{a-1} \simeq \tau_{\leq a-1}T_a$ and $T_* \simeq \tau_{\leq 0}T_a$.

(2) $\mathrm{Ext}_a^{\mathrm{cn},\otimes}$ is spanned by the diagrams $T\colon \mathcal{E}_a^{\mathrm{cn}} \longrightarrow \mathcal{V}$ such that:
   - (a) the restriction to $\{a \to (a-1) \to *\}$ is contained in $\mathrm{Trun}_a^{\otimes}$,
   - (b) the restriction along $\kappa_{a*}\colon \Delta^1 \times \{a, a-1\} \longrightarrow \mathcal{E}_a^{\mathrm{cn}}$ (Definition 4.4) is a pullback square in which all maps induce isomorphisms on 0-truncations,
   - (c') the restriction to Fin_* defines an $\mathbb{E}_\infty$-groupoid object (Definition 2.29).

(3) $\mathrm{Ext}_a^{\otimes}$ is spanned by the diagrams $T\colon \mathcal{E}_a \longrightarrow \mathcal{V}$ satisfying conditions (a) and (b) above, as well as:
   - (c) the restriction to $\mathcal{S}_*^{\mathrm{fin}}$ defines an object in $\mathcal{T}^{\geq 0}\mathcal{V} \subseteq \mathcal{T}\mathcal{V} = \mathrm{Exc}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V})$.

Lemma 4.27. *Let $\mathcal{V}$ be a complete Grothendieck prestable ∞-category and $T\colon \mathcal{E}_a^{\mathrm{cn}} \longrightarrow \mathcal{V}$ a diagram. Then the following are equivalent:*

(1) T is contained in $\mathrm{Ext}_a^{\mathrm{cn}}$.

(2) T is left Kan extended from its restriction to $\{a \to (a-1) \to *\}$, and this restriction is contained in Trun_a.

Proof. Recall that $j\colon \Delta^2 \hookrightarrow \mathcal{E}_a^{\mathrm{cn}}$ denotes the inclusion of the full subcategory on $a, (a-1)$ and $*$. For any diagram $F\colon \Delta^2 \longrightarrow \mathcal{V}$ of the form $F(a) \to F(a-1) \to F(*)$ and a finite pointed set S with basepoint s_0, the left Kan extension $j_!F(S)$ can be computed as the pushout

$$\begin{tikzcd} \bigoplus_{s\in S} F(a) \arrow[r] \arrow[d] & F(a) \arrow[d] \\ F(*) \oplus \bigoplus_{s\in S\setminus\{s_0\}} F(a-1) \arrow[r] & j_!F(S). \end{tikzcd}$$

Here the vertical functor is given by $F(a) \to F(*)$ on the summand labeled by the basepoint of S and by $F(a) \to F(a-1)$ on the summand labeled by each other point of S. Indeed, the above colimit coincides with the colimit of

$$\Delta^2 \times_{\mathcal{E}_a^{\mathrm{cn}}} \big(\mathcal{E}_a^{\mathrm{cn}}\big)_{/S} \longrightarrow \Delta^2 \xrightarrow{F} \mathcal{V}$$

where one can use the explicit description of the (ordinary) category $\mathcal{E}_a^{\mathrm{cn}}$ to identify the comma category. Using that $\mathcal{V}$ is a prestable (and in particular additive) ∞-category, this implies that

$$j_!F(S) = F(*) \oplus \bigoplus_{s\in S\setminus\{s_0\}} \mathrm{cof}\big(F(a) \to F(a-1)\big). \tag{4.5}$$

This formula shows that the restriction $j_!F\big|_{\mathrm{Fin}_*}$ is a Segal $\mathbb{E}_\infty$-groupoid and that the square

$$\begin{tikzcd} j_!F(a) \arrow[r] \arrow[d] & j_!F(*) \arrow[d] \\ j_!F(a-1) \arrow[r] & j_!F(S^0) \end{tikzcd}$$

is coCartesian, and hence also Cartesian since $\mathcal{V}$ is prestable. It follows that (2) implies (1). For the converse, if $T \in \mathcal{E}_a^{\mathrm{cn}}$, then the natural map $\epsilon\colon j_!j^*T \longrightarrow T$ is an equivalence at the objects $a, (a-1)$ and $*$ because j is fully faithful. In light of Remark 4.21, the map $F(a) \amalg_{F(a-1)} F(*) \longrightarrow F(S^0)$ is an equivalence so that ϵ is also an equivalence at S^0. Since both $j_!j^*F$ and F restrict to Segal $\mathbb{E}_\infty$-groupoids on Fin_*, it follows that ϵ is also an equivalence at all other $S \in \mathrm{Fin}_*$. □

Lemma 4.28. *Let $\mathcal{V}$ be a complete Grothendieck prestable ∞-category with a closed SM structure. Then restriction along the maps (4.4) induces equivalences of ∞-operads*

$$\mathrm{ev}_{(0,a)}\colon \mathrm{Ext}_a^{\otimes} \xrightarrow{\sim} \mathrm{Ext}_a^{\mathrm{cn},\otimes} \xrightarrow{\sim} \mathrm{Trun}_a^{\otimes} \xrightarrow{\sim} (\tau_{\leq a}\mathcal{V})^{\otimes}.$$

Proof. Restriction along the functors in (4.4) defines SM functors between the ∞-categories of $\mathcal{V}$-valued diagrams, with the levelwise tensor product, which preserve the full sub-∞-operads from Definition 4.26. We will check that each of the restriction functors is an equivalence.

Step 1. Using that $\tilde{i}\colon \mathcal{E}_a^{\mathrm{cn}} \longrightarrow \mathcal{E}_a$ is the pushout of the inclusion $i\colon \mathrm{Fin}_* \longrightarrow \mathcal{S}_*^{\mathrm{fin}}$, it follows that there is a pullback square of ∞-operads

$$\begin{tikzcd} \mathrm{Ext}_a^{\otimes} \arrow[r, "\tilde{i}^*"] \arrow[d] & \mathrm{Ext}_a^{\mathrm{cn},\otimes} \arrow[d] \\ (\mathcal{T}^{\geq 0}\mathcal{V})^{\otimes} \arrow[r, "i^*=\Omega^\infty"] & \mathrm{Gpd}_{\mathbb{E}_\infty}(\mathcal{V})^{\otimes}. \end{tikzcd}$$

Here the ∞-operads in the bottom row are full sub-∞-operads of $\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{V})^{\otimes_{\mathrm{lev}}}$ and $\mathrm{Fun}(\mathrm{Fin}_*, \mathcal{V})^{\otimes_{\mathrm{lev}}}$, respectively. Corollary 2.31 implies that these bottom two ∞-operads are in fact SM ∞-categories and that Ω^∞ is a SM equivalence between them. Consequently, $\tilde{i}^*$ is an equivalence of ∞-operads as well.

Step 2. Let $j\colon \Delta^2 \hookrightarrow \mathcal{E}_a^{\mathrm{cn}}$ denote the inclusion of the full subcategory $\{a \to (a-1) \to *\}$. To see that $j^*\colon \mathrm{Ext}_a^{\mathrm{cn},\otimes} \longrightarrow \mathrm{Trun}_a^{\otimes}$ is an equivalence of ∞-operads, we will show that it is essentially surjective and fully faithful, i.e. it induces equivalences on spaces of multi-morphisms [Bar18, Proposition 7.17]. Essential surjectivity follows from Lemma 4.27: indeed, each object $F \in \mathrm{Trun}_a$ arises as the restriction of its left Kan extension $j_!F \in \mathrm{Ext}_a^{\mathrm{cn}}$.

To check that j^* is fully faithful, let $T_1, \ldots, T_n$ and T_0 be objects in $\mathrm{Ext}_a^{\mathrm{cn}}$, and let us abbreviate $X_i = T_i(a)$ and $Y_i = T_i(a-1)$. The condition that $T_i \in \mathrm{Ext}_a^{\mathrm{cn}}$ then implies that X_i is a-truncated and that

$$Y_i \simeq \tau_{\leq a-1}X_i, \qquad T_i(*) = \pi_0 X_i, \qquad \mathrm{cof}\big(T_i(a) \to T_i(a-1)\big) \simeq \Sigma^{a+1}\pi_a X_i. \tag{4.6}$$

We now need to show that restriction along j induces an equivalence

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{E}_a^{\mathrm{cn}}, \mathcal{V})}\big(T_1 \otimes_{\mathrm{lev}} \cdots \otimes_{\mathrm{lev}} T_n, T_0\big) \longrightarrow \mathrm{Map}_{\mathrm{Fun}(\Delta^2, \mathcal{V})}\big(j^*T_1 \otimes_{\mathrm{lev}} \cdots \otimes_{\mathrm{lev}} j^*T_n, j^*T_0\big). \tag{4.7}$$

By adjunction, the map (4.7) is obtained by applying $\mathrm{Map}_{\mathrm{Fun}(\mathcal{E}_a^{\mathrm{cn}},\mathcal{V})}(-,T_0)$ to the counit map $\epsilon\colon j_!j^*\big(T_1\otimes_{\mathrm{lev}}\cdots\otimes_{\mathrm{lev}}T_n\big)\longrightarrow T_1\otimes_{\mathrm{lev}}\cdots\otimes_{\mathrm{lev}}T_n$. We claim that ϵ is given pointwise by an $(a+1)$-connective map in $\mathcal{V}$. This implies that (4.7) is an equivalence, because $(a+1)$-connective maps induce equivalences on $(a+1)$-truncations and T_0 takes values in $(a+1)$-truncated objects, by equations (4.5) and (4.6).

It thus remains to verify that each component of the natural transformation ϵ is $(a+1)$-connective. This is clear for the components of the natural transformation ϵ at the objects a, $(a-1)$ and $*$ in $\mathcal{E}_a^{\mathrm{cn}}$, where the counit is an equivalence (since j is fully faithful). We will prove by induction on $k\geq 0$ that the component of ϵ at the finite pointed set $\langle k\rangle$ with $k+1$ elements is $(a+1)$-connective. The case $k=0$ has already been treated, and for $k\geq 1$ consider the following commuting diagram

$$\begin{tikzcd}
X_1\otimes\cdots\otimes X_n \arrow[r]\arrow[d] & j_!j^*\big(T_1\otimes\cdots\otimes T_n\big)(\langle k-1\rangle) \arrow[r, "\epsilon"]\arrow[d] & T_1(\langle k-1\rangle)\otimes\cdots\otimes T_n(\langle k-1\rangle) \arrow[d]\\
Y_1\otimes\cdots\otimes Y_n \arrow[r] & j_!j^*\big(T_1\otimes\cdots\otimes T_n\big)(\langle k\rangle) \arrow[r, "\epsilon"] & T_1(\langle k\rangle)\otimes\cdots\otimes T_n(\langle k\rangle).
\end{tikzcd}$$

Formula (4.5) shows that the left square is a pushout, so that the pushout Q for the right cospan is equivalent to the pushout for the total cospan. Lemma 2.13 then implies that the natural map $Q\longrightarrow T_0(\langle k\rangle)\otimes\cdots\otimes T_n(\langle k\rangle)$ is $(a+1)$-connective. On the other hand, the map $j_!j^*\big(T_1\otimes_{\mathrm{lev}}\cdots\otimes_{\mathrm{lev}}T_n\big)(\langle k\rangle)\longrightarrow Q$ is the pushout of the counit map ϵ at $\langle k-1\rangle$, which was $(a+1)$-connective by inductive hypothesis. We conclude that ϵ is a natural transformation given at each object by an $(a+1)$-connective map, as desired.

Step 3. Finally, let us show that $\mathrm{ev}_a\colon \mathrm{Trun}_a^{\otimes}\longrightarrow(\tau_{\leq a}\mathcal{V})^{\otimes}$ is an equivalence of ∞-operads. Note that the objects of Trun_a are simply given by sequences $\sigma=\big[X\to\tau_{\leq a-1}X\to\tau_{\leq 0}X\big]$ with $X\in\tau_{\leq a}\mathcal{V}$. In particular, ev_a is essentially surjective. To see that it is a fully faithful map of ∞-operads, i.e. that each

$$\mathrm{Map}_{\mathrm{Trun}_a^{\otimes}}\big(\sigma_1,\dots,\sigma_n;\sigma_0\big)\longrightarrow \mathrm{Map}_{(\tau_{\leq a}\mathcal{V})^{\otimes}}\big(\mathrm{ev}_a(\sigma_1),\dots,\mathrm{ev}_a(\sigma_n);\mathrm{ev}_a(\sigma_0)\big)$$

is an equivalence, it suffices to verify the following: for each diagram in $\tau_{\leq a}\mathcal{V}$ of the form

$$\begin{tikzcd}
X_1\otimes\cdots\otimes X_n \arrow[r]\arrow[d] & (\tau_{\leq a-1}X_1)\otimes\cdots\otimes(\tau_{\leq a-1}X_n) \arrow[r]\arrow[d, dotted] & (\tau_{\leq 0}X_1)\otimes\cdots\otimes(\tau_{\leq 0}X_n) \arrow[d, dotted]\\
X_0 \arrow[r] & \tau_{\leq a-1}X_0 \arrow[r] & \tau_{\leq 0}X_0
\end{tikzcd}$$

there exists a contractible space of dotted extensions, as indicated. This follows from the fact that the first horizontal map is a-connective and the second is 1-connective. □

Proof of Proposition 4.23. Let us inductively define a tower of ∞-operads $\mathrm{PoStr}_{\leq a}^{\mathrm{cn}}(\mathcal{V})^{\otimes}$ by setting $\mathrm{PoStr}_{\leq 0}^{\mathrm{cn}}(\mathcal{V})^{\otimes}=(\tau_{\leq 0}\mathcal{V})^{\otimes}$ and taking pullbacks

(4.8)
$$\begin{tikzcd}
\mathrm{PoStr}_{\leq a}^{\mathrm{cn}}(\mathcal{V})^{\otimes} \arrow[r]\arrow[d] & \mathrm{Ext}_a^{\otimes} \arrow[r, "\mathrm{ev}_a"]\arrow[d, "\mathrm{ev}_{a-1}"] & (\tau_{\leq a}\mathcal{V})^{\otimes}\\
\mathrm{PoStr}_{\leq a-1}^{\mathrm{cn}}(\mathcal{V})^{\otimes} \arrow[r, "\mathrm{ev}_{a-1}"] & (\tau_{\leq a-1}\mathcal{V})^{\otimes}. &
\end{tikzcd}$$

Note that each $\mathrm{ev}_a\colon \mathrm{PoStr}^{\mathrm{cn}}_{\leq a}(\mathcal{V})^{\otimes} \longrightarrow (\tau_{\leq a}\mathcal{V})^{\otimes}$ is an equivalence of ∞-operads: by inductive hypothesis the first top horizontal arrow in (4.8) will be an equivalence, and the second map is an equivalence by Lemma 4.27. Furthermore, step 3 of the proof of Lemma 4.27 shows that the map of ∞-operads $(\tau_{\leq a}\mathcal{V})^{\otimes} \simeq \mathrm{Ext}^{\otimes}_a \longrightarrow (\tau_{\leq a-1}\mathcal{V})^{\otimes}$ is given by the localization $\tau_{\leq a-1}\colon \tau_{\leq a}\mathcal{V} \longrightarrow \tau_{\leq a-1}\mathcal{V}$ with its canonical SM structure. We thus obtain a natural diagram

$$\begin{tikzcd}
\mathrm{PoStr}^{\mathrm{cn}}(\mathcal{V})^{\otimes} \arrow[r] \arrow[d, "\mathrm{ev}_{\infty}"'] & \dots \arrow[r] & \mathrm{PoStr}^{\mathrm{cn}}_{\leq 2}(\mathcal{V})^{\otimes} \arrow[r] \arrow[d, "\sim"', "\mathrm{ev}_2"] & \mathrm{PoStr}^{\mathrm{cn}}_{\leq 1}(\mathcal{V})^{\otimes} \arrow[r] \arrow[d, "\sim"', "\mathrm{ev}_1"] & \mathrm{PoStr}^{\mathrm{cn}}_{\leq 0}(\mathcal{V})^{\otimes} \arrow[d, "\sim"', "\mathrm{ev}_0"] \\
\mathcal{V}^{\otimes} \arrow[r] & \dots \arrow[r] & (\tau_{\leq 2}\mathcal{V})^{\otimes} \arrow[r, "\tau_{\leq 1}"] & (\tau_{\leq 1}\mathcal{V})^{\otimes} \arrow[r, "\tau_{\leq 0}"] & (\tau_{\leq 0}\mathcal{V})^{\otimes}
\end{tikzcd}$$

Since $\mathcal{V}$ was a complete Grothendieck prestable ∞-category (so that Postnikov towers are convergent), the bottom row exhibits $\mathcal{V}^{\otimes}$ as the limit of the $(\tau_{\leq a})^{\otimes}$. Using this and unraveling the definitions (cf. Construction 4.5), we then have an equivalence $\mathrm{PoStr}^{\mathrm{cn}}(\mathcal{V})^{\otimes} \simeq \mathcal{V}^{\otimes} \times_{\lim_a (\tau_{\leq a}\mathcal{V})^{\otimes}} \lim_a \mathrm{PoStr}^{\mathrm{cn}}_{\leq a}(\mathcal{V})^{\otimes}$. Since this is the pullback of a span consisting of two equivalences, we conclude that $\mathrm{ev}_{\infty}\colon \mathrm{PoStr}^{\mathrm{cn}}(\mathcal{V})^{\otimes} \longrightarrow \mathcal{V}^{\otimes}$ is an equivalence, as desired. $\square$

5. Postnikov structures on enriched categories

In the previous section we have seen how multiplicative Postnikov structures give rise to multiplicative Postnikov structures on ∞-categories of algebras over operads (Proposition 4.14). The purpose of this section is to prove that similarly, a multiplicative Postnikov structure on a symmetric monoidal ∞-category $\mathcal{V}$ induces a multiplicative Postnikov structure on the ∞-category of $\mathcal{V}$-enriched ∞-categories.

