Lie algebroids in derived differential topology

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Contents

Chapter 1

Introduction

This thesis studies the role of Lie algebroids in deformation theory and derived differential topology. More precisely, we discuss the homotopy theory of 'derived Lie algebroids' and show that derived Lie algebroids arise naturally as the objects classifying formal deformation problems. We use this to examine the relationship between higher stacks and their Lie algebroids in the setting of (derived) differential topology, leading to a version of the Van Est isomorphism.

A theorem of Van Est. A basic principle in Lie theory asserts that there is an intimate relationship between Lie groups and their associated Lie algebras, especially when the Lie groups in question are highly connected. For example, a classical result of Van Est [28] provides an isomorphism

$$H^n(G, \mathbb{R}) \xrightarrow{\cong} H^n(\mathfrak{g}, \mathbb{R})$$

between the cohomology of an n-connected Lie group G and the cohomology of its Lie algebra $\mathfrak{g}$.

The Lie algebra $\mathfrak{g}$, and hence the Van Est isomorphism, can be thought of in two different ways. On the one hand, we can identify $\mathfrak{g}$ with the Lie algebra of left invariant vector fields on the Lie group G. The cohomology of $\mathfrak{g}$ can then be identified with the cohomology of the G-invariant de Rham complex of G. This complex is equivalent to the (smooth) *homotopy* G-invariant de Rham complex of G, which can be computed using a cosimplicial complex given in degree k by the forms on G^{k+1} that are tangent to the last copy of G.

The Van Est homomorphism then arises from the inclusion of the constant $\mathbb{R}$-valued functions on G into its de Rham complex, by passing to (smooth) homotopy G-invariants

$$\mathbb{R}^{\mathrm{h}G} \longrightarrow \Omega^\bullet(G)^{\mathrm{h}G} \simeq \Omega^\bullet(G)^G.$$

Alternatively, we can identify the Lie algebra $\mathfrak{g}$ with the tangent space to G at the identity and think of the Van Est map as differentiation. This point of view admits a natural description in terms of classifying spaces: the cohomology groups of G can be identified with the sets of homotopy classes of maps $BG \longrightarrow K(\mathbb{R}, n)$ from the classifying space of G into an Eilenberg-Maclane space. In the present case, both G and $\mathbb{R}$ carry a differentiable structure, which induces a differentiable structure on BG and $K(\mathbb{R}, n)$. This differentiable structure can be made explicit by realizing both objects by (higher) stacks, or by presenting them by simplicial manifolds.

For example, the classifying space BG can be presented by a simplicial manifold whose only nontrivial simplicial homotopy group is $\pi_1(BG) = G$, considered as a Lie group. The nerve of G is the canonical choice for such a simplicial manifold. Similarly, $K(\mathbb{R}, n)$ can be presented by a simplicial manifold whose only nontrivial homotopy group is $\pi_n(K(\mathbb{R}, n)) = \mathbb{R}$. Both of these simplicial manifolds admit a canonical basepoint and $H^n(G, \mathbb{R})$ can be identified with the set of pointed maps

$$\alpha\colon BG \longrightarrow K(\mathbb{R}, n)$$

up to pointed simplicial homotopy. Similarly, the Lie algebra cohomology group $H^n(\mathfrak{g}, \mathbb{R})$ can be identified with the set of maps $\mathfrak{g} \longrightarrow K(\mathbb{R}, n-1)$ in the homotopy category of simplicial Lie algebras, where $K(\mathbb{R}, n-1)$ has zero bracket (see e.g. [81]). The Van Est homomorphism therefore yields a map

$$\big[BG, K(\mathbb{R}, n)\big]_* \longrightarrow \big[\mathfrak{g}, K(\mathbb{R}, n-1)\big] \cong H^n(\mathfrak{g}, \mathbb{R}) \tag{1.0.1}$$

between (pointed) homotopy classes of maps. The Van Est homomorphism sends a map $\alpha\colon BG \longrightarrow K(\mathbb{R}, n)$, to an explicit Lie algebra cocycle, which only depends on the n-th derivative of α at the basepoint of BG (see [28, Section 11]). We can therefore think of (1.0.1) as differentiating a map $BG \longrightarrow K(\mathbb{R}, n)$ at the basepoint and passing to loop spaces.

The purpose of this thesis is to describe an analogue of the Van Est theorem that applies not only to the 'classifying spaces' BG and $K(\mathbb{R}, n)$, but also to homotopical variants thereof (see Theorem II). For instance, the classifying space BGL_n of vector bundles admits a natural homotopical analogue, given by the classifying space for chain complexes of vector bundles (or rather, perfect complexes [96, 97]).

Similarly, instead of the classifying space of a Lie group G, one can consider the classifying space of its n-connective cover, or a geometric refinement thereof. Such a space classifies principal G-bundles, equipped with trivializations of various of their characteristic classes. For example, the classifying space of the String-group classifies oriented real vector bundles, together with a trivialization of their second Stiefel-Whitney class and fractional first Pontryagin class $\frac{1}{2}p_1$ (see e.g. [44, 93, 84]).

In a different direction, one can replace Lie groups by Lie groupoids, for which there is an analogue of the Van Est theorem [20]. The classifying space of a Lie groupoid $\mathcal{G} \rightrightarrows M$ does not have a canonical basepoint, but an entire manifold M of basepoints.

Each of these 'spaces' is an example of a higher (geometric) stack [89, 97]. In various recent works [39, 105, 77, 8, 104], such higher stacks have been studied in terms higher Lie groupoids, simplicial manifolds satisfying a smooth variant of the usual Kan conditions for simplicial sets. As a first result, let us mention that such higher stacks give rise to Lie algebroids:

Theorem 0. *Let $f\colon M \longrightarrow X$ be a map from a manifold to a higher (derived) stack and consider the canonical map*

$$T_{M/X} = \mathrm{Hom}_{\mathcal{O}_M}(L_{M/X}, \mathcal{O}_M) \longrightarrow T_M$$

from the derived tangent bundle to the fibers of f. This map is the anchor map of a (derived) Lie algebroid over M.

Theorem 0 is not as straightforward as the usual construction of Lie algebras out of Lie groups (or Lie algebroids out of Lie groupoids). Indeed, suppose that $* \longrightarrow BG$ is the basepoint of the classifying stack of a *higher* Lie group, i.e. a stack with a compatible group structure. To mimic the first description of the Lie algebra $\mathfrak{g}$ mentioned above, we would have to provide a strict model for the Lie algebra of vector fields on the stack $\mathcal{G}$. On the other hand, the second description requires a strictly associative multiplication on G, providing a (strict) adjoint representation on T_eG.

Because of the homotopical natural of higher stacks, such strict structures are not readily available. For example, higher Lie groupoids do not come with explicit multiplication maps, but instead admit local lifts that guarantee the existence of a homotopically unique composition. In fact, a map of stacks $M \longrightarrow X$ need not even be representable by a map from M to a higher Lie groupoid.

For these reasons, we will describe a more homotopy-theoretic method for constructing Lie algebroids. This method is based on (derived) deformation theory.

Deformation theory. Generally speaking, a formal deformation problem is a question of the following form: if E_0 is a certain mathematical object, what are the infinitesimal families of such objects around E_0? The study of such deformation problems has a long history, originating (at least) in the work of Fröhlicher and Nijenhuis [31] and Kodaira and Spencer [55] on the deformation theory of complex manifolds. It turns out that in many situations, the possible infinitesimal deformations of E_0 can be classified explicitly in terms of Lie-algebraic data.

This has lead to the following principle, due to Deligne and Drinfeld (tracing back to Quillen's work on rational homotopy theory [81]):

> Any reasonable formal deformation problem over a field of characteristic zero is controlled by a differential graded Lie algebra.

The Lie algebra of a Lie group arises naturally from this principle. Indeed, let G be a Lie group and consider G as a G-torsor. All infinitesimal families of G-torsors around G are trivial, but there are many infinitesimal families of *automorphisms* of the G-torsor G. For example, a first order family of automorphisms

$$\{\phi_\epsilon\}_{\epsilon^2=0}\colon G \longrightarrow G$$

is simply a left-invariant vector field on G. Because of this, the dg-Lie algebra that governs the deformations of the G-torsor G is simply the Lie algebra of left invariant vector fields on G (cf. Proposition 6.4.30).

The above principle has been developed further by Kontsevich [56], Hinich [40], Manetti [65], Pridham [75] and Lurie [61] (among many others), in terms of *derived* deformation theory. In terms of algebra, the passage to derived deformation theory can be described as follows. Consider an object E_0 defined over a field k, such as a module, an algebra or a variety. An infinitesimal deformation of E_0 is given by an object E defined over a local Artin k-algebra A, whose fiber of k is equivalent to E_0. In addition to such deformations of E_0 over Artin algebras, one can often study derived deformations of E_0 along dg-Artin algebras as well.

The collection of such derived deformations of E_0 can be organized into a functor

$$\mathrm{dgArt}_k \longrightarrow \mathcal{S}$$

sending each dg-Artin algebra A to the space of deformations of E over A. In this way, 'reasonable' deformation problems determine functors that satisfy a version of the Schlessinger conditions [83], called *formal moduli problems*. The above principle can then be formulated more precisely as an equivalence between the ∞-category of formal moduli problems and the ∞-category of dg-Lie algebras [61].

The first half of the thesis is devoted to an extension of the above result to the case where the field k is replaced by a commutative dg-algebra A. There is a natural notion of a formal moduli problem over such a dg-algebra A, given by a diagram of spaces satisfying a version of the Schlessinger conditions. The indexing category of such a formal moduli problem consists of certain derived nilpotent extensions $A' \longrightarrow A$, which need not admit a section. The main result is the following:

Theorem I. *Let A be a cofibrant commutative dg-algebra over a field of characteristic zero with the property that $\pi_i(A) = 0$ for $i \gg 0$. Then there is an equivalence between the ∞-category of derived Lie algebroids over A and the ∞-category of formal moduli problems over A.*

Among other things, this requires a reasonable homotopy theory of derived Lie algebroids over a commutative dg-algebra A. We provide such a homotopy theory by proving that the category of (unbounded) dg-Lie algebroids over A admits a semi-model structure whose weak equivalences are the quasi-isomorphisms.

Derived differential topology. Although the formulation and proof of Theorem I are essentially algebraic, it should be considered as a geometric statement: if E_0 is a certain mathematical object defined over a space $\mathrm{Spec}(A)$, then the (derived) infinitesimal families of such objects around E_0 are controlled by a Lie algebroid over $\mathrm{Spec}(A)$.

The object E_0 is often classified by a map $f\colon \mathrm{Spec}(A) \longrightarrow X$ into a certain moduli space, so that a family of objects around E_0 is classified by a family of maps around f. For example, the G-torsor G is classified by the basepoint $* \longrightarrow BG$ of the classfying space of G. One can therefore rephrase Theorem I somewhat informally as follows:

> Let A be as in Theorem I and let $f\colon M = \mathrm{Spec}(A) \longrightarrow X$ be a map to a moduli space (over a field of characteristic zero). Then a formal neighbourhood of M in X is controlled by a derived Lie algebroid over M.

This perspective also appears in [33], where Lie algebroids are essentially *defined* as certain formal (derived, stacky) nilpotent thickenings of $\mathrm{Spec}(A)$.

To make the above assertion more precise, the moduli space X has to be considered as an object in *derived geometry*. In an algebro-geometric setting, this means that X should be a derived (Artin) stack, for which there is a well-developed theory due to Toën and Vezzosi (see e.g. [97]) and Lurie [63], among others. Given a map $f\colon \mathrm{Spec}(A) \longrightarrow X$ into a derived Artin stack, the spaces of infinitesimal deformations

$$\begin{tikzcd} \mathrm{Spec}(A) \arrow[r, "f"] \arrow[d] & X \\ \mathrm{Spec}(A') \arrow[ur, dotted] & \end{tikzcd}$$

can be organized into a formal moduli problem over A. This formal moduli problem is classified by a derived Lie algebroid over A.

Theorem 0 is proven by adapting this simple argument to the setting of *derived differential topology*. There are various approaches to derived differential topology, due to Spivak [92], Joyce [51] and Borisov and Noel [14], which are all based on the theory of $\mathcal{C}^\infty$-rings (see e.g. [68]). These works mostly concentrate on applications of derived differential topology to intersection theory, and therefore only consider 'quasi-smooth' spaces, arising as derived intersections of two smooth manifolds.

To study deformation theory, we need to make use of spaces that are significantly more singular than these derived intersections. For this reason, the second half of this thesis contains a rather extensive account of derived differential topology: we extended the theory from quasi-smooth derived manifolds to more general derived stacks, with emphasis on their infinitesimal (deformation-theoretic) aspects. This follows the lines of the aforementioned work of Toën, Vezzosi and Lurie in derived algebraic geometry.

The infinitesimal theory of stacks in derived differential topology is essentially the same as in derived algebraic geometry. In particular, any map $f\colon M \longrightarrow X$ from a (derived) manifold to a derived stack gives rise to a formal moduli problem over M. Using a version of Theorem I for $\mathcal{C}^\infty$-rings, such a formal moduli problem determines a derived Lie algebroid over M.

With a reasonable theory of derived differential topology in place, we can address the promised analogue of the Van Est theorem:

Theorem II. *Let $p\colon M \longrightarrow X$ be a smooth surjection from a smooth manifold to a smooth (higher) stack and let $M \longrightarrow Y$ be any map into a derived n-stack. If the fibers of p are n-connected, then the map*

$$\mathrm{Map}_{M/}(X, Y) \longrightarrow \mathrm{Map}_{\mathrm{LieAlgd}_M}(T_{M/X}, T_{M/Y})$$

is an equivalence. In other words, any map of Lie algebroids $T_{M/X} \longrightarrow T_{M/Y}$ integrates to a map of stacks (under M).

When $\mathcal{G} \rightrightarrows M$ is an ordinary Lie groupoid and $p\colon M \longrightarrow X = M/\mathcal{G}$ is the induced map to the quotient stack, the fibers of p can be identified with the source fibers of $\mathcal{G}$. Taking $Y = K(\mathbb{R}, n)$, the above result then reduces to the Van Est isomorphism.

To some extend, the methods leading to Theorem II are more interesting than the result itself and make use of two particular (analytic) features of (derived) differential topology. Because the Lie algebroid $T_{M/X}$ describes the formal neighbourhood of M inside X, the problem of integrating a map between Lie algebroids comes down to an extension problem

$$\begin{tikzcd} M \arrow[r] \arrow[dr] & X^\wedge_M \arrow[r] \arrow[d] & X \arrow[d, dotted, "\exists ?"] \\ & Y^\wedge_M \arrow[r] & Y \end{tikzcd}$$

where $X^\wedge_M$ is the *formal completion* of X at M. This extension problem can be addressed *locally* on M and X and then involves a descent argument to patch together various local extensions. Finding a local extension essentially reduces to solving a parallel transport equation; the analytical nature of differential topology ensures that such differential equations have a solution.

The problem of patching together local extensions is greatly simplified by another property of differential topology: the fibers of a smooth map are locally equivalent to $\mathbb{R}^n$, and in particular locally contractible. This implies that the fibers of the smooth map $p\colon M \longrightarrow X$ have a well-behaved theory of locally constant sheaves, which is controlled by their underlying homotopy type.

The obstructions to patching together local extensions are therefore controlled by the homotopy types of the fibers of p. Analyzing these obstructions leads to Theorem II. In fact, we prove a variant of Theorem II where M is allowed to be more singular, e.g. a derived intersection of two smooth manifolds.

Contents

This thesis is outlined as follows. Chapters 2 – 4 are essentially purely algebraic and discuss the homotopy theory of Lie algebroids and its relation to deformation theory. Chapters 5 – 7 study the role of Lie algebroids in derived differential topology.

Chapter 2 provides some background material on the homotopical algebra that we will employ throughout this thesis. We start by briefly recalling the language of abstract homotopy theory, which involves ∞-categories and model categories. This is a vast subject that cannot be done justice in such a short introductory paragraph, but we have nonetheless tried to convey some of the basic ideas that we will use.

We then discuss the homotopy theory of (derived) $\mathcal{C}^\infty$-rings, which provide the basic objects on which derived differential topology is built. Most importantly, the actual *derived* part of the theory of $\mathcal{C}^\infty$-rings largely reduces to commutative algebra. We furthermore review the theory of deformations of algebraic structures along square zero extensions, as outlined for example in [62, Section 7.4], and show that such deformation problems are naturally organized into *formal moduli problems.*

Chapter 3 discusses the homotopy theory of Lie algebroids and their representations. Our main result asserts that the category of differential graded Lie algebroids can be endowed with a semi-model structure having good homotopical properties (Theorem 3.1.10 and

Theorem 3.1.15). We also discuss an equivalent homotopy theory for L_∞-algebroids and their representations, which correspond to the representations up-to-homotopy of [3]. Although we do not really need this from an abstract point of view, we included it since L_∞-algebroids arise frequently in the literature.

The main results of this thesis are contained in **Chapter 4**. The most important result is Theorem I, reformulated slightly more explicitly as Theorem 4.2.1 in the setting of $\mathcal{C}^\infty$-rings. The proof is based on the proof for Lie algebras over a field due to Lurie [61] and uses a version of Koszul duality for Lie algebroids. We furthermore prove an equivalence between the homotopy theories of connective representations of a Lie algebroid and connective quasi-coherent modules over the corresponding formal moduli problem (Theorem 4.3.1).

By our discussion in Chapter 2, the deformation theory of algebras gives rise to examples of formal moduli problems. We show that the corresponding Lie algebroids can be explicitly identified as certain 'Atiyah Lie algebroids', consisting of differential operators acting by derivations with respect to the algebra structure (Theorem 4.4.1).

In **Chapter 5**, we develop the basic theory of *derived differential topology*, i.e. the geometry of derived $\mathcal{C}^\infty$-rings. The main purpose of this chapter is to extend the theory of derived intersections of smooth manifolds (studied in [92], among others) to a theory that describes more general kinds of singular spaces and stacks.

Our approach follows a general program for constructing geometric objects from algebraic objects, such as $\mathcal{C}^\infty$-rings, as outlined in [60]. We start by studying spectra of (derived) $\mathcal{C}^\infty$-rings, which form the affine local models of derived manifolds, as well as module sheaves over them. This requires some extra attention because different $\mathcal{C}^\infty$-rings (and modules over them) may have equivalent spectra (and associated sheaves). We then recall the general notion of a higher Lie groupoid and a (geometric) stack in the particular setting of derived manifolds. To make sure that there is a well-behaved theory of higher Lie groupoids and stacks, we need to make use of a version of the inverse function theorem for derived manifolds, which we prove in Chapter 6.

Finally, we show that in derived differential topology, there is a well-behaved theory of locally constant sheaves along the fibers of a smooth map $p\colon Y \longrightarrow X$. This theory is controlled by the homotopy types of the fibers of p (Theorem 5.3.19) and can be applied to give a sheaf-theoretic construction of the 'source n-connected cover' of a Lie groupoid (Example 5.3.27).

In **Chapter 6** we discuss the infinitesimal structure of derived stacks. There is an extensive treatment of infinitesimal properties of derived stacks in [97, 63, 33], which can be carried over directly to derived differential topology. Using this, together with a sheaf version of Theorem 4.2.1 (Corollary 6.3.15), we show that any map $M \longrightarrow X$ from a derived manifold to a derived stack gives rise to a sheaf of Lie algebroids over M.

We also study some infinitesimal properties of the entire moduli space of derived stacks (Theorem 6.4.3). These properties were also studied using a somewhat different method in [76]. In particular, we obtain that the deformations of a stack $p\colon X \longrightarrow M$ over a derived manifold are controlled by a certain Lie algebroid over M. This Lie algebroid can informally be thought of as the Lie algebroid of p-related vector fields on M and X.

Finally, **Chapter 7** is devoted to a proof of Theorem II, appearing as Theorem 7.2.1. This uses that the Lie algebroid $T_{M/X}$ of a map $p\colon M \longrightarrow X$ is closely related to the formal completion $X^\wedge_M$: when p is locally of finite presentation, we show that Lie algebroid maps $T_{M/X} \longrightarrow T_{M/Y}$ can equivalently be thought of as maps $X^\wedge_M \longrightarrow Y$ from the formal completion (Proposition 7.1.6). We conclude by sketching some simple applications of Theorem II, for example to the integrability of L_∞-algebras (Example 7.3.43).

Conventions

Most objects considered in this text are of a homotopical nature, which means that they are naturally organized into ∞-categories. We will freely make use of the theory of ∞-categories, as developed by Joyal [47] and Lurie [59].

The ∞-category $\mathcal{S}$ of *spaces* is the homotopy coherent nerve of the simplicially enriched category of Kan complexes. For two objects x, y in an ∞-category $\mathcal{C}$, we let $\mathrm{Map}_{\mathcal{C}}(x, y) \in \mathcal{S}$, or simply $\mathrm{Map}(x, y)$, denote their mapping space (i.e. Kan complex). We will always refer to a set equipped with a topology as a *topological space*, to avoid confusion with the interpretation of 'spaces' in terms of Kan complexes.

For $X \in \mathcal{C}$, we denote by $\mathcal{C}/X$ the ∞-category of objects in $\mathcal{C}$ over (i.e. with a map to) X, as described e.g. in [59, Section 1.2.9]. In particular, objects in CAlg/A are commutative algebras over A, conflicting with the usual meaning of 'algebras over A' as A-algebras.

Many of the ∞-categories considered in the text arise from model categories of differential graded objects. To simplify the notation, the passage from such a model category to its associated ∞-category is indicated by *removing* the mentioning of 'dg-'. For example, the model category $\mathrm{Mod}_A^{\mathrm{dg}}$ of (unbounded) dg-modules over a dg-algebra A gives rise to an ∞-category Mod_A of modules over the algebra A. This means that terms like 'modules', 'algebras' and 'Lie algebroids' generally refer to homotopical objects, while the non-homotopical variants are referred to as '*discrete* algebras'.