5.1. **Recollections on enriched ∞-categories.** Let us briefly recall some elements of the theory of enriched ∞-categories developed by Gepner–Haugseng [GH15].

Definition 5.1. For each space X, let us write $\mathcal{O}_X$ for the universal $(X \times X)$-coloured (symmetric) ∞-operad receiving a map from $\Delta^{\mathrm{op}}_X \longrightarrow \Delta^{\mathrm{op}} \longrightarrow \mathrm{Fin}_*$, where Δ^{op}_X is the generalized nonsymmetric ∞-operad from [GH15, Definition 4.1.1]. By [GH15, Corollary 3.7.8, Corollary 4.2.8], one can model $\mathcal{O}_X$ explicitly by the symmetrization of the simplicial operad from [GH15, Definition 4.2.4].

When the space X is a point, one recovers the associative operad $\mathcal{O}_* = \mathbb{E}_1$. The operads $\mathcal{O}_X$ depend functorially on the space X, so that we obtain a functor

$$\mathcal{O}_{(-)}\colon \mathcal{S} \longrightarrow (\mathrm{Op}_{\infty})_{/\mathbb{E}_1} \longrightarrow \mathrm{Op}_{\infty}\,.$$

If $\mathcal{V}$ is a monoidal category, then an $\mathcal{O}_X$-algebra in $\mathcal{V}$ can be informally described as follows: an algebra consists of objects $\mathrm{Map}(x, y) \in \mathcal{V}$, depending functorially on $(x, y) \in X \times X$, together with composition operations satisfying obvious associativity conditions.

Definition 5.2. We will refer to the ∞-category $\mathrm{Alg}_{\mathcal{O}_X}(\mathcal{V})$ as the ∞-category of **$\mathcal{V}$-enriched categorical algebras with space of objects** X. These ∞-categories depend (contravariantly) functorially on X and we define the ∞-category of **categorical algebras**

$$\mathrm{Ob}\colon \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) = \textstyle\int_{X \in \mathcal{S}} \mathrm{Alg}_{\mathcal{O}_X}(\mathcal{V}) \longrightarrow \mathcal{S} \tag{5.1}$$

to be the domain of the corresponding Cartesian fibration [GH15, Definition 4.3.1]. If $\mathcal{V}$ is a presentable monoidal ∞-category, then $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$ is presentable as well [GH15, Proposition 4.3.5].

For later purposes, we will mainly be interested in a refinement of this construction for *symmetric* monoidal $\mathcal{V}$.

Proposition 5.3. *Let $\mathcal{S}^{\times} \longrightarrow \mathrm{Fin}_*$ denote the Cartesian ∞-operad associated to the ∞-category of spaces. Then there exists a natural functor*

$$\mathrm{Alg}_{\mathrm{Cat}}\colon \mathrm{SMCat}_{\infty}^{\mathrm{lax}} \longrightarrow \mathrm{SMCat}_{\infty/\mathcal{S}^{\times}}^{\mathrm{lax,big}}$$

that sends each SM ∞-category $\mathcal{V}$ to the ∞-category $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$ of categorical algebras, together with a SM structure such that the tensor product of categorical algebras with spaces of objects X and Y is a categorical algebra with space of objects $X \times Y$.

Let us point out that the results from [GH15] only provide functoriality of $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$ with respect to (strong) SM functors in $\mathcal{V}$. Since the proof of Proposition 5.3 is rather technical, we will postpone it to Appendix A and instead record two further consequences (which are also proven in Appendix A). First, note that Proposition 5.3 asserts in particular that $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$ inherits a symmetric monoidal structure from $\mathcal{V}$, whose underlying tensor product functor can be identified as follows:

Lemma 5.4. *Let $\mathcal{V}$ be a SM ∞-category. Then the tensor product*

$$\begin{tikzcd} \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \times \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \arrow[r, "\otimes"] \arrow[d] & \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \arrow[d] \\ \mathcal{S} \times \mathcal{S} \arrow[r, "\times"] & \mathcal{S} \end{tikzcd}$$

arises as the unstraightening of the natural transformation of functors $\mathcal{S}^{\mathrm{op}} \times \mathcal{S}^{\mathrm{op}} \longrightarrow \mathrm{Cat}$ given at (X, Y) by

$$\mathrm{Alg}_{\mathcal{O}_X}(\mathcal{V}) \times \mathrm{Alg}_{\mathcal{O}_Y}(\mathcal{V}) \longrightarrow \mathrm{Alg}_{\mathcal{O}_{X\times Y}}(\mathcal{V}) \times \mathrm{Alg}_{\mathcal{O}_{X\times Y}}(\mathcal{V}) \xrightarrow{\otimes} \mathrm{Alg}_{\mathcal{O}_{X\times Y}}(\mathcal{V}) \tag{5.2}$$

where the first functor restricts along the maps $\mathcal{O}_X \leftarrow \mathcal{O}_{X\times Y} \rightarrow \mathcal{O}_Y$ and the second functor arises from the SM structure on algebras in $\mathcal{V}$ [Lur17, Example 3.2.4.4].

Informally, this means that given two categorical algebras $\mathbb{C}$, $\mathbb{D}$ with spaces of objects X, Y, their tensor product $\mathbb{C} \otimes \mathbb{D}$ has space of objects $X \times Y$ and mapping objects

$$\mathrm{Map}_{\mathbb{C}\otimes\mathbb{D}}\big((x_0, y_0), (x_1, y_1)\big) = \mathrm{Map}_{\mathbb{C}}(x_0, x_1) \otimes \mathrm{Map}_{\mathbb{D}}(y_0, y_1).$$

In particular, the unit is given by the categorical algebra $[0]_{1_{\mathcal{V}}}$ with a single object $*$ and with $1_{\mathcal{V}}$ as endomorphisms.

Remark 5.5. If $\mathcal{V}$ is a SM ∞-category, then the natural map (5.2) can also be identified with the composite map $\mathrm{Alg}_{\mathcal{O}_X}(\mathcal{V}) \times \mathrm{Alg}_{\mathcal{O}_Y}(\mathcal{V}) \longrightarrow \mathrm{Alg}_{\mathcal{O}_{X\times Y}}(\mathcal{V} \times \mathcal{V}) \longrightarrow \mathrm{Alg}_{\mathcal{O}_{X\times Y}}(\mathcal{V})$ where the first map is the "exterior product" from [GH15, Proposition 3.6.14, Proposition 4.3.11] and the second map is the image under $\mathrm{Alg}_{\mathcal{O}_{X\times Y}}(-)$ of the lax monoidal functor $\otimes_{\mathcal{V}}\colon \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V}$. Consequently, the functor $\otimes\colon \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \times \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \longrightarrow \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$ from Proposition 5.3 is naturally equivalent to the tensor product functor from [GH15, Corollary 4.3.13]. In particular, we find that $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$ is a presentable SM ∞-category if $\mathcal{V}$ is a presentable SM ∞-category, i.e. the monoidal structure is **closed** [GH15, Corollary 4.3.16].

Proposition 5.6. *For each ∞-category $\mathcal{I}$, there is a commuting square depending functorially on $\mathcal{I}$*

$$\begin{tikzcd} \mathrm{SMCat}^{\mathrm{lax}} \arrow[r, "\mathrm{Alg}_{\mathrm{Cat}}"] \arrow[d, "\mathrm{Fun}(\mathcal{I},-)"'] & \mathrm{SMCat}^{\mathrm{lax,big}}_{/\mathcal{S}^\times} \arrow[d, "\mathrm{Fun}(\mathcal{I},-)\times_{\mathrm{Fun}(\mathcal{I},\mathcal{S}^\times)}\mathcal{S}^\times"] \\ \mathrm{SMCat}^{\mathrm{lax}} \arrow[r, "\mathrm{Alg}_{\mathrm{Cat}}"'] & \mathrm{SMCat}^{\mathrm{lax,big}}_{/\mathcal{S}^\times} \end{tikzcd}$$

where the vertical functors use the levelwise tensor product from Construction 2.6.

In other words, for each ∞-category $\mathcal{I}$ there is a natural monoidal equivalence

$$\mathrm{Alg}_{\mathrm{Cat}}(\mathrm{Fun}(\mathcal{I},\mathcal{V})) \simeq \mathrm{Fun}(\mathcal{I}, \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})) \times_{\mathrm{Fun}(\mathcal{I},\mathcal{S})} \mathcal{S}.$$

When $\mathcal{I}$ is weakly contractible, the constant diagram functor $\mathcal{S} \longrightarrow \mathrm{Fun}(\mathcal{I},\mathcal{S})$ is fully faithful, so that we can rephrase this as follows: there is a natural (SM) fully faithful embedding $\mathrm{Alg}_{\mathrm{Cat}}(\mathrm{Fun}(\mathcal{I},\mathcal{V})) \hookrightarrow \mathrm{Fun}(\mathcal{I}, \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}))$ whose essential image consists of $\mathcal{I}$-diagrams of categorical algebras whose underlying diagram of objects is constant.

For any presentable monoidal ∞-category $\mathcal{V}$, we then define the ∞-category of **$\mathcal{V}$-enriched ∞-categories** $\mathrm{Cat}(\mathcal{V})$ to be the full subcategory $\mathrm{Cat}(\mathcal{V}) \subseteq \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$ of **complete** categorical algebras. More precisely, there is a functor

$$J[-]\colon \Delta \longrightarrow \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$$

sending $[n]$ to the categorical algebra with object set $\{0,\dots,n\}$, all mapping objects being $1_{\mathcal{V}}$ and all compositions being equivalences. We will abbreviate $J = J[1]$. Every categorical algebra $\mathbb{C}$ then defines a simplicial space

$$\Delta^{\mathrm{op}} \longrightarrow \mathcal{S};\ \ [n] \longmapsto \mathrm{Map}_{\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})}\big(J[n], \mathbb{C}\big).$$

This simplicial space is a Segal groupoid [GH15, Corollary 5.2.7] and $\mathbb{C}$ is defined to be complete if this Segal groupoid is essentially constant. Note that the above Segal space only depends on the **underlying space-valued categorical algebra**, i.e. the categorical algebra in $\mathcal{S}$ obtained by applying the lax monoidal functor $\mathrm{Map}(1_{\mathcal{V}}, -)\colon \mathcal{V} \longrightarrow \mathcal{S}$ to all mapping objects [GH15, Proposition 5.1.11]. Furthermore, the space $\mathrm{Map}(J[n], \mathbb{C}) \subseteq \mathrm{Map}([n]_{1_{\mathcal{V}}}, \mathbb{C})$ is a union of path components in the space of n-composable sequences of arrows in $\mathbb{C}$ [GH15, Proposition 5.1.17].

The inclusion of $\mathcal{V}$-enriched ∞-categories into categorical algebras is part of an adjoint pair

$$\begin{tikzcd} \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \arrow[r, shift left, "(-)^\wedge"] \arrow[r, phantom, "\perp"] & \mathrm{Cat}(\mathcal{V}) \arrow[l, hook, shift left] \end{tikzcd}$$

whose left adjoint is called **completion** [GH15, Theorem 5.6.6]. When $\mathcal{V}$ is presentable symmetric monoidal, this is a symmetric monoidal localization [GH15, Proposition 5.7.14] (using Remark 5.5). Finally, let us recall that the completion functor realizes $\mathrm{Cat}(\mathcal{V})$ as the localization of $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$ at the **Dwyer–Kan (DK) equivalences**, i.e. the fully faithful and essentially surjective functors in the following sense:

Definition 5.7. We will say that a map of categorical algebras $f\colon \mathbb{C} \longrightarrow \mathbb{D}$ is:

(1) **fully faithful** if for every two objects $x, y \in \mathrm{Ob}(\mathbb{C})$, the map
$$f\colon \mathrm{Map}_{\mathbb{C}}(x, y) \longrightarrow \mathrm{Map}_{\mathbb{D}}\big(f(x), f(y)\big)$$
is an equivalence in $\mathcal{V}$. Equivalently, f is a Cartesian arrow for the Cartesian fibration (5.1).
(2) **essentially surjective** if the map
$$\mathrm{Map}(\{0\}, \mathbb{C}) \times_{\mathrm{Map}(\{0\},\mathbb{D})} \mathrm{Map}(J, \mathbb{D}) \longrightarrow \mathrm{Map}(\{1\}, \mathbb{D})$$
is surjective on π_0. Here the mapping spaces are taken in the ∞-category $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$.
(3) an **isofibration** if the induced map
$$\mathrm{Map}(J, \mathbb{C}) \longrightarrow \mathrm{Map}(J, \mathbb{D}) \times_{\mathrm{Map}(\{1\},\mathbb{D})} \mathrm{Map}(\{1\}, \mathbb{C})$$
is surjective on π_0.

Remark 5.8. Let $F\colon \mathcal{V} \longrightarrow \mathcal{W}$ be a SM left adjoint functor between presentable SM ∞-categories, with (lax SM) right adjoint G. Then $\mathrm{Alg}_{\mathrm{Cat}}(G)\colon \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{W}) \longrightarrow \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$ preserves underlying space-valued categorical algebras. Indeed, this follows from the equivalence of lax SM functors $\mathrm{Map}_{\mathcal{W}}(1_{\mathcal{W}}, -) \simeq \mathrm{Map}_{\mathcal{V}}(1_{\mathcal{V}}, G(-))$, which is right adjoint to the equivalence of SM functors between $\mathcal{S} \xrightarrow{1_{\mathcal{V}}\otimes -} \mathcal{V} \xrightarrow{F} \mathcal{W}$ and $1_{\mathcal{W}} \otimes -\colon \mathcal{S} \longrightarrow \mathcal{W}$, where $1_{\mathcal{W}} \otimes -$ denotes the unique SM functor preserving colimits (and likewise for $\mathcal{V}$). In particular, the right adjoint $\mathrm{Alg}_{\mathrm{Cat}}(G)\colon \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{W}) \longrightarrow \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$ detects completeness of categorical algebras, as well as essential surjectivity and being an isofibration for maps between these.

5.2. **The cube and tower lemmas.** Throughout, let $\mathcal{V}$ be a monoidal ∞-category. The purpose of this section is to record two kinds of ('homotopy') limits of categorical algebras that are preserved by the completion functor $(-)^{\wedge}\colon \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \longrightarrow \mathrm{Cat}(\mathcal{V})$. The results and arguments are very analogous to the usual way of computing homotopy limits of categories in terms of the canonical model structure on categories.

Lemma 5.9. *Consider a commutative square of categorical algebras*

$$\begin{tikzcd} \mathbb{C}' \arrow[r, "g'"] \arrow[d, "p"'] & \mathbb{D}' \arrow[d, "q"] \\ \mathbb{C} \arrow[r, "g"'] & \mathbb{D} \end{tikzcd} \tag{5.3}$$

such that g' is essentially surjective, g is fully faithful and p is an isofibration. Then
$$\mathrm{Map}(\{0\}, \mathbb{C}') \underset{\mathrm{Map}(\{0\},\mathbb{D}')}{\times} \mathrm{Map}(J, \mathbb{D}') \longrightarrow \mathrm{Map}(\{0\}, \mathbb{C}) \underset{\mathrm{Map}(\{0\},\mathbb{D})}{\times} \mathrm{Map}(J, \mathbb{D}) \underset{\mathrm{Map}(\{1\},\mathbb{D})}{\times} \mathrm{Map}(\{1\}, \mathbb{D}')$$
is surjective on path components.

Informally, this means that for any object $d \in \mathbb{D}'$, each lift-up-to-equivalence of $q(d)$ to $\mathbb{C}$ refines to a lift-up-to-equivalence of d to $\mathbb{C}'$.

Proof. By [GH15, Proposition 5.1.11], we may as well assume that $\mathcal{V} = \mathcal{S}$. Explicitly, suppose that we are given objects $c \in \mathbb{C}$ and $d' \in \mathbb{D}'$ together with an equivalence $\alpha\colon g(c) \xrightarrow{\simeq} d = q(d')$ in $\mathbb{D}$, that is, a map from J. We then need to find an object $c' \in \mathbb{C}'$ lying over c and an equivalence $\alpha'\colon g'(c') \xrightarrow{\simeq} d'$ in $\mathbb{D}'$ lying over α.

To begin, g' being essentially surjective provides an object $t' \in \mathbb{C}'$ and an equivalence $\gamma'\colon g'(t') \xrightarrow{\simeq} d'$ in $\mathbb{D}'$. One can then complete $q(\gamma')$ and α to a commutative triangle

$$\begin{tikzcd} & q(g'(t')) \arrow[dr, "q(\gamma')"] & \\ g(c) \arrow[ur, "\delta"] \arrow[rr, "\alpha"] & & d \end{tikzcd} \tag{5.4}$$

in $\mathbb{D}$ for some equivalence $\delta\colon g(c) \longrightarrow q(g'(t'))$. Using the commuting square (5.3), we can identify $q(g'(t')) \simeq g(p(t'))$ in the space of objects of $\mathbb{D}$. Because g is fully-faithful, every map from $g(c) \longrightarrow g(p(t'))$ lifts to a unique map $c \longrightarrow p(t')$ in $\mathbb{C}$, so that there exists an equivalence $\varepsilon\colon c \longrightarrow p(t')$ lying over δ. Since p is an isofibration we may lift ε to an equivalence $\varepsilon'\colon c' \longrightarrow t'$ for some $c' \in \mathbb{C}'$ lying over c. We may then complete $g'(\varepsilon')$ and γ' to a commutative diagram

$$\begin{tikzcd} & g'(t') \arrow[dr, "\gamma'"] & \\ g'(c') \arrow[ur, "g'(\varepsilon')"] \arrow[rr, "\alpha'"] & & d' \end{tikzcd} \tag{5.5}$$

for some equivalence $\alpha'\colon g'(c') \longrightarrow d'$ in $\mathbb{D}'$, since $\mathrm{Map}(J[-], \mathbb{C})$ is a Segal groupoid object. Because the image of triangle (5.5) under $q\colon \mathbb{D}' \longrightarrow \mathbb{D}$ agrees with triangle (5.4) on the inner horn, it follows that $q(\alpha')$ is homotopic to α. This yields the desired data of c' and $\alpha'\colon g'(c') \longrightarrow d'$ so that the proof is complete. ☐

Lemma 5.10 (Cube lemma). *Consider a map of Cartesian squares in* $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$

$$\begin{tikzcd} \mathbb{P} \arrow[r] \arrow[d] & \mathbb{C}' \arrow[d, "p"] \\ \mathbb{C}'' \arrow[r, "h"'] & \mathbb{C} \end{tikzcd} \quad \overset{\begin{pmatrix} f & g' \\ g'' & g \end{pmatrix}}{\Longrightarrow} \quad \begin{tikzcd} \mathbb{Q} \arrow[r] \arrow[d] & \mathbb{D}' \arrow[d, "q"] \\ \mathbb{D}'' \arrow[r, "k"'] & \mathbb{D} \end{tikzcd} \tag{5.6}$$

such that p is an isofibration. If the components $g\colon \mathbb{C} \longrightarrow \mathbb{D}$, $g'\colon \mathbb{C}' \longrightarrow \mathbb{D}'$ and $g''\colon \mathbb{C}'' \longrightarrow \mathbb{D}''$ are Dwyer–Kan equivalences, then the same holds for $f\colon \mathbb{P} \longrightarrow \mathbb{Q}$.