We use chain conventions for differential graded objects, i.e. the differential ∂ has degree -1. Homology groups are denoted by $\pi_n(-)$ and an object is n-(co)connective if its homology groups vanish in degrees $i < n$ (resp. $i > n$). The n-fold suspension $V[n]$ of a chain complex V is given by

$$V[n]_m = V_{m-n} \qquad \partial_{V[n]} = (-1)^n \partial_V$$

an its cone is denoted $V[n, n+1]$. We use $\mathrm{Hom}(x, y)$ to denote mapping complexes, rather than mapping spaces.

A *stack* will always mean a higher geometric (i.e. Artin) stack. A functor satisfying a homotopical version of descent is referred to as a sheaf (with values in some ∞-category $\mathcal{C}$). By default, a sheaf is a sheaf of spaces, rather than a sheaf of sets.

Chapter 2

Preliminaries

The purpose of this section is to recall some of the algebraic and homotopy theoretic language that will be employed throughout the text. Since we will be interested in (derived) *differential* topology, the basic algebraic objects under consideration are $\mathcal{C}^\infty$-rings, rather than ordinary commutative rings. We will give a brief account of the theory of derived $\mathcal{C}^\infty$-rings in Section 2.2, based on the homotopy theory of dg-$\mathcal{C}^\infty$-rings described in [18].

We will concentrate on algebraic aspects of derived $\mathcal{C}^\infty$-rings and postpone the discussion of geometry based on derived $\mathcal{C}^\infty$-rings to Chapter 5. A particularly relevant part of this algebraic story concerns deformations of algebraic structures along square zero extensions of $\mathcal{C}^\infty$-rings (Section 2.3). The abstract behaviour of such deformations can be axiomatized by the notion of a *formal moduli problem*, which will play an important role in the rest of the text.

Almost all of the objects considered in this thesis have a homotopy-theoretic flavour, which means that they can be organized in terms of model categories [80] or ∞-categories [47, 59]. Section 2.1 is supposed to give a brief account of the homotopy-theoretic language employed throughout the thesis.

2.1 Higher category theory

Categories provide a convenient way of organizing a collection of mathematical objects, together with the structure-preserving morphisms between them. These morphisms can often be used to study the mathematical objects themselves, and in particular allow us to decide when two objects are isomorphic.

In homotopy theory, one usually classifies objects not up to isomorphism, but up to a weaker notion of equivalence. For example, the category of topological spaces and continuous maps between them is supplemented by *homotopies* between maps, allowing one to study spaces up to *homotopy equivalence*, rather than homeomorphism. Similarly, chain complexes are often studied up to chain homotopy equivalence or up to quasi-isomorphism.

The language of ∞-categories is intended to describe these kinds of situations. On the one hand, one can think of an ∞-category as a category, together with a collection of 'weak equivalences' between its objects [6]. On the other hand, one can think of an ∞-category as a collection of objects, together with the data of maps between them, homotopies between these maps and higher homotopies between these homotopies. This data of homotopies and higher homotopies admits a simple combinatorial description using simplices, resulting in the notion of a Kan complex.

2.1.1 Kan complexes. Recall that a simplicial set is a diagram $X\colon \mathbf{\Delta}^{\mathrm{op}} \longrightarrow \mathrm{Set}$ indexed by the category of finite nonempty linear orders, which can be depicted as

$$X_0 \;\substack{\longleftarrow \\ \dashrightarrow \\ \longleftarrow}\; X_1 \;\substack{\longleftarrow \\ \dashrightarrow \\ \longleftarrow \\ \dashrightarrow \\ \longleftarrow}\; X_2 \;\substack{\longleftarrow \\ \dashrightarrow \\ \longleftarrow \\ \dashrightarrow \\ \longleftarrow \\ \dashrightarrow \\ \longleftarrow}\; X_3 \quad \cdots .$$

For each $k \geq 0$, the k-simplex $\Delta[k]$ is the simplicial set whose value on a linear order $[n] = \{0 < \cdots < n\}$ is given by the set of order-preserving maps $[n] \longrightarrow [k]$. If X is a simplicial set, then X_k can be identified with the set of maps $\Delta[k] \longrightarrow X$, i.e. the set of k-simplices of X. The k-simplex $\Delta[k]$ has a collection of horns $\Lambda^i[k] \subseteq \Delta[k]$ (for each $i = 0, \ldots, k$), whose n-simplices are the maps $[n] \longrightarrow [k]$ whose image misses a vertex different from i.

Definition 2.1.1. A simplicial set $X \colon \mathbf{\Delta}^{\mathrm{op}} \longrightarrow \mathrm{Set}$ is called a *Kan complex* if every horn in X can be extended to a simplex, i.e. if each solid diagram

$$\begin{tikzcd} \Lambda^i[k] \arrow[r] \arrow[d] & X \\ \Delta[k] \arrow[ur, dotted] & \end{tikzcd}$$

admits a dotted extension, as indicated. It is called an *n-groupoid* if this dotted extension is unique for all $k > n$ and all $i = 0, \ldots, k$.

Remark 2.1.2. The notion of a Kan complex admits many variations, replacing sets by other types of mathematical structures, like groups and rings. When one replaces sets by geometric objects like smooth manifolds, it makes sense to require that there exist (at least locally) smoothly varying choices of horn-fillers. We will come back to this in Section 5.2.

The horn-filling conditions provide rules for composing and inverting arrows and homotopies. For example, a horn $\Lambda^0[2] \longrightarrow X$ can be interpreted as a tuple of arrows $f \colon x_0 \longrightarrow x_1$ and $h \colon x_0 \longrightarrow x_2$, while an extension of this horn to a 2-simplex is given by an arrow $g \colon x_1 \longrightarrow x_2$, together with a homotopy between gf and h. In particular, the horn-filling conditions provide the existence of a composite (up to homotopy) hf^{-1}.

Example 2.1.3. Let X be a Kan complex and K a simplicial set and consider the simplicial set X^K whose k-simplices are maps $K \times \Delta[k] \longrightarrow X$. Then X^K is a Kan complex, which is an n-groupoid if X was an n-groupoid. When X, Y and Z are Kan complexes, there is an obvious composition map $Z^Y \times Y^X \longrightarrow Z^X$. One can think of Y^X as a combinatorial model for the mapping space between two Kan complexes: its vertices are maps $X \longrightarrow Y$ and its 1-simplices are homotopies $X \times \Delta[1] \longrightarrow Y$ between these maps.

Example 2.1.4. Recall that the singular complex $\mathrm{Sing}(T)$ of a topological space T is the simplicial set whose k-simplices are maps $\Delta^k \longrightarrow T$ from the topological k-simplex into T. This simplicial set is a Kan complex and any Kan complex is homotopy equivalent to the singular complex of a topological space. Because of this, we will refer to Kan complexes as *spaces*, to 0-simplices as points and 1-simplices as paths. The Kan complex $\mathrm{Sing}(T)$ is homotopy equivalent to an n-groupoid if and only if the homotopy groups of T vanish for all $m > n$.

Example 2.1.5. Let $X \colon \mathcal{I} \longrightarrow \mathrm{Kan}$ be a diagram of Kan complexes indexed by a category $\mathcal{I}$. The limit of X need not be a Kan complex and even if it is a Kan complex, the construction need not be homotopy invariant. To see why the ordinary limit is usually ill-behaved, note that a simplex x in the limit of X is given by a tuple of simplices $x_i \in X_i$ which are identified by the structure maps of the diagram. Instead of requiring simplices to be equal, one should ask them to be (coherently) homotopic.

This leads to a modification of the notion a limit, given by the *homotopy limit* of the diagram X, whose vertices are given (for example) by the following data:

(0) for each $i_0 \in \mathcal{I}$, a vertex $x_{i_0} \in X_{i_0}$.
(1) for each $\alpha \colon i_0 \longrightarrow i_1$ in $\mathcal{I}$, a path $x_\alpha \colon \Delta[1] \longrightarrow X_{i_1}$ from $\alpha(x_{i_0})$ to x_{i_1}.

(n) for each sequence $\alpha\colon i_0 \longrightarrow i_1 \longrightarrow \dots \longrightarrow i_n$ in $\mathcal{I}$, an n-simplex $x_\alpha\colon \Delta[n] \longrightarrow X_{i_n}$ whose faces are the images in X_{i_n} of the $(n-1)$-simplices $x_{\alpha\setminus i_k}$.

There is a similar description of the higher simplices of the homotopy limit. For example, the homotopy limit of a diagram $* \xrightarrow{x} X \xleftarrow{y} *$ can be identified with the space of paths from x to y in X.

2.1.2 ∞-categories. An ∞-category can be thought of as a category which is enriched over spaces (i.e. Kan complexes) in a homotopy coherent way. In particular, categories strictly enriched over Kan complexes give rise to ∞-categories. For example, the category of Kan complexes itself determines an ∞-category $\mathcal{S}$ called the ∞-category of spaces.

Unfortunately, categories strictly enriched over Kan complexes tend to be too rigid to efficiently perform categorical constructions. Because of this, one tends to work with more flexible notions of ∞-categories like quasi-categories (originally due to Boardman and Vogt [13]):

Definition 2.1.6. By an ∞-category $\mathcal{C}$ we will mean a *quasi-category* $\mathcal{C}$, i.e. a simplicial set such that every horn $\Lambda^i[k] \longrightarrow \mathcal{C}$ with $0 < i < k$ admits an extension to a k-simplex in $\mathcal{C}$. A functor of ∞-categories $\mathcal{C} \longrightarrow \mathcal{D}$ is simply a map of simplicial sets.

Where the horn-filling conditions for a Kan complex provide rules for composing and inverting arrows, the horn-filling conditions for a quasi-category only provide rule for composing arrows: for example, a horn $\Lambda^1[2] \longrightarrow \mathcal{C}$ determines a tuple of arrows $f\colon c \longrightarrow d$ and $g\colon d \longrightarrow e$ in $\mathcal{C}$ and an extension to the 2-simplex provides a map $h\colon c \longrightarrow e$, together with a homotopy $h \simeq gf$.

Example 2.1.7. The nerve of a category $\mathcal{C}$ (whose n-simplices are the n-tuples of composable arrows in $\mathcal{C}$) is an ∞-category, which we identify with $\mathcal{C}$.

Example 2.1.8. Let $\mathcal{C}$ be a quasi-category and K a simplicial set. Then there is a functor ∞-category $\mathrm{Fun}(K, \mathcal{C})$ whose k-simplices are maps $\Delta[k] \times K \longrightarrow \mathcal{C}$.

Example 2.1.9. Let $c_0, \dots, c_n$ be objects in an ∞-category $\mathcal{C}$ and consider the sub-simplicial set $\mathrm{Map}(c_0, \dots, c_n) \subseteq \mathrm{Fun}(\Delta[n], \mathcal{C})$ whose k-simplices are maps $\Delta[k] \times \Delta[n] \longrightarrow \mathcal{C}$ whose restriction to $\Delta[k] \times \{i\}$ is constant on c_i. Then $\mathrm{Map}(c_0, \dots, c_n)$ turns out to be a Kan complex (see [59]). Restricting to various edges of $\Delta[n]$ determines a diagram of Kan complexes

$$\mathrm{Map}(c_0, c_1) \times \cdots \times \mathrm{Map}(c_{n-1}, c_n) \longleftarrow \mathrm{Map}(c_0, \dots, c_n) \longrightarrow \mathrm{Map}(c_0, c_n).$$

The left map turns out to be a homotopy equivalence, so that the above zig-zag yields a composition map from the space of tuples of maps $c_i \longrightarrow c_{i+1}$ to the space of maps $c_0 \longrightarrow c_n$. In this way, one obtains a category weakly enriched over Kan complexes (a Segal category [86, 27]), from which $\mathcal{C}$ can be recovered up to equivalence (see e.g. [49]).

There is a well-developed theory of ∞-categories due to Joyal [47] and Lurie [59], which encompasses analogues of many constructions from category theory. For example, associated to a functor $f\colon \mathcal{D} \longrightarrow \mathcal{C}$ is an ∞-category $\mathcal{C}/f$ of cones over f, as well as the notion of a limit of f (a homotopy terminal object in $\mathcal{C}/f$).

Although constructions of ∞-categories are performed concretely at the level of simplicial sets, they often have universal properties that can also be understood at the level of mapping spaces. For example, given a diagram $f\colon \mathcal{D} \longrightarrow \mathcal{C}$, the limit and colimit of f can be characterized by the universal property that for each $c \in \mathcal{C}$, the maps

$$\mathrm{Map}_{\mathcal{C}}(c, \lim_{\mathcal{D}} f(d)) \xrightarrow{\simeq} \lim_{\mathcal{D}} \mathrm{Map}_{\mathcal{C}}(c, f(d))$$
$$\mathrm{Map}_{\mathcal{C}}(\mathrm{colim}_{\mathcal{D}} f(d), c) \xrightarrow{\simeq} \lim_{\mathcal{D}} \mathrm{Map}_{\mathcal{C}}(f(d), c)$$

are homotopy equivalences (see e.g. [59, Proposition 5.1.3.2]). The latter limit is taken in the ∞-category of spaces and can be described concretely by the homotopy limit construction of Example 2.1.5. For example, it follows from this that the colimit of a diagram $* \longleftarrow X \longrightarrow *$ of spaces is just the suspension of X.

Similarly, one can use mapping spaces to characterize adjunctions between ∞-categories (see [59, Proposition 5.2.2.8]): given functors $f\colon \mathcal{C} \longrightarrow \mathcal{D}$ and $g\colon \mathcal{D} \longrightarrow \mathcal{C}$, one may say that g is right adjoint to f if there exists a natural equivalence of functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \longrightarrow \mathcal{S}$

$$\mathrm{Map}_{\mathcal{D}}(f(c), d) \xrightarrow{\ \simeq\ } \mathrm{Map}_{\mathcal{C}}(c, g(d)).$$

These characterizations in terms of mapping spaces imply that left adjoint functors preserve colimits, while right adjoint functors preserve limits (see also [59, Proposition 5.2.3.5] for a more precise proof).

2.1.3 Fibrations. A piece of (∞-) category theory that we will repeatedly use concerns families of ∞-categories indexed by another ∞-category. The basic example to keep in mind is the category of modules over a ring A: each map $\phi\colon A \longrightarrow B$ induces a functor $\phi^*\colon \mathrm{Mod}_A \longrightarrow \mathrm{Mod}_B$ given by extension of scalars. A convenient way to organize this functoriality is to consider the category Mod of *all* rings and modules: this category has objects (A, M) consisting of a ring A and a module M over it, and morphisms $(A, M) \longrightarrow (B, N)$ given by a map of rings $A \longrightarrow B$ and an A-linear map $M \longrightarrow N$. There is an obvious projection functor

$$\pi\colon \mathrm{Mod} \longrightarrow \mathrm{Rings};\ \ (A, M) \longmapsto A$$

whose fiber over A is the category of A-modules. For any map of rings $\phi\colon A \longrightarrow B$ and an A-module M, the A-linear map $M \longrightarrow \phi^*M = B \otimes_A M$ gives a map $(A, M) \longrightarrow (B, \phi^*M)$ in Mod with the following universal property: precomposition with this map induces a bijection

$$\Big\{(B, \phi^*M) \longrightarrow (B, N) \text{ in } \pi^{-1}(B)\Big\} \xrightarrow{\ \cong\ } \Big\{(A, M) \xrightarrow{f} (B, N) \text{ s.t. } \pi(f) = \phi\Big\}.$$

The functor ϕ^* can be completely recovered from the arrows $(A, M) \longrightarrow (B, \phi^*M)$: one uses them to define the functor ϕ^* on objects and invokes the universal property to define the functor on morphisms.

Generally, a functor $\mathcal{C} \longrightarrow \mathrm{Cat}_\infty$ to (∞-) categories can be described equivalently by a fibration $\mathcal{X} \longrightarrow \mathcal{C}$ with ample supply of such 'universal lifts' of arrows in $\mathcal{C}$, in the following sense:

Definition 2.1.10 (informal)**.** Let $\pi\colon \mathcal{X} \longrightarrow \mathcal{C}$ be a functor of ∞-categories. Given an arrow $\alpha\colon c \longrightarrow d$ in $\mathcal{C}$, we will say that an arrow $\tilde{\alpha}\colon x \longrightarrow y$ in $\mathcal{X}$ is a *locally cocartesian lift* if $\pi(\tilde{\alpha}) = \alpha$ and precomposition with $\tilde{\alpha}$ defines an equivalence on mapping spaces

$$\tilde{\alpha}^*\colon \mathrm{Map}_{\mathcal{X}_d}(y, z) \longrightarrow \mathrm{Map}_{\mathcal{X}}(x, z) \times_{\mathrm{Map}_{\mathcal{C}}(c,d)} \{\alpha\}.$$

whose domain is the mapping space in the fiber $\mathcal{X}_d$ over d.

The functor π is a *locally cocartesian fibration* if for any arrow $\alpha\colon c \longrightarrow d$ in $\mathcal{C}$ and any $x \in \mathcal{X}_c$, there exists a locally cocartesian lift $\tilde{\alpha}\colon x \longrightarrow y$. It is called a *cocartesian fibration* if in addition, the locally cocartesian arrows are closed under composition.

If $\pi\colon \mathcal{X} \longrightarrow \mathcal{C}$ is a locally cocartesian fibration, then every arrow $\alpha\colon c \longrightarrow d$ determines a functor between the fibers $\alpha_!\colon \mathcal{X}_c \longrightarrow \mathcal{X}_d$. On objects, this sends $x \in \mathcal{X}_c$ to y, where $x \longrightarrow y$ is a locally cocartesian lift of α. To make sure that the resulting functors compose, i.e. that $\alpha_!\beta_! \simeq (\alpha\beta)_!$, one needs the fact that locally cocartesian arrows compose. If this is the case (i.e. if π is a cocartesian fibration), then this construction gives rise to a functor $\mathcal{C} \longrightarrow \mathrm{Cat}_\infty$. For a more detailed and precise exposition, see [59, Chapter 3].

2.1.4 Relative categories and model categories. A useful method for producing ∞-categories is by localizing a $(\infty-)$category at a set of morphisms.

Definition 2.1.11. Let $(\mathcal{C}, W)$ be a relative $(\infty-)$category, i.e. a $(\infty-)$category $\mathcal{C}$ together with a set W of maps in $\mathcal{C}$. The *localization* of $\mathcal{C}$ at W is a functor $u\colon \mathcal{C} \longrightarrow \mathcal{C}[W^{-1}]$ with the universal property that for any ∞-category $\mathcal{D}$, the functor

$$u^*\colon \mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{D}) \longrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

is fully faithful, with essential image consisting of those functors $f\colon \mathcal{C} \longrightarrow \mathcal{D}$ that send maps in W to equivalences in $\mathcal{D}$.

In other words, $\mathcal{C}[W^{-1}]$ is the universal ∞-category in which the maps in W are turned into homotopy equivalences. A functor $\mathcal{C}[W^{-1}] \longrightarrow \mathcal{D}[V^{-1}]$ can be obtained from a functor $\mathcal{C} \longrightarrow \mathcal{D}$ sending W to V.

Remark 2.1.12. Any (small) relative ∞-category $(\mathcal{C}, W)$ has a localization; in terms of categories enriched over simplicial sets, this is the Dwyer-Kan simplicial localization [25].

Remark 2.1.13. The universal property of localization implies that it is functorial. In terms of cocartesian fibrations, this can be described more precisely by the following result of Hinich [42], which we will repeatedly use: let $\pi\colon \mathcal{X} \longrightarrow \mathcal{C}$ be a cocartesian fibration of ∞-categories and let W_c be a collection of arrows in $\mathcal{X}_c$ for every c. Each map $\alpha\colon c \longrightarrow d$ in $\mathcal{C}$ determines a functor $\alpha_!\colon \pi^{-1}(c) \longrightarrow \pi^{-1}(d)$. If these functors send W_c to W_d, then inverting the arrows in all W_c produces a map of cocartesian fibrations

$$\begin{tikzcd}
\mathcal{X} \arrow[rr, "f"] \arrow[dr, "\pi"'] & & \mathcal{X}[\cup W_c^{-1}] \arrow[dl] \\
& \mathcal{C}. &
\end{tikzcd}$$

The functor f preserves locally cocartesian lifts and is given on fibers by the localization map $\mathcal{X}_c \longrightarrow \mathcal{X}_c[W_c^{-1}]$. Furthermore, if V is a set of arrows α in $\mathcal{C}$ for which $\alpha_!\colon \mathcal{X}_c[W_c^{-1}] \longrightarrow \mathcal{X}_d[W_d^{-1}]$ is an equivalence, then there exists a cocartesian fibration $\mathcal{Y} \longrightarrow \mathcal{C}[V^{-1}]$ whose restriction along $\mathcal{C} \longrightarrow \mathcal{C}[V^{-1}]$ is equivalent to $\mathcal{X}[\cup W_c^{-1}] \longrightarrow \mathcal{C}$.

Given a relative category $(\mathcal{C}, W)$, one can ask whether constructions in the underlying ∞-category $\mathcal{C}[W^{-1}]$ can be performed at the level of $\mathcal{C}$ itself. A particularly structured example where this is the case is provided by (Quillen) *model categories* [80]. In fact, we will need to make use of the following slight variant of the notion of a model category:

Definition 2.1.14 ([91], [29])**.** A (left) *semi-model category* is a bicomplete category $\mathcal{M}$ equipped with wide subcategories of weak equivalences, cofibrations and fibrations, subject to the following conditions:

(1) The weak equivalences have the two out of three property and the weak equivalences, fibrations and cofibrations are stable under retracts.

(2) The cofibrations have the left lifting property with respect to the trivial fibrations. The trivial cofibrations *with cofibrant domain* (i.e. with a domain X for which the map $\emptyset \longrightarrow X$ is a cofibration) have the left lifting property with respect to the fibrations.

(3) Every map can be factored functorially into a cofibration, followed by a trivial fibration. Every map *with cofibrant domain* can be factored functorially into a trivial cofibration followed by a fibration.

(4) The fibrations and trivial fibrations are stable under transfinite composition, products and base change.