Corollary 5.11. *The completion functor* $(-)^\wedge\colon \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \longrightarrow \mathrm{Cat}(\mathcal{V})$ *sends pullback squares with one leg being an isofibration to pullback squares.*

Proof. Apply Lemma 5.10 to the case where the maps g, g' and g'' exhibit $\mathbb{D}, \mathbb{D}'$ and $\mathbb{D}''$ as the completions of $\mathbb{C}, \mathbb{C}'$ and $\mathbb{C}''$ respectively (in this case $\mathbb{Q} \simeq \mathbb{D}' \times_{\mathbb{D}} \mathbb{D}''$ is automatically complete). ☐

Proof of Lemma 5.10. To show that f is fully faithful, let $x, y \in \mathbb{P}$ be two objects, and consider the induced map of squares

$$\begin{tikzcd} \mathrm{Map}_{\mathbb{P}}(x,y) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathbb{C}'}(x,y) \arrow[d, "p_*"] \\ \mathrm{Map}_{\mathbb{C}''}(x,y) \arrow[r] & \mathrm{Map}_{\mathbb{C}}(x,y) \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} \mathrm{Map}_{\mathbb{Q}}(x,y) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathbb{D}'}(x,y) \arrow[d, "q_*"] \\ \mathrm{Map}_{\mathbb{D}''}(x,y) \arrow[r] & \mathrm{Map}_{\mathbb{D}}(x,y). \end{tikzcd}$$

Both squares are Cartesian in $\mathcal{V}$ and by assumption the three maps associated to g, g' and g'' are equivalences, so that the map $f_*\colon \mathrm{Map}_{\mathbb{P}}(x,y) \longrightarrow \mathrm{Map}_{\mathbb{Q}}(x,y)$ is an equivalence as well.

Let us now show that f is essentially surjective as a map of categorical algebras. Essential surjectivity is detected on the level of the underlying space-valued categorical algebras [GH15, Proposition 5.1.11]. We may hence assume that $\mathcal{V} = \mathcal{S}$. Let $y \in \mathbb{Q}$ be an object and let $d' \in \mathbb{D}', d \in \mathbb{D}, d'' \in \mathbb{D}''$ be its images. Since $g''\colon \mathbb{C}'' \longrightarrow \mathbb{D}''$ is essentially surjective there exists an object $c'' \in \mathbb{C}''$ and an equivalence $\alpha''\colon g''(c'') \xrightarrow{\simeq} d''$ in $\mathbb{D}''$. Let $c := h(c'') \in \mathbb{C}$. Applying Lemma 5.9 to the image

$$\alpha\colon g(c) \simeq k(g''(c'')) \xrightarrow{\simeq} k(d'') \simeq d \simeq q(d')$$

of α'' in $\mathbb{D}$ we deduce the existence of an object $c' \in \mathbb{C}'$ lying over c and an equivalence $\alpha'\colon g'(c') \longrightarrow d'$ in $\mathbb{D}'$ lying over α. The compatible triple (c, c', c'') now determines an object $x \in \mathbb{P}$ while the compatible triple $(\alpha, \alpha', \alpha'')$ determines an equivalence $g(x) \xrightarrow{\simeq} y$ in $\mathbb{Q}$. $\square$

Lemma 5.12 (Tower lemma)**.** *Consider a natural transformation between limit cones in* $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$

$$\begin{tikzcd}
\mathbb{P} \arrow[r] \arrow[d, "f"'] & \dots \arrow[r] & \mathbb{C}_2 \arrow[r, "p_2"] \arrow[d, "g_2"'] & \mathbb{C}_1 \arrow[r, "p_1"] \arrow[d, "g_1"'] & \mathbb{C}_0 \arrow[d, "g_0"] \\
\mathbb{Q} \arrow[r] & \dots \arrow[r] & \mathbb{D}_2 \arrow[r, "q_2"'] & \mathbb{D}_1 \arrow[r, "q_1"'] & \mathbb{D}_0 .
\end{tikzcd}$$

Suppose that all p_i for $i \geq 1$ are isofibrations and all g_i for $i \geq 0$ are Dwyer–Kan equivalences. Then f is a Dwyer–Kan equivalence as well.

Corollary 5.13. *The completion functor* $(-)^\wedge\colon \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \longrightarrow \mathrm{Cat}(\mathcal{V})$ *sends limits of towers of isofibrations to limits.*

Proof. Apply Lemma 5.12 to the case where the maps g_i exhibit $\mathbb{D}_i$ as the completion of $\mathbb{C}_i$ (in which case $\mathbb{Q}$ is automatically complete). $\square$

Proof of Lemma 5.12. To show that f is fully faithful, let $x, y \in \mathbb{P}$ be two objects, and consider the induced map of towers

$$\begin{tikzcd}
\mathrm{Map}_{\mathbb{P}}(x,y) \arrow[r] \arrow[d, "f_*"'] & \dots \arrow[r] & \mathrm{Map}_{\mathbb{C}_1}(x,y) \arrow[r] \arrow[d, "(g_1)_*"'] & \mathrm{Map}_{\mathbb{C}_0}(x,y) \arrow[d, "(g_0)_*"] . \\
\mathrm{Map}_{\mathbb{Q}}(x,y) \arrow[r] & \dots \arrow[r] & \mathrm{Map}_{\mathbb{D}_1}(x,y) \arrow[r] & \mathrm{Map}_{\mathbb{D}_0}(x,y)
\end{tikzcd}$$

Then both towers are limit towers in $\mathcal{V}$ and by assumption the $(g_i)_*$ are equivalences, so that the map $f_*\colon \mathrm{Map}_{\mathbb{P}}(x,y) \longrightarrow \mathrm{Map}_{\mathbb{Q}}(x,y)$ is an equivalence as well.

Let us now show that f is essentially surjective as a map of categorical algebras. We may again assume that $\mathcal{V} = \mathcal{S}$ [GH15, Proposition 5.1.11]. Let $y \in \mathbb{Q}$ be an object and let $d_i \in \mathbb{D}_i$ be its images. Since $g_0\colon \mathbb{C}_0 \longrightarrow \mathbb{D}_0$ is essentially surjective, there exists an object $c_0 \in \mathbb{C}_0$ and an equivalence $\alpha_0\colon g_0(c_0) \xrightarrow{\simeq} d_0$ in $\mathbb{D}_0$. Applying Lemma 5.9 to α_0 and $d_1 \in \mathbb{D}_1$, we deduce the existence of an object $c_1 \in \mathbb{C}_1$ lying over c_0 and an equivalence $\alpha_1\colon g_1(c_1) \longrightarrow d_1$ in $\mathbb{D}_1$ lying over α_1. Proceeding inductively, we obtain compatible sequences of objects $c_i \in \mathbb{C}_i$ and equivalences $\alpha_i\colon g_i(c_i) \longrightarrow d_i$. These determine an object $x \in \mathbb{P}$ and an equivalence $g(x) \xrightarrow{\simeq} y$ in $\mathbb{Q}$. $\square$

5.3. **Postnikov structures on enriched ∞-categories.** We now turn to our main result, providing Postnikov structures on $\mathcal{V}$-enriched ∞-categories from (certain) multiplicative Postnikov structures on $\mathcal{V}$.

Definition 5.14. Let $\mathcal{V}$ be a SM ∞-category. We will say that a map $f\colon X \longrightarrow Y$ in $\mathcal{V}$ is an **external π_0-isomorphism** if the induced map of spaces $\mathrm{Map}_{\mathcal{V}}(1_{\mathcal{V}}, X) \longrightarrow \mathrm{Map}_{\mathcal{V}}(1_{\mathcal{V}}, Y)$ induces an isomorphism on π_0.

We will say that a Postnikov structure on an object $T\colon \mathcal{E} \longrightarrow \mathcal{V}$ is **externally π_0-constant** if it sends every map in $\mathcal{E}$ to an external π_0-isomorphism in $\mathcal{V}$. A Postnikov structure $\Phi\colon \mathcal{V} \longrightarrow \mathrm{Fun}(\mathcal{E}, \mathcal{V})$ on $\mathcal{V}$ is externally π_0-constant if it sends each object $X \in \mathcal{V}$ to an externally π_0-constant Postnikov structure on X.

Remark 5.15. If $T \in \mathrm{PoStr}(\mathcal{V})$ is externally π_0-constant then each $K_a \in \mathcal{T}\mathcal{V}$ has the property that the induced parametrized spectrum $E_a = \mathrm{Map}_{\mathcal{V}}(1_{\mathcal{V}}, K_a) \in \mathcal{T}\mathcal{S}$ is 0-connected (or 1-connective), i.e. each fiber is a 1-connective spectrum. Indeed, for each $n \geq 0$, the map $E_a(S^n) = \Omega^{\infty-n}(E_a) \longrightarrow \pi(E_a)$ induces an isomorphism on π_0 by assumption and a surjection on π_1 since it admits a section, so that its fibers are all connected. It follows that the fiber $\Omega^{\infty-n}(E_a)_x$ is the connected delooping of $\Omega^{\infty-n+1}(E_a)_x$ for each $n \geq 0$ and $E_{a,x}$ is a 1-connective spectrum.

Example 5.16. The usual Postnikov structure on spaces (Example 4.15) is externally π_0-constant: for every space X, the resulting Postnikov structure is even constant after applying $\tau_{\leq 1}$. For more general ∞-toposes (Example 4.17), the Postnikov structure is typically *not* externally π_0-constant: even though all maps induce isomorphisms on π_0-sheaves, on global sections they typically do not induce bijections on π_0. For example, for any finite CW-complex X, abelian group A and $n \geq 2$, the map of constant sheaves $\mathcal{K}(A, n) \longrightarrow \tau_{\leq n-1}\mathcal{K}(A, n) \simeq *$ in $\mathrm{Sh}_{\infty}(X)$ induces an isomorphism on 1-truncations, but at the level of π_0 of the global sections we obtain $\mathrm{H}^n(X; A) \longrightarrow *$, which need not be an isomorphism.

Example 5.17. The canonical Postnikov structure on $\mathrm{Sp}^{\geq 0}$ is externally π_0-constant. Indeed, for each $E \in \mathrm{Sp}^{\geq 0}$, the image of its Postnikov structure under $\mathrm{Map}_{\mathrm{Sp}^{\geq 0}}(\mathbb{S}, -)$ is simply the Postnikov structure on the space $\Omega^{\infty}(E)$, but extended down to dimension 0, see Example 4.19. All spaces appearing in the Postnikov structure for $\Omega^{\infty}(E)$ have isomorphic π_0.

More generally, let $\mathcal{V}$ be a stable, presentable SM ∞-category with a left complete t-structure such that the connective part $\mathcal{V}^{\geq 0}$ is closed under finite tensor products. If the mapping spectrum functor $\mathrm{Map}(1_{\mathcal{V}}, -)\colon \mathcal{V} \longrightarrow \mathrm{Sp}$ sends $\mathcal{V}^{\geq 0}$ to $\mathrm{Sp}^{\geq 0}$, then the Postnikov structure on $\mathcal{V}^{\geq 0}$ from Example 4.23 is externally π_0-constant. This is notably the case when $\mathcal{V} = \mathrm{Mod}_R$ with R a connective ring spectrum (with the usual t-structure).

Theorem 5.18. *Let $\mathcal{V}$ be a SM ∞-category equipped with a multiplicative Postnikov structure $\Phi\colon \mathcal{V} \longrightarrow \mathrm{Fun}(\mathcal{E}, \mathcal{V})$. If Φ is externally π_0-constant, then the composite*

$$\mathrm{Cat}(\mathcal{V}) \subseteq \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \xrightarrow{\Phi_*} \mathrm{Alg}_{\mathrm{Cat}}(\mathrm{Fun}(\mathcal{E}, \mathcal{V})) \longrightarrow \mathrm{Fun}(\mathcal{E}, \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})) \xrightarrow{(-)^{\wedge}} \mathrm{Fun}(\mathcal{E}, \mathrm{Cat}(\mathcal{V}))$$

defines a multiplicative Postnikov structure Φ_{Cat} on $\mathrm{Cat}(\mathcal{V})$. Furthermore, this Postnikov structure Φ_{Cat} is itself externally π_0-constant.

Slightly informally (i.e. up to Dwyer–Kan equivalence), the Postnikov structure $\Phi_{\mathrm{Cat}}(\mathbb{C})$ of a $\mathcal{V}$-enriched ∞-category $\mathbb{C}$ is obtained by applying Φ to all mapping objects in $\mathbb{C}$. To see that this still yields a Postnikov structure after completion, we will make use of the cube and tower lemmas (Lemma 5.10 and 5.12), for which we will need Φ to be externally π_0-constant.

Note that Theorem 5.18 can be applied repeatedly:

Definition 5.19. Let $\mathcal{V}$ be a presentable SM ∞-category. Then the presentable SM ∞-category of **$\mathcal{V}$-enriched (∞, n)-categories** is defined inductively as

$$\mathrm{Cat}_n(\mathcal{V}) := \mathrm{Cat}(\mathrm{Cat}_{n-1}(\mathcal{V})).$$

For later purposes, let us record the following:

Lemma 5.20. *Let $\mathcal{V}$ and $\mathcal{W}$ be presentable SM ∞-categories and $L\colon \mathcal{V} \rightleftarrows \mathcal{W} : \iota$ a reflective (symmetric) monoidal localization. This induces a reflective monoidal localization of presentable SM ∞-categories*

$$\mathrm{Cat}_n(L)\colon\ \mathrm{Cat}_n(\mathcal{V}) \underset{\perp}{\rightleftarrows} \mathrm{Cat}_n(\mathcal{W})\colon\ \mathrm{Cat}_n(\iota).$$

The essential image of ι_* consists of those $\mathcal{V}$-enriched (∞, n)-categories whose mapping objects are (in the essential image of) $\mathcal{W}$-enriched $(\infty, n-1)$-categories.

Proof. An inductive application of [GH15, Corollary 5.7.12, Proposition 5.7.16] provides the presentable SM structure on $\mathrm{Cat}_n(\mathcal{V})$ and $\mathrm{Cat}_n(\mathcal{W})$. The induced reflective monoidal localization arises from an inductive application of [GH15, Proposition 5.7.18]. ☐

An inductive application of Theorem 5.18 then immediately yields the following:

Corollary 5.21. *Let $\mathcal{V}$ be a presentable SM ∞-category equipped with a multiplicative, externally π_0-constant Postnikov structure $\Phi\colon \mathcal{V} \longrightarrow \mathrm{Fun}(\mathcal{E}, \mathcal{V})$. Then there is an induced multiplicative Postnikov structure*

$$\Phi_{\mathrm{Cat}_n}\colon\ \mathrm{Cat}_n(\mathcal{V})^{\otimes} \longrightarrow \mathrm{PoStr}\left(\mathrm{Cat}_n(\mathcal{V})\right)^{\otimes}$$

on the ∞-category of $\mathcal{V}$-enriched (∞, n)-categories. Explicitly, Φ_{Cat_n} is given (up to n-categorical Dwyer–Kan equivalence) by applying Φ to objects of n-morphisms.

We will now turn to the proof of Theorem 5.18. Our strategy will be to first prove a version of Theorem 5.18 for categorical algebras and then descend to enriched ∞-categories. The case of categorical algebras follows readily from the following observation:

Lemma 5.22. *Let $\mathcal{I}$ be a weakly contractible ∞-category, $\mathcal{I}^{\triangleleft}$ its cone and $\mathcal{V}$ a monoidal ∞-category with $\mathcal{I}$-indexed limits. Consider the natural functor*

$$\phi\colon\ \mathrm{Alg}_{\mathrm{Cat}}(\mathrm{Fun}(\mathcal{I}^{\triangleleft}, \mathcal{V})) \xrightarrow{\ \sim\ } \mathrm{Fun}(\mathcal{I}^{\triangleleft}, \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})) \times_{\mathrm{Fun}(\mathcal{I}^{\triangleleft}, \mathcal{S})} \mathcal{S} \longrightarrow \mathrm{Fun}(\mathcal{I}^{\triangleleft}, \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}))$$

where the first functor is the natural equivalence from Proposition 5.6. If $\mathbb{C}$ is a categorical algebra in $\mathrm{Fun}(\mathcal{I}^{\triangleleft}, \mathcal{V})$ whose mapping objects belong to the full subcategory of limit cones, then $\phi(\mathbb{C})\colon \mathcal{I}^{\triangleleft} \longrightarrow \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$ is a limit cone as well.

Proof. By naturality, the equivalences of Proposition 5.6 fit into a commuting square where the horizontal functors restrict along $\mathcal{I} \hookrightarrow \mathcal{I}^{\triangleleft}$

$$\begin{tikzcd}
\mathrm{Alg}_{\mathrm{Cat}}(\mathrm{Fun}(\mathcal{I}^{\triangleleft}, \mathcal{V})) \arrow[r] \arrow[d, "\simeq"'] & \mathrm{Alg}_{\mathrm{Cat}}(\mathrm{Fun}(\mathcal{I}, \mathcal{V})) \arrow[d, "\simeq"] \\
\mathrm{Fun}(\mathcal{I}^{\triangleleft}, \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})) \times_{\mathrm{Fun}(\mathcal{I}^{\triangleleft}, \mathcal{S})} \mathcal{S} \arrow[r] & \mathrm{Fun}(\mathcal{I}, \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})) \times_{\mathrm{Fun}(\mathcal{I}, \mathcal{S})} \mathcal{S}.
\end{tikzcd}$$

This induces a commuting square between the right adjoints of the two horizontal functors. The top right adjoint is a fully faithful embedding whose essential image consists precisely of categorical algebras enriched over limit cones. To compute the bottom right adjoint, consider the diagram

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{J}^{\triangleleft}, \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})) \arrow[r] \arrow[d] & \mathrm{Fun}(\mathcal{J}^{\triangleleft}, \mathcal{S}) \arrow[d] & \mathcal{S} \arrow[l] \arrow[d, equal] \\
\mathrm{Fun}(\mathcal{J}, \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})) \arrow[r] & \mathrm{Fun}(\mathcal{J}, \mathcal{S}) & \mathcal{S} \arrow[l]
\end{tikzcd}$$

where the vertical functors restrict along $\mathcal{J} \longrightarrow \mathcal{J}^{\triangleleft}$. The horizontal functors then commute with the right adjoints to the restriction functors as well: for the left square, this uses that the forgetful functor $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \longrightarrow \mathcal{S}$ preserves limits, and for the right square, this uses that $\mathcal{J}$ is contractible, so that constant $\mathcal{J}^{\triangleleft}$-diagrams are limit cones. The right adjoint to $\mathrm{Fun}(\mathcal{J}^{\triangleleft}, \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})) \times_{\mathrm{Fun}(\mathcal{J}^{\triangleleft}, \mathcal{S})} \mathcal{S} \longrightarrow \mathrm{Fun}(\mathcal{J}, \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})) \times_{\mathrm{Fun}(\mathcal{J}, \mathcal{S})} \mathcal{S}$ is then the fiber product of the three right adjoint functors [Lur17, Corollary 4.7.4.18]. In particular, the projection onto the first factor commutes with these right adjoints. It follows that the composite ϕ intertwines the right adjoints to restriction along $\mathcal{J} \hookrightarrow \mathcal{J}^{\triangleleft}$, which yields the result. ☐

Proof of Theorem 5.18. We have to verify that Φ_{Cat} is a lax symmetric monoidal section of the (lax) symmetric monoidal functor $\mathrm{ev}_\infty\colon \mathrm{Fun}(\mathcal{E}, \mathrm{Cat}(\mathcal{V})) \longrightarrow \mathcal{V}$, and that it takes values in the full subcategory $\mathrm{PoStr}(\mathrm{Cat}(\mathcal{V})) \subseteq \mathrm{Fun}(\mathcal{E}, \mathrm{Cat}(\mathcal{V}))$ of Postnikov structures.

For the first assertion, consider the commuting diagram

$$\begin{tikzcd}
\mathrm{Cat}(\mathcal{V}) \arrow[r, hook] \arrow[d, "="'] & \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \arrow[r, "\Phi_*"] \arrow[d, "="'] & \mathrm{Alg}_{\mathrm{Cat}}(\mathrm{Fun}(\mathcal{E}, V)) \arrow[r, "\varphi"] \arrow[d, "\mathrm{ev}_\infty"'] & \mathrm{Fun}(\mathcal{E}, \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})) \arrow[r, "(-)^\wedge"] \arrow[d, "\mathrm{ev}_\infty"] & \mathrm{Fun}(\mathcal{E}, \mathrm{Cat}(\mathcal{V})) \arrow[d, "\mathrm{ev}_\infty"] \\
\mathrm{Cat}(\mathcal{V}) \arrow[r, hook] & \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \arrow[r, "="'] & \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \arrow[r, "="'] & \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \arrow[r, "(-)^\wedge"'] & \mathrm{Cat}(\mathcal{V}).
\end{tikzcd}$$

All arrows are lax SM functors. For the first and last horizontal arrows (in both rows), this follows from [GH15, Proposition 5.7.14] and for Φ_*, this follows from Proposition 5.3. The functor φ is the composite of the SM equivalence from Proposition 5.6 and the SM projection $\mathrm{Fun}(\mathcal{E}, \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})) \times_{\mathrm{Fun}(\mathcal{E}, \mathcal{S})} \mathcal{S} \longrightarrow \mathrm{Fun}(\mathcal{E}, \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}))$. Furthermore, the second square commutes since Φ_* is a multiplicative Postnikov structure and all other squares commute by naturality in the ∞-category $\mathcal{E}$. This provides a lax SM section of ev_∞ because the composition of the bottom row is naturally equivalent to the identity via the (lax SM) counit of the monoidal adjunction $(-)^\wedge\colon \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \underset{\perp}{\rightleftarrows} \mathrm{Cat}(\mathcal{V}) :\iota$ [GH15, Proposition 5.7.14].

For the second assertion, it suffices to verify that for a $\mathcal{V}$-enriched category $\mathbb{C}$, its image

$$\mathbb{T}^\wedge := \Phi_{\mathrm{Cat}}(\mathbb{C})\colon \mathcal{E} \longrightarrow \mathrm{Cat}(\mathcal{V})$$

defines a Postnikov structure in $\mathrm{Cat}(\mathcal{V})$. By construction, $\mathbb{T}^\wedge$ is the levelwise completion of the diagram which applies the multiplicative Postnikov structure Φ to all mapping objects

$$\mathbb{T} := \phi\big(\Phi_*(\mathbb{C})\big)\colon \mathcal{E} \longrightarrow \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}).$$

For each of the cone diagrams $\mathcal{J}_\alpha^{\triangleleft} \longrightarrow \mathcal{E}$ from Remark 4.7 (with $\mathcal{J}_\alpha$ contractible), $\mathbb{T}\big|_{\mathcal{J}^{\triangleleft}}$ is a limit cone in $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$ by Lemma 5.22. Consequently, $\mathbb{T}$ defines a Postnikov structure in $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$. We have to show that these cones remain limit cones upon applying the completion functor objectwise. This will follow from Corollary 5.11 and Corollary 5.13 once we verify that $\mathbb{T}$ sends every arrow in $\mathcal{E}$ to an isofibration.

To this end, consider a map $\mathbb{T}(i) \longrightarrow \mathbb{T}(j)$ of categorical algebras induced by $i \to j$ in $\mathcal{E}$. By construction, f induces the identity on spaces of objects and for every two objects $x, y \in \mathbb{T}(i)$, the map

$$\mathrm{Map}\left(1_{\mathcal{V}}, \mathrm{Map}_{\mathbb{T}(i)}(x, y)\right) \longrightarrow \mathrm{Map}\left(1_{\mathcal{V}}, \mathrm{Map}_{\mathbb{T}(j)}(x, y)\right)$$

is a π_0-isomorphism since the Postnikov structure Φ is π_0-constant (Definition 5.14). The condition that $\mathbb{T}(i) \to \mathbb{T}(j)$ is an isofibration is determined at the level of the underlying $\mathcal{S}$-enriched categorical algebras, so we may as well assume that $\mathcal{V} = \mathcal{S}$. For every object x in $\mathbb{T}(i)$ and every arrow $\alpha\colon x \to y$ in $\mathbb{T}(j)$, we then have a lift of α to a map $\tilde{\alpha}\colon x \longrightarrow y$ in $\mathbb{T}(i)$ (note that $\mathbb{T}(i) \longrightarrow \mathbb{T}(j)$ is the identity on objects), which is furthermore unique up to homotopy. If α is an equivalence, then $\tilde{\alpha}$ is an equivalence by [GH15, Proposition 5.1.15]: indeed, using that such lifts of arrows to $\mathbb{T}(i)$ are unique up to homotopy, a homotopy inverse of $\tilde{\alpha}$ is provided by a lift of the homotopy inverse of α to $\mathbb{T}(i)$. We conclude that $\mathbb{T}(i) \to \mathbb{T}(j)$ is an isofibration.

Finally, we have to verify that $\mathbb{T}^\wedge$ is a π_0-constant Postnikov structure. Note that the monoidal unit of $\mathrm{Cat}(\mathcal{V})$ is the completion $1_{\mathrm{Cat}(\mathcal{V})} = [0]^\wedge_{1_{\mathcal{V}}}$ of the unit object of $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$, given by the operad map $\mathcal{O}_* \simeq \mathrm{Ass} \longrightarrow \mathcal{V}^\otimes$ encoding the unit (i.e. initial) associative algebra $1_{\mathcal{V}}$ in $\mathcal{V}$. In particular, the functor

$$\mathrm{Map}_{\mathrm{Cat}(\mathcal{V})}(1_{\mathrm{Cat}(\mathcal{V})}, -)\colon\ \mathrm{Cat}(\mathcal{V}) \longrightarrow \mathcal{S}$$

is equivalent to the functor sending a $\mathcal{V}$-enriched ∞-category to its space of objects. To see that $\mathbb{T}^\wedge$ is π_0-constant, it therefore suffices to verify that the diagram of object spaces

$$\mathrm{Ob}(\mathbb{T}^\wedge)\colon \mathcal{E} \xrightarrow{\mathbb{T}^\wedge} \mathrm{Cat}(\mathcal{V}) \xrightarrow{\mathrm{Ob}} \mathcal{S}$$

sends each map $i \to j$ in $\mathcal{E}$ to a map with 0-connected fibers (in particular, a π_0-isomorphism). Let us pick an object $x \in \mathbb{T}^\wedge(j)$ and verify that the fiber $\mathrm{Ob}(\mathbb{T}^\wedge(i))_x$ is connected. The object x determines a map $x\colon [0]_{1_{\mathcal{V}}} \longrightarrow 1_{\mathrm{Cat}(\mathcal{V})} \longrightarrow \mathbb{T}^\wedge(j)$. Since the functor $\mathbb{T}(j) \longrightarrow \mathbb{T}^\wedge(j)$ induces a π_0-surjection on objects by [GH15, Theorem 5.6.2], this composite map factors over $\mathbb{T}(j)$. Taking pullbacks along these maps, we obtain a commuting square of categorical algebras

$$\begin{tikzcd}
\mathbb{T}(i) \times_{\mathbb{T}(j)} [0]_{1_{\mathcal{V}}} \arrow[r] \arrow[d] & \mathbb{T}^\wedge(i) \times_{\mathbb{T}^\wedge(j)} 1_{\mathrm{Cat}(\mathcal{V})} \arrow[d] \\
{[0]_{1_{\mathcal{V}}}} \arrow[r] & 1_{\mathrm{Cat}(\mathcal{V})}
\end{tikzcd}$$

Since $\mathbb{T}(i) \longrightarrow \mathbb{T}(j)$ was an isofibration, the top and bottom horizontal maps are Dwyer–Kan equivalences (Cube Lemma 5.10) and the left vertical map is an isofibration. Lemma 5.9, together with the fact that the right column consists of *complete* categorical algebras, then implies that $\mathbb{T}(i) \times_{\mathbb{T}(j)} [0]_{1_{\mathcal{V}}} \longrightarrow \mathbb{T}^\wedge(i) \times_{\mathbb{T}^\wedge(j)} [0]_{1_{\mathcal{V}}}$ induces a π_0-surjection on spaces of objects. Since $\mathbb{T}(i) \longrightarrow \mathbb{T}(j)$ is constant on objects, the domain of this map has a contractible space of objects. Consequently, $\mathrm{Ob}\left(\mathbb{T}^\wedge(i) \times_{\mathbb{T}^\wedge(j)} [0]_{1_{\mathcal{V}}}\right) \simeq \mathrm{Ob}(\mathbb{T}^\wedge(i))_x$ is connected, as desired. □

6. Local systems on (∞, n)-categories

In this section, we spell out the contents of Theorem 5.18 and Corollary 5.21 in the setting of (∞, n)-categories. There are many equivalent models for the ∞-category of (∞, n)-categories, one of which is the ∞-category $\mathrm{Cat}_n(\mathcal{S})$ of $\mathcal{S}$-enriched (∞, n)-categories of Definition 5.19 [Hau15, Corollary 7.21]. This model is particularly well-adapted to definitions that proceed by induction on mapping objects, such as the following:

Definition 6.1. An $(m,0)$-category is defined to be an m-truncated space. For any $0 \leq n \leq m$, an (∞,n)-category $\mathcal{C}$ is called an (m,n)**-category** if each mapping $(\infty,n-1)$-category is an $(m-1,n-1)$-category.

In light of [GH15, Corollary 6.1.10], this coincides with [GH15, Definition 6.1.1].

Lemma 6.2. *The fully faithful inclusion $\iota\colon \mathrm{Cat}_{(m,n)} \subseteq \mathrm{Cat}_{(\infty,n)}$ of the full subcategory of (m,n)-categories admits a left adjoint sending an (∞,n)-category $\mathcal{C}$ to its **homotopy** (m,n)**-category** $\mathrm{ho}_{(m,n)}\,\mathcal{C}$. The adjoint pair $\mathrm{ho}_{(m,n)}\colon \mathrm{Cat}_{(\infty,n)} \rightleftarrows \mathrm{Cat}_{(m,n)} :\iota$ has the canonical structure of a reflective monoidal localization.*

Proof. Consider the reflective localization $\tau_{\leq m-n}\colon \mathcal{S} \rightleftarrows \tau_{\leq m-n}\mathcal{S} :\iota$. Since truncation preserve products, this is a reflective monoidal localization. Consequently, it induces a reflective monoidal localization $\mathrm{Cat}_n(\tau_{\leq m-n})\colon \mathrm{Cat}_n(\mathcal{S}) \rightleftarrows \mathrm{Cat}_n(\tau_{\leq m-n}\mathcal{S}) : \mathrm{Cat}_n(\iota)$ by Lemma 5.20. Note that (by induction) the essential image of ι_* is precisely the full subcategory $\mathrm{Cat}_{(m,n)} \subseteq \mathrm{Cat}_{(\infty,n)}$. It follows that there is a left adjoint $\mathrm{ho}_{(m,n)}\colon \mathrm{Cat}_{(\infty,n)} \longrightarrow \mathrm{Cat}_{(m,n)}$ (equivalent to $\mathrm{Cat}_n(\tau_{\leq m-n})$), which has the canonical structure of a monoidal localization. $\square$

The universal properties of the homotopy (m,n)-categories imply that they fit into a tower

$$\mathcal{C} \longrightarrow \dots \longrightarrow \mathrm{ho}_{(m,n)}(\mathcal{C}) \longrightarrow \mathrm{ho}_{(m-1,n)}(\mathcal{C}) \longrightarrow \dots \longrightarrow \mathrm{ho}_{(n+1,n)}(\mathcal{C}). \tag{6.1}$$

By induction on n, one sees that this tower is convergent. Indeed, using that $\mathrm{Cat}_{(\infty,n+1)} \subseteq \mathrm{Alg}_{\mathrm{Cat}}(\mathrm{Cat}_{(\infty,n)})$ preserves limits, this follows from the fact that:

(a) at the level of spaces of objects, the tower induces isomorphisms on π_0 so that $\mathcal{C} \longrightarrow \lim_m \mathrm{ho}_{(m,n+1)}(\mathcal{C})$ is essentially surjective.
(b) the map $\mathrm{Map}_{\mathcal{C}}(c,d) \longrightarrow \lim_m \mathrm{Map}_{\mathrm{ho}_{(m,n+1)}(\mathcal{C})}(c,d) \simeq \lim_m \mathrm{ho}_{(m,n)}\big(\mathrm{Map}_{\mathcal{C}}(c,d)\big)$ is an equivalence for each $c,d \in \mathcal{C}$ by inductive hypothesis.

Theorem 5.18 and Corollary 5.21 then yield the following more precise statement of Theorem 1.1:

Theorem 6.3. *For each $n \geq 1$, the tower of natural transformations (6.1) refines to a multiplicative Postnikov structure on $\mathrm{Cat}_{(\infty,n)}$.*

Proof. By Lemma 6.2, the reflective monoidal localization

$$\mathrm{ho}_{(m,n)}\colon \mathrm{Cat}_{(\infty,n)} = \mathrm{Cat}_n(\mathcal{S}) \underset{\mathrm{Cat}_n(\iota)}{\overset{\mathrm{Cat}_n(\tau_{\leq m-n})}{\rightleftarrows}} \mathrm{Cat}_n(\mathcal{S}_{\leq m-n}) = \mathrm{Cat}_{(m,n)} :\iota$$

arises from the reflective monoidal localization $\tau_{\leq m-n}\colon \mathcal{S} \rightleftarrows \mathcal{S}_{\leq m-n} :\iota$ via Lemma 5.20. Now let $\Phi\colon \mathcal{S} \longrightarrow \mathrm{Fun}(\mathcal{E},\mathcal{S})$ be the multiplicative, π_0-constant Postnikov structure on spaces refining the classical Postnikov tower (Example 4.15). By Theorem 5.18 and Corollary 5.21, this induces a multiplicative Postnikov structure Φ_{Cat_n} on $\mathrm{Cat}_{(\infty,n)}$.