An adjunction $F\colon \mathcal{M} \leftrightarrows \mathcal{N}\colon G$ between two semi-model categories is a *Quillen adjunction* if the right adjoint G preserves fibrations and trivial fibrations. It is a Quillen equivalence when a map $X \longrightarrow G(Y)$ is a weak equivalence if and only if its adjoint map $F(X) \longrightarrow Y$ is a weak equivalence, for any cofibrant $X \in \mathcal{M}$ and fibrant $Y \in \mathcal{N}$.

Remark 2.1.15. We refer to [91, 29] for the basic theory of semi-model categories. Let us remark that in a semi-model category, only the cofibrations and trivial fibrations determine each other via the lifting property; in particular, a semi-model structure is only determined by its weak equivalences and fibrations.

One can use a semi-model structure on $\mathcal{M}$ to get a grasp on the associated ∞-category $\mathcal{M}[W^{-1}]$. For example, the mapping spaces of $\mathcal{M}[W^{-1}]$ can be computed in terms of fibrant and cofibrant resolutions in $\mathcal{M}$ [91], the *homotopy limit* and colimit of a diagram in $\mathcal{M}$ can be used to compute the limit and colimit of its image in $\mathcal{M}[W^{-1}]$ and Quillen adjunctions (equivalences) induce adjunctions (equivalences) between ∞-categorical localizations [42, Proposition 1.5.1].

Many important examples of ∞-categories arise from localizing (semi-)model categories:

Example 2.1.16. The ∞-category $\mathcal{S}$ of spaces arises from the Kan-Quillen model structure on simplicial sets and the ∞-category Cat_∞ of ∞-categories arises from the Joyal model structure on simplicial sets.

Example 2.1.17. If A is a ring, the category $\mathrm{Mod}_A^{\mathrm{dg}}$ of unbounded chain complexes of discrete A-modules can be endowed with the *projective model structure*, in which a map is a weak equivalence (fibration) if it is a quasi-isomorphism (surjection). There is a similar model structure for dg-modules over a dg-algebra A. We will refer to the associated ∞-category Mod_A as the ∞-category of modules over A.

If A is a dg-algebra in non-negative degrees, then there is a similar projective model structure on the category $\mathrm{Mod}_A^{\mathrm{dg},\geq 0}$ of non-negatively graded dg-A-modules. This models the ∞-category $\mathrm{Mod}_A^{\geq 0}$ of *connective* modules over A. The inclusion $\mathrm{Mod}_A^{\mathrm{dg},\geq 0} \longrightarrow \mathrm{Mod}_A^{\mathrm{dg}}$ is a left Quillen functor. Its right adjoint $\tau_{\geq 0}$ sends an unbounded chain complex M to the chain complex

$$Z_0(M) \longleftarrow M_1 \longleftarrow M_2 \longleftarrow \dots$$

with the zero-cycles in degree 0. We refer to $\tau_{\geq 0}M$ as the *connective cover* of M.

A model structure on a category $\mathcal{M}$ can often be used to construct model structures on categories constructed out of $\mathcal{M}$ by means of *transfer*. For example, the category of diagrams $F\colon \mathcal{I} \longrightarrow \mathcal{M}$ indexed by a small category $\mathcal{I}$ often carries the *projective* model structure, in which a map $F \longrightarrow G$ is an equivalence (resp. fibration) when the map $F(i) \longrightarrow G(i)$ is a weak equivalence (fibration) in $\mathcal{M}$ for every $i \in \mathcal{I}$.

To show that this indeed defines a (semi-)model structure, one has to impose some set-theoretic conditions on the model category $\mathcal{M}$, which allow one to verify the factorization axioms of Definition 2.1.14 by means of Quillen's small object argument [80].

Definition 2.1.18. A semi-model category $\mathcal{M}$ is called *tractable* if its underlying category is locally presentable and if there exists sets of maps with cofibrant domain I and J (called generating cofibrations and generating trivial cofibrations) such that a map has the right lifting property against I (resp. J) if and only if it is a trivial fibration (resp. a fibration).

Lemma 2.1.19 (cf. [29, Proposition 12.1.4])**.** *Consider an adjunction between locally presentable categories*

$$F\colon \mathcal{M} \rightleftarrows \mathcal{N}\colon G$$

and suppose that $\mathcal{M}$ carries a tractable semi-model structure, with generating (trivial) cofibrations I and J. Define a map in $\mathcal{N}$ to be a weak equivalence (fibration) if its image under G is a weak equivalence (fibration) in $\mathcal{M}$. It is a cofibration if it has the left lifting property against the trivial fibrations.

This determines a tractable semi-model structure on $\mathcal{N}$, whose generating (trivial) cofibrations are given by $F(I)$ and $F(J)$, as soon as the following condition holds:

- ($\star$) *Let $f\colon A \longrightarrow B$ be a map in $\mathcal{N}$ with cofibrant domain, obtained as a transfinite composition of pushouts of maps in $F(J)$. Then f is a weak equivalence.*

Proof. The factorization axioms follow from the small object argument. The only nontrivial thing to check is the lifting axiom for trivial cofibrations between cofibrant objects against fibrations. If $A \longrightarrow B$ is a trivial cofibration between cofibrant objects, we can factor $A \longrightarrow A' \longrightarrow B$ as an iterated pushout of maps in $F(J)$, followed by a fibration. By condition ($\star$), the map $A \longrightarrow A'$ is a weak equivalence, so that $A' \longrightarrow B$ is a trivial fibration and the map $A \longrightarrow B$ is a retract of the map $A \longrightarrow A'$. The latter has the lifting property against the fibrations by definition. □

Example 2.1.20. If $\mathcal{M}$ is a tractable semi-model category and $\mathcal{C}$ is a small category, then the diagram category $\mathrm{Fun}(\mathcal{C}, \mathcal{M})$ carries the projective semi-model structure, transferred along the right (and left) adjoint functor $\mathrm{ev}\colon \mathrm{Fun}(\mathcal{C}, \mathcal{M}) \longrightarrow \prod_{c\in\mathcal{C}} \mathcal{M}$.

Remark 2.1.21. Condition ($\star$) automatically holds if every cofibration between cofibrant objects $A \longrightarrow B$ admits a natural weak equivalence to a fibration $f'\colon A' \longrightarrow B'$. Indeed, when f is a transfinite composition of maps in $F(J)$, one can use the right lifting property against the fibration f' to find a map $r\colon B \longrightarrow A^{\mathrm{f}}$ such that rf and $f'r$ are both weak equivalences. The 2-out-of-6 property of the weak equivalences then implies that f is a weak equivalence.

Example 2.1.22. Suppose that $\mathcal{P}$ is a (coloured, symmetric) dg-operad over a field k of characteristic zero and that A is a commutative dg-k-algebra. Let $\mathcal{P}\mathrm{Alg}_A^{\mathrm{dg}}$ be the category of dg-$\mathcal{P}$-algebras in dg-A-modules. This carries a model structure, obtained from projective model structure by transfer along the free-forgetful adjunction

$$\mathrm{Free}\colon \mathrm{Mod}_A^{\mathrm{dg}} \rightleftarrows \mathcal{P}\mathrm{Alg}_A^{\mathrm{dg}}\colon U.$$

Indeed, every dg-$\mathcal{P}$-algebra B gives rise to a dg-$\mathcal{P}$-algebra $B[t, dt]$ of B-valued polynomial differential forms on $\Delta[1]$. The diagonal then factors as a weak equivalence, followed by fibration [15]

$$B \xrightarrow{\sim} B[t, dt] \xrightarrow{(\mathrm{ev}_{t=0}, \mathrm{ev}_{t=1})} B \times B.$$

Using this, one can factor every map of dg-$\mathcal{P}$-algebras $A \longrightarrow B$ as a weak equivalence $A \longrightarrow A \times_B B[t, dt]$, followed by a fibration $A \times_B B[t, dt] \longrightarrow B$. The associated ∞-category

$$\mathcal{P}\mathrm{Alg}_A = \mathcal{P}\mathrm{Alg}_A^{\mathrm{dg}}[W^{-1}]$$

is the ∞-category of *unbounded $\mathcal{P}$-algebras.* For instance, there is a model structure on the category $\mathrm{CAlg}_A^{\mathrm{dg}}$ of commutative dg-k-algebras whose underlying ∞-category CAlg_A is the ∞-category of *unbounded commutative A-algebras.*

Example 2.1.23. Suppose that the dg-operad $\mathcal{P}$ and the commutative dg-algebra A are concentrated in non-negative degrees. Then there is a similar model structure on the category $\mathcal{P}\mathrm{Alg}_A^{\geq 0, \mathrm{dg}}$ of non-negatively graded dg-$\mathcal{P}$-algebras over A, which presents the ∞-category $\mathcal{P}\mathrm{Alg}_A^{\geq 0}$ of *connective* $\mathcal{P}$-algebras over A.

Let us conclude with the following property of a tractable semi-model category $\mathcal{M}$, which asserts that *any* diagram in the ∞-category $\mathcal{M}[W^{-1}]$ can be studied at the level of $\mathcal{M}$:

Proposition 2.1.24 ([62, Proposition 1.3.4.25]). *Let $f\colon \mathcal{I} \longrightarrow \mathcal{J}$ be a functor between two small (ordinary) categories and let $\mathcal{M}$ be a tractable semi-model category. Consider the commuting square of ∞-categories*

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{J}, \mathcal{M})[W^{-1}] \arrow[r, "\sim"] \arrow[d, "f^*"'] & \mathrm{Fun}(\mathcal{J}, \mathcal{M}[W^{-1}]) \arrow[d, "f^*"] \\
\mathrm{Fun}(\mathcal{I}, \mathcal{M})[W^{-1}] \arrow[r, "\sim"'] & \mathrm{Fun}(\mathcal{I}, \mathcal{M}[W^{-1}])
\end{tikzcd}$$

in which ∞-categories on the left are obtained by formally inverting the pointwise weak equivalences between diagrams. Then the horizontal functors are equivalences of locally presentable ∞-categories (see [59, Chapter 5] for a definition).

In other words, any diagram in $\mathcal{M}[W^{-1}]$ can be rectified (in a homotopically unique way) to a diagram in $\mathcal{M}$, so that statements about diagrams in $\mathcal{M}$ can always be checked for rectified diagrams in the model category $\mathcal{M}$ itself. For example, the functor of ∞-categories

$$f^*\colon \mathrm{Fun}\big(\mathcal{J}, \mathcal{M}[W^{-1}]\big) \longrightarrow \mathrm{Fun}\big(\mathcal{I}, \mathcal{M}[W^{-1}]\big)$$

admits both a left and a right adjoint, which can be presented at the model-categorical level by the derived functors of left and right Kan extension. To compute the (co)limit of a diagram in $\mathcal{M}[W^{-1}]$, it therefore suffices to rectify it to a diagram in $\mathcal{M}$ and compute the homotopy (co)limit of this diagram.

Proposition 2.1.24 is proven in [62] for combinatorial model categories, rather than semi-model categories. The case of semi-model categories follows immediately from the fact that Quillen equivalent semi-model categories give rise to Quillen equivalent semi-model categories of diagrams, combined with following observation:

Lemma 2.1.25 ([23]). *Let $\mathcal{M}$ be a tractable semi-model category. Then there exists a left proper, simplicial, combinatorial model category $\mathcal{N}$ and a Quillen equivalence*

$$F\colon \mathcal{N} \rightleftarrows \mathcal{M}\colon G.$$

This follows from the following slightly more general statement:

Proposition 2.1.26 ([23]). *Let $\mathcal{M}_{(-)}\colon \mathcal{C} \longrightarrow \mathrm{ModCat}^{\mathrm{L}}$ be a (small) diagram of tractable semi-model categories. Then there exists the following data:*

(1) A diagram of (small) categories $\mathcal{G}_{(-)}\colon \mathcal{C} \longrightarrow \mathrm{Cat}$. This determines a diagram of model categories

$$\widehat{\mathcal{G}}\colon \mathcal{C} \longrightarrow \mathrm{ModCat}^{\mathrm{L}};\ \ c \longmapsto \widehat{\mathcal{G}}_c := \mathrm{Fun}\big(\mathcal{G}_c^{\mathrm{op}}, \mathrm{sSet}\big)$$

sending each $c \in \mathcal{C}$ to simplicial presheaves on $\mathcal{G}_c$, equipped with the projective model structure. An arrow $\alpha\colon c \longrightarrow d$ is sent to the Quillen pair taking left Kan extension and restriction along $\mathcal{G}_c^{\mathrm{op}} \longrightarrow \mathcal{G}_d^{\mathrm{op}}$.

(2) A natural transformation of left Quillen functors $\gamma_!\colon \widehat{\mathcal{G}} \longrightarrow \mathcal{M}$.

(3) For each $c \in \mathcal{C}$, a set of maps W_c in $\widehat{\mathcal{G}}_c$ such that $\gamma_!$ descends to a Quillen equivalence from the left Bousfield localization at W_c

$$\gamma_!\colon L_{W_c}\widehat{\mathcal{G}}_c \longrightarrow \mathcal{M}_c.$$

We therefore obtain a diagram $\mathcal{C} \longrightarrow \mathrm{ModCat}^{\mathrm{L}}; c \mapsto L_{W_c}\mathrm{Fun}\big(\mathcal{G}_c^{\mathrm{op}}, \mathrm{sSet}\big)$ which is naturally Quillen equivalent to $\mathcal{M}$.

Remark 2.1.27. Our proof makes extensive use of cosimplicial objects. To this end, let us recall that if $\mathcal{M}$ is a tractable semi-model category, then the category $\mathcal{M}^{\Delta} = \mathrm{Fun}(\Delta, \mathcal{M})$ of cosimplicial objects in $\mathcal{M}$ comes endowed with the Reedy model structure. We will denote the constant diagram on an object $Y \in \mathcal{M}$ by $\mathrm{cst}(Y)$.

Note that a cosimplicial object $X\colon \Delta \longrightarrow \mathcal{M}$ can equivalently be considered as a left adjoint functor $X(-)\colon \mathrm{sSet} \longrightarrow \mathcal{M}^{\mathrm{op}}$. This left adjoint functor is left Quillen if and only the cosimplicial object X is a *cosimplicial resolution*: it is Reedy cofibrant and weakly equivalent to a constant diagram.

If $X \in \mathcal{M}^{\Delta}$ is a cosimplicial resolution, we will denote by $X^{(n)} \in \mathcal{M}^{\Delta}$ the cosimplicial object corresponding to the left adjoint functor $\mathrm{sSet} \longrightarrow \mathcal{M}^{\mathrm{op}}; K \mapsto X(K \times \Delta[n])$. This determines a cosimplicial object $X^{(\bullet)}\colon \Delta \longrightarrow \mathcal{M}^{\Delta}$, which is itself a cosimplicial resolution of X.

Proof. Given the data (1) - (3), the conclusion follows as soon as we know that for each map $\alpha\colon c \longrightarrow d$, the right Quillen functor $\widehat{\mathcal{G}}_d \longrightarrow \widehat{\mathcal{G}}_c$ sends W_d-local objects to W_c-local objects. This follows from the commuting square of homotopy categories and right adjoint functors

$$\begin{tikzcd}
\mathrm{ho}(\mathcal{M}_d) \arrow[r, hook] \arrow[d] & \mathrm{ho}(\widehat{\mathcal{G}}_d) \arrow[d] \\
\mathrm{ho}(\mathcal{M}_c) \arrow[r, hook] & \mathrm{ho}(\widehat{\mathcal{G}}_c)
\end{tikzcd}$$

and the fact that the horizontal functors are fully faithful inclusions whose essential images consist precisely of W_d-local objects and W_c-local objects.

Data (1). Since each $\mathcal{M}_c$ is tractable, one can use the small object argument to show that there is a regular cardinal κ_c with the following properties [23, Proposition 2.3]:

- every object of $\mathcal{M}$ is the κ_c-filtered colimit of κ_c-small objects.
- all functorial factorizations preserve κ_c-filtered colimits and factor a map between κ_c-small objects into maps between κ_c-small objects.
- if $X(-) \longrightarrow Y(-)$ is a natural weak equivalence (fibration) between κ_c-filtered diagrams, then the map $\mathrm{colim}_i\, X(i) \longrightarrow \mathrm{colim}_i\, Y(i)$ is a weak equivalence (fibration) as well.

The last condition implies that κ_c-filtered colimits are already homotopy colimits. Since $\mathcal{C}$ is a small category, we can find a regular cardinal κ larger than all κ_c, with the additional property that for every $c \longrightarrow d$ in $\mathcal{C}$, the left adjoint $\mathcal{M}_c \longrightarrow \mathcal{M}_d$ preserves κ-small objects.

Now let $\mathcal{G}_c$ denote the full subcategory of $\mathcal{M}_c^{\Delta}$ consisting of cosimplicial resolutions that take values in κ-small objects of $\mathcal{M}_c$. By construction, this has has the following properties:

(a) the categories $\mathcal{G}_c$ assemble into a diagram $\mathcal{G}_{(-)}\colon \mathcal{C} \longrightarrow \mathrm{Cat}$, consisting of full subcategories of $\mathcal{M}_{(-)}^{\Delta}$.

(b) any map in $\mathcal{G}_c$ can be factored functorially (inside $\mathcal{G}_c \subseteq \mathcal{M}_c^{\boldsymbol{\Delta}}$) into a Reedy (trivial) cofibration followed by Reedy (trivial) fibration. Every κ-small object in $\mathcal{M}_c$ admits a cofibrant resolution in $\mathcal{G}_c$.

(c) for every object $X \in \mathcal{G}_c \subseteq \mathcal{M}_c^{\boldsymbol{\Delta}}$, the functor

$$\mathcal{M}^{\mathrm{fib}} \longrightarrow \mathrm{sSet};\ Y \longmapsto \mathrm{Map}_{\mathcal{M}^{\boldsymbol{\Delta}}}\big(X^{(\bullet)}, \mathrm{cst}(Y)\big).$$

preserves homotopy colimits of κ-filtered diagrams. In fact, the n-simplices are simply given by maps $X([n]) \longrightarrow Y$ in $\mathcal{M}$; this shows that the above functor preserves κ-filtered colimits, which are also *homotopy* colimits.

Data (2). There is a canonical natural transformation

$$\gamma\colon \mathcal{G}_{(-)} \times \Delta \longrightarrow \mathcal{M}_{(-)}$$

sending a tuple $(X, [n])$ in $\mathcal{G}_c \times \Delta$ to the object $X([n])$ in $\mathcal{M}_c$. Because each $X \in \mathcal{G}_c$ is a cosimplicial resolution, it follows from [24, Proposition 3.4] that this induces a family of Quillen pairs

$$\gamma_!\colon \mathrm{Fun}(\mathcal{G}_c^{\mathrm{op}}, \mathrm{sSet}) \rightleftarrows \mathcal{M}_c\colon \gamma^* \tag{2.1.28}$$

depending functorially on $c \in \mathcal{C}$. Here the left hand side is endowed with the projective model structure. Unraveling the definitions, for any $Y \in \mathcal{M}_c$, the simplicial presheaf $\gamma^*(Y)$ is given by the functor $X \mapsto \mathrm{Map}_{\mathcal{M}^{\boldsymbol{\Delta}}}\big(X^{(\bullet)}, \mathrm{cst}(Y)\big)$ (see Remark 2.1.27 for the notation).

Data (3). Fix an object $c \in \mathcal{C}$. We claim that the derived counit map of (2.1.28) is a weak equivalence. Assuming this, it follows from [23, Proposition 3.2] that there exists a set of maps W_c in $\mathrm{Fun}(\mathcal{G}_c^{\mathrm{op}}, \mathrm{sSet})$ such that (2.1.28) becomes a Quillen equivalence after taking the left Bousfield localization of $\mathrm{Fun}(\mathcal{G}_c^{\mathrm{op}}, \mathrm{sSet})$ at W_c.

Let us denote by $\mathcal{K}$ the class of (weak equivalence classes of) objects $Y \in \mathcal{M}_c$ for which the derived counit map $\mathbb{L}\gamma_!\mathbb{R}\gamma^*(Y) \longrightarrow Y$ is a weak equivalence. Since $\mathbb{R}\gamma^*$ preserves κ-filtered homotopy colimits by point (c) above, the class $\mathcal{K}$ is closed under κ-filtered homotopy colimits. Every object in $\mathcal{M}_c$ is a κ-filtered (homotopy) colimit of a diagram of κ-small fibrant-cofibrant objects. It therefore suffices to show that $\mathcal{K}$ contains all κ-small Y.

Let $Y \in \mathcal{M}_c$ be a κ-small fibrant-cofibrant object. Then $\mathrm{cst}(Y) \in \mathcal{M}_c^{\boldsymbol{\Delta}}$ is Reedy fibrant and there exists a Reedy trivial fibration $\tilde{Y} \longrightarrow \mathrm{cst}(Y)$ with Reedy fibrant-cofibrant domain. Let $\tilde{Y}^{\Delta[\bullet]}\colon \Delta^{\mathrm{op}} \longrightarrow \mathcal{M}_c^{\boldsymbol{\Delta}}$ be a fibrant-cofibrant simplicial resolution of $\tilde{Y}$. By point (b) above, we can assume that every $\tilde{Y}^{\Delta[n]}$ is an object in $\mathcal{G}_c \subseteq \mathcal{M}_c^{\boldsymbol{\Delta}}$.

Now consider the diagram of presheaves $\mathcal{G}_c^{\mathrm{op}} \longrightarrow \mathrm{sSet}$ whose value on $(X, [n]) \in \mathcal{G}_c \times \Delta$ is given by

$$\begin{tikzcd}
 & \mathrm{Map}_{\mathcal{M}_c^{\boldsymbol{\Delta}}}\big(X, \tilde{Y}\big) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{M}_c^{\boldsymbol{\Delta}}}\big(X, \tilde{Y}^{\Delta[n]}\big) \arrow[d, "\sim"] \\
\mathrm{Map}_{\mathcal{M}_c^{\boldsymbol{\Delta}}}\big(X^{(n)}, \mathrm{cst}(Y)\big) & \mathrm{Map}_{\mathcal{M}_c^{\boldsymbol{\Delta}}}\big(X^{(n)}, \tilde{Y}\big) \arrow[l, "\sim"'] \arrow[r, "\sim"'] & \mathrm{Map}_{\mathcal{M}_c^{\boldsymbol{\Delta}}}\big(X^{(n)}, \tilde{Y}^{\Delta[n]}\big)
\end{tikzcd} \tag{2.1.29}$$

The maps labeled by $\sim$ are pointwise weak equivalences of simplicial presheaves on $\mathcal{G}_c$. Indeed, this follows because mapping out of $X^{(\bullet)}$ preserves weak equivalences between fibrant objects and mapping into $\tilde{Y}^{\Delta[\bullet]}$ preserves weak equivalences between cofibrant objects in $\mathcal{M}_c^{\boldsymbol{\Delta}}$.

The simplicial presheaf $X \mapsto \mathrm{Map}_{\mathcal{M}_c^{\boldsymbol{\Delta}}}\big(X, \tilde{Y}^{\Delta[\bullet]}\big)$ is weakly equivalent to the homotopy colimit of representable presheaves

$$\operatorname*{hocolim}_{[n]\in\Delta^{\mathrm{op}}} \mathrm{Map}_{\mathcal{M}_c^{\boldsymbol{\Delta}}}\big(-, \tilde{Y}^{\Delta[n]}\big).$$

Applying the left adjoint $\gamma_!$ to Diagram (2.1.29), we can identify the derived counit map $\mathbb{L}\gamma_!\mathbb{R}\gamma^*(Y) \longrightarrow Y$ with the zig-zag of maps

$$\mathrm{hocolim}_{[n]\in\Delta^{\mathrm{op}}} \tilde{Y}([n]) \xleftarrow{\ \sim\ } \tilde{Y}([0]) \longrightarrow Y.$$

The first map is a weak equivalence since the cosimplicial object $\tilde{Y}$ is homotopically constant and the second map is a weak equivalence since $\tilde{Y}$ was a resolution of Y. We conclude that the derived counit is an equivalence for all κ-compact Y. $\square$

2.2 Derived $\mathcal{C}^\infty$-rings

In this section we collect some basic algebraic results about derived $\mathcal{C}^\infty$-rings. Our preferred model for derived $\mathcal{C}^\infty$-rings is given by dg-$\mathcal{C}^\infty$-rings (due to Carchedi-Roytenberg [18]), which emphasizes the close connection to (derived) commutative algebra.

2.2.1 DG-$\mathcal{C}^\infty$-rings. Recall that a $\mathcal{C}^\infty$-ring (in sets) is a set A together with an operation $\phi_*\colon A^{\times k} \longrightarrow A^{\times m}$ for every $\mathcal{C}^\infty$-function $\phi\colon \mathbb{R}^k \longrightarrow \mathbb{R}^m$, such that for any further $\mathcal{C}^\infty$-function $\psi\colon \mathbb{R}^m \longrightarrow \mathbb{R}^n$, the diagram

$$\begin{tikzcd} A^{\times k} \arrow[d, "\phi_*"'] \arrow[dr, "(\psi\circ\phi)_*"] & \\ A^{\times m} \arrow[r, "\psi_*"'] & A^{\times n} \end{tikzcd}$$

commutes. Alternatively, a $\mathcal{C}^\infty$-ring in sets is a product preserving functor from the category of smooth manifolds $\mathbb{R}^n$ (for $n \geq 0$) to the category of sets (see e.g. [68] for a textbook account). Any $\mathcal{C}^\infty$-ring has an underlying $\mathbb{R}$-algebra structure, which is determined by the maps ϕ_* where ϕ is a polynomial function.

Definition 2.2.1 ([18])**.** A dg-$\mathcal{C}^\infty$-ring A is a non-negatively graded commutative dg-algebra over $\mathbb{R}$, together with a compatible structure of a $\mathcal{C}^\infty$-ring on the set A_0. A map of dg-$\mathcal{C}^\infty$-rings $A \longrightarrow B$ is a map of dg-algebras such that $A_0 \longrightarrow B_0$ is a map of $\mathcal{C}^\infty$-rings in sets. Let $\mathcal{C}^\infty\mathrm{Alg}^{\mathrm{dg}}$ denote the category of dg-$\mathcal{C}^\infty$-rings.

Example 2.2.2. Let M be a smooth manifold and let $f_1, ..., f_n\colon M \longrightarrow \mathbb{R}$ be $\mathcal{C}^\infty$-functions on M. Then there is a dg-$\mathcal{C}^\infty$-ring $A = \mathcal{C}^\infty(M)[\eta_1, ..., \eta_n]$, given by the polynomial algebra over $\mathcal{C}^\infty(M)$ with generators $\eta_1, ..., \eta_n$ of degree 1, satisfying $\partial\eta_i = f_i$. The dg-$\mathcal{C}^\infty$-ring A serves as a model for the derived zero locus of the function $f\colon M \longrightarrow \mathbb{R}^n$. In particular, $\pi_0(A)$ is given by the $\mathcal{C}^\infty$-ring $\mathcal{C}^\infty(M)/(f_1, ..., f_n)$ and A is quasi-isomorphic to $\mathcal{C}^\infty(f^{-1}(0))$ when 0 is a regular value of f.

Remark 2.2.3. The work [18] also defines a non-connective version of the notion of a dg-$\mathcal{C}^\infty$-ring. We will make no use of this notion, and always use 'dg-$\mathcal{C}^\infty$-ring' to refer to a dg-$\mathcal{C}^\infty$-ring in the sense of Definition 2.2.1.

Proposition 2.2.4 ([18])**.** *The category $\mathcal{C}^\infty\mathrm{Alg}^{\mathrm{dg}}$ carries a tractable model structure, in which a map is a weak equivalence (resp. a fibration) if it is a quasi-isomorphism (resp. a surjection in all nonzero degrees).*

Definition 2.2.5. The ∞-category of $\mathcal{C}^\infty$-rings is the ∞-category underlying the model category $\mathcal{C}^\infty\mathrm{Alg}^{\mathrm{dg}}$

$$\mathcal{C}^\infty\mathrm{Alg} := \mathcal{C}^\infty\mathrm{Alg}^{\mathrm{dg}}[W^{-1}].$$

We will refer to objects of $\mathcal{C}^\infty\mathrm{Alg}$ simply as (derived) $\mathcal{C}^\infty$-rings. To avoid confusion, we will always refer to a $\mathcal{C}^\infty$-ring in sets as a *discrete* $\mathcal{C}^\infty$-ring.

Remark 2.2.6. The inclusion $\mathcal{C}^\infty\mathrm{Alg}_{\mathrm{disc}} \longrightarrow \mathcal{C}^\infty\mathrm{Alg}^{\mathrm{dg}}$ of the discrete $\mathcal{C}^\infty$-rings into the dg-$\mathcal{C}^\infty$-rings is a right Quillen functor, where $\mathcal{C}^\infty\mathrm{Alg}_{\mathrm{disc}}$ carries the model structure whose weak equivalences are the isomorphisms. This induces a fully faithful functor $\mathcal{C}^\infty\mathrm{Alg}_{\mathrm{disc}} \longrightarrow \mathcal{C}^\infty\mathrm{Alg}$, with left adjoint given by the functor taking π_0.

The model structure on $\mathcal{C}^\infty\mathrm{Alg}^{\mathrm{dg}}$ is defined by transfer along the free-forgetful adjunction

$$\mathrm{Free}\colon \mathrm{Mod}_{\mathbb{R}}^{\geq 0,\mathrm{dg}} \rightleftarrows \mathcal{C}^\infty\mathrm{Alg}^{\mathrm{dg}}\colon U$$

to non-negatively graded chain complexes of vector spaces. Consequently, the generating cofibrations of $\mathcal{C}^\infty\mathrm{Alg}^{\mathrm{dg}}$ are of the form

$$\mathrm{Free}(0) \longrightarrow \mathrm{Free}(\mathbb{R}) \qquad \mathrm{Free}(\mathbb{R}[k]) \longrightarrow \mathrm{Free}(\mathbb{R}[k, k+1]).$$

The first map can be identified with the map $\mathbb{R} \longrightarrow \mathcal{C}^\infty(\mathbb{R})$ and the second map is given (for $k \geq 1$) by the map of polynomial algebras $\mathbb{R}[x] \longrightarrow \mathbb{R}[x, \xi]$, where x is a generator of degree k and ξ is a generator of degree $k+1$ satisfying $\partial\xi = x$. When $k = 0$, the second map is given by the inclusion $\mathcal{C}^\infty(\mathbb{R}) \longrightarrow \mathcal{C}^\infty(\mathbb{R})[\xi]$ where $\partial\xi$ is the identity function on $\mathbb{R}$.

Lemma 2.2.7. *The forgetful functor $\mathcal{C}^\infty\mathrm{Alg}^{\mathrm{dg}} \longrightarrow \mathrm{CAlg}_{\mathbb{R}}^{\geq 0,\mathrm{dg}}$ to connective commutative dg-algebras over $\mathbb{R}$ is a right Quillen functor. The induced right adjoint functor $\mathcal{C}^\infty\mathrm{Alg} \longrightarrow \mathrm{CAlg}_{\mathbb{R}}^{\geq 0}$ preserves pushouts along maps of $\mathcal{C}^\infty$-rings that induce a surjection on π_0.*

Proof. Clearly the forgetful functor is right Quillen. Let $A \longrightarrow B$ and $A \longrightarrow C$ be maps between cofibrant objects of $\mathcal{C}^\infty\mathrm{Alg}^{\mathrm{dg}}$ and suppose that $A \longrightarrow B$ induces a surjection on π_0. To compute the homotopy pushout $B \coprod_A^h C$, it suffices to replace the map $A \longrightarrow B$ by a cofibration $A \longrightarrow \widetilde{B}$, followed by a weak equivalence $\widetilde{B} \longrightarrow B$. Such a cofibration may be constructed by means of 'adding cells to kill a cycle': we construct a sequence of cofibrations $A = A^{(-1)} \longrightarrow A^{(0)} \longrightarrow A^{(1)} \longrightarrow \cdots \longrightarrow B$ such that the map $A^{(n)} \longrightarrow B$ induces an isomorphism on homotopy groups in degree $< n$ and a surjection on homotopy groups in degree n. Given $A^{(n)} \longrightarrow B$, we define $A^{(n+1)}$ as the pushout

$$\begin{tikzcd}
\mathrm{Free}(\bigoplus_\alpha \mathbb{R}[n]) \arrow[r] \arrow[d] & A^{(n)} \arrow[d] & \\
\mathrm{Free}(\bigoplus_\alpha \mathbb{R}[n, n+1]) \arrow[r] & A^{(n+1)} \arrow[r] & B
\end{tikzcd}$$

where the sum runs over all classes $\alpha \in \pi_{n+1}(B, A^{(n)})$. Such classes can be represented by a cycle $\mathbb{R}[n] \longrightarrow A^{(n)}$ together with a null-homotopy $\mathbb{R}[n, n+1] \longrightarrow B$ of its image in B. One can easily verify that $A^{(n+1)}$ has the desired property, so that the map $\widetilde{B} := \mathrm{colim}\, A^{(n)} \longrightarrow B$ is a weak equivalence while $A \longrightarrow \widetilde{B}$ is a cofibration.

When the map $\pi_0(A) \longrightarrow \pi_0(B)$ is surjective, we may assume that $A^{(0)} = A$. In that case, the cofibration $A \longrightarrow \widetilde{B}$ is given by a map of the form $A \longrightarrow A[x_i]$, where the variables x_i are concentrated in degrees ≥ 1. In particular, the map $A \longrightarrow \widetilde{B}$ induces an isomorphism of $\mathcal{C}^\infty$-rings in degree 0, so that the pushout $\widetilde{B} \coprod_A C$ is given by the usual tensor product $\widetilde{B} \otimes_A C$. Since the map $A \longrightarrow \widetilde{B}$ is also a cofibration of commutative dg-algebras it follows that

$$B \coprod_A^h C \simeq \widetilde{B} \coprod_A C \cong \widetilde{B} \otimes_A C \cong B \otimes_A^h C.$$

where the last equivalence uses that $\mathrm{CAlg}_{\mathbb{R}}^{\geq 0,\mathrm{dg}}$ is left proper. $\square$

Remark 2.2.8. Let $B \longleftarrow A \longrightarrow C$ be a diagram in $\mathcal{C}^\infty\mathrm{Alg}^{\mathrm{dg}}$ and assume that $A \longrightarrow B$ induces a surjection on π_0. Lemma 2.2.7 implies that the homotopy pushout $B \coprod_A^h C$ can be computed as the usual pushout after replacing the map $A \longrightarrow B$ by an equivalent cofibration which induces an isomorphism in degree 0. In particular, one does not have to replace A and C by cofibrant objects.

Corollary 2.2.9. *The forgetful functor of ∞-categories $U\colon \mathcal{C}^\infty\mathrm{Alg} \longrightarrow \mathrm{CAlg}_{\mathbb{R}}^{\geq 0}$ preserves all colimits of diagrams indexed by filtered categories and $\mathbf{\Delta}^{\mathrm{op}}$.*

Proof. The right Quillen functor $\mathcal{C}^\infty\mathrm{Alg}^{\mathrm{dg}} \longrightarrow \mathrm{CAlg}_{\mathbb{R}}^{\geq 0,\mathrm{dg}}$ preserves filtered colimits. Since the quasi-isomorphisms are closed under filtered colimits, it follows that the functor U preserves all filtered (homotopy) colimits.

To see that U preserves colimits of simplicial diagrams, let us start by considering some special cases. For each $B \in \mathcal{C}^\infty\mathrm{Alg}$ and each simplicial set K, let $B \otimes K\colon \mathbf{\Delta}^{\mathrm{op}} \longrightarrow \mathcal{C}^\infty\mathrm{Alg}$ be the simplicial object given in degree k by a coproduct of copies of B indexed by K_k. The functor U preserves the colimits of $B \otimes \Delta[n]$ and $B \otimes \Lambda^0[n]$: indeed, these diagrams extend to augmented simplicial diagrams (given in degree -1 by B) with extra degeneracies (provided by a contracting homotopy of $\Delta[n]$ and $\Lambda^0[n]$). It follows that the colimits of these diagrams are just B and are preserved by any functor (see [59, Lemma 6.1.3.16]).

Next, let us show by induction that U preserves the colimits of all simplicial objects $B \otimes \partial\Delta[n+1]$ for $n \geq 0$. This is clear for $n = 0$, in which case it is simply the constant diagram on $B \coprod B$. Note that we are *not* asserting that $U(B \otimes K)$ is given in degree k by a coproduct of commutative algebras.

For higher n, consider the two pushout squares of simplicial objects in $\mathcal{C}^\infty\mathrm{Alg}$

$$\begin{tikzcd} B \otimes \Lambda^0[n+1] \arrow[r] \arrow[d] & B \otimes \Delta[0] \arrow[d] \\ B \otimes \partial\Delta[n+1] \arrow[r] & B \otimes (\Delta[n]/\partial\Delta[n]) \end{tikzcd} \qquad \begin{tikzcd} B \otimes \partial\Delta[n] \arrow[r] \arrow[d] & B \otimes \Delta[0] \arrow[d] \\ B \otimes \Delta[n] \arrow[r] & B \otimes (\Delta[n]/\partial\Delta[n]) \end{tikzcd}$$

The top horizontal arrows induce (degreewise) surjections on π_0, so the induced maps on colimits also induce surjections on π_0. By Lemma 2.2.7, the above squares, as well as the induced square of colimits, remain pushout squares after applying U.

The map $B \otimes \Lambda^0[n+1] \longrightarrow B \otimes \Delta[0]$ induces an equivalence on colimits, both before and after applying U. It follows that U preserves the colimit of $B \otimes \partial\Delta[n+1]$ if and only if it preserves the colimit of $B \otimes (\Delta[n]/\partial\Delta[n])$. In turn, it follows from the second square that U preserves the colimit of $B \otimes (\Delta[n]/\partial\Delta[n])$ if it preserves the colimit of $B \otimes \partial\Delta[n]$. One can therefore proceed by induction.

Finally, let $A\colon \mathbf{\Delta}^{\mathrm{op}} \longrightarrow \mathcal{C}^\infty\mathrm{Alg}$ be an arbitrary diagram. The simplex category $\mathbf{\Delta}^{\mathrm{op}}$ has a cofinal subcategory $\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{inj}}$, consisting of only the injective maps between linear orders [59, Lemma 6.5.3.7]. It therefore suffices to prove that the forgetful functor U preserves the colimit of the underlying semisimplicial diagram $\underline{A}$ of A. This diagram can be obtained as the colimit of a sequence of skeleta $\mathrm{sk}_0(\underline{A}) \longrightarrow \mathrm{sk}_1(\underline{A}) \longrightarrow \cdots \longrightarrow \underline{A}$, where $\mathrm{sk}_n(\underline{A})$ is the left Kan extension of the restriction $\underline{A}\big|\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{inj},\leq n}$. Each map $\mathrm{sk}_n(\underline{A}) \longrightarrow \mathrm{sk}_{n+1}(\underline{A})$ fits into a pushout diagram of semisimplicial diagrams

$$\begin{tikzcd} A_{n+1}\underline{\otimes}\partial\Delta[n+1] \arrow[r] \arrow[d] & \mathrm{sk}_n(\underline{A}) \arrow[d] \\ A_{n+1}\underline{\otimes}\Delta[n+1] \arrow[r] & \mathrm{sk}_{n+1}(\underline{A}). \end{tikzcd}$$

Here $A_{n+1}\underline{\otimes}K$ is the semisimplicial object given in degree k by a coproduct of A_{n+1} indexed by the nondegenerate k-simplices of K. In particular, its left Kan extension along $\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{inj}} \longrightarrow \mathbf{\Delta}^{\mathrm{op}}$ is just $A_{n+1}\otimes K$. The top horizontal map induces surjections on π_0 in each degree: indeed, the map is given by equivalences in degrees $> n$ and in degrees $k \leq n$, the map $\coprod A_{n+1} \longrightarrow A_k$ admits a section, provided by a choice of degeneracy. Consequently, the above square remains a pushout of semisimplicial objects after applying U.

Since U preserves the above pushout square, as well as the colimits of three of its constituents, it follows that it also preserves the colimit of $\mathrm{sk}_{n+1}(\underline{A})$. Finally, we conclude that U preserves the colimit of $\underline{A}$ since it preserves colimits of sequences. $\square$

Corollary 2.2.10. *Let $\mathcal{T} \subseteq \mathcal{C}^\infty\mathrm{Alg}$ be the full (discrete) subcategory on the $\mathcal{C}^\infty$-rings $\mathcal{C}^\infty(\mathbb{R}^n)$ for $n \geq 0$. Then $\mathcal{T}$ provides a set of compact projective generators for the ∞-category $\mathcal{C}^\infty\mathrm{Alg}$ (in the sense of [59, Definition 5.5.8.23]). In particular, the ∞-category $\mathcal{C}^\infty\mathrm{Alg}$ is equivalent to the ∞-category of functors $\mathcal{T}^{\mathrm{op}} \longrightarrow \mathcal{S}$ that preserve finite products.*

Proof. There is a Quillen equivalence between the model category $\mathrm{CAlg}^{\geq 0,\mathrm{dg}}$ of non-negatively graded dg-$\mathbb{R}$-algebras and the model category of simplicial $\mathbb{R}$-algebras, via the Dold-Kan correspondence (see [81] or [30, Chapter 6]). This implies that the ∞-category $\mathrm{CAlg}^{\geq 0}_{\mathbb{R}}$ has a set of compact projective generators, given by the discrete polynomial algebras $\mathbb{R}[x_1, ..., x_n]$ with $n \geq 0$ (see [59, Proposition 5.5.8.25]). Since the forgetful functor $\mathcal{C}^\infty\mathrm{Alg} \longrightarrow \mathrm{CAlg}^{\geq 0}_{\mathbb{R}}$ detects equivalences and preserves sifted (homotopy) colimits (by Corollary 2.2.9), its left adjoint sends a set of compact projective generators to a set of compact projective generators. The result now follows from the fact that this left adjoint sends a polynomial algebra $\mathbb{R}[x_1, ..., x_n]$ to $\mathcal{C}^\infty(\mathbb{R}^n)$. □

Remark 2.2.11. Corollary 2.2.10 implies that the homotopy theory of dg-$\mathcal{C}^\infty$-rings is equivalent to the homotopy theory of $\mathcal{C}^\infty$-rings in simplicial sets. Indeed, by [59, Corollary 5.5.9.3], the model structure on $\mathcal{C}^\infty$-rings in simplicial sets is also a model for the ∞-category of product-preserving functors $\mathcal{T}^{\mathrm{op}} \longrightarrow \mathcal{S}$. In particular, our differential graded approach to 'derived $\mathcal{C}^\infty$-rings' is equivalent to the simplicial approach from [92, 14].

Remark 2.2.12. The conclusions of Corollary 2.2.10 and Remark 2.2.11 apply to more general types of algebraic theories $\mathcal{T}$. If $\mathcal{T}$ is a Fermat theory [21] extending the theory of commutative $\mathbb{Q}$-algebras, then the ∞-category of $\mathcal{T}$-algebras in spaces can be presented by a (transferred) model structure on non-negatively graded dg-algebras A endowed with a compatible $\mathcal{T}$-algebra structure on A_0. The only nontrivial thing to check is that this transferred model structure exists; all proofs in this section then carry over. To construct the model structure, one uses that for any dg-$\mathcal{T}$-algebra A, the free dg-$\mathcal{T}$-algebra $A\{t\}$ with an extra generator has t as a regular element.

2.2.2 Cotangent complex. A module over a $\mathcal{C}^\infty$-ring is simply a module over its underlying commutative $\mathbb{R}$-algebra. If A is a dg-$\mathcal{C}^\infty$-ring, we define $\mathrm{Mod}^{\mathrm{dg}}_A$ to be the category of (unbounded) dg-modules over A, with the usual projective model structure, and Mod_A to be its associated ∞-category. A map $f\colon A \longrightarrow B$ of dg-$\mathcal{C}^\infty$-rings induces a Quillen adjunction

$$f^*\colon \mathrm{Mod}^{\mathrm{dg}}_A \rightleftarrows \mathrm{Mod}^{\mathrm{dg}}_B \colon f_*$$

where f_* is given by restriction of scalars and $f^*(E) = B \otimes_A E$. The notation follows geometric conventions: we think of $f^*(E)$ as the restriction of E along $f\colon \mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$. When f is a weak equivalence, the above adjunction is a Quillen equivalence.