Note that we can view Φ_{Cat_n} as a diagram $\mathcal{E} \longrightarrow \mathrm{Fun}^{\otimes,\mathrm{lax}}(\mathrm{Cat}_{(\infty,n)}, \mathrm{Cat}_{(\infty,n)})$ of (lax symmetric monoidal) endofunctors of $\mathrm{Cat}_{(\infty,n)}$, by adjunction with the Boardman–Vogt tensor product (see Remark 2.7). Forgetting about the k-invariants, the underlying tower of Φ_{Cat_n} is given by the tower of functors

$$\mathrm{id} \longrightarrow \dots \longrightarrow \mathrm{Cat}_n(\tau_{\leq a}) \longrightarrow \mathrm{Cat}_n(\tau_{\leq a-1}) \longrightarrow \dots \longrightarrow \mathrm{Cat}_n(\tau_{\leq 1}).$$

Lemma 6.2 identifies this with the natural tower of homotopy (m, n)-categories (6.1), as desired. $\square$

In other words, for each (∞, n)-category $\mathcal{C}$ and $a \geq 2$, there exists a natural parametrized spectrum object

$$\mathrm{H}\pi_a(\mathcal{C}) \in \mathcal{T}_{\mathrm{ho}_{(n+1,n)}(\mathcal{C})}\big(\mathrm{Cat}_{(\infty,n)}\big)$$

and a pullback square of (∞, n)-categories

$$\begin{tikzcd}
\mathrm{ho}_{(n+a,n)}\,\mathcal{C} \arrow[r] \arrow[d] & \mathrm{ho}_{(n+1,n)}\,\mathcal{C} \arrow[d, "0"] \\
\mathrm{ho}_{(n+a-1,n)}\,\mathcal{C} \arrow[r, "k_a"'] & \Omega^\infty\big(\Sigma^{a+1}\mathrm{H}\pi_a(\mathcal{C})\big).
\end{tikzcd}$$

The proof of Theorem 6.3 is not completely satisfying because these parametrized spectra $\mathrm{H}\pi_a(\mathcal{C})$ are defined somewhat implicitly. In the remainder of this section, we will explain how (as the notation suggests) the parametrized spectra $\mathrm{H}\pi_a(\mathcal{C})$ can be considered as the Eilenberg–Maclane spectra associated to **local systems of abelian groups** on $\mathrm{ho}_{(n+1,n)}(\mathcal{C})$, as considered in [Lur09b].

6.1. **Tangent bundle of enriched ∞-categories.** Our first goal will be to construct a t-orientation (Definition 2.20) on the tangent bundle to $\mathcal{V}$-enriched ∞-categories, using a version of Proposition 2.25. To this end, we will need a description of the tangent bundle to enriched ∞-categories along the lines of Proposition 2.17:

Theorem 6.4. *Let $\mathcal{V}$ be a differentiable presentable SM ∞-category such that $1_{\mathcal{V}}$ is compact. Then there exists a natural equivalence of SM ∞-categories*

$$\begin{tikzcd}
\mathrm{Cat}(\mathcal{T}\mathcal{V}) \arrow[rr, "\mathcal{L}", "\sim"'] \arrow[dr, "\mathrm{Cat}(\pi_{\mathcal{V}})"'] & & \mathcal{T}\,\mathrm{Cat}(\mathcal{V}) \arrow[dl, "\pi"] \\
& \mathrm{Cat}(\mathcal{V}) &
\end{tikzcd} \tag{6.2}$$

where $\mathrm{Cat}(\pi_{\mathcal{V}})$ is induced by the (monoidal) tangent projection $\pi_{\mathcal{V}} \colon \mathcal{T}\mathcal{V} \longrightarrow \mathcal{V}$ and π is the tangent projection for $\mathrm{Cat}(\mathcal{V})$.

Recall from [Lur17, Definition 6.1.1.6] that a presentable ∞-category $\mathcal{V}$ is differentiable if the sequential colimit functor $\mathrm{colim}\colon \mathrm{Fun}(\mathbb{N}, \mathcal{V}) \longrightarrow \mathcal{V}$ is left exact. In particular, any compactly generated ∞-category is differentiable. To apply Theorem 6.4 inductively, let us record the following observation:

Lemma 6.5. *Let $\mathcal{V}$ be a presentable monoidal ∞-category which is differentiable and such that $1_{\mathcal{V}}$ is compact. Then $\mathrm{Cat}(\mathcal{V})$ is differentiable and $1_{\mathrm{Cat}(\mathcal{V})}$ (i.e. the image of the categorical algebra with one object with endomorphism algebra $1_{\mathcal{V}}$) is compact as well.*

Proof. By [Lur17, Remark 4.1.8.9], there exists a monoidal model category $\mathbf{V}$ presenting $\mathcal{V}$ of the following form: one can construct a simplicial monoid $A \in \mathrm{Alg}(\mathrm{sSet})$ and take $\mathbf{V} = \mathrm{sSet}_{/A}$, with monoidal structure given by $(X \to A) \otimes (Y \to A) = (X \times Y \to A \times A \to A)$ and model structure given by a left Bousfield localization of the covariant model structure. In particular, $\mathbf{V}$ is simplicial and combinatorial, its cofibrations are the monomorphisms, all objects are cofibrant and weak equivalences are stable under filtered colimits. Then $\mathrm{Cat}(\mathcal{V})$ arises from the model category $\mathrm{Cat}^{\mathrm{strict}}(\mathbf{V})$ on $\mathbf{V}$-enriched categories [Hau15] and it then follows from [HNP18, Corollary 3.1.12] that $\mathrm{Cat}(\mathcal{V})$ is again differentiable.

To see that $1_{\mathrm{Cat}(\mathcal{V})}$ is compact, note that the corepresentable functor $\mathrm{Map}(1_{\mathrm{Cat}(\mathcal{V})}, -)$ can be identified with the functor taking spaces of objects. This functor decomposes as

$$\mathrm{Cat}(\mathcal{V}) \longrightarrow \mathrm{Cat}(\mathcal{S}) \simeq \mathrm{Cat}_\infty \xrightarrow{\mathrm{Core}} \mathcal{S}$$

where the first functor is induced by the lax monoidal functor $\mathrm{Map}_{\mathcal{V}}(1_{\mathcal{V}}, -)\colon \mathcal{V} \longrightarrow \mathcal{S}$ and the second functor takes the core (or maximal sub-∞-groupoid). Taking cores preserves filtered colimits (as is easily checked using quasicategories). To see that $\mathrm{Cat}(\mathcal{V}) \longrightarrow \mathrm{Cat}(\mathcal{S})$ preserves filtered colimits, we use model categories. Since $\mathbf{V}$ is simplicial and monoidal, there is a monoidal Quillen pair $1_{\mathbf{V}} \otimes -\colon \mathrm{sSet} \rightleftarrows \mathbf{V} : \mathrm{Map}_{\mathbf{V}}(1_{\mathbf{V}}, -)$. Applying these functors on mapping objects yields a Quillen pair on enriched categories. In light of [Lur09, Proposition 5.3.1.16], it now suffices to verify that the right Quillen functor $U\colon \mathrm{Cat}^{\mathrm{strict}}(\mathbf{V}) \longrightarrow \mathrm{Cat}^{\mathrm{strict}}(\mathrm{sSet})$ preserves filtered homotopy colimits.

Let $\mathbb{C}_\bullet\colon \mathcal{I} \longrightarrow \mathrm{Cat}^{\mathrm{strict}}(\mathbf{V})$ be a projectively cofibrant filtered diagram with colimit $\mathbb{C}_\infty$. To see that the natural map $\mathrm{hocolim}\, U(\mathbb{C}_\bullet) \longrightarrow U(\mathbb{C}_\infty)$ is a weak equivalence, note that the Dwyer–Kan equivalences of simplicial categories are closed under filtered colimits. Consequently, the homotopy colimit can simply be computed by the ordinary colimit and it suffices to verify that $\mathrm{colim}\, U(\mathbb{C}_\bullet) \longrightarrow U(\mathbb{C}_\infty)$ is a Dwyer–Kan equivalence. At the level of objects, the map is simply an isomorphism and for each $c, d \in \mathrm{colim}\, U(\mathbb{C}_\bullet)$ arising from some $U(\mathbb{C}_i)$, the induced map on mapping spaces is given by

$$\mathrm{colim}_{j\in\mathcal{I}_{i/}} \mathrm{Map}_{\mathbf{V}}\big(1_{\mathbf{V}}, \mathbb{C}_j(c,d)\big) \longrightarrow \mathrm{Map}_{\mathbf{V}}\big(1_{\mathbf{V}}, \mathrm{colim}_{j\in\mathcal{I}_{i/}} \mathbb{C}_j(c,d)\big).$$

This map is a weak equivalence of simplicial sets because $\mathrm{Map}_{\mathbf{V}}(1_{\mathbf{V}}, -)$ preserves filtered homotopy colimits and because weak equivalences in both sSet and $\mathbf{V}$ are closed under filtered colimits. ☐

The proof of Theorem 6.4 requires a few preliminary observations:

Proposition 6.6. *Let $\mathcal{V}$ be a presentable monoidal ∞-category and $\pi_{\mathcal{V}}\colon \mathcal{T}\mathcal{V} \longrightarrow \mathcal{V}$ its tangent projection. Then the inclusions of complete objects and the completion functors fit into pullback squares*

$$\begin{tikzcd}
\mathrm{Cat}(\mathcal{T}\mathcal{V}) \arrow[r, hook] \arrow[d, "\mathrm{Cat}(\pi_{\mathcal{V}})"'] & \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}\mathcal{V}) \arrow[r, "(-)^\wedge"] \arrow[d, "\mathrm{Alg}_{\mathrm{Cat}}(\pi_{\mathcal{V}})"'] & \mathrm{Cat}(\mathcal{T}\mathcal{V}) \arrow[d, "\mathrm{Cat}(\pi_{\mathcal{V}})"] \\
\mathrm{Cat}(\mathcal{V}) \arrow[r, hook] & \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \arrow[r, "(-)^\wedge"] & \mathrm{Cat}(\mathcal{V})
\end{tikzcd}$$

in which the vertical functors are all Cartesian and coCartesian fibrations.

Proof. Recall from Lemma 2.11 that the monoidal functor $\pi_{\mathcal{V}}\colon \mathcal{T}\mathcal{V} \longrightarrow \mathcal{V}$ has a (strong) monoidal fully faithful *left* adjoint $\mathrm{cst}\colon \mathcal{V} \longrightarrow \mathcal{T}\mathcal{V}$ taking constant diagrams, and that cst is also a monoidal fully faithful *right* adjoint to $\pi_{\mathcal{V}}$. Considering $\pi_{\mathcal{V}}$ as a right adjoint functor, Remark 5.8 then implies that $\mathrm{Alg}_{\mathrm{Cat}}(\pi_{\mathcal{V}})\colon \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}\mathcal{V}) \longrightarrow \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$ preserves and detects completeness, so that the left square commutes and is Cartesian. On the other hand, considering $\pi_{\mathcal{V}}$ as a monoidal left adjoint, we find that the right square commutes: taking right adjoints, this comes down to $\mathrm{cst}\colon \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \longrightarrow \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}\mathcal{V})$ preserving complete objects.

Next, the two monoidal adjunctions $(\mathrm{cst}, \pi_{\mathcal{V}})$ and $(\pi_{\mathcal{V}}, \mathrm{cst})$ induce adjunctions

$$\begin{tikzcd} \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \arrow[r, hook, shift left, "\mathrm{Alg}_{\mathrm{Cat}}(\mathrm{cst})"] \arrow[r, phantom, "\perp"] & \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}\mathcal{V}) \arrow[l, shift left, "\mathrm{Alg}_{\mathrm{Cat}}(\pi_{\mathcal{V}})"] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}\mathcal{V}) \arrow[r, shift left, "\mathrm{Alg}_{\mathrm{Cat}}(\pi_{\mathcal{V}})"] \arrow[r, phantom, "\perp"] & \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \arrow[l, hook, shift left, "\mathrm{Alg}_{\mathrm{Cat}}(\mathrm{cst})"] \end{tikzcd}$$

in which $\mathrm{Alg}_{\mathrm{Cat}}(\mathrm{cst})$ is fully faithful. This implies that $\mathrm{Alg}_{\mathrm{Cat}}(\pi_{\mathcal{V}})$ preserves limits and colimits and by [CDH+20, Lemma 2.6.1], it is both a Cartesian and coCartesian fibration. Since the left square was a pullback, this implies that $\mathrm{Cat}(\pi_{\mathcal{V}})$ is a Cartesian and coCartesian fibration as well and that the inclusion $\mathrm{Cat}(\mathcal{T}\mathcal{V}) \hookrightarrow \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}\mathcal{V})$ preserves Cartesian and coCartesian arrows.

It remains to verify that the right square is Cartesian. To see this, we claim that the following conditions are equivalent for a map $\alpha\colon \mathbb{C} \longrightarrow \mathbb{D}$ in $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}\mathcal{V})$:

(a) α is a DK-equivalence.
(b) α is an $\mathrm{Alg}_{\mathrm{Cat}}(\pi_{\mathcal{V}})$-coCartesian lift of a DK-equivalence in $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$.
(c) α is an $\mathrm{Alg}_{\mathrm{Cat}}(\pi_{\mathcal{V}})$-Cartesian lift of a DK-equivalence in $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$.

Assuming this, it follows that $\mathrm{Alg}_{\mathrm{Cat}}(\pi_{\mathcal{V}})$ classifies a diagram $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \longrightarrow \mathrm{Cat}^{\mathrm{L}}$ sending each DK-equivalence to an adjoint equivalence: over a fixed DK-equivalence in $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$, the $\mathrm{Alg}_{\mathrm{Cat}}(\pi_{\mathcal{V}})$-Cartesian and $\mathrm{Alg}_{\mathrm{Cat}}(\pi_{\mathcal{V}})$-coCartesian arrows coincide, so that the proof of [Lur09, Proposition 5.2.2.8] shows that the unit and counit of the induced adjunction are equivalences. Since the DK-equivalences in $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}\mathcal{V})$ are precisely the $\mathrm{Alg}_{\mathrm{Cat}}(\pi_{\mathcal{V}})$-(co)Cartesian lifts of DK-equivalences in $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$, it then follows from [Hin16, Proposition 2.1.4] that the right square is a Cartesian square.

To see the claim, let us write $\alpha_0\colon \mathbb{C}_0 \longrightarrow \mathbb{D}_0$ for the image of α in $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$ and let $\alpha_0'\colon \mathbb{C}_0' \longrightarrow \mathbb{D}_0'$ be the image of α_0 under $\mathrm{Alg}_{\mathrm{Cat}}(\mathrm{cst})$. The unit and the counit of the adjoint pairs above then determine a commuting diagram in $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}\mathcal{V})$

$$\begin{tikzcd} \mathbb{C}_0' \arrow[r, "\epsilon"] \arrow[d, "\alpha_0'"'] & \mathbb{C} \arrow[r, "\eta"] \arrow[d, "\alpha"] & \mathbb{C}_0' \arrow[d, "\alpha_0'"] \\ \mathbb{D}_0' \arrow[r, "\epsilon"] & \mathbb{D} \arrow[r, "\eta"] & \mathbb{D}_0'. \end{tikzcd}$$

Since $\mathrm{Alg}_{\mathrm{Cat}}(\mathrm{cst})$ and $\mathrm{Alg}_{\mathrm{Cat}}(\pi_{\mathcal{V}})$ both preserve DK-equivalences (having right adjoints preserving complete objects), α_0' is a DK-equivalence in $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}\mathcal{V})$ if and only if α_0 is a DK-equivalence in $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$. Furthermore, the left square is a pushout if and only if α is $\mathrm{Alg}_{\mathrm{Cat}}(\pi_{\mathcal{V}})$-coCartesian and the right square is a pullback if and only if α is $\mathrm{Alg}_{\mathrm{Cat}}(\pi_{\mathcal{V}})$-Cartesian, by [CDH+20, Lemma 2.6.1] and its opposite.

Using this, (b) implies (a) because DK-equivalences are stable under pushout. Conversely, if α is a DK-equivalence, then α_0' is a DK-equivalence as well. Consequently, $\mathbb{D}_0' \amalg_{\mathbb{C}_0'} \mathbb{C} \longrightarrow \mathbb{D}$ is both a DK-equivalence and an equivalence on spaces of objects (since its image in $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$ is an equivalence), and hence an equivalence.

Dually, (c) is equivalent to α_0' being a DK-equivalence and the right square being a pullback. Since fully faithful maps are stable under pullback, this implies that α is fully faithful. Since essential surjectivity is detected on the underlying space-valued categorical algebra, which in turn is determined by the underlying $\mathcal{V}$-enriched categorical algebra, we find that α is essentially surjective as well. Conversely, if α is a DK-equivalence, then α_0' is a DK-equivalence as well. Consequently, the map $\mathbb{C} \longrightarrow \mathbb{C}_0' \times_{\mathbb{D}_0'} \mathbb{D}$ is fully faithful and an equivalence on spaces of objects, and is hence an equivalence. □

Lemma 6.7. *Let $\mathcal{V}$ be a presentable SM ∞-category and let $\mathcal{T}\mathcal{V}$ be its tangent bundle with the square zero monoidal structure. Then there is a natural SM equivalence*

$$\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}\mathcal{V}) \simeq \mathcal{T}\,\mathrm{Alg}_{\mathrm{Cat}}(V) \times_{\mathcal{T}\mathcal{S}} \mathcal{S}$$

between $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}\mathcal{V})$ and the full subcategory of excisive functors $\mathbb{C}_\bullet\colon \mathcal{S}^{\mathrm{fin}}_* \longrightarrow \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$ which are constant at the level of objects.