Recall that any unbounded module E over a commutative dg-algebra A gives rise to a split square zero extension $A \oplus \tau_{\geq 0}E$ of A by the connective cover of E. When A is a dg-$\mathcal{C}^\infty$-ring, the degree zero part $A_0 \oplus Z_0(E)$ of this square zero extension carries the canonical structure of a $\mathcal{C}^\infty$-ring, depending functorially on A and E [19, Proposition 2.42]. We therefore obtain a functor

$$A \oplus \tau_{\geq 0}(-)\colon \mathrm{Mod}^{\mathrm{dg}}_A \longrightarrow \mathcal{C}^\infty\mathrm{Alg}^{\mathrm{dg}}/A \tag{2.2.13}$$

depending naturally on A. This functor is easily seen to be a right Quillen functor.

Remark 2.2.14. The right Quillen functor (2.2.13) induces a right adjoint functor of ∞-categories $\mathrm{Mod}_A \longrightarrow \mathcal{C}^\infty\mathrm{Alg}/A$. This right adjoint can be characterized by a universal property: it realizes Mod_A as the universal stable ∞-category equipped with a right adjoint functor to $\mathcal{C}^\infty\mathrm{Alg}/A$ (see e.g. [62, Section 7.3] for a discussion). Indeed, Mod_A has a similar universal property with respect to the ∞-category $\mathrm{CAlg}^{\geq 0}_{\mathbb{R}}/A$ of commutative algebras over

A. The result then follows from the fact that the functor $\mathcal{C}^\infty\mathrm{Alg}/A \longrightarrow \mathrm{CAlg}_{\mathbb{R}}^{\geq 0}/A$ induces an equivalence between *suspensions* $A \coprod_B A$ of $\mathcal{C}^\infty$-rings over A and suspensions of commutative algebras over A (by an argument analogous to Lemma 2.2.7).

Explicitly, the left adjoint of (2.2.13) sends a map of dg-$\mathcal{C}^\infty$-rings $B \longrightarrow A$ to $A \otimes_B \Omega_B$, where Ω_B is the dg-module of $\mathcal{C}^\infty$-algebraic Kähler differentials of B. This is simply the quotient of the free dg-B-module generated by elements $d_{\mathrm{dR}}b$ for $b \in B$, subject to relations of the form

$$\begin{aligned} d_{\mathrm{dR}}(a \cdot b) &= (-1)^{ab} b \cdot d_{\mathrm{dR}}a + a \cdot d_{\mathrm{dR}}b \\ d_{\mathrm{dR}}\big(\phi_*(b_1, ..., b_n)\big) &= \sum_i \left(\frac{\partial \phi}{\partial x_i}\right)_* (b_1, ..., b_n) \cdot d_{\mathrm{dR}}(b_i) \\ d_{\mathrm{dR}}(\partial a) &= \partial(d_{\mathrm{dR}}a) \end{aligned}$$

where $\phi\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ is a smooth function and $b_i \in B_0$.

Definition 2.2.15. Let A be a $\mathcal{C}^\infty$-ring. The *cotangent complex* L_A of A is the value on A of the derived functor of the left Quillen functor $\big[B \longrightarrow A\big] \mapsto A \otimes_B \Omega_B$. More generally, if $\phi\colon A \longrightarrow B$ is a map of dg-$\mathcal{C}^\infty$-rings, the relative cotangent complex $L_{B/A}$ of ϕ is the cofiber of the natural map of B-modules $B \otimes_A L_A \longrightarrow L_B$.

Remark 2.2.16. Alternatively, the relative cotangent complex $L_{B/A}$ can be characterized by the following universal property: for every B-module M, the space of maps $\mathrm{Map}_B(L_{B/A}, M)$ is naturally equivalent to the space of dotted sections

$$\begin{tikzcd} A \arrow[r, "{(\phi,0)}"] \arrow[d, "\phi"'] & B \oplus M \arrow[d] \\ B \arrow[r, "="'] \arrow[ur, dotted] & B. \end{tikzcd}$$

Example 2.2.17. Let $U \subseteq \mathbb{R}^n$ be the open subset and consider the map of $\mathcal{C}^\infty$-rings $f\colon \mathcal{C}^\infty(\mathbb{R}^n) \longrightarrow \mathcal{C}^\infty(U)$. Then $L_{\mathcal{C}^\infty(U)/\mathcal{C}^\infty(\mathbb{R}^n)} \simeq 0$. Indeed, let χ be a characteristic function for U, i.e. $U = \chi^{-1}(\mathbb{R} \setminus \{0\})$. Then the map f can be presented by the cofibration

$$\mathcal{C}^\infty(\mathbb{R}^n) \longrightarrow \mathcal{C}^\infty(\mathbb{R}^{n+1})[\eta]$$

with $\partial\eta = \chi(x_1, \ldots, x_n) \cdot y - 1$, which is a regular element of $\mathcal{C}^\infty(\mathbb{R}^{n+1})$. Since $\mathcal{C}^\infty(\mathbb{R}^{n+1})[\eta]$ is free, its cotangent complex is a free dg-module on generators $d_{\mathrm{dR}}(x_i), d_{\mathrm{dR}}(y)$ and $d_{\mathrm{dR}}(\eta)$, with

$$\partial(d_{\mathrm{dR}}\eta) = d_{\mathrm{dR}}(\chi(x)y - 1) = \chi \cdot d_{\mathrm{dR}}y + y \cdot d_{\mathrm{dR}}\chi.$$

The map $\mathcal{C}^\infty(\mathbb{R}^{n+1})[\eta] \otimes_{\mathcal{C}^\infty(\mathbb{R}^n)} L_{\mathcal{C}^\infty(\mathbb{R}^n)} \longrightarrow L_{\mathcal{C}^\infty(\mathbb{R}^{n+1})[\eta]}$ is the inclusion of the free module on $d_{\mathrm{dR}}x_i$. The second term of the above formula vanishes in the quotient and one finds that $L_{\mathcal{C}^\infty(U)/\mathcal{C}^\infty(\mathbb{R}^n)} \simeq 0$.

Example 2.2.18. Let M be a smooth manifold. Then the cotangent complex of $\mathcal{C}^\infty(M)$ is equivalent the module $\Omega^1(M)$ of 1-forms on M. Indeed, for any open subspace $U \subseteq \mathbb{R}^n$, it follows from Example 2.2.17 that $L_U \simeq \Omega^1(U)$. A general manifold M can be realized as a retract of an open subspace U of some $\mathbb{R}^n$. It follows that the natural map $L_{\mathcal{C}^\infty(M)} \longrightarrow \Omega^1(M)$ is a retract of the map

$$\mathcal{C}^\infty(M) \otimes_{\mathcal{C}^\infty(U)} L_{\mathcal{C}^\infty(U)} \longrightarrow \mathcal{C}^\infty(M) \otimes_{\mathcal{C}^\infty(U)} \Omega^1(U).$$

But this map is a weak equivalence, so the map $L_{\mathcal{C}^\infty(M)} \longrightarrow \Omega^1(M)$ is a weak equivalence as well.

Similarly, suppose that $p\colon N \longrightarrow M$ is a surjective submersion. Then

$$L_{\mathcal{C}^\infty(N)/\mathcal{C}^\infty(M)} \simeq \Omega^1(N/M)$$

is equivalent to the module of smooth 1-forms along the fibers of p.

Example 2.2.19. Let $f = (f_1, \dots, f_n)\colon M \longrightarrow \mathbb{R}^n$ be any $\mathcal{C}^\infty$-function and consider the dg-$\mathcal{C}^\infty$-ring $A = \mathcal{C}^\infty(M)[\eta_1, \dots, \eta_n]$ of Example 2.2.2. Then the cotangent complex L_A is given by

$$\Omega^1(M)[\eta_1, \dots, \eta_n] \oplus A\big\langle d_{\mathrm{dR}}\eta_1, \dots, d_{\mathrm{dR}}\eta_n\big\rangle \qquad \partial(d_{\mathrm{dR}}\eta_i) = d_{\mathrm{dR}}(f_i) \in \Omega^1(M).$$

In particular, the relative cotangent complex $L_{A/\mathcal{C}^\infty(M)}$ is given by the free A-module $A\big\langle d_{\mathrm{dR}}\eta_1, \dots, d_{\mathrm{dR}}\eta_n\big\rangle$. This witnesses the fact that the derived zero locus of f is of codimension n inside M.

Classically, the cotangent complex of a discrete ring A plays an important role in the classification of square-zero extensions of A: for every discrete A-module I, there is a bijection between the set of square-zero extensions of A by I and the first Ext-group $\mathrm{Ext}^1(L_A, I)$. When trying to generalize this correspondence to the derived setting, one runs into the problem of giving a rigorous definition of a square zero extension $p\colon A' \longrightarrow A$: rather than saying that two elements in the kernel of p multiply to zero, one needs to provide (coherent) homotopies that witness this. For this reason, one usually *defines* square zero extensions in terms the cotangent complex. In the setting of $\mathcal{C}^\infty$-rings, this leads to the following definition:

Definition 2.2.20 ([97, Definition 1.2.1.6], [62, Definition 7.4.1.6])**.** Let $A \in \mathcal{C}^\infty\mathrm{Alg}$ and let $I \in \mathrm{Mod}_A^{\geq 0}$. We will say that a map $p\colon A' \longrightarrow A$ in $\mathcal{C}^\infty\mathrm{Alg}$ is a *square zero extension* of A by I if it fits into a pullback square in $\mathcal{C}^\infty\mathrm{Alg}/A$ of the form

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "p"'] & A \arrow[d, "0"] \\ A \arrow[r, "\eta"'] & A \oplus I[1]. \end{tikzcd}$$

Here the map 0 is the image of $0 \longrightarrow I[1]$ under the functor (2.2.13) and η is any section of the structure map $A \oplus I[1] \longrightarrow A$.

Any map of A-modules $\eta\colon L_A \longrightarrow I[1]$ determines a map $\eta\colon A \longrightarrow A \oplus I[1]$. In turn, this yields a square zero extension $A_\eta \longrightarrow A$ of A by I, given by the pullback $A \times_{A \oplus I[1]} A$. Conversely, any square zero extension $p\colon A' \longrightarrow A$ is classified by some $\eta\colon L_A \longrightarrow \mathrm{fib}(p)$. However, this map η is *not* determined uniquely by the map p (in fact, there is no canonical way to make $\mathrm{fib}(p)$ into an A-module).

Because of this, it is not so clear how to recognize square zero extensions. The following result gives a model-categorical method for producing square zero extensions:

Lemma 2.2.21. *Let $p\colon A' \longrightarrow A$ be a* strict *square zero extension of dg-$\mathcal{C}^\infty$-rings, i.e. p is surjective and elements in the kernel of p square to zero. Then p determines a square zero extension in the ∞-category $\mathcal{C}^\infty\mathrm{Alg}$.*

Proof. By pulling back p along a trivial fibration, we may assume that the dg-$\mathcal{C}^\infty$-ring A is freely generated (without differential) by a set of generators $\{x_i\}$. Since $p\colon A' \longrightarrow A$ is a square zero extension in the usual (non-homotopical) sense, the kernel $I = \ker(p)$ has the structure of a dg-A-module. Since A is free, the map $p\colon A' \longrightarrow A$ admits a section $s\colon A \longrightarrow A'$ at the level of graded objects, so that $A' \cong A \oplus I$ without the differential. The differential takes the form $\partial(a, v) = (\partial_A a, \partial_I v + \eta(a))$, for some $\mathcal{C}^\infty$-derivation $\eta\colon A \longrightarrow I[1]$.

A straightforward computation now shows that $p\colon A' \longrightarrow A$ fits into a pullback square of the form

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & A \oplus I[0,1] \arrow[d, "q"] \\ A \arrow[r, "\eta"'] & A \oplus I[1] \end{tikzcd} \tag{2.2.22}$$

where the bottom map arises from the derivation η and the right vertical map is induced by the surjective map $I[0,1] \longrightarrow I[1]$ from the path space of $I[1]$. It follows that the above square is a homotopy pullback square, which realizes A' as a square zero extension of A. □

Remark 2.2.23. Let $p\colon A' \longrightarrow A$ be square zero extension in $\mathcal{C}^\infty\mathrm{Alg}$, classified by a map $\eta\colon A \longrightarrow A \oplus I[1]$. One can choose a cofibrant dg-$\mathcal{C}^\infty$-ring presenting A, together with a dg-module over it that presents I. In that case, the map η arises from a map of dg-$\mathcal{C}^\infty$-rings, so that the map p can be modeled by the (homotopy) pullback of η along the path fibration q as in (2.2.22). It follows that any square zero extension in the ∞-category $\mathcal{C}^\infty\mathrm{Alg}$ can be modeled by a strict square zero extension of dg-$\mathcal{C}^\infty$-rings.

Corollary 2.2.24. *Let $p\colon A' \longrightarrow A$ be a map of k-truncated objects in $\mathcal{C}^\infty\mathrm{Alg}$ such that p induces an isomorphism $\pi_m(A') \longrightarrow \pi_m(A)$ for all $m < k$ and a surjection $\pi_k(A') \longrightarrow \pi_k(A)$ with kernel I. Then p is a square zero extension if $k > 0$ or if $k = 0$ and $I^2 = 0$ in the $\mathcal{C}^\infty$-ring $\pi_0(A')$.*

Proof. We can model the map p by a map between dg-$\mathcal{C}^\infty$-rings which are concentrated in degrees $[0,k]$. In that case, $I \subseteq \pi_k(A') \cong Z_k(A')$ is just an ideal in A' and the map $A'/I \longrightarrow A$ is a weak equivalence. But then the map $A' \longrightarrow A$ is just modeled by the map $A' \longrightarrow A'/I$, which is a strict square zero extension under the given conditions. □

In particular, it follows that for any $A \in \mathcal{C}^\infty\mathrm{Alg}$, the maps in the Postnikov tower $\tau_{\leq n}A \longrightarrow \tau_{\leq n-1}A$ are square zero extensions for $n \geq 1$. A simple inductive argument now shows:

Corollary 2.2.25. *A map $f\colon A \longrightarrow B$ in $\mathcal{C}^\infty\mathrm{Alg}$ is an equivalence if and only if $\pi_0(A) \longrightarrow \pi_0(B)$ is an isomorphism of discrete $\mathcal{C}^\infty$-rings and $\mathrm{L}_{A/B} \simeq 0$.*

As another example of the power of the cotangent complex, one can observe that the cotangent complex largely controls the finiteness properties of $\mathcal{C}^\infty$-rings:

Definition 2.2.26. A connective module E over a $\mathcal{C}^\infty$-ring A is *finitely presented* if it is contained in the smallest subcategory of $\mathrm{Mod}_A^{\geq 0}$ which contains the free A-module A and is closed under finite colimits.

A map of $\mathcal{C}^\infty$-rings $f\colon A \longrightarrow B$ is *finitely presented* if it is contained in the smallest subcategory of $A/\mathcal{C}^\infty\mathrm{Alg}$ which contains the free $\mathcal{C}^\infty$-ring $A\{t\} = A \coprod \mathcal{C}^\infty(\mathbb{R})$ and is closed under finite colimits.

Remark 2.2.27. Let A be a cofibrant object of $\mathcal{C}^\infty\mathrm{Alg}^{\mathrm{dg}}$ and let $B = A\{x_i\}$ be a dg-$\mathcal{C}^\infty$-ring which is obtained by freely adding generators x_i of degrees ≥ 0, with possibly nontrivial differential. Then the map $A \longrightarrow B$ is a cofibration, whose associated map in $\mathcal{C}^\infty\mathrm{Alg}$ is finitely presented if there are only finitely many generators. Conversely, any finitely presented map in $\mathcal{C}^\infty\mathrm{Alg}$ can be modeled by such a map. A similar remark applies to finitely presented A-modules.

Lemma 2.2.28 ([62, Theorem 7.4.3.18])**.** *Let $f\colon A \longrightarrow B$ be a map of $\mathcal{C}^\infty$-rings. Then the following two assertions are equivalent:*

(1) f is finitely presented.

(2) The map of discrete $\mathcal{C}^\infty$-rings $\pi_0(A) \longrightarrow \pi_0(B)$ is finitely presented in the usual sense and $L_{B/A}$ is an finitely presented B-module.

Proof. Using Remark 2.2.27, one easily sees that (1) implies (2). For the converse, we reduce to the case of commutative algebras proven in [62] (see also [97, Proposition 2.2.2.4]) as follows: since $\pi_0(A) \longrightarrow \pi_0(B)$ is finitely presented, there exists a factorization

$$A \longrightarrow A\{x_1, \dots, x_n\} \xrightarrow{\tilde{f}} B$$

where $\tilde{f}$ induces a surjection on π_0. It suffices to show that $\tilde{f}$ is finitely presented. Since $\tilde{f}$ induces a surjection on π_0, it can be modeled by a cofibration given by only adding polynomial generator in degrees ≥ 1. This implies that the $\mathcal{C}^\infty$-algebraic cotangent complex $L_{B/A\{x_1,\dots,x_n\}}$ is equivalent to the usual cotangent complex in the sense of algebra.

Since $L_{B/A\{x_1,\dots,x_n\}}$ is finitely presented, the proof in loc. cit. shows that $\tilde{f}$ can be obtained by adding finitely many generators in degrees ≥ 1. This means that $\tilde{f}$ is finitely presented as a map of $\mathcal{C}^\infty$-rings. □

2.3 Deformation theory

Square zero extensions play an important role in *deformation theory*, tracing back at least to the work of Kodaira and Spencer [55]. As a motivating example, let us recall the classical theory of deformations of modules over square zero extensions. If A is a (discrete) commutative algebra and E is a (discrete) A-module, then a *deformation* of E along a map of commutative algebras $A' \longrightarrow A$ is an A'-module E', together with an isomorphism

$$E' \otimes_{A'} A \xrightarrow{\cong} E.$$

For a general map of rings $A' \longrightarrow A$, there is no way to classify deformations of M, since A' may be much bigger than A.

However, if $A' \longrightarrow A$ is a square zero extension of A by an A-module I (i.e. an infinitesimal thickening in terms of geometry), then deformations of E over A' can be classified concretely in terms of cohomological data: there is an obstruction class $[\mathrm{ob}] \in \mathrm{Ext}^2_A(E, E \otimes_A I)$, which vanishes if and only if there exists a deformation of E. In that case, the set of isomorphism classes of such deformations is a torsor over $\mathrm{Ext}^1_A(E, E \otimes_A I)$ (see e.g. [46, Chapitre IV] for a textbook account).

This kind of cohomological classification of deformations admits a particularly useful and natural description in the derived setting of Definition 2.2.20 (see e.g. [62, Section 7.4.2] for deformations of commutative algebras). The aim of this section is to recall this description and to recall how formal deformation theory, i.e. the theory of deformations along nilpotent extensions, can be organized conveniently in terms of so-called *formal moduli problems*. For more discussion, we refer to [61], which we follow closely. For later applications, we will phrase everything in terms of extensions of $\mathcal{C}^\infty$-rings, although the $\mathcal{C}^\infty$-structure plays essentially no role.

2.3.1 Deformations of modules. Let us start by giving an account of the deformation theory of modules in terms of derived $\mathcal{C}^\infty$-rings.

Definition 2.3.1. Let $p\colon A' \longrightarrow A$ be a map of $\mathcal{C}^\infty$-rings and let E be a connective A-module. A *deformation* of E along p is a connective A'-module E', together with the data of an A-linear equivalence

$$p^*E' := A \otimes_{A'} E' \xrightarrow{\sim} E.$$

Given a deformation E' of E over A' and a map of $\mathcal{C}^\infty$-rings over A

$$A' \xrightarrow{f} A'' \xrightarrow{p} A$$

the A''-module f^*E' is a deformation of E over A''. It follows that the possible deformations of E over A' depend functorially on A'. To make this a bit more precise, let us consider the following construction:

Construction 2.3.2. Consider the (pseudo-)functor

$$\mathcal{C}^\infty\mathrm{Alg}^{\mathrm{dg}} \longrightarrow \mathrm{ModCat}^{\mathrm{L}};\ \ A \longmapsto \mathrm{Mod}_A^{\geq 0,\mathrm{dg}}$$

sending each dg-$\mathcal{C}^\infty$-ring A to the category of connective dg-A-modules, equipped with the projective model structure. A map $f\colon A \longrightarrow B$ is sent to the left Quillen functor $f^* = B \otimes_A (-)$.

Since a weak equivalence of dg-$\mathcal{C}^\infty$-rings induces a Quillen equivalence between module categories, we obtain a functor of ∞-categories

$$\mathrm{Mod}^{\geq 0}\colon \mathcal{C}^\infty\mathrm{Alg} \longrightarrow \mathrm{Pr}^{\mathrm{L}};\ \ A \longmapsto \mathrm{Mod}_A^{\geq 0}$$

after formally inverting the quasi-isomorphisms. If E is a connective A-module, let

$$\mathrm{Def}_E\colon \mathcal{C}^\infty\mathrm{Alg}/A \longrightarrow \widehat{\mathrm{Cat}}_\infty;\ \ A' \longmapsto \mathrm{Mod}_{A'}^{\geq 0} \times_{\mathrm{Mod}_A} \{E\} \tag{2.3.3}$$

be the fiber over E of the canonical map from the restriction of the diagram $\mathrm{Mod}^{\geq 0}$ along $\mathcal{C}^\infty\mathrm{Alg}/A \longrightarrow \mathcal{C}^\infty\mathrm{Alg}$ and the constant diagram with value Mod_A. For each A', the (locally small) ∞-category $\mathrm{Def}_E(A')$ is the *∞-category of deformations of* E over A'.

In terms of the functor Def_E (2.3.3), the classification of deformations of an A-module E along a square zero extension arises from the following observation:

Proposition 2.3.4. *Consider a pullback square of $\mathcal{C}^\infty$-rings over A*

$$\begin{tikzcd} A'_\eta \arrow[r, "f"] \arrow[d, "p"'] & A \arrow[d, "0"] \\ A' \arrow[r, "\eta"'] & A \oplus I[1]. \end{tikzcd} \tag{2.3.5}$$

describing A'_η as a square zero extension of A' by an A-module I. Then the images of this square

$$\begin{tikzcd} \mathrm{Mod}_{A'_\eta}^{\geq 0} \arrow[r] \arrow[d, "p^*"'] & \mathrm{Mod}_A^{\geq 0} \arrow[d, "0^*"] \\ \mathrm{Mod}_{A'}^{\geq 0} \arrow[r, "\eta^*"'] & \mathrm{Mod}_{A\oplus I[1]}^{\geq 0} \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Def}_E(A'_\eta) \arrow[r] \arrow[d, "p^*"'] & \mathrm{Def}_E(A) \arrow[d, "0^*"] \\ \mathrm{Def}_E(A') \arrow[r, "\eta^*"'] & \mathrm{Def}_E(A \oplus I[1]). \end{tikzcd}$$

are pullback diagrams of ∞-categories.

Remark 2.3.6. In particular, it follows that for every square zero extension $p\colon A_\eta \longrightarrow A$, the functor $p^*\colon \mathrm{Mod}_{A_\eta}^{\geq 0} \longrightarrow \mathrm{Mod}_A^{\geq 0}$ detects equivalences. Indeed, it is the pullback of the functor

$$0^*\colon \mathrm{Mod}_A^{\geq 0} \longrightarrow \mathrm{Mod}_{A\oplus I[1]}^{\geq 0}$$

which detects equivalences because it admits a retraction.

Corollary 2.3.7. *Let A be a $\mathcal{C}^\infty$-ring and E be a connective A-module. Let $\eta\colon A \longrightarrow A \oplus I[1]$ classify a square zero extension $p\colon A_\eta \longrightarrow A$ of A by a connective A-module I. Then the ∞-category $\mathrm{Def}_E(A_\eta)$ of deformations of E is equivalent to the space of sections of the natural map of A-modules*

$$r\colon \eta_*\big(E \otimes_A (A \oplus I[1])\big) \longrightarrow \eta_*(E \otimes_A (A \oplus 0)) = E. \tag{2.3.8}$$

Proof. Let $q\colon A \oplus I[1] \longrightarrow A$ be the obvious projection, so that $q \circ \eta = q \circ 0 = \mathrm{id}_A$. Since $\mathrm{Def}_E(A) \simeq \{E\}$ is contractible, Proposition 2.3.4 identifies $\mathrm{Def}_E(A_\eta)$ with the space of equivalences $\eta^*E \xrightarrow{\sim} 0^*E$ in $\mathrm{Def}_E(A \oplus I[1])$. This space of equivalences can be identified with the fiber of the map

$$\mathrm{Map}_{A\oplus I[1]}(\eta^*E, 0^*E) \xrightarrow{q^*} \mathrm{Map}_A(q^*\eta^*E, q^*0^*E) \simeq \mathrm{Map}_A(E, E)$$

over the identity of E. Indeed, any map $\eta^*E \longrightarrow 0^*E$ in this fiber is automatically an equivalence, by Remark 2.3.6. Using that the functors η^* and 0^* admit right adjoints (given by restriction of scalars), one can then identify $\mathrm{Def}_E(A'_\eta)$ with the fiber of the map

$$\mathrm{Map}_A(E, \eta_*0^*E) \longrightarrow \mathrm{Map}_A(E, \eta_*q_*q^*0^*E) \simeq \mathrm{Map}_A(E, E)$$

over the identitity of E. In other words, $\mathrm{Def}_E(A_\eta)$ is given by the space of sections of the map

$$r\colon \eta_*0^*E \longrightarrow \eta_*q_*q^*0^*E$$

obtained by applying η_* to the unit map of the adjoint pair (q^*, q_*). Unraveling the definitions, the unit map $0^*E \longrightarrow q_*q^*E$ is the canonical $A \oplus I[1]$-linear map

$$0^*E = E \otimes_A (A \oplus I[1]) \longrightarrow q_*q^*0^*E = q_*E = E$$

which takes the quotient by the submodule $E \otimes_A I[1]$. $\square$

Remark 2.3.9. The map $q\colon A \oplus I[1] \longrightarrow A$ fits into a commuting diagram of $A \oplus I[1]$-modules

$$\begin{tikzcd} A \oplus I[1] \arrow[r] \arrow[d, "q"'] & A \arrow[r] \arrow[d, "0"] & 0 \arrow[d] \\ A \arrow[r] & A \oplus I[2] \arrow[r] & I[2]. \end{tikzcd}$$

The left square is a pullback square of commutative algebras and the right square is a pullback square of $A \oplus I[1]$-modules. The bottom $A \oplus I[1]$-linear map $\theta\colon A \longrightarrow I[2]$ is *not* null-homotopic, but it does admit a null-homotopy after restricting scalars along $0\colon A \longrightarrow A \oplus I[1]$.

Taking the tensor product with 0^*E, we obtain a fiber sequence of $A \oplus I[1]$-modules of the form

$$0^*E = E \otimes_A (A \oplus I[1]) \longrightarrow E \xrightarrow{E \otimes_A \theta} E \otimes_A I[2].$$

Restricting scalars along η, we find that the space of A-sections of the map r (2.3.8) is equivalent to the space of null-homotopies of the A-linear map

$$\theta_E := \eta_*(E \otimes_A \theta)\colon E \longrightarrow E \otimes_A I[2].$$

In particular, the homotopy class of θ_E determines an obstruction class $\mathrm{ob} := [\theta_E]$ in $\mathrm{Ext}^2_A(E, E \otimes_A I)$, which vanishes if and only if there exists a section of r, or equivalently, if there exists a deformation of E along $A_\eta \longrightarrow A$. If the map θ_E is null-homotopic, then the space of null-homotopies is a torsor over the loop space of $\mathrm{Map}(E, E \otimes_A I[2])$. But this is just the space $\mathrm{Map}(E, E \otimes_A I[1])$, whose zeroth homotopy group is $\mathrm{Ext}^1_A(E, E \otimes_A I)$. Corollary 2.3.8 therefore reproduces the standard cohomological classification of deformations of E along square zero extensions.

To prove Proposition 2.3.4, note that the functor $\mathrm{Mod}^{\geq 0}$ factors as

$$\mathcal{C}^\infty\mathrm{Alg} \longrightarrow \mathrm{CAlg}_{\mathbb{R}}^{\geq 0} \xrightarrow{\mathrm{Mod}^{\geq 0}} \mathrm{Pr}^{\mathrm{L}}.$$

The image of the pullback square (2.3.5) under the forgetful functor to commutative $\mathbb{R}$-algebras is a pullback square of (connective) commutative algebras over A, which realizes A'_η as a square zero extension of A' in commutative algebras as well. We can therefore forget about $\mathcal{C}^\infty$-ring structures and work at the level of commutative algebras.

Construction 2.3.10. Let $\mathrm{Mod}^{\geq 0,\mathrm{dg}}$ the category with

- objects given by tuples (A, M) where A is a connective commutative dg-algebra and M is a connective A-module.
- morphisms $(A, M) \longrightarrow (B, N)$ given by a map of dg-algebras $A \longrightarrow B$ and an A-linear map $M \longrightarrow N$.

This category carries a model structure, in which a map is a weak equivalence (fibration) if it is a quasi-isomorphism (surjection in degrees > 0). The obvious projection $\pi\colon \mathrm{Mod}^{\geq 0,\mathrm{dg}} \longrightarrow \mathrm{CAlg}_{\mathbb{R}}^{\geq 0,\mathrm{dg}}$ is a cocartesian fibration which preserves weak equivalences, fibrations and cofibrations.

Remark 2.3.11. In fact, the projection $\pi\colon \mathrm{Mod}^{\geq 0,\mathrm{dg}} \longrightarrow \mathrm{CAlg}_{\mathbb{R}}^{\geq 0,\mathrm{dg}}$ is a model fibration in the sense of [37], which is classified by the (pseudo-)functor

$$\mathrm{CAlg}_{\mathbb{R}}^{\geq 0,\mathrm{dg}} \longrightarrow \mathrm{ModCat}^{\mathrm{L}}; \;\; A \longmapsto \mathrm{Mod}_A^{\geq 0,\mathrm{dg}}.$$

This functor sends weak equivalences of commutative dg-algebras to Quillen equivalences. By [37, Proposition 3.1.2], the associated functor of ∞-categories

$$\pi\colon \mathrm{Mod}^{\geq 0} = \mathrm{Mod}^{\geq 0,\mathrm{dg}}[W^{-1}] \longrightarrow \mathrm{CAlg}_{\mathbb{R}}^{\geq 0,\mathrm{dg}} = \mathrm{CAlg}_{\mathbb{R}}^{\geq 0}$$

is a cartesian and cocartesian fibration, classified by the functor $\mathrm{CAlg}_{\mathbb{R}}^{\geq 0} \longrightarrow \mathrm{Pr}^{\mathrm{L}}$ sending $A \mapsto \mathrm{Mod}_A^{\geq 0}$.

Lemma 2.3.12. *Consider a square $\Delta[1] \times \Delta[1] \longrightarrow \mathrm{Mod}^{\geq 0}$ and its image in $\mathrm{CAlg}_{\mathbb{R}}^{\geq 0}$, which we may depict as*

$$\begin{tikzcd} E' \arrow[r] \arrow[d] & E \arrow[d, "\tilde{0}"] & & A'_\eta \arrow[r, "f"] \arrow[d, "p"'] & A \arrow[d, "0"] \\ F' \arrow[r, "\tilde{\eta}"'] & F & \overset{\pi}{\longmapsto} & A' \arrow[r, "\eta"'] & A \oplus I[1] \end{tikzcd} \tag{2.3.13}$$

Suppose that the square in $\mathrm{CAlg}_{\mathbb{R}}^{\geq 0}$ is cartesian, realizing A'_η as a square zero extension of A' by the A-module I. If $\tilde{\eta}$ and $\tilde{0}$ are π-cocartesian arrows in $\mathrm{Mod}^{\geq 0}$, then the following are equivalent:

(1) The left square is cartesian in $\mathrm{Mod}^{\geq 0}$. In other words, the A'_η-module E' is the pullback $F' \times_F E$, all of whose constituents are considered as A'_η-modules by restriction of scalars.
(2) All edges in the left square are π-cocartesian.

Proof. The map $\tilde{0}$ is cocartesian, so that its underlying map of chain complexes is given by $E \longrightarrow E \otimes_A (A \oplus I[1]) = F$. This map induces a surjection on π_0, so that the square of modules (2.3.13) is cartesian in $\mathrm{Mod}^{\geq 0}$ if and only if it is cartesian in the ∞-category Mod of *unbounded* modules.

Let us now assume (2). Then the square (2.3.13) is equivalent to the square of modules

$$\begin{tikzcd} E' = E' \otimes_{A'_\eta} A'_\eta \arrow[r] \arrow[d] & E' \otimes_{A'_\eta} A = f^*E' \arrow[d] \\ p^*E' = E' \otimes_{A'_\eta} A' \arrow[r] & E' \otimes_{A'_\eta} (A \oplus E[1]) = 0^* f^* E'. \end{tikzcd} \tag{2.3.14}$$

Since the functor $E' \otimes_{A'_\eta} (-)$ preserves homotopy pullback squares of unbounded A'_η-modules, (1) follows.

Conversely, suppose that (1) holds, so that the square (2.3.13) is a pullback square of (unbounded) chain complexes. The square (2.3.14) maps naturally to the original square (2.3.13) and assertion (2) asserts that this natural transformation is a natural equivalence. Taking the pointwise cofiber of this natural transformation yields another cartesian (and cocartesian) square of chain complexes

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & E/f^*E' = C \arrow[d] \\ C' = F'/p^*E' \arrow[r] & F/0^*f^*E' = C''. \end{tikzcd}$$

Each of these complexes is connective, being a cofiber of connective modules. There are equivalences $C'' \simeq \eta^*(C')$, $C'' \simeq 0^*(C)$ and $C'' \simeq C \oplus C'$. By the last equivalence, it suffices to show that $C'' \simeq 0$.

Suppose that C'' is nonzero and let k be the smallest integer such that $\pi_k(C'') \neq 0$. In this case, C and C' are k-connective and the map $C \longrightarrow C'' = 0^*(C)$ induces an isomorphism on homotopy groups

$$\pi_k(C) \xrightarrow{\cong} \pi_k(C) \otimes_{\pi_0(A)} \pi_0(A \oplus I[1]) \xrightarrow{\cong} \pi_k(C'').$$

It follows that $\pi_k(C') = \pi_k(C'')/\pi_k(C) = 0$. But since $C'' \simeq \eta^*(C')$, this implies that

$$0 = \pi_k(C') \otimes_{\pi_0(A')} \pi_0(A \oplus I[1]) \cong \pi_k(C'')$$

so that C'' is indeed null-homotopic. $\square$

Proof (of Proposition 2.3.4). The assertion about Def_E follows from the assertion about the functor $\mathrm{Mod}^{\geq 0}$, since Def_E is the pullback (2.3.3) of $\mathrm{Mod}^{\geq 0}$ along a map between constant diagrams of ∞-categories. Let $\pi\colon \mathrm{Mod}^{\geq 0} \longrightarrow \mathrm{CAlg}_{\mathbb{R}}^{\geq 0}$ be the cocartesian fibration classifying the functor

$$\mathrm{Mod}^{\geq 0}\colon \mathrm{CAlg}_{\mathbb{R}}^{\geq 0} \longrightarrow \mathrm{Cat}_\infty$$

and let $\chi\colon \Delta[1] \times \Delta[1] \longrightarrow \mathrm{CAlg}_{\mathbb{R}}^{\geq 0}$ classify the image of the pullback square (2.3.5) under the forgetful functor from $\mathcal{C}^\infty$-rings.

We have to verify that the image of χ under $\mathrm{Mod}^{\geq 0}$ is a cartesian square of ∞-categories. To see this, let us consider $\mathrm{Mod}^{\geq 0}$ as a marked simplicial set over $\mathrm{CAlg}_{\mathbb{R}}^{\geq 0}$, whose marked edges are the π-cocartesian edges. There is a zig-zag of marked simplicial sets over $\mathrm{CAlg}_{\mathbb{R}}^{\geq 0}$

$$\begin{tikzcd} \left(\Lambda^2[2]\right)^\sharp \arrow[r] \arrow[dr, "\chi_0"'] & \left(\Delta[1] \times \Delta[1]\right)^\sharp \arrow[d, "\chi"] & \{0,0\} \arrow[l, "\sim"'] \arrow[dl, "A'_\eta"] \\ & \mathrm{CAlg}_{\mathbb{R}}^{\geq 0} & \end{tikzcd} \tag{2.3.15}$$

where χ_0 classifies the restriction $A' \longrightarrow A \oplus I[1] \longleftarrow A$ of (2.3.3) and the right map is marked anodyne. Let $\mathcal{E}$ and $\mathcal{D}$ be the ∞-categories of cocartesian sections of π over χ and χ_0. In other words, a map of simplicial sets $K \longrightarrow \mathcal{E}$ corresponds to a diagram of marked simplicial sets

$$\begin{tikzcd}
 & & \mathrm{Mod}^{\geq 0} \arrow[d, "\pi"] \\
K^\flat \times \big(\Delta[1] \times \Delta[1]\big)^\sharp \arrow[r] \arrow[urr, "\tilde{\chi}"] & \big(\Delta[1] \times \Delta[1]\big)^\sharp \arrow[r, "\chi"'] & \mathrm{CAlg}_{\mathbb{R}}^{\geq 0}.
\end{tikzcd}$$

Restriction of sections along the horizontal maps from (2.3.15) produces functors

$$\mathcal{D} \longleftarrow \mathcal{E} \longrightarrow \mathrm{Mod}_{A'_\eta} \tag{2.3.16}$$

The right restriction functor is a trivial fibration because the right map in (2.3.15) was marked anodyne. On the other hand, Lemma 2.3.12 shows that a cocartesian section over the square is exactly a functor $\tilde{\chi}\colon \Delta[1] \times \Delta[1] \longrightarrow \mathrm{Mod}^{\geq 0}$ such that

(a) $\tilde{\chi}$ is a right Kan extension of its restriction to $\Lambda^2[2]$.

(b) the restriction of $\tilde{\chi}$ to $\Lambda^2[2]$ is a cocartesian lift.

It then follows from [59, Proposition 4.3.2.15] that the restriction functor $\mathcal{E} \longrightarrow \mathcal{D}$ is a trivial fibration as well.

The diagram of marked simplicial sets (2.3.15) presents the natural map of corepresentable functors

$$h_A \coprod\nolimits_{h_{A[\epsilon_n]}} h_{A'} \longrightarrow h_{A'_\eta} \tag{2.3.17}$$

in the ∞-category of functors $\mathrm{CAlg}_{\mathbb{R}}^{\geq 0} \longrightarrow \mathrm{Cat}_\infty$. Under this identification, the zig-zag of restriction functors (2.3.16) is the image of (2.3.17) upon taking mapping categories into $\mathrm{Mod}^{\geq 0}\colon \mathrm{CAlg}_{\mathbb{R}}^{\geq 0} \longrightarrow \mathrm{Cat}_\infty$. It follows that the natural map

$$\mathrm{Mod}_{A'_\eta}^{\geq 0} \longrightarrow \mathrm{Mod}_{A'}^{\geq 0} \times_{\mathrm{Mod}_{A \oplus I[1]}^{\geq 0}} \mathrm{Mod}_A^{\geq 0}$$

is an equivalence, which concludes the proof. □

2.3.2 Deformations of algebras. The deformation theory of modules serves as the basis for the deformation theory of other algebraic objects, like algebras over (dg-)operads. For our purposes, it will be important to understand the deformation theory of $\mathcal{C}^\infty$-rings as well.

Definition 2.3.18. Let A be a dg-$\mathcal{C}^\infty$-ring. A *connective dg-$\mathcal{P}$-algebra R over A* is one of the following three examples:

(a) A connective algebra over a connective (I-coloured, symmetric) dg-operad $\mathcal{P}$ over $\mathbb{R}$ (see Example 2.1.23).

(b) A diagram of dg-$\mathcal{C}^\infty$-rings $A \longrightarrow R_\bullet$ indexed by a category $\mathcal{I}$.

(c) A diagram of $\mathcal{C}^\infty$-rings $A \longrightarrow R_\bullet \longrightarrow A$ equipped with a retraction, indexed by a category $\mathcal{I}$.

In other words, we slightly expand the notion of algebras over operads to also include the last two $\mathcal{C}^\infty$-algebraic examples.

For each of the above three examples, let $\mathcal{P}\mathrm{Alg}_A^{\geq 0,\mathrm{dg}}$ denote the category of (non-negatively graded) dg-$\mathcal{P}$-algebras over A. This category comes equipped with a model structure whose weak equivalences (resp. fibrations) are the quasi-isomorphisms (resp. surjections in degrees > 0). In the $\mathcal{C}^\infty$-algebraic cases this arises from the model structure on $\mathcal{C}^\infty$-rings. Every map $f\colon A \longrightarrow B$ of $\mathcal{C}^\infty$-rings induces a commuting square of right Quillen functors

$$\begin{tikzcd} \mathcal{P}\mathrm{Alg}_B^{\geq 0,\mathrm{dg}} \arrow[r, "f_*"] \arrow[d, "\mathrm{forget}"'] & \mathcal{P}\mathrm{Alg}_A^{\geq 0,\mathrm{dg}} \arrow[d, "\mathrm{forget}"] \\ \prod_I \mathrm{Mod}_B^{\geq 0,\mathrm{dg}} \arrow[r, "f_*"'] & \prod_I \mathrm{Mod}_A^{\geq 0,\mathrm{dg}}. \end{tikzcd} \tag{2.3.19}$$

The top functor is part of a Quillen pair $f^*\colon \mathcal{P}\mathrm{Alg}_A^{\geq 0,\mathrm{dg}} \leftrightarrows \mathcal{P}\mathrm{Alg}_B^{\geq 0,\mathrm{dg}}\colon f_*$, given by

(a) $f_*(R) = R$ and $f^*(R) = B \otimes_A R$ for algebras over operads.

(b) $f_*(R_\bullet) = R_\bullet$ and $f^*(R_\bullet) = B \amalg_A R_\bullet$ for $\mathcal{C}^\infty$-rings under A.

(c) $f_*(R_\bullet \to A) = R_\bullet \times_A B$ and $f^*(A \to R_\bullet \to A) = (B \to B \amalg_A R_\bullet \to B)$ for augmented $\mathcal{C}^\infty$-rings.

The vertical functors in (2.3.19) send a $\mathcal{P}$-algebra to the underlying collection of modules, indexed by the set I of colours of the operad $\mathcal{P}$, resp. the set of objects of the category $\mathcal{I}$. In case (c), the forgetful functor sends a retract diagram $A \longrightarrow R_\bullet \longrightarrow A$ to the collection of kernels $R_i \times_A \{0\}$. In all cases, the forgetful functors detect weak equivalences.