Proof. Recall the equivalence $\mathrm{Alg}_{\mathrm{Cat}}(\mathrm{Fun}(\mathcal{S}^{\mathrm{fin}}_*, \mathcal{V})) \simeq \mathrm{Fun}(\mathcal{S}^{\mathrm{fin}}_*, \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})) \times_{\mathrm{Fun}(\mathcal{S}^{\mathrm{fin}}_*, \mathcal{S})} \mathcal{S}$ from Proposition 5.6. By Lemma 5.22, this restricts to an equivalence on full subcategories of excisive functors. □

Lemma 6.8. *Let $\mathcal{V}$ be a presentable monoidal ∞-category. Then there exists a natural relative adjunction*

$$\begin{tikzcd}
\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}\mathcal{V}) \arrow[rr, hook, shift left, "\varphi"] \arrow[rd, "\mathrm{Alg}_{\mathrm{Cat}}(\pi_{\mathcal{V}})"'] \arrow[rr, phantom, "\perp"] & & \mathcal{T}\,\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \arrow[ll, shift left, "\psi"] \arrow[ld, "\pi"] \\
& \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}). &
\end{tikzcd}$$

Proof. Using Lemma 6.7, we define φ as the projection onto the first factor

$$\begin{tikzcd}
\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}\mathcal{V}) \simeq \mathcal{T}\,\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \times_{\mathcal{T}\mathcal{S}} \mathcal{S} \arrow[r, "\varphi"] \arrow[d, "\mathrm{Alg}_{\mathrm{Cat}}(\pi_{\mathcal{V}})"'] & \mathcal{T}\,\mathrm{Alg}_{\mathrm{Cat}}(V) \arrow[d, "\pi"] \\
\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \simeq \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \times_{\mathcal{S}} \mathcal{S} \arrow[r, "\sim"] & \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}).
\end{tikzcd} \tag{6.3}$$

Here the square commutes by naturality of the equivalence from Proposition 5.6 with respect to restriction along $\{*\} \hookrightarrow \mathcal{S}^{\mathrm{fin}}_*$. Note that φ is the base change along the Cartesian fibration $\mathcal{T}\,\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \longrightarrow \mathcal{T}\mathcal{S}$ of the fully faithful functor $\mathrm{cst}\colon \mathcal{S} \longrightarrow \mathcal{T}\mathcal{S}$ that is left adjoint to the tangent projection. The opposite of [Lur09, Corollary 5.2.7.11] then implies that φ is fully faithful and admits a right adjoint ψ. This right adjoint can be described as follows: given $\mathbb{C}_\bullet\colon \mathcal{S}^{\mathrm{fin}}_* \longrightarrow \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$, $\psi(\mathbb{C}_\bullet)\colon \mathcal{S}^{\mathrm{fin}}_* \longrightarrow \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$ sends each space T to the full sub-categorical algebra of $\mathbb{C}_T$ obtained by restricting the objects along the canonical map $\mathrm{Ob}(\mathbb{C}_*) \longrightarrow \mathrm{Ob}(\mathbb{C}_T)$. In other words, the counit map $\psi(\mathbb{C}_T) \longrightarrow \mathbb{C}_T$ is a Cartesian lift of the map of spaces $\mathrm{Ob}(\mathbb{C}_*) \longrightarrow \mathrm{Ob}(\mathbb{C}_T)$. It follows from this description that ψ commutes with evaluation at $*$ as well, so that ϕ and ψ form a relative adjunction. □

Proof of Theorem 6.4. To define the functor $\mathcal{L}$, consider the composition

$$\mathcal{T}((-)^\wedge) \circ \varphi\colon\ \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}\mathcal{V}) \hookrightarrow \mathcal{T}\,\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \longrightarrow \mathcal{T}\,\mathrm{Cat}(\mathcal{V}).$$

Here φ is the functor from Lemma 6.8 and the last square arises from the adjoint pair $(-)^\wedge\colon \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \rightleftarrows \mathrm{Cat}(\mathcal{V}) : \iota$ by taking tangent bundles. Note that φ sends DK-equivalences in $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}\mathcal{V})$ to $\mathcal{S}^{\mathrm{fin}}_*$-diagrams of DK-equivalences in $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$, which in turn are sent to equivalences by $\mathcal{T}((-)^\wedge)$. Consequently, the above composite induces a functor $\mathcal{L}\colon \mathrm{Cat}(\mathcal{T}\mathcal{V}) \longrightarrow \mathcal{T}\,\mathrm{Cat}(\mathcal{V})$ from the localization at the DK-equivalences.

Since $(-)^\wedge\colon \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}\mathcal{V}) \longrightarrow \mathrm{Cat}(\mathcal{T}\mathcal{V})$ is a monoidal localization, $\mathcal{L}$ inherits a natural SM structure from the composite $\mathcal{T}((-)^\wedge) \circ \varphi\colon \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}\mathcal{V})$. Here $\mathcal{T}((-)^\wedge)$ inherits its SM structure from $(-)^\wedge$, and φ corresponds under the SM equivalence of Proposition 5.6 to the projection (6.3), which is a SM functor.

To show that $\mathcal{L}$ is an equivalence, observe that (since $(-)^\wedge\colon \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}\mathcal{V}) \longrightarrow \mathrm{Cat}(\mathcal{T}\mathcal{V})$ has a section), $\mathcal{L}$ coincides with the top horizontal composite

$$\begin{tikzcd} \mathrm{Cat}(\mathcal{T}\mathcal{V}) \arrow[r, hook] \arrow[d] & \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}\mathcal{V}) \arrow[r, hook, "\varphi"] \arrow[d] & \mathcal{T}\,\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \arrow[r, "\mathcal{T}((-)^\wedge)"] \arrow[d] & \mathcal{T}\,\mathrm{Cat}(\mathcal{V}) \arrow[d] \\ \mathrm{Cat}(\mathcal{V}) \arrow[r, hook] & \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \arrow[r, "="'] & \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \arrow[r, "(-)^\wedge"] & \mathrm{Cat}(\mathcal{V}). \end{tikzcd} \tag{6.4}$$

Here all vertical functors are coCartesian fibrations, where we use Proposition 6.6 for the left two. The first two horizontal functors preserve coCartesian arrows by Proposition 6.6 and by Lemma 6.8 and [Lur17, Proposition 7.3.2.6]. The last functor $\mathcal{T}((-)^\wedge)$ preserves coCartesian arrows for formal reasons: for any adjoint pair $F\colon \mathcal{C} \rightleftarrows \mathcal{D} :G$, the induced adjoint pair $\mathcal{T}F\colon \mathcal{T}\mathcal{C} \rightleftarrows \mathcal{T}\mathcal{D} :\mathcal{T}G$ covering F and G has left adjoint $\mathcal{T}F$ preserving coCartesian arrows and right adjoint $\mathcal{T}G$ preserving Cartesian arrows.

Having proven that $\mathcal{L}$ is a map between coCartesian fibrations preserving coCartesian arrows, it suffices to verify that $\mathcal{L}$ induces an equivalence on fibers. Let us therefore fix a $\mathcal{V}$-enriched category $\mathbb{C} \in \mathrm{Cat}(\mathcal{V})$ and let us write $X = \mathrm{Ob}(\mathbb{C})$ for its space of objects. Since the left square in (6.4) was Cartesian, it suffices to verify that φ and completion induce an equivalence $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}\mathcal{V}) \times_{\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})} \{\mathbb{C}\} \longrightarrow \mathcal{T}_{\mathbb{C}}\,\mathrm{Cat}(\mathcal{V})$. This follows essentially from [HNP18, Section 3.1].

Indeed, note that the equivalence from Lemma 6.7 induces an equivalence on fibers

$$\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}\mathcal{V}) \times_{\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})} \{\mathbb{C}\} \simeq \mathcal{T}_{\mathbb{C}}\,\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \times_{\mathcal{T}_X \mathcal{S}} \{X\}.$$

Recall that under the equivalence of Lemma 6.7, the functor φ was simply given by projection onto the first factor. It will therefore suffice to verify that the composite

$$\begin{tikzcd} \mathcal{T}_{\mathbb{C}}\,\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \times_{\mathcal{T}_X \mathcal{S}} \{X\} \arrow[r, shift left, "\varphi=\pi_1"] & \mathcal{T}_{\mathbb{C}}\,\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \arrow[l, shift left, "\psi"] \arrow[r, shift left, "\mathcal{T}_{\mathbb{C}}((-)^\wedge)"] & \mathcal{T}_{\mathbb{C}}\,\mathrm{Cat}(\mathcal{V}) \arrow[l, shift left, "\mathcal{T}_{\mathbb{C}}(\iota)"] \end{tikzcd} \tag{6.5}$$

is an equivalence. As indicated, this composite admits a right adjoint: ψ is the right adjoint from Lemma 6.8, restricting the space of objects to X, and $\mathcal{T}(\iota)$ is induced by the canonical inclusion $\iota\colon \mathrm{Cat}(\mathcal{V}) \longrightarrow \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$.

Now note that the above diagram arises upon stabilization from the following diagram of adjunctions between ∞-categories of retractive objects over $\mathbb{C}$:

$$\begin{tikzcd} F\colon\ \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})_{\mathbb{C}//\mathbb{C}} \times_{\mathcal{S}_{X//X}} \{X\} \arrow[r, shift left, "\pi_1"] & \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})_{\mathbb{C}//\mathbb{C}} \arrow[l, shift left, "\psi'"] \arrow[r, shift left, "(-)^\wedge"] & \mathrm{Cat}(\mathcal{V})_{\mathbb{C}//\mathbb{C}}\colon G. \arrow[l, shift left, "\iota"] \end{tikzcd}$$

Here the right adjoint ψ' exists by (the opposite of) [Lur09, Corollary 5.2.7.11]. Explicitly, it takes the full sub-categorical algebra with objects X, i.e. the counit map $\psi'(\mathbb{D}) \longrightarrow \mathbb{D}$ is a Cartesian lift of the map of spaces $X = \mathrm{Ob}(\mathbb{C}) \longrightarrow \mathrm{Ob}(\mathbb{D})$ (cf. Definition 5.7). Upon stabilization, this induces the right adjoint pair in (6.5) by definition. For the left adjoint pair, note that $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})_{\mathbb{C}//\mathbb{C}} \times_{\mathcal{S}_{X//X}} \{X\}$ is a fiber product of pointed ∞-categories along left exact functors, and that stabilization preserves such fiber products. Consequently, the projection onto the first factor induces the functor ϕ, and since stabilization sends an adjoint pair of left exact functors to an adjunction between stable ∞-categories, its right adjoint ψ' induces the functor ψ at the stable level.

We now follow the same proof as [HNP18, Proposition 3.1.9]: applying (the ∞-categorical analogue of) [HNP19c, Corollary 2.39], it suffices to verify that the unit and counit of (F, G) become equivalences upon taking loop spaces. For the unit map, let $\mathbb{C} \to \mathbb{D} \to \mathbb{C}$ be a retract diagram of categorical algebras with spaces of objects X. Then $\mathbb{D} \longrightarrow GF(\mathbb{D})$ is the natural map obtained by decomposing $\mathbb{D} \longrightarrow \mathbb{D}^\wedge$ (essentially uniquely) into a map that is the identity on objects, followed by a fully faithful map. Since $\mathbb{D} \longrightarrow \mathbb{D}^\wedge$ is itself fully faithful, the unit is itself already an equivalence.

For the counit, let $\mathbb{C} \longrightarrow \mathbb{D} \longrightarrow \mathbb{C}$ be a retract diagram of $\mathcal{V}$-enriched categories. Then the counit map $\epsilon_\mathbb{D}\colon FG(\mathbb{D}) = \psi'(\mathbb{D})^\wedge \longrightarrow \mathbb{D}$ is the natural map from the completion of the full sub-categorical algebra $\psi'(\mathbb{D}) \longrightarrow \mathbb{D}$ with objects X. Since $\psi'(\mathbb{D}) \longrightarrow \mathbb{D}$ is fully faithful, $\epsilon_\mathbb{D}$ is fully faithful as well. Consequently, the base change $FG(\mathbb{D}) \times_\mathbb{D} \mathbb{C} \longrightarrow \mathbb{C}$ is fully faithful as well. This map has a canonical section (since we are working in $\mathrm{Cat}(\mathcal{V})_{\mathbb{C}//\mathbb{C}}$) and is hence also essentially surjective. Since all categorical algebras involved were complete, it follows that $FG(\mathbb{D}) \times_\mathbb{D} \mathbb{C} \simeq \mathbb{C}$ is the zero object in $\mathrm{Cat}(\mathcal{V})_{\mathbb{C}//\mathbb{C}}$. Using this, the looping of the counit map $\Omega_{/\mathbb{C}}(\epsilon_\mathbb{D})\colon \mathbb{C} \times_{FG(\mathbb{D})} \mathbb{C} \longrightarrow \mathbb{C} \times_\mathbb{D} \mathbb{C}$ can be identified with $\mathbb{C} \times_{FG(\mathbb{D})} \mathbb{C} \longrightarrow \mathbb{C} \times_{FG(\mathbb{D})} FG(\mathbb{D}) \times_\mathbb{D} \mathbb{C}$, which is the base change of an equivalence. It follows that the counit is an equivalence upon taking loop space objects, so that (6.5) is indeed an equivalence. $\square$

Proposition 6.9. *Let $\mathcal{V}$ be presentable monoidal ∞-category and suppose that $\mathcal{T}\mathcal{V}$ carries a monoidal t-orientation. Then the full subcategories $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}^{\geq 0}\mathcal{V})$ and $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}^{\leq 0}\mathcal{V})$ define a t-orientation on the stable Cartesian fibration $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}\mathcal{V}) \longrightarrow \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$.*

Proof. Theorem 6.4 and Proposition 6.6 imply that $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}\mathcal{V}) \longrightarrow \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$ is a stable Cartesian fibration, being the base change of such. For a fixed space of objects, the restrictions

$$\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}^{\geq 0}\mathcal{V}) \times_\mathcal{S} \{X\} \simeq \mathrm{Alg}_{\mathcal{O}_X}(\mathcal{T}^{\geq 0}\mathcal{V}), \qquad \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}^{\leq 0}\mathcal{V}) \times_\mathcal{S} \{X\} \simeq \mathrm{Alg}_{\mathcal{O}_X}(\mathcal{T}^{\leq 0}\mathcal{V}) \tag{6.6}$$

coincide with the t-orientation on $\mathcal{O}_X$-algebras from Proposition 2.25. In particular, this implies that $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}^{\geq 0}\mathcal{V})$ and $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}^{\leq 0}\mathcal{V})$ restrict to a t-structure on the fiber over a fixed categorical algebra $\mathbb{C}$.

For condition (1) of Definition 2.20, let $f\colon \mathbb{C} \longrightarrow \mathbb{D}$ be a map in $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$, $\mathbb{E} \in \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}^{\leq 0}\mathcal{V})$ an object living over $\mathbb{D}$ and $f^*\mathbb{E} \longrightarrow \mathbb{E}$ the Cartesian lift of f. To see that $f^*\mathbb{E} \in \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}^{\leq 0}\mathcal{V})$, factor f as a map $g\colon \mathbb{C} \longrightarrow \mathbb{D}'$ which is the identity on objects, followed by a fully faithful map $h\colon \mathbb{D}' \longrightarrow \mathbb{D}$. Then $h^*(\mathbb{E}) \longrightarrow \mathbb{E}$ is fully faithful; in particular, if all mapping objects of $\mathbb{E}$ are contained in $\mathcal{T}^{\leq 0}\mathcal{V}$, the same holds for $h^*(\mathbb{E})$. The map $f^*(\mathbb{E}) \longrightarrow h^*(\mathbb{E})$ is then a Cartesian arrow in $\mathrm{Alg}_{\mathcal{O}_{\mathrm{Ob}(\mathbb{C})}}(\mathcal{T}\mathcal{V})$, so that Proposition 2.25 implies that $f^*(\mathbb{E}) \in \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}^{\leq 0}\mathcal{V})$. $\square$

Corollary 6.10. *Let $\mathcal{V}$ be a differentiable presentable SM ∞-category such that $1_\mathcal{V}$ is compact and suppose that $\mathcal{T}\mathcal{V}$ carries a monoidal t-orientation. Under the equivalence $\mathcal{T}\,\mathrm{Cat}(\mathcal{V}) \simeq \mathrm{Cat}(\mathcal{T}\mathcal{V})$ from Theorem 6.4, the full subcategories*

$$\mathcal{T}^{\geq 0}\,\mathrm{Cat}(\mathcal{V}) \simeq \mathrm{Cat}(\mathcal{T}^{\geq 0}\mathcal{V}) \qquad \mathcal{T}^{\leq 0}\,\mathrm{Cat}(\mathcal{V}) \simeq \mathrm{Cat}(\mathcal{T}^{\leq 0}\mathcal{V})$$

then determine a monoidal t-orientation of the tangent bundle $\mathcal{T}\,\mathrm{Cat}(\mathcal{V})$, with heart $\mathcal{T}^\heartsuit\,\mathrm{Cat}(\mathcal{V}) \simeq \mathrm{Cat}(\mathcal{T}^\heartsuit\mathcal{V})$.

Proof. Using the left pullback square from Proposition 6.6, this is simply the base change of the t-orientation from Proposition 6.9. It is a monoidal t-structure because $\mathrm{Cat}(-)$ preserves symmetric monoidal functors and fully faithful functors, so that $\mathcal{T}^{\geq 0}\,\mathrm{Cat}(\mathcal{V}) \simeq \mathrm{Cat}(\mathcal{T}^{\geq 0}\mathcal{V}) \subseteq \mathcal{T}\,\mathrm{Cat}(\mathcal{V})$ is closed under the tensor product. $\square$

Remark 6.11. In the setting of Corollary 6.10, let $\mathbb{C}$ be $\mathcal{V}$-enriched category together with a π_0-surjection of spaces $X \longrightarrow \mathrm{Ob}(\mathbb{C})$. Let $\mathbb{C}_X \hookrightarrow \mathbb{C}$ be the induced fully faithful functor, which realizes $\mathbb{C}$ as the completion of $\mathbb{C}_X$. Theorem 6.4 and Proposition 6.6 then provide equivalences of stable ∞-categories

$$\mathcal{T}_{\mathbb{C}}\,\mathrm{Cat}(\mathcal{V}) \simeq \mathrm{Alg}_{\mathrm{Cat}}(\mathcal{T}\mathcal{V}) \times_{\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})} \{\mathbb{C}_X\} \simeq \mathrm{Alg}_{\mathcal{O}_X}(\mathcal{T}\mathcal{V}) \times_{\mathrm{Alg}_{\mathcal{O}_X}(\mathcal{V})} \{\mathbb{C}_X\}.$$

In the presence of a t-orientation, the proof of Proposition 6.9 (see Diagram (6.6)) shows that this identifies $\mathcal{T}^{\heartsuit}_{\mathbb{C}}\,\mathrm{Cat}(\mathcal{V})$ with the fiber $\mathrm{Alg}_{\mathcal{O}_X}(\mathcal{T}^{\heartsuit}\mathcal{V}) \times_{\mathrm{Alg}_{\mathcal{O}_X}(\mathcal{V})} \{\mathbb{C}_X\}$.