Let $\mathcal{P}\colon \prod \mathrm{Mod}_A^{\geq 0,\mathrm{dg}} \longrightarrow \mathcal{P}\mathrm{Alg}_A^{\geq 0,\mathrm{dg}}$ denote the left adjoint to the forgetful functor, taking *free* $\mathcal{P}$-algebras. The various commuting squares (2.3.19) determine a natural transformation

$$\begin{tikzcd} \mathcal{C}^\infty\mathrm{Alg}^{\mathrm{dg,cof}} \arrow[r, bend left, "\prod \mathrm{Mod}^{\geq 0,\mathrm{dg}}"] \arrow[r, bend right, "\mathcal{P}\mathrm{Alg}^{\geq 0,\mathrm{dg}}"'] \arrow[r, phantom, "\Downarrow \mathcal{P}"] & \mathrm{ModCat}^{\mathrm{L}} \end{tikzcd}$$

between diagrams of (combinatorial) model categories and left Quillen functors. The diagram $\mathcal{P}\mathrm{Alg}^{\geq 0,\mathrm{dg}}$ sends weak equivalences between cofibrant dg-$\mathcal{C}^\infty$-rings to Quillen equivalences, so that we obtain a natural transformation between diagrams of presentable ∞-categories and left adjoint functors

$$\begin{tikzcd} \mathcal{C}^\infty\mathrm{Alg} \arrow[r, bend left, "\prod \mathrm{Mod}^{\geq 0}"] \arrow[r, bend right, "\mathcal{P}\mathrm{Alg}^{\geq 0}"'] \arrow[r, phantom, "\Downarrow \mathcal{P}"] & \mathrm{Pr}^{\mathrm{L}}. \end{tikzcd} \tag{2.3.20}$$

Our goal is to prove the following analogue of Proposition 2.3.4:

Proposition 2.3.21. *Let $\eta\colon A' \longrightarrow A \oplus I[1]$ classify a square zero extension $A'_\eta \longrightarrow A'$ by an A-module I, so that we have a pullback square of the form (2.3.5). If the map η induces a surjection*

$$\pi_0(A') \longrightarrow \pi_0(A \oplus I[1]) \cong \pi_0(A)$$

then $\mathcal{P}\mathrm{Alg}^{\geq 0}$ *sends the pullback square* (2.3.5) *to a pullback square of* ∞*-categories*

$$\begin{tikzcd}
\mathcal{P}\mathrm{Alg}^{\geq 0}_{A'_\eta} \arrow[r] \arrow[d] & \mathcal{P}\mathrm{Alg}^{\geq 0}_{A} \arrow[d] \\
\mathcal{P}\mathrm{Alg}^{\geq 0}_{A'} \arrow[r] & \mathcal{P}\mathrm{Alg}^{\geq 0}_{A\oplus I[1]}.
\end{tikzcd}$$

To prove Proposition 2.3.21, let us recall the following from [62, Definition 4.7.5.16]:

Definition 2.3.22. A commuting square of ∞-categories

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "G"'] & \mathcal{D} \arrow[d, "G'"] \\
\mathcal{C}' \arrow[r, "F'"'] & \mathcal{D}'
\end{tikzcd} \tag{2.3.23}$$

is called *right adjointable* if the functors F and F' admit right adjoints U and U', and the Beck-Chevalley map

$$G \circ U \longrightarrow U' \circ G$$

is an equivalence. It is *left adjointable* if the square of opposite categories is right adjointable.

Let us denote by $\mathrm{Fun}^{\mathrm{RAd}}(\Delta[1], \mathrm{Cat}_\infty) \subseteq \mathrm{Fun}(\Delta[1], \mathrm{Cat}_\infty)$ be the subcategory whose objects are the left adjoint functors $F\colon \mathcal{C} \longrightarrow \mathcal{D}$, with maps between them given by those squares (2.3.23) that are right adjointable.

Remark 2.3.24. Suppose that (2.3.23) is a square of left adjoint functors. Then (2.3.23) is right adjointable if and only if the square of right adjoint functors

$$\begin{tikzcd}
\mathcal{D}' \arrow[r, "H'"] \arrow[d, "U'"'] & \mathcal{D} \arrow[d, "U"] \\
\mathcal{C}' \arrow[r, "H"'] & \mathcal{C}
\end{tikzcd}$$

is left adjointable. If this is the case, then U and U' intertwine the adjoint pair (G, H) and the adjoint pair (G', H'), in the sense that $GU \simeq U'G$ and $HU' \simeq UH'$.

It follows that the (co)units of (G', H') and (G, H) are related by

$$U'(\epsilon') \simeq \epsilon \circ U' \quad \text{and} \quad U(\eta') \simeq \eta \circ U.$$

When U and U' detect equivalences, it follows that (G', H') is an equivalence if (G, H) is an equivalence.

Lemma 2.3.25. *Let* $f\colon A \longrightarrow B$ *be a map of* $\mathcal{C}^\infty$*-rings which induces a surjection on* π_0*. Then the square of left adjoints*

$$\begin{tikzcd}
\prod \mathrm{Mod}^{\geq 0}_{A} \arrow[r, "\mathcal{P}"] \arrow[d, "f^*"'] & \mathcal{P}\mathrm{Alg}^{\geq 0}_{A} \arrow[d, "f^*"] \\
\prod \mathrm{Mod}^{\geq 0}_{B} \arrow[r, "\mathcal{P}"'] & \mathcal{P}\mathrm{Alg}^{\geq 0}_{B}
\end{tikzcd}$$

is right adjointable.

Proof. Let us write $\underline{R} = \mathrm{forget}(R)$ for the image of a $\mathcal{P}$-algebra under the forgetful functor. We have to show that for any $R \in \mathcal{P}\mathrm{Alg}^{\geq 0}_A$, the natural B-linear map

$$\beta\colon B \otimes_A \underline{R} \longrightarrow \underline{f^*(R)}$$

is an equivalence. This is immediate for algebras over operads, since $f^*(R) = B \otimes_A R$, endowed with its natural $\mathcal{P}$-algebra structure.

For a diagram $A \longrightarrow R_\bullet$ of $\mathcal{C}^\infty$-rings under A, β is the natural B-linear map $B \otimes_A R_i \longrightarrow B \coprod_A R_i$. This map is a weak equivalence by Lemma 2.2.7. For a diagram $A \longrightarrow R_\bullet \longrightarrow A$ of augmented $\mathcal{C}^\infty$-rings, the map β arises from the maps $B \otimes_A R_i \longrightarrow B \coprod_A R_i$ by taking fibers over $0 \in B$, and is hence an equivalence as well. $\square$

Proof (of Proposition 2.3.21). Let $\chi\colon \Delta[1] \times \Delta[1] \longrightarrow \mathcal{C}^\infty\mathrm{Alg}$ denote the pullback square of $\mathcal{C}^\infty$-rings

$$\begin{tikzcd} A'_\eta \arrow[r] \arrow[d] & A \arrow[d, "0"] \\ A' \arrow[r, "\eta"'] & A \oplus I[1]. \end{tikzcd}$$

By assumption, every map in this square induces a surjection on π_0. Lemma 2.3.25 then implies that the restriction of the natural transformation (2.3.20) to the square χ gives rise to a functor

$$\Delta[1] \times \Delta[1] \longrightarrow \mathrm{Fun}^{\mathrm{RAd}}(\Delta[1], \mathrm{Cat}_\infty);\ B \longmapsto \Big(\mathcal{P}\colon \prod \mathrm{Mod}^{\geq 0}_B \longrightarrow \mathcal{P}\mathrm{Alg}^{\geq 0}_B\Big).$$

The inclusion $\mathrm{Fun}^{\mathrm{RAd}}(\Delta[1], \mathrm{Cat}_\infty) \subseteq \mathrm{Fun}(\Delta[1], \mathrm{Cat}_\infty)$ preserves limits by [62, Corollary 4.7.4.18]. This means that the natural square of ∞-categories

$$\begin{tikzcd} \prod \mathrm{Mod}^{\geq 0}_{A'_\eta} \arrow[r, "\mathcal{P}"] \arrow[d] & \mathcal{P}\mathrm{Alg}^{\geq 0}_{A'_\eta} \arrow[d] \\ \prod \Big(\mathrm{Mod}^{\geq 0}_{A'} \times_{\mathrm{Mod}^{\geq 0}_{A \oplus I[1]}} \mathrm{Mod}^{\geq 0}_A\Big) \arrow[r, "\mathcal{P}"'] & \mathcal{P}\mathrm{Alg}^{\geq 0}_{A'} \times_{\mathcal{P}\mathrm{Alg}^{\geq 0}_{A \oplus I[1]}} \mathcal{P}\mathrm{Alg}^{\geq 0}_A \end{tikzcd}$$

is right adjointable. The left vertical functor is a product of equivalences by Proposition 2.3.4. The right adjoint of the free functor $\mathcal{P}$ is just the forgetful functor, which detects equivalences. It then follows from Remark 2.3.24 that the right vertical functor is an equivalence as well. $\square$

Proposition 2.3.21 can be used to classify deformations of algebras along square zero extensions. To see this, let R be a $\mathcal{P}$-algebra over a $\mathcal{C}^\infty$-ring A and consider the functor

$$\mathrm{Def}_R\colon \mathcal{C}^\infty\mathrm{Alg}/A \longrightarrow \widehat{\mathrm{Cat}}_\infty;\ A' \longmapsto \mathcal{P}\mathrm{Alg}^{\geq 0}_{A'} \times_{\mathrm{PAlg}^{\geq 0}_A} \{R\} \tag{2.3.26}$$

sending each $A' \longrightarrow A$ to the ∞-category of *deformations* of R to a $\mathcal{P}$-algebra over A'. It follows immediately from Proposition 2.3.21 that for any $\eta\colon A' \longrightarrow A \oplus I[1]$ which induces a surjection on π_0, there is an equivalence

$$\mathrm{Def}_R(A'_\eta) \xrightarrow{\sim} \mathrm{Def}_R(A') \times_{\mathrm{Def}_R(A \oplus I[1])} \{R\}.$$

In the special case where $\eta\colon A \longrightarrow A \oplus I[1]$ classifies a square zero extension of A itself, the argument of Corollary 2.3.7 provides a simple description of $\mathrm{Def}_R(A_\eta)$: it is equivalent to the space of sections of the canonical map

$$r\colon \eta_* 0^*(R) \longrightarrow R \tag{2.3.27}$$

in the category of $\mathcal{P}$-algebras over A.

Example 2.3.28. Suppose that R is an algebra over a dg-operad $\mathcal{P}$. As in Remark 2.3.9, one sees that the $\mathcal{P}$-algebra $0^*(R) = R \otimes_A (A \oplus I[1])$ is a (non-split) square zero extension of R by the R-module $R \otimes_A I[1]$. It follows that the restriction $\eta_* 0^*(R) \longrightarrow R$ is a square zero extension of $\mathcal{P}$-algebras over A, which is classified by a certain map of operadic R-modules

$$\theta_R \colon L_R \longrightarrow R \otimes_A I[2].$$

Here L_R denotes the (operadic) cotangent complex of R, which corepresents the functor $\mathrm{Der}^{\mathcal{P}}_A(R, -)$ sending an R-module E to the space of (A-linear) $\mathcal{P}$-algebra derivations with coefficients in E.

The space of sections of r (2.3.27) can therefore be identified with the space of null-homotopies of the map θ_R. In particular, the homotopy class of θ_R determines an obstruction class

$$\mathrm{ob} = [\theta] \in \mathrm{Ext}^2_R(L_R, R \otimes_A I)$$

which vanishes if and only if there exists a deformation of R over A_η.

Example 2.3.29 (cf. [62, Section 7.4.2])**.** Let $A \longrightarrow R$ be a map of $\mathcal{C}^\infty$-rings. Unwinding the definitions, the functor Def_R sends $f \colon A' \longrightarrow A$ to the ∞-category of pushout squares of $\mathcal{C}^\infty$-rings

$$\begin{tikzcd} A' \arrow[r, "f"] \arrow[d, dotted] & A \arrow[d] \\ R' \arrow[r, dotted] & R. \end{tikzcd}$$

In this case, the $\mathcal{C}^\infty$-ring $0^*(R) \simeq R \oplus R \otimes_A I[1]$ is a square zero extension of R by $R \otimes_A I[1]$. The space of sections of r (2.3.27) can then be identified with the space of sections in $\mathcal{C}^\infty\mathrm{Alg}$

$$\begin{tikzcd} A \arrow[r, "\eta"] \arrow[d] & A \oplus I[1] \arrow[r] \arrow[d] & A \arrow[d] \\ R \arrow[r, dotted] \arrow[rr, bend right] & R \oplus (R \otimes_A I[1]) \arrow[r] & R \end{tikzcd}$$

making the entire diagram commute. This can be identified further as follows: the composite map $A \xrightarrow{\eta} A \oplus I[1] \longrightarrow R \oplus (R \otimes_A I[1])$ is classified by a map of R-modules $R \otimes_A L_A \longrightarrow R \otimes_A I[1]$. Using this, one finds that the space of dotted sections is equivalent to the space of dotted lifts in the ∞-category of R-modules

$$\begin{tikzcd} L_{R/A}[-1] \arrow[r] \arrow[d] & R \otimes_A L_A \arrow[r] \arrow[d] & R \otimes_A I[1]. \\ 0 \arrow[r] & L_R \arrow[ur, dotted] & \end{tikzcd}$$

Using the left pushout square, the space of such dotted lifts is equivalent to the space of null-homotopies of the total horizontal map $\theta \colon L_{R/A}[-1] \longrightarrow R \otimes_A I[1]$. Again, the obstruction to finding a deformation of $A \longrightarrow R$ along $A_\eta \longrightarrow A$ is then given by a class

$$\mathrm{ob} = [\theta] \in \mathrm{Ext}^2(L_{R/A}, R \otimes_A I).$$

If this class vanishes, the space of such deformations is a torsor over the loop space of $\mathrm{Map}(L_{R/A}[-1], R \otimes_A I[1])$, which is equivalent to $\mathrm{Map}(L_{R/A}, R \otimes_A I[1])$. This retrieves (a $\mathcal{C}^\infty$-algebraic version of) the usual first order obstruction to extending a commutative algebra (see e.g. [46, Chapitre III] or [55] for the original analytic analogue).

2.3.3 Axiomatic approach: formal moduli problems. The examples from the previous sections show how the usual cohomological classification of deformations over a square zero extension A'_η arises naturally from the equivalence

$$\mathrm{Def}_R(A'_\eta) \simeq \mathrm{Def}_R(A') \times_{\mathrm{Def}_R(A\oplus I[1])} \{R\}$$

between the ∞-category of deformations of R over A'_η and the ∞-category of deformations R' over A', together with an equivalence between $\eta^*(R')$ and the trivial deformation $0^*(R)$ over $A \oplus I[1]$. This behaviour of deformations with respect to square zero extensions can be axiomatized in terms of so-called *formal moduli problems*, which describe deformations over small (or 'nilpotent') extensions of A.

Remark 2.3.30. The description of deformation problems in terms of functors on nilpotent dg-extensions of algebras has a long history with many contributors (see e.g. [56], [65], [75]). Our presentation closely follows the recent discussion in [61] (which treats the case where A is a field), which is closely related to the work of Hinich in [40].

Definition 2.3.31. If A is a $\mathcal{C}^\infty$-ring, we let $A[\epsilon_n]$ be the $\mathcal{C}^\infty$-ring $A\{x_n\}/(x_n^2)$, where x_n has degree n. Equivalently, $A[\epsilon_n]$ is the split square zero extension $A \oplus A[n]$.

Definition 2.3.32. The ∞-category $\mathcal{C}^\infty\mathrm{Alg}^{\mathrm{sm}}/A$ of *small extensions* of A is the smallest full subcategory of $\mathcal{C}^\infty\mathrm{Alg}/A$ which contains A and is stable under homotopy pullbacks along the maps $0\colon A \longrightarrow A[\epsilon_n]$ for $n \geq 1$. In other words, $B \longrightarrow A$ is a small extension if it factors as finite composition of maps (over A)

$$B = B_n \longrightarrow B_{n-1} \longrightarrow \ldots \longrightarrow B_0 = A$$

where each $B_i \longrightarrow B_{i-1}$ is a square zero extension of B_{i-1} by a shifted copy of A (Definition 2.2.20).

Example 2.3.33. Let K be a local Artin algebra over $\mathbb{R}$ with residue field $\mathbb{R}$. Then the tensor product $A \otimes_{\mathbb{R}} K \longrightarrow A$ determines a small extension of A. Indeed, using Artinian induction, one finds that $B \longrightarrow A$ factors as a finite composition of surjective maps $A \otimes_{\mathbb{R}} K_i \longrightarrow A \otimes_{\mathbb{R}} K_{i-1}$, whose kernel $A \otimes_{\mathbb{R}} \mathfrak{m}_i$ is a free A-module of rank 1, which squares to zero in $A \otimes_{\mathbb{R}} K_i$. Any such strict square zero extension fits into a homotopy diagram

$$\begin{tikzcd} A \otimes_{\mathbb{R}} K_i \arrow[r] \arrow[d] & A \arrow[d] \\ A \otimes_{\mathbb{R}} K_{i-1} \arrow[r] & A \oplus A[1] \end{tikzcd}$$

In fact, the same argument shows that $A \otimes_{\mathbb{R}} K$ is a small algebra over A when K is a (connective) dg-Artin algebra, since the Postnikov tower of K can be refined by a tower of small extensions of the field $\mathbb{R}$ by copies of $\mathbb{R}[n]$ (see Section 2.2.2 or [61]). For example, the $\mathcal{C}^\infty$-rings $A\{x_i\}/(x_i^n)$, where x_i has degree $i \geq 0$, are small extensions of A.

Definition 2.3.34. Let A be a $\mathcal{C}^\infty$-ring. A functor $X\colon \mathcal{C}^\infty\mathrm{Alg}^{\mathrm{sm}}/A \longrightarrow \mathcal{S}$ is said to be a *formal moduli problem* if

(a) X sends every pullback square

$$\begin{tikzcd} A'_\eta \arrow[r] \arrow[d] & A \arrow[d, "0"] \\ A' \arrow[r, "\eta"'] & A[\epsilon_n] \end{tikzcd}$$

with $n \geq 1$ to a pullback square of spaces.

(b) $X(A)$ is contractible.

Let $\mathrm{FMP}_A \subseteq \mathrm{Fun}(\mathcal{C}^\infty\mathrm{Alg}^{\mathrm{sm}}/A, \mathcal{S})$ be the full subcategory of formal moduli problems.

Condition (b) asserts that there is a unique way of deforming an object over the identity map $A \longrightarrow A$. Axiom (a) can be thought of as a version of the Schlessinger conditions [83], and encodes the behaviour of deformations encountered in the previous sections.

Example 2.3.35. Let R be a (connective) $\mathcal{P}$-algebra over A. The restriction of the functor (2.3.26) to the small extensions of A defines a functor

$$\mathrm{Def}_R\colon \mathcal{C}^\infty\mathrm{Alg}^{\mathrm{sm}}/A \longrightarrow \mathcal{S}$$

A priori, the functor Def_R from (2.3.26) takes values in locally small ∞-categories. Using that $\mathrm{Def}_R\big(A[\epsilon_n]\big) \simeq \Omega\mathrm{Def}_R\big(A[\epsilon_{n+1}]\big)$ is a (small) space, it follows from Artinian induction that the above functor takes values in spaces. By Proposition 2.3.21 (see also the discussion above Example 2.3.28), the functor Def_R is a formal moduli problem.

Example 2.3.36. Let $B \longrightarrow A$ be a map of $\mathcal{C}^\infty$-rings and consider the functor

$$\mathrm{Spf}(B)\colon \mathcal{C}^\infty\mathrm{Alg}^{\mathrm{sm}}/A \longrightarrow \mathcal{S};\ \ A' \longmapsto \mathrm{Map}_{/A}(B, A').$$

This functor preserves all limits of $\mathcal{C}^\infty$-rings over A and therefore determines a formal moduli problem, which we will call the *formal spectrum* of B. This determines a right adjoint functor

$$\mathrm{Spf}\colon \big(\mathcal{C}^\infty\mathrm{Alg}/A\big)^{\mathrm{op}} \longrightarrow \mathrm{FMP}_A. \tag{2.3.37}$$

Similarly, any unbounded commutative dg-algebra B determines a formal moduli problem $\mathrm{Spf}(B) = \mathrm{Map}_{\mathrm{CAlg}_{\mathbb{R}}/A}(B, -)$. The construction of the formal spectrum fits into an adjunction

$$\mathcal{O}\colon \mathrm{FMP}_A \rightleftarrows \big(\mathrm{CAlg}_{\mathbb{R}}/A\big)^{\mathrm{op}}\colon \mathrm{Spf}.$$

The left adjoint functor $\mathcal{O}$ sends a formal moduli problem to its 'function algebra'. The connective cover $\tau_{\geq 0}\mathcal{O}(X)$ has the structure of a $\mathcal{C}^\infty$-ring over A; this $\mathcal{C}^\infty$-ring is just the value on X of the left adjoint to (2.3.37).

The last example illustrates that formal moduli problems can be considered as (formal) geometric objects, which behave dual to commutative rings (or $\mathcal{C}^\infty$-rings). On the other hand, we will see in Chapter 4 that formal moduli problems can also be considered as (Lie) algebraic objects. As preliminary evidence for this, let us give the following example:

Example 2.3.38. Let X be a formal moduli problem and consider the spaces $X\big(A[\epsilon_n]\big)$ for $n \geq 0$. Since $A[\epsilon_n] \simeq A \times_{A[\epsilon_{n+1}]} A$, it follows that $X\big(A[\epsilon_n]\big) \simeq \Omega X\big(A[\epsilon_{n+1}]\big)$. In other words, the spaces $X(A[\epsilon_n])$ form an Ω-spectrum, called the *tangent complex* of X and denoted by T_X. More precisely, consider the composite functor

$$\mathcal{S}_*^{\mathrm{fin}} \xrightarrow{A \oplus C_*(-,A)} \mathcal{C}^\infty\mathrm{Alg}^{\mathrm{sm}}/A \xrightarrow{\quad X \quad} \mathcal{S}.$$

sending a finite pointed space K to the square zero extension of A by the reduced chains $C_*(K, A)$ with coefficients in A. This is a reduced excisive functor, which is classified by the spectrum T_X. For example, $T_{\mathrm{Spf}(B)} \simeq \mathrm{Map}_B(L_B, A)$ is the spectrum underlying the (derived) module of derivations of B with values in A.