6.2. **Local systems of abelian groups on (∞,n)-categories.** Applying Corollary 6.10 inductively, starting with the t-structure on parametrized spectra from Example 2.24, we obtain the following:

Corollary 6.12. *Let $\mathcal{C}$ be an (∞,n)-category. Then the tangent ∞-category $\mathcal{T}_{\mathcal{C}}\,\mathrm{Cat}_{(\infty,n)}$ carries a t-structure, in which an object E is (co)connective if and only if for any two objects $x,y \in \mathcal{C}$, the functor*

$$\mathrm{Map}_{(-)}(x,y)\colon \mathcal{T}_{\mathcal{C}}\,\mathrm{Cat}_{(\infty,n)} \longrightarrow \mathcal{T}_{\mathrm{Map}_{\mathcal{C}}(x,y)}\,\mathrm{Cat}_{(\infty,n-1)}$$

sends E to a (co)connective object.

Applying this inductively, one finds the following inductive description of the heart of the t-orientation on $\mathcal{T}\,\mathrm{Cat}_{(\infty,n)}$:

Definition 6.13 (cf. [Lur09b, Definition 3.5.10])**.** The ∞-category of **local systems of abelian groups on $(\infty,0)$-categories** is defined to be the domain of the Cartesian fibration

$$\mathrm{Loc}_{(\infty,0)} \longrightarrow \mathcal{S}$$

classified by the functor $\mathcal{S}^{\mathrm{op}} \longrightarrow \mathrm{Cat}_\infty$ sending a space X to the category of local systems $\mathrm{Fun}\big(\Pi_1(X),\mathrm{Ab}\big)$. This carries a symmetric monoidal structure given by the Cartesian product. For $n \geq 1$, we define the symmetric monoidal ∞-category of **local systems of abelian groups on (∞,n)-categories** to be the domain of the Cartesian fibration

$$\mathrm{Loc}_{(\infty,n)} = \mathrm{Cat}\big(\mathrm{Loc}_{(\infty,n-1)}\big) \longrightarrow \mathrm{Cat}\big(\mathrm{Cat}_{(\infty,n-1)}\big) = \mathrm{Cat}_{(\infty,n)}\,.$$

Note that $\mathrm{Loc}_{(\infty,n)}$ inherits a symmetric monoidal structure from $\mathrm{Loc}_{(\infty,n-1)}$, such that the projection to $\mathrm{Cat}_{(\infty,n)}$ is symmetric monoidal.

For each $\mathcal{C}$, let us denote the fiber of $\mathrm{Loc}_{(\infty,n)}$ over $\mathcal{C}$ by $\mathrm{Loc}_{(\infty,n)}(\mathcal{C})$ and refer to it as the **abelian category of local systems on** $\mathcal{C}$. Note that $\mathrm{Loc}_{(\infty,n)}(\mathcal{C})$ is indeed an (ordinary) abelian category by the following immediate consequence of Corollary 6.10:

Corollary 6.14. *There are equivalences of ∞-categories over $\mathrm{Cat}_{(\infty,n)}$*

$$\mathrm{Loc}_{(\infty,n)} \simeq \mathrm{Cat}(\mathrm{Loc}_{(\infty,n-1)}) \simeq \mathrm{Cat}(\mathcal{T}^{\heartsuit}\,\mathrm{Cat}_{(\infty,n-1)}) \simeq \mathcal{T}^{\heartsuit}\,\mathrm{Cat}_{(\infty,n)}\,.$$

Given an (∞,n)-category $\mathcal{C}$, Remark 6.11 now implies that a local system $\mathcal{A}$ on $\mathcal{C}$ is given by the datum of map of ∞-operads

$$\begin{tikzcd} & & \mathrm{Loc}^{\otimes}_{(\infty,n-1)} \arrow[d] \\ \mathcal{O}_X \arrow[r] \arrow[urr, dotted, "\mathcal{A}"] & \mathcal{O}_{\mathrm{Ob}(\mathcal{C})} \arrow[r, "\mathcal{C}"'] & \mathrm{Cat}^{\otimes}_{(\infty,n-1)} \end{tikzcd}$$

for any choice of π_0-surjection of spaces $X \longrightarrow \mathrm{Ob}(\mathcal{C})$.

Remark 6.15. For the canonical choice $X = \mathrm{Ob}(\mathcal{C})$, the datum of a local system $\mathcal{A}$ on an (∞, n)-category $\mathcal{C}$ can therefore be described informally by the as follows:

(0) for each $x, y \in \mathcal{C}$, a local system $\mathcal{A}_{x,y}$ over the $(\infty, n-1)$-category of maps $\mathcal{C}(x, y)$.

(i) a map of local systems for each triple $x, y, z \in \mathcal{C}$ and a map of abelian groups for each $x \in \mathcal{C}$

$$m_{x,y,z} \colon p_1^*\mathcal{A}_{y,z} \times p_0^*\mathcal{A}_{x,y} \longrightarrow c^*\mathcal{A}_{x,z}, \qquad u_x \colon \mathbb{Z} \longrightarrow e^*\mathcal{A}_{x,x}.$$

Here $c\colon \mathcal{C}(y, z) \times \mathcal{C}(x, y) \longrightarrow \mathcal{C}(x, z)$ is the composition, $p_0 \colon \mathcal{C}(y, z) \times \mathcal{C}(x, y) \longrightarrow \mathcal{C}(y, z)$ and $p_1 \colon \mathcal{C}(y, z) \times \mathcal{C}(x, y) \longrightarrow \mathcal{C}(x, y)$ are the projections and $e\colon * \longrightarrow \mathcal{C}(x, x)$ is the unit.

(ii) an associativity condition for each quadruple $w, x, y, z \in \mathcal{C}$ and left and right unitality conditions for each tuple $x, y \in \mathcal{C}$, given by the commutativity of the following diagrams:

$$\begin{tikzcd} & p_2^*\mathcal{A}_{z,w} \times p_1^*\mathcal{A}_{y,z} \times p_0^*\mathcal{A}_{x,y} \arrow[dl, "m_{w,x,z}\circ(m_{x,y,z}\times\mathrm{id})"'] \arrow[dr, "m_{w,y,z}\circ(\mathrm{id}\times m_{w,x,y})"] & \\ (c\circ(\mathrm{id}\times c))^*\mathcal{A}_{x,w} \arrow[rr, "\alpha^*", "\cong"'] & & (c\circ(c\times\mathrm{id}))^*\mathcal{A}_{x,w} \end{tikzcd}$$

where α^* arises from the associator α of $\mathcal{C}$ by naturality of base change, and

$$\begin{tikzcd} & \mathcal{A}_{x,y} \arrow[dl, "m_{x,x,y}\circ(\mathrm{id}\times u_x)"'] \arrow[d, "="] \arrow[dr, "m_{y,y,x}\circ(u_y\times\mathrm{id})"] & \\ (c\circ(\mathrm{id}\times e))^*\mathcal{A}_{x,y} \arrow[r, "\rho^*", "\cong"'] & \mathcal{A}_{x,y} & (c\circ(e\times\mathrm{id}))^*\mathcal{A}_{x,y} \arrow[l, "\lambda^*"', "\cong"] \end{tikzcd}$$

where the bottom maps arise from the left and right unit equivalences λ and ρ.

Note that there are no higher coherences because the local systems over each $\mathcal{C}(x, y)$ form an ordinary 1-category. Definition 6.13 therefore gives a precise formulation of the (informal) definition of local systems on (∞, n)-categories appearing in [Lur09b, Definition 3.5.10].

Remark 6.16. Taking $X \longrightarrow \mathrm{Ob}(\mathcal{C})$ to be a π_0-surjection from a *set*, the datum of a local system over $\mathcal{C}$ can also be identified with a section of the map of operads $\mathcal{L}_{\mathcal{C},X} := \mathrm{Loc}^{\otimes}_{(\infty,n-1)} \times_{\mathrm{Cat}^{\otimes}_{(\infty,n-1)}} \mathcal{O}_X \longrightarrow \mathcal{O}_X$. Now note that $\mathrm{Loc}^{\otimes}_{(\infty,n-1)} \longrightarrow \mathrm{Cat}^{\otimes}_{(\infty,n-1)}$ induces maps on mapping spaces with *discrete fibers*. Since X is a set, both $\mathcal{O}_X$ and $\mathcal{L}_{\mathcal{C},X}$ are therefore *ordinary* operads and a section $\mathcal{O}_X \longrightarrow \mathcal{L}_{\mathcal{C},X}$ is given by choosing images of objects and multi-morphisms satisfying a certain associativity condition, but *no higher coherences*. In this situation, the informal description of a local system from Remark 6.15, for objects taken in the set X, is *exactly* the data of a local system on $\mathcal{C}$.

The inductive construction of the Postnikov structure in Theorem 6.3 shows that all parametrized spectra appearing in it are contained in the heart of the t-structure on $\mathcal{T}\,\mathrm{Cat}_{(\infty,n)}$. We therefore obtain the following result (which appears without proof as [Lur09b, Claim 3.5.18]):

Corollary 6.17. *For every (∞, n)-category $\mathcal{C}$, the parametrized spectrum $\mathrm{H}\pi_a(\mathcal{C})$ of Theorem 6.3 is the Eilenberg–Maclane spectrum associated to*

$$\pi_a(\mathcal{C}) \in \mathrm{Loc}_{(\infty,n)}(\mathcal{C}) = \mathcal{T}^{\heartsuit}_{\mathrm{ho}_{(n+1,n)}(\mathcal{C})}\,\mathrm{Cat}_{(\infty,n)}\,.$$

In terms of Remark 6.15, it is the Eilenberg–Maclane spectrum of the local system of abelian groups on $\mathrm{ho}_{(n+1,n)}\,\mathcal{C}$ *given inductively by* $\pi_a(\mathcal{C})_{x,y} = \pi_a \operatorname{Map}_{\mathcal{C}}(x, y)$*, for any* $x, y \in \mathcal{C}$*.*

Appendix A. Symmetric monoidal structure on categorical algebras

In this appendix, we provide the proofs of Proposition 5.3, Lemma 5.4 and Proposition 5.6 about the symmetric monoidal structure on $\mathrm{Cat}_{\mathrm{Alg}}(\mathcal{V})$. The key ingredient of these proofs will be Construction A.3: given an ∞-category $\mathcal{C}$ with finite products, this produces a diagram $F_\Psi\colon \mathcal{D} \longrightarrow \mathrm{SMCat}^{\mathrm{lax}}_{/\mathcal{C}^\times}$ of SM ∞-categories over $\mathcal{C}^\times$ from the data of a (suitable) diagram $\Psi\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \longrightarrow \mathrm{SMCat}^{\mathrm{lax}}_\infty$. Let us start with a preliminary observation:

Definition A.1. For an ∞-category $\mathcal{D}$, let us write

$$\mathrm{coCart}(\mathcal{D})^{\mathrm{lax}} \subseteq \mathrm{Cat}_{\infty/\mathcal{D}}, \qquad \mathrm{Cart}(\mathcal{D})^{\mathrm{opl}} \subseteq \mathrm{Cat}_{\infty/\mathcal{D}}$$

for the full subcategories spanned by the coCartesian and Cartesian fibrations, respectively. Furthermore, let us denote by $\mathrm{Fun}(\mathcal{D}, \mathrm{SMCat}^{\mathrm{lax}}_\infty)^{\mathrm{strong}} \hookrightarrow \mathrm{Fun}(\mathcal{D}, \mathrm{SMCat}^{\mathrm{lax}}_\infty)$ the wide subcategory whose morphisms $\mu\colon F \longrightarrow G$ are natural transformations such that for each $d \in \mathcal{D}$, the map $\mu_d\colon F(d) \longrightarrow G(d)$ is a strong (as opposed to lax) SM functor.

Lemma A.2. *For any ∞-category $\mathcal{D}$, there is a natural (wide) subcategory inclusion*

$$\mathrm{Fun}(\mathcal{D}, \mathrm{SMCat}^{\mathrm{lax}})^{\mathrm{strong}} \hookrightarrow \mathrm{CAlg}\left(\mathrm{Cart}(\mathcal{D}^{\mathrm{op}})^{\mathrm{opl}}\right)$$

where we take commutative algebras with respect to the fiber product over $\mathcal{D}$.

Proof. We will use unstraightening to identify both categories with (non-full) subcategories of $\mathrm{Cat}_{\infty/\mathcal{D}^{\mathrm{op}}\times\mathrm{Fin}_*}$ and then show that one is naturally included in the other. First, note that $\mathrm{Fun}(\mathcal{D}, \mathrm{SMCat}^{\mathrm{lax}})^{\mathrm{strong}}$ is a subcategory of $\mathrm{Fun}(\mathcal{D}, \mathrm{Cat}_{\infty/\mathrm{Fin}_*})$. By [HHLN23, Corollary 2.3.4], unstraightening to a Cartesian fibration over $\mathcal{D}$ then provides an equivalence between $\mathrm{Fun}(\mathcal{D}, \mathrm{SMCat}^{\mathrm{lax}})^{\mathrm{strong}}$ and the following subcategory of $\mathrm{Cat}_{\infty/\mathcal{D}^{\mathrm{op}}\times\mathrm{Fin}_*}$:

1. Objects are maps $p = (p_1, p_2)\colon \mathcal{E} \longrightarrow \mathcal{D}^{\mathrm{op}} \times \mathrm{Fin}_*$ such that p_1 is a Cartesian fibration, p_2 is a coCartesian fibration, p_1 sends p_2-coCartesian arrows to equivalences and p_2 sends p_1-Cartesian arrows to equivalences. Furthermore, for each $d \in \mathcal{D}$, the fiber $\mathcal{E}_d \longrightarrow \mathrm{Fin}_*$ is a SM ∞-category and for each $\alpha\colon d \to d'$ in $\mathcal{D}$, the change of fiber functor $\alpha^*\colon \mathcal{E}_{d'} \longrightarrow \mathcal{E}_d$ preserves p_2-coCartesian lifts of inert morphisms in Fin_*.
2. Morphisms are commuting triangles

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "f"] \arrow[dr, "(p_1,p_2)=p"'] & & \mathcal{E}' \arrow[dl, "q=(q_1,q_2)"] \\ & \mathcal{D}^{\mathrm{op}} \times \mathrm{Fin}_* & \end{tikzcd} \tag{A.1}$$

such that f sends all p_1-Cartesian arrows to q_1-Cartesian arrows and all p_2-coCartesian arrows to q_2-coCartesian arrows.

Similarly, we can view $\mathrm{CAlg}(\mathrm{Cart}(\mathcal{D}^{\mathrm{op}})^{\mathrm{opl}})$ as a subcategory of $\mathrm{Fun}(\mathrm{Fin}_*, \mathrm{Cat}_{\infty/\mathcal{D}^{\mathrm{op}}})$. By [HHLN23, Corollary 2.3.4], unstraightening to a coCartesian fibration over Fin_* then provides an equivalence between $\mathrm{CAlg}(\mathrm{Cart}(\mathcal{D}^{\mathrm{op}})^{\mathrm{opl}})$ and the following subcategory of $\mathrm{Cat}_{\infty/\mathcal{D}^{\mathrm{op}}\times\mathrm{Fin}_*}$:

(1') Objects are maps $p = (p_1, p_2)\colon \mathcal{E} \longrightarrow \mathcal{D}^{\mathrm{op}} \times \mathrm{Fin}_*$ such that p_1 is a Cartesian fibration, p_2 is a coCartesian fibration, p_1 sends p_2-coCartesian arrows to equivalences and p_2 sends p_1-Cartesian arrows to equivalences. Furthermore, for each $\langle n\rangle$ in Fin_*, the Segal maps induce an equivalence $\mathcal{E}_{\langle n\rangle} \simeq \mathcal{E}_{\langle 1\rangle} \times_{\mathcal{D}^{\mathrm{op}}} \cdots \times_{\mathcal{D}^{\mathrm{op}}} \mathcal{E}_{\langle 1\rangle}$ of Cartesian fibrations over $\mathcal{D}^{\mathrm{op}}$.

(2') Morphisms are commuting triangles (A.1) such that f sends all p_2-coCartesian arrows to q_2-coCartesian arrows.

Notice that conditions (1) and (1') are equivalent. Indeed, consider the Segal map $g\colon \mathcal{E}_{\langle n\rangle} \simeq \mathcal{E}_{\langle 1\rangle} \times_{\mathcal{D}^{\mathrm{op}}} \cdots \times_{\mathcal{D}^{\mathrm{op}}} \mathcal{E}_{\langle 1\rangle}$ between categories over $\mathcal{D}^{\mathrm{op}}$. Then g preserves Cartesian arrows over $\mathcal{D}^{\mathrm{op}}$ if and only if for each $\alpha\colon d \to d'$ in $\mathcal{D}$, the change of fiber functor $\alpha^*\colon \mathcal{E}_{d'} \longrightarrow \mathcal{E}_d$ preserves coCartesian lifts of inert morphisms in Fin_*. When this is the case, the Segal map is an equivalence if and only if it induces an equivalence between the fibers over each $d \in \mathcal{D}^{\mathrm{op}}$, i.e. iff each $\mathcal{E}_d$ is an SM ∞-category. We therefore obtain two subcategories with the same objects, while on morphisms the condition (2) is clearly stronger than (2'). This yields the desired wide subcategory inclusion. $\square$

Construction A.3. Let $\mathcal{C}$ be an ∞-category with finite products, $\mathcal{D}$ an ∞-category and consider a functor $\Psi\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \longrightarrow \mathrm{SMCat}^{\mathrm{lax}}$ that sends each arrow in $\mathcal{C}^{\mathrm{op}}$ to a strong SM functor. We will construct from Ψ a natural functor $F_\Psi\colon \mathcal{D} \longrightarrow \mathrm{SMCat}^{\mathrm{lax}}_{/\mathcal{C}^\times}$ where $F_\Psi(d) \longrightarrow \mathcal{C}^\times$ is a SM functor whose underlying functor is the Cartesian fibration classified by $\Psi(-,d)\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Cat}_\infty$.

To do this, note that by adjunction and Lemma A.2, we obtain a natural functor

$$\Psi\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Fun}\left(\mathcal{D}, \mathrm{SMCat}^{\mathrm{lax}}_\infty\right)^{\mathrm{strong}} \longrightarrow \mathrm{CAlg}\left(\mathrm{Cart}(\mathcal{D}^{\mathrm{op}})^{\mathrm{opl}}\right).$$

By [Lur17, Theorem 2.4.3.18], this defines a map of ∞-operads

$$(\mathcal{C}^{\mathrm{op}})^{\amalg} \longrightarrow \mathrm{Cart}(\mathcal{D}^{\mathrm{op}})^{\mathrm{opl},\times} \simeq \mathrm{coCart}(\mathcal{D})^{\mathrm{lax},\times}$$

from the coCartesian ∞-operad $(\mathcal{C}^{\mathrm{op}})^{\amalg}$ to the Cartesian operad $\mathrm{coCart}(\mathcal{D})^{\mathrm{lax},\times}$. Here the equivalence of Cartesian operads arises from the equivalence sending a Cartesian fibration $\mathcal{E} \longrightarrow \mathcal{D}^{\mathrm{op}}$ to the opposite coCartesian fibration $\mathcal{E}^{\mathrm{op}} \longrightarrow \mathcal{D}$.

Since the target of the above map is a Cartesian ∞-operad, this is uniquely determined by an $(\mathcal{C}^{\mathrm{op}})^{\amalg}$-monoid object $(\mathcal{C}^{\mathrm{op}})^{\amalg} \longrightarrow \mathrm{coCart}(\mathcal{D})^{\mathrm{lax}}$. The unstraightening of this functor determines a functor

$$p = (p_1, p_2)\colon \mathcal{X}_\Psi^{\circ,\otimes} \longrightarrow (\mathcal{C}^{\mathrm{op}})^{\amalg} \times \mathcal{D} \tag{A.2}$$

with the following properties:

(1) $p_1\colon \mathcal{X}_\Psi^{\circ,\otimes} \longrightarrow (\mathcal{C}^{\mathrm{op}})^{\amalg}$ is a coCartesian fibration and p_2 sends p_1-coCartesian arrows to equivalences.

(2) For each $\langle n\rangle \in \mathrm{Fin}_*$, the n inert maps $\sigma_i\colon \langle n\rangle \longrightarrow \langle 1\rangle$ induce an equivalence $\mathcal{X}^{\circ,\otimes}_{\Psi,\langle n\rangle} \longrightarrow \mathcal{X}^{\circ,\otimes}_{\Psi,\langle 1\rangle} \times_{\mathcal{D}} \cdots \times_{\mathcal{D}} \mathcal{X}^{\circ,\otimes}_{\Psi,\langle 1\rangle}$.

(3) For each $\langle n\rangle \in \mathrm{Fin}_*$, the map between fibers over $\langle n\rangle$

$$p\colon \mathcal{X}^{\circ,\otimes}_{\Psi,\langle n\rangle} \longrightarrow (\mathcal{C}^{\mathrm{op}})^{\amalg}_{\langle n\rangle} \times \mathcal{D} \simeq (\mathcal{C}^{\mathrm{op}})^{\times n} \times \mathcal{D}$$

is a coCartesian fibration, classified by the functor sending $(c_1, \ldots, c_n, d)$ to $\Psi(c_1, d)^{\mathrm{op}} \times \cdots \times \Psi(c_n, d)^{\mathrm{op}}$.

Here (2) is equivalent to $(\mathcal{C}^{\mathrm{op}})^{\amalg} \longrightarrow \mathrm{coCart}(\mathcal{D})^{\mathrm{lax}}$ being a monoid object, after which (3) is equivalent to the fact that the underlying functor $\mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{coCart}(\mathcal{D})^{\mathrm{lax}}$ corresponds to $\Psi\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \longrightarrow \mathrm{Cat}$ under unstraightening over $\mathcal{D}$.

In particular, these conditions imply that for each $d \in \mathcal{D}$, the map between fibers $p_1\colon p_2^{-1}(d) \longrightarrow (\mathcal{C}^{\mathrm{op}})^{\amalg}$ is a coCartesian fibration of ∞-operads and that each map $d \to d'$ induces a map of ∞-operads $p_2^{-1}(d) \longrightarrow p_2^{-1}(d')$ over $(\mathcal{C}^{\mathrm{op}})^{\amalg}$. Unraveling the definitions, the coCartesian fibration $p_1\colon p_2^{-1}(d) \longrightarrow (\mathcal{C}^{\mathrm{op}})^{\amalg}$ arises as the coCartesian unstraightening of the functor

$$\mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Cat}; \quad c \longmapsto \Psi(c, d)^{\mathrm{op}}$$

with lax monoidal structure maps given by

$$\Psi(c, d)^{\mathrm{op}} \times \Psi(c', d)^{\mathrm{op}} \longrightarrow \Psi(c \times c', d)^{\mathrm{op}} \times \Psi(c \times c', d)^{\mathrm{op}} \longrightarrow \Psi(c \times c', d)^{\mathrm{op}}. \tag{A.3}$$

Here the first map restricts along the maps $c \leftarrow c \times c' \to c'$ and the second map uses the SM structure on $\Psi(c \times c', d)^{\mathrm{op}}$. Similarly, unwinding the construction shows that the coCartesian fibration $p_1^{-1}(c_1, \ldots, c_n) \longrightarrow \mathcal{D}$ is classified by the functor sending d to $\Psi(c_1, d)^{\mathrm{op}} \times \cdots \times \Psi(c_n, d)^{\mathrm{op}}$.

Postcomposing with the map $(\mathcal{C}^{\mathrm{op}})^{\amalg} \longrightarrow \mathrm{Fin}_*$, one can view (A.2) as a map of coCartesian fibrations over Fin_*. Let us take the induced map of fiberwise opposite coCartesian fibrations, i.e. the coCartesian fibrations classifying the Fin_*-diagram of opposite ∞-categories [BGN18]. This yields a diagram of the form

$$\begin{tikzcd} \mathcal{X}_{\Psi}^{\otimes} \arrow[rr, "q=(q_1, q_2)"] \arrow[dr, "r"'] & & \mathcal{C}^{\times} \times \mathcal{D}^{\mathrm{op}} \arrow[dl] \\ & \mathrm{Fin}_* & \end{tikzcd}$$

where $\mathcal{C}^{\times} \to \mathrm{Fin}_*$ is the Cartesian operad associated to the ∞-category with products $\mathcal{C}$ (which is the fiberwise opposite of $(\mathcal{C}^{\mathrm{op}})^{\amalg}$, cf. [Lur17, Variant 2.4.3.12]). Here the map q has the following properties (which correspond to the properties of p under taking fiberwise opposites over Fin_*):

(1) q is a map of coCartesian fibrations over Fin_* preserving coCartesian arrows.
(2) For each $\langle n \rangle \in \mathrm{Fin}_*$, the n inert maps $\sigma_i\colon \langle n \rangle \longrightarrow \langle 1 \rangle$ induce an equivalence $\mathcal{X}_{\Psi,\langle n \rangle}^{\otimes} \longrightarrow \mathcal{X}_{\Psi,\langle 1 \rangle}^{\otimes} \times_{\mathcal{D}} \cdots \times_{\mathcal{D}} \mathcal{X}_{\Psi,\langle 1 \rangle}^{\circ,\otimes}$.
(3) For each $\langle n \rangle$ in Fin_*, the map on fibers $\mathcal{X}_{\Psi,\langle n \rangle}^{\otimes} \longrightarrow \mathcal{C}_{\langle n \rangle}^{\times} \times \mathcal{D}^{\mathrm{op}} \simeq \mathcal{C}^{\times n} \times \mathcal{D}^{\mathrm{op}}$ is a Cartesian fibration classified by the functor sending $(c_1, \ldots, c_n, d) \longmapsto \Psi(c_1, d) \times \cdots \times \Psi(c_n, d)$.

In particular, the map $(r, q_2)\colon \mathcal{X}_{\Psi}^{\otimes} \longrightarrow \mathrm{Fin}_* \times \mathcal{D}^{\mathrm{op}}$ has the property that (1) r is a coCartesian fibration and that q_2 sends r-coCartesian arrows to equivalences in $\mathcal{D}^{\mathrm{op}}$ and (2) for each $\langle n \rangle \in \mathrm{Fin}_*$, the map $q_2\colon \mathcal{X}_{\Psi,\langle n \rangle}^{\otimes} \longrightarrow \mathcal{D}^{\mathrm{op}}$ is a Cartesian fibration. It follows from [HHLN23, Proposition 2.3.3] that $q_2\colon \mathcal{X}_{\Psi}^{\otimes} \longrightarrow \mathcal{D}^{\mathrm{op}}$ is a Cartesian fibration and that q is a map of Cartesian fibrations over $\mathcal{D}^{\mathrm{op}}$ (preserving Cartesian arrows). We can therefore apply straightening over $\mathcal{D}$, and the above three conditions then imply that the straightening determines the desired functor $F_{\Psi}\colon \mathcal{D} \longrightarrow \mathrm{SMCat}_{/\mathcal{C}^{\times}}^{\mathrm{lax}}$.

Proof of Proposition 5.3. Recall that there is a functor $\mathrm{Op}_{\infty}^{\mathrm{op}} \times \mathrm{Op}_{\infty} \longrightarrow \mathrm{Op}_{\infty}$ sending $(\mathcal{O}, \mathcal{P})$ to the ∞-operad of algebras $\mathrm{Alg}_{\mathcal{O}}(\mathcal{P})^{\otimes}$. This takes values in $\mathrm{SMCat}_{\infty}^{\mathrm{lax}}$ if $\mathcal{P}$ is an SM ∞-category [Lur17, Example 3.2.4.4], in which case restriction along $\mathcal{O} \longrightarrow \mathcal{O}'$ determines a

SM functor $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{P}) \longrightarrow \mathrm{Alg}_{\mathcal{O}}(\mathcal{P})$. We can thus apply Construction A.3 to the functor $\mathcal{S}^{\mathrm{op}} \times \mathrm{SMCat}^{\mathrm{lax}}_{\infty} \longrightarrow \mathrm{SMCat}^{\mathrm{lax}}_{\infty}$ sending $X \longmapsto \mathrm{Alg}_{\mathcal{O}_X}(\mathcal{V})$. □

Proof of Lemma 5.4. Unraveling Construction A.3, the structure of the SM functor $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V}) \longrightarrow \mathcal{S}$ arises as the straightening over Fin_* of the map $q_1\colon \mathrm{Alg}^{\otimes}_{\mathrm{Cat}}(\mathcal{V}) = q_2^{-1}(\mathcal{V}) \longrightarrow \mathcal{S}^{\times}$. By construction, the straightening of q_1 is the pointwise opposite of the straightening of $p_1\colon p_2^{-1}(\mathcal{V}) \longrightarrow (\mathcal{S}^{\mathrm{op}})^{\amalg}$ over Fin_* by taking fiberwise opposites over Fin_*. Consequently, the tensor product on $\mathrm{Alg}_{\mathrm{Cat}}(\mathcal{V})$ is arises as the unstraightening of the opposite of the natural transformation (A.3), as desired. □

Proof of Proposition 5.6. Construction A.3 has the following general property: for any $f\colon \mathcal{D}' \longrightarrow \mathcal{D}$ and $\Psi\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \longrightarrow \mathrm{SMCat}^{\mathrm{lax}}$, the functor $F_{\Psi\circ(\mathrm{id}\times f)}\colon \mathcal{D}' \longrightarrow \mathrm{SMCat}^{\mathrm{lax}}_{/\mathcal{C}^{\times}}$ is naturally equivalent to the functor $F_{\psi} \circ f$. Consequently, the left-bottom composite in Proposition 5.6 arises by applying Construction A.3 to the functor $\Psi_{\mathcal{I}}\colon \mathcal{S}^{\mathrm{op}} \times \mathrm{SMCat}^{\mathrm{lax}} \longrightarrow \mathrm{SMCat}^{\mathrm{lax}}$ sending $(X, \mathcal{V})$ to $\mathrm{Alg}_{\mathcal{O}_X}\left(\mathrm{Fun}(\mathcal{I}, \mathcal{V})\right)^{\otimes}$.

Now notice that $\Psi_{\mathcal{I}}$ is equivalent to the functor sending $(X, \mathcal{V})$ to $\mathrm{Fun}(\mathcal{I}, \mathrm{Alg}_{\mathcal{O}_X}(\mathcal{V}))$ with the levelwise tensor product (by adjunction to the Boardman–Vogt tensor product, see Remark 2.7). The result will therefore follow from the following general claim about Construction A.3: for any $\Psi\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \longrightarrow \mathrm{SMCat}^{\mathrm{lax}}$ and any ∞-category $\mathcal{I}$, applying Construction A.3 to the functor $\Psi_{\mathcal{I}}(c, d) = \mathrm{Fun}(\mathcal{I}, \Psi(c, d))$ results in the composite functor

$$F_{\Psi_{\mathcal{I}}}\colon \mathcal{D} \xrightarrow{F_{\Psi}} \mathrm{SMCat}^{\mathrm{lax}}_{\infty/\mathcal{C}^{\times}} \xrightarrow{\mathrm{Fun}(\mathcal{I},-)\times_{\mathrm{Fun}(\mathcal{I},\mathcal{C}^{\times})}\mathcal{C}^{\times}} \mathrm{SMCat}^{\mathrm{lax}}_{\infty/\mathcal{C}^{\times}}. \tag{A.4}$$

To see this, recall the inclusion $\mathrm{Fun}(\mathcal{D}, \mathrm{SMCat}^{\mathrm{lax}}) \hookrightarrow \mathrm{CAlg}\left(\mathrm{Cart}(\mathcal{D}^{\mathrm{op}})^{\mathrm{opl}}\right)$ from Lemma A.2, which was given by unstraightening over $\mathcal{D}$. Under this inclusion, applying $\mathrm{Fun}(\mathcal{I}, -)$ pointwise corresponds to sending a Cartesian fibration $\mathcal{E}^{\mathrm{op}} \longrightarrow \mathcal{D}^{\mathrm{op}}$ to the Cartesian fibration $\mathrm{Fun}(\mathcal{I}, \mathcal{E}^{\mathrm{op}}) \times_{\mathrm{Fun}(\mathcal{I},\mathcal{D}^{\mathrm{op}})} \mathcal{D}^{\mathrm{op}} \longrightarrow \mathcal{D}^{\mathrm{op}}$. This implies that the monoid object determined by $\Psi_{\mathcal{I}}$ is given by the composite

$$(\mathcal{C}^{\mathrm{op}})^{\amalg} \longrightarrow \mathrm{coCart}(\mathcal{D})^{\mathrm{lax}} \longrightarrow \mathrm{coCart}(\mathcal{D})^{\mathrm{lax}}$$

where the first functor is the monoid object associated to Ψ and the second functor sends a coCartesian fibration $\mathcal{E} \longrightarrow \mathcal{D}$ to $\mathrm{Fun}(\mathcal{I}^{\mathrm{op}}, \mathcal{E}) \times_{\mathrm{Fun}(\mathcal{I}^{\mathrm{op}},\mathcal{D})} \mathcal{D} \longrightarrow \mathcal{D}$ (note that we took opposite categories to pass from Cartesian to coCartesian fibrations). Next, applying the same reasoning to the unstraightening over $(\mathcal{C}^{\mathrm{op}})^{\amalg}$, we obtain that

$$\mathcal{X}^{\circ,\otimes}_{\Psi_{\mathcal{I}}} \simeq \mathrm{Fun}\left(\mathcal{I}^{\mathrm{op}}, \mathcal{X}^{\circ,\otimes}_{\Psi}\right) \times_{\mathrm{Fun}(\mathcal{I}^{\mathrm{op}},(\mathcal{C}^{\mathrm{op}})^{\amalg}\times\mathcal{D})} (\mathcal{C}^{\mathrm{op}})^{\amalg} \times \mathcal{D}.$$

Taking fiberwise opposite coCartesian fibrations over Fin_*, one then obtains an equivalence of Cartesian fibrations over $\mathcal{D}^{\mathrm{op}}$

$$\mathcal{X}^{\otimes}_{\Psi_{\mathcal{I}}} \simeq \mathrm{Fun}\left(\mathcal{I}, \mathcal{X}^{\otimes}_{\Psi}\right) \times_{\mathrm{Fun}(\mathcal{I},\mathcal{C}^{\times}\times\mathcal{D}^{\mathrm{op}})} \mathcal{C}^{\times} \times \mathcal{D}^{\mathrm{op}}.$$

Under straightening over $\mathcal{D}^{\mathrm{op}}$, this equivalence provides the desired identification of $F_{\Psi_{\mathcal{I}}}$ as in (A.4). □

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