In fact, the above functor can be refined by a reduced excisive functor

$$\mathrm{Mod}_A^{\mathrm{f.p.},\geq 0} \xrightarrow{A \oplus (-)} \mathcal{C}^\infty\mathrm{Alg}^{\mathrm{sm}}/A \xrightarrow{\quad X \quad} \mathcal{S}$$

from the ∞-category of finitely presented (connective) A-modules (Definition 2.2.26). This functor is classified by an A-module T_X (cf. Example 4.2.10), whose underlying spectrum is the tangent complex.

Remark 2.3.39. A simple inductive argument shows that a map $X \longrightarrow Y$ of formal moduli problems is an equivalence if and only $X(A[\epsilon_n]) \longrightarrow Y(A[\epsilon_n])$ is an equivalence for all $n \geq 0$. Equivalently, this means that the map of tangent complexes $T_X \longrightarrow T_Y$ is an equivalence. In other words, the assignment $X \mapsto T_X$ behaves like a forgetful functor.

Example 2.3.40. Consider the $\mathcal{C}^\infty$-ring

$$B = A[x, y]/(x^2, y^2, xy) = A[\epsilon_m] \times_A A[\epsilon_n]$$

with x of degree m and y of degree n. Geometrically, one can think of $\mathrm{Spec}(B)$ as the wedge sum of two (derived) infinitesimal lines, parametrized by $\mathrm{Spec}(A)$. The $\mathcal{C}^\infty$-ring B admits a square zero extension $B_\eta = A[x, y]/(x^2, y^2)$, which one can think of as the infinitesimal *square* spanned by these two lines. This extension is classified by a map

$$\eta\colon B \longrightarrow A[\epsilon_{m+n+1}].$$

The value of a formal moduli problem X on η gives a certain operation

$$X(\eta)\colon \pi_{-m}(T_X) \times \pi_{-n}(T_X) \cong \pi_0 X(B) \longrightarrow \pi_0 X(A[\epsilon_{m+n+1}]) \cong \pi_{-m-n-1}(T_X)$$

which describes the obstructions to extending a deformation from the wedge of two (infinitesimal) lines to the infinitesimal square spanned by them. We will see in Example 4.2.25 that this operation defines a Lie algebra structure on the homotopy groups $\pi_{*+1}(T_X)$.

Chapter 3

Homotopical algebra for Lie algebroids

This chapter treats the homotopy theory of Lie algebroids over a (derived) $\mathcal{C}^\infty$-ring. More precisely, we will show that the categories of (unbounded) dg-Lie algebroids and L_∞-algebroids over a dg-$\mathcal{C}^\infty$-ring admit Quillen equivalent semi-model structures. Furthermore, we show that these semi-model structures have a rather 'algebraic' behaviour, in the sense that certain types of homotopy colimits can be computed at the level of chain complexes.

We discuss some elementary properties of these model structures in Section 3.1 and verify the existence of the semi-model structures in Section 3.2. Most importantly, we provide an explicit cofibrant replacement of a dg-Lie algebroid in Section 3.1.3, which can be used to describe the space of maps between two dg-Lie algebroids in terms of 'nonlinear maps' between them. Section 3.3 contains a brief discussion of the homotopy theory of representations over Lie algebroids.

3.1 Homotopy theories of Lie algebroids

3.1.1 DG-Lie algebroids and L_∞-algebroids. Let A be a dg-$\mathcal{C}^\infty$-ring and let $T_A := \mathrm{Hom}_A(\Omega_A, A)$ be the chain complex of $\mathcal{C}^\infty$- derivations of A, i.e. of derivations $v\colon A \longrightarrow A$ of commutative dg-algebras with the property that

$$v(\phi(a_1,\dots,a_n)) = \sum_i \frac{\partial\phi}{\partial x_i}(a_1,\dots,a_n)\cdot v(a_i)$$

for any $\mathcal{C}^\infty$-function $\phi\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ and any $a_i \in A_0$. Recall that T_A has the structure of a dg-A-module, given by pointwise multiplication in A, as well as the structure of a dg-Lie-algebra over $\mathbb{R}$, with Lie bracket given by the commutator bracket.

Definition 3.1.1. A *dg-Lie algebroid* $\mathfrak{g}$ over A is an (unbounded) dg-A-module $\mathfrak{g}$, equipped with a ($\mathbb{R}$-linear) dg-Lie algebra structure and an *anchor map* $\rho\colon \mathfrak{g} \longrightarrow T_A$ such that

(1) ρ is both a map of dg-A-modules and dg-Lie algebras.

(2) the failure of the Lie bracket to be A-bilinear is governed by the Leibniz rule

$$[X, a\cdot Y] = (-1)^{Xa}a[X,Y] + \rho(X)(a)\cdot Y.$$

Let $\mathrm{LieAlgd}_A^{\mathrm{dg}}$ be the category of dg-Lie algebroids over A, with maps between them given by A-linear maps over T_A that preserve the Lie bracket.

Example 3.1.2. Given a (possibly unbounded) dg-A-module E, there is an *Atiyah dg-Lie algebroid* $\mathrm{At}(E)$ over A, which can be described as follows: an element of $\mathrm{At}(E)$ (of degree

n) is a tuple (v, ∇_v) consisting of a derivation $v\colon A \longrightarrow A$ (of degree n), together with a $\mathbb{R}$-linear map $\nabla_v\colon E \longrightarrow E$ (of degree n) such that

$$\nabla_v(a \cdot e) = v(a) \cdot e + (-1)^{|a|\cdot n} a \cdot \nabla_v(e)$$

for all $a \in A$ and $e \in E$. This becomes a dg-A-module under pointwise multiplication and a dg-Lie algebra under the commutator bracket. The anchor map is the obvious projection $\mathrm{At}(E) \longrightarrow T_A$ sending (v, ∇_v) to v.

Example 3.1.3. Suppose that R is a $\mathcal{P}$-algebra in $\mathrm{Mod}_A^{\geq 0,\mathrm{dg}}$, in the sense of Definition 2.3.18. Then there is a sub dg-Lie algebroid

$$\mathrm{At}_{\mathcal{P}}(R) \subseteq \mathrm{At}(R)$$

consisting of the tuples (v, ∇_v) where ∇_v is a $\mathcal{P}$-algebra derivation. For example, if $\phi\colon A \longrightarrow R$ is a map of dg-$\mathcal{C}^\infty$-rings, this produces a dg-Lie algebroid whose elements are tuples (v, w) of derivations $v\colon A \longrightarrow A$ and $w\colon R \longrightarrow R$ which are compatible, in the sense that $\phi \circ v = w \circ \phi$.

Example 3.1.4. Let $p\colon N \longrightarrow M$ be a surjective submersion and let $A = \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(N) = R$ be the induced diagram of discrete $\mathcal{C}^\infty$-rings. Then $\mathrm{At}_{\mathcal{C}^\infty}(R)$ is the usual Atiyah Lie algebroid of p, which has the property that Lie algebroid maps $TM \longrightarrow \mathrm{At}_{\mathcal{C}^\infty}(R)$ correspond to flat connections on p.

Similarly, suppose that the map p admits a section $s\colon M \longrightarrow N$, yielding a retraction $R \longrightarrow A$. Then the associated Lie algebroid $\mathrm{At}_{\mathcal{C}^\infty}(R)$ has the property that Lie algebroid maps $TM \longrightarrow \mathrm{At}_{\mathcal{C}^\infty}(R)$ are flat connections on N for which the section s is horizontal.

In certain situations, it can be convenient to slightly weaken the Lie algebra structure of a dg-Lie algebroid to an L_∞-structure. Recall that an L_∞-structure on a chain complex $\mathfrak{g}$ is given by a collection of maps $[-, ..., -]\colon \mathfrak{g}^{\otimes n} \longrightarrow \mathfrak{g}$ of graded anti-symmetric maps of (homological) degree $n-2$, one for each $n \geq 2$. These maps have to satisfy a sequence of Jacobi identities

$$J^k(X_1, \cdots, X_k) = 0,$$

one for each $k \geq 2$, where $J^k(X_1, \cdots, X_k)$ is the $\mathbb{R}$-th *Jacobiator*:

$$\sum_{i+j=k} (-1)^{ij} \sum_{\sigma \in \mathrm{UnSh}(i,j)} (-1)^\sigma \pm \big[[X_{\sigma(1)}, \cdots, X_{\sigma(i)}], X_{\sigma(i+1)}, \cdots, X_{\sigma(i+j)}\big]. \tag{3.1.5}$$

Here $\pm$ denotes the usual Koszul sign due to the permutation of the variables X_i and the 1-ary bracket $[-]\colon \mathfrak{g} \longrightarrow \mathfrak{g}$ is given by $[X] = -\partial X$.

There is a map of operads $L_\infty \longrightarrow \mathrm{Lie}$, realizing a Lie algebra as an L_∞-algebra whose n-ary brackets vanish for $n \geq 3$. In particular, T_A can be considered as an L_∞-algebra.

Definition 3.1.6. An *L_∞-algebroid* over A is a dg-A-module $\mathfrak{g}$, equipped with the structure of a ($\mathbb{R}$-linear) L_∞-algebra and an *anchor map* $\rho\colon \mathfrak{g} \longrightarrow T_A$, such that

(1) ρ is both a map of dg-A-modules and of L_∞-algebras.

(2) the brackets satisfy the Leibniz rules

$$\begin{aligned} [X, a \cdot Y] &= (-1)^{aX} a[X, Y] + \rho(X)(a) \cdot Y \\ [X_1, ..., a \cdot X_n] &= (-1)^{an}(-1)^{a(X_1 + ... + X_{n-1})} a[X_1, ..., X_n] \qquad n \geq 3. \end{aligned}$$

Let $\mathrm{L}_\infty\mathrm{Algd}_A^{\mathrm{dg}}$ be the category of L_∞-algebroids over A, with maps between them given by A-linear maps over T_A that preserve the L_∞-structure.

Remark 3.1.7. There is a more general notion of L_∞-algebroid in the literature (see [45, 54, 103]), which allows the anchor map to be a *nonlinear* map of L_∞-algebras. We will not use this generalization.

Example 3.1.8. Let $\mathfrak{g}$ is an L_∞-algebroid and let

$$\alpha \in \overline{C}^*(\mathfrak{g}) = \mathrm{Hom}_A(\mathrm{Sym}_A^{\geq 1}\mathfrak{g}, A)$$

be a cycle in its reduced Chevalley-Eilenberg complex (see Definition 4.1.16) of (homological) degree $-n$. Then $\mathfrak{g}\oplus A[n-2] \longrightarrow \mathfrak{g}$ is a map of L_∞-algebroids, where the nontrivial brackets of the domain are

$$[X, a] = X(a) \qquad [X_1, \dots, X_k] = \big([X_1, \dots, X_k]_{\mathfrak{g}}, \alpha(X_1, \dots, X_k)\big).$$

For example, let $A = \mathcal{C}^\infty(M)$ be the ring of smooth functions on a smooth manifold and let $\omega \in \Omega^n_{\mathrm{cl}}(M)$ be a closed n-form, for $n \geq 3$. Then the chain complex $T_A \oplus A[n-2]$ carries an L_∞-algebroid structure with binary bracket given by $[X, f] = X(f)$ for $X \in T_A$ and $f \in A[n-2]$ and n-ary bracket given by $[X_1, ..., X_n] = \omega(X_1, ..., X_n)$ for $X_1, ..., X_n \in T_A$.

Example 3.1.9 (Action L_∞-algebroids)**.** Let $\rho\colon \mathfrak{g} \longrightarrow T_A$ be a map of L_∞-algebras over $\mathbb{R}$. Then $A \otimes \mathfrak{g}$ has the structure of an L_∞-algebroid, with anchor map given by the A-linear extension of ρ and with brackets given by

$$\begin{aligned} [a \otimes X, b \otimes Y] &= \pm ab \otimes [X, Y] + a \cdot \rho(X)(b) \otimes Y - (\pm) b \cdot \rho(Y)(a) \otimes X \\ [a_1 \otimes X_1, ..., a_n \otimes X_n] &= \pm a_1 ... a_n \otimes [X_1, ..., X_n]. \end{aligned}$$

where $\pm$ is the usual Koszul sign. The only nontrivial condition to verify is the Jacobi identity: by an explicit computation, one can show that the Jacobiators $J^n(X_1, \dots, X_n)$ depend A-multilinearly on $X_i \in A \otimes \mathfrak{g}$. One can therefore reduce to the case where all X_i are contained in $\mathfrak{g}$, where the Jacobi identities hold by assumption (cf. Lemma 3.1.40).

By construction, maps of L_∞-algebroids $A \otimes \mathfrak{g} \longrightarrow \mathfrak{h}$ are in bijective correspondence to L_∞-algebra maps $\mathfrak{g} \longrightarrow \mathfrak{h}$ over T_A. Furthermore, $A \otimes \mathfrak{g}$ is a dg-Lie algebroid whenever $\mathfrak{g}$ is a dg-Lie algebra.

The categories of dg-Lie algebroids and L_∞-algebroids over A fit into a commuting diagram

$$\begin{tikzcd} \mathrm{LieAlgd}_A^{\mathrm{dg}} \arrow[r] \arrow[d] & \mathrm{L}_\infty\mathrm{Algd}_A^{\mathrm{dg}} \arrow[r] \arrow[d] & \mathrm{Mod}_A^{\mathrm{dg}}/T_A \arrow[d] \\ \mathrm{Lie}_{\mathbb{R}}^{\mathrm{dg}}/T_A \arrow[r] & \mathrm{L}_\infty\mathrm{Alg}_{\mathbb{R}}^{\mathrm{dg}}/T_A \arrow[r] & \mathrm{Mod}_{\mathbb{R}}^{\mathrm{dg}}/T_A. \end{tikzcd}$$

The vertical functors forget the A-module structure, the left two horizontal functors are inclusions and the right two horizontal functors forget the L_∞-structure. Each of these forgetful functors admits a left adjoint. The left adjoint to $\mathrm{LieAlgd}_A^{\mathrm{dg}} \longrightarrow \mathrm{Mod}_A^{\mathrm{dg}}/T_A$ is described in [53] and the left adjoints to the forgetful functors

$$\mathrm{LieAlgd}_A^{\mathrm{dg}} \longrightarrow \mathrm{Lie}_{\mathbb{R}}^{\mathrm{dg}}/T_A \qquad \mathrm{L}_\infty\mathrm{Algd}_A^{\mathrm{dg}} \longrightarrow \mathrm{L}_\infty\mathrm{Alg}_{\mathbb{R}}^{\mathrm{dg}}/T_A$$

are both given by the 'action Lie algebroid' construction of Example 3.1.9.

Theorem 3.1.10. *The following two categories admit a right proper, tractable semi-model structure, in which a map is a weak equivalence (resp. a fibration) if and only if it is a quasi-isomorphism (a degreewise surjection):*

(a) the category $\mathrm{LieAlgd}_A^{\mathrm{dg}}$ *of dg-Lie algebroids over* A*.*

(b) the category $\mathrm{L}_\infty\mathrm{Algd}_A^{\mathrm{dg}}$ *of* L_∞*-algebroids over* A*.*

Example 3.1.11. Let R be a fibrant-cofibrant $\mathcal{P}$-algebra in the sense of Definition 2.3.18. Then the Atiyah Lie algebroid $\mathrm{At}_{\mathcal{P}}(R)$ of Example 3.1.3 is a fibrant dg-Lie algebroid. Indeed, if R is a cofibrant algebra over a dg-operad, then R is (up to a retraction) free on a graded vector space V, so that there is an isomorphism of graded vector spaces $\mathrm{At}(R) \cong T_A \oplus \mathrm{Hom}_k(V, R)$.

Similarly, if R is a diagram of $\mathcal{C}^\infty$-rings under A (resp. with a retraction $R \longrightarrow A$), the anchor maps of the associated Atiyah Lie algebroids are base changes of the maps

$$\mathrm{Der}(R, R) \longrightarrow \mathrm{Der}(A, R) \qquad \mathrm{Der}(R, R) \longrightarrow \mathrm{Der}(A, R) \times_{\mathrm{Der}(A,A)} \mathrm{Der}(R, A).$$

These maps are fibrations when $A \longrightarrow R$ is a (projective) cofibration and when furthermore the retraction $R \longrightarrow A$ is a projective fibration.

The quasi-isomorphisms and surjections do not define a genuine model structure on dg-Lie algebroids. Indeed, the following example demonstrates that dg-Lie algebroids may fail to have a fibrant replacement:

Example 3.1.12. In general, there is no genuine model structure on dg-Lie algebroids with the above weak equivalences and fibrations. For example, consider the free $\mathcal{C}^\infty$-rings

$$A = \mathbb{R}\{x, y\} = \mathcal{C}^\infty(\mathbb{R}^2) \qquad\qquad B = A/(x - y) \cong \mathbb{R}\{x\}.$$

Let $\mathfrak{g}$ be the free A-linear Lie algebra generated by the A-module $B^{\oplus 2}$. Equivalently, $\mathfrak{g}$ is the free B-linear Lie algebra on two generators e_1, e_2, considered as a Lie algebra over A. Suppose that the zero map $\mathfrak{g} \longrightarrow T_A$ factors over a transitive dg-Lie algebroid

$$\mathfrak{g} \xrightarrow{\ \iota\ } \mathfrak{h} \xrightarrow{\ \rho\ } \!\!\!\!\rightarrow T_A.$$

We claim that ι can never be a quasi-isomorphism. To see this, let $v \in \mathfrak{h}$ be an element in $\mathfrak{h}$ and consider the following two equalities in $\mathfrak{h}$:

$$[x \cdot \iota(e_1), [y \cdot \iota(e_2), v]] = xy \cdot [\iota(e_1), [\iota(e_2), v]] - xv(y) \cdot \iota\big([e_1, e_2]\big)$$
$$[y \cdot \iota(e_1), [x \cdot \iota(e_2), v]] = xy \cdot [\iota(e_1), [\iota(e_2), v]] - yv(x) \cdot \iota\big([e_1, e_2]\big)$$

The left hand sides agree by definition of $\mathfrak{g}$. If we let v be an element such that $\rho(v) = \partial/\partial y$, then

$$\iota\Big(x \cdot [e_1, e_2]\Big) = 0.$$

This means that the kernel of $\pi_0(\iota) \colon \pi_0(\mathfrak{g}) \longrightarrow \pi_0(\mathfrak{h})$ always contains the (nonzero) element $x \cdot [e_1, e_2]$.

Definition 3.1.13. Let A be a dg-$\mathcal{C}^\infty$-ring or a commutative dg-$\mathbb{R}$-algebra. The ∞-category of Lie algebroids over A

$$\mathrm{LieAlgd}_A = \mathrm{LieAlgd}_{A^c}^{\mathrm{dg}}[W^{-1}]$$

is the ∞-category associated to the semi-model category of dg-Lie algebroids over a cofibrant replacement A^c of A. By Proposition 3.1.20, one may equivalently model $\mathrm{LieAlgd}_A$ by the semi-model category of L_∞-algebroids over A^c.

Remark 3.1.14. In the above definition, we need to replace A by a cofibrant dg-$\mathcal{C}^\infty$-ring because the tangent bundle T_A is only homotopy invariant when A is cofibrant (see also Section 3.1.4).

Rather than invoking Quillen's path object argument, our proof of Theorem 3.1.10 depends on an analysis of pushouts along generating trivial cofibrations. Such pushouts of dg-Lie algebroids (and L_∞-algebroids) have a similar structure as pushouts of algebras over an operad, but require some extra care because one may add generators that act nontrivially on A. For this reason, we postpone the proof of Theorem 3.1.10 to Section 3.2, where we will also prove the following result:

Theorem 3.1.15. *The forgetful functors*

$$U\colon \mathrm{LieAlgd}_A^{\mathrm{dg}} \longrightarrow \mathrm{Mod}_A^{\mathrm{dg}}/T_A \qquad U\colon \mathrm{L}_\infty\mathrm{Algd}_A^{\mathrm{dg}} \longrightarrow \mathrm{Mod}_A^{\mathrm{dg}}/T_A$$

are right Quillen functors with the following two properties:

(a) they preserve cofibrant objects, i.e. any cofibrant dg-Lie-algebroid is cofibrant as a dg-A-module.

(b) they preserve sifted homotopy colimits. More precisely, for any homotopy *sifted category* $\mathcal{J}$ *and any projectively cofibrant diagram* $\mathfrak{g}\colon \mathcal{J} \longrightarrow \mathrm{LieAlgd}_A^{\mathrm{dg}}$*, the natural map*

$$\mathrm{hocolim}_{\mathcal{J}}\, U(\mathfrak{g}) \longrightarrow U(\mathrm{colim}_{\mathcal{J}}\, \mathfrak{g})$$

is a weak equivalence of dg-A-modules over T_A*.*

Remark 3.1.16. Recall that a category $\mathcal{J}$ is said to be homotopy sifted if it is non-empty and if the diagonal functor $\Delta\colon \mathcal{J} \longrightarrow \mathcal{J}\times\mathcal{J}$ is homotopy cofinal, i.e. for each $(i,j)\in\mathcal{J}$ the comma category $(i,j)/\Delta$ is weakly contractible. Examples of homotopy sifted categories are filtered categories and $\mathbf{\Delta}^{\mathrm{op}}$.

The fact that the functor $U\colon \mathrm{LieAlgd}_A^{\mathrm{dg}} \longrightarrow \mathrm{Mod}_A^{\mathrm{dg}}/T_A$ forgets certain algebraic structure (the Lie bracket) is encoded categorically in the fact that U detects isomorphisms and preserves all ordinary colimits indexed by sifted categories (like filtered colimits and reflexive coequalizers). Part (b) of Theorem 3.1.15 asserts that this point of view persists at the homotopical level. In particular, the ∞-category $\mathrm{LieAlgd}_A$ is locally presentable and the forgetful functor

$$\mathrm{LieAlgd}_A \longrightarrow \mathrm{Mod}_{\mathbb{R}}/T_A$$

preserves limits and sifted colimits (and detect equivalences).

3.1.2 Immediate properties.

As an application of Theorem 3.1.15, let us consider the inclusion

$$i\colon \mathrm{Lie}_A^{\mathrm{dg}} \longrightarrow \mathrm{LieAlgd}_A^{\mathrm{dg}};\ \ \mathfrak{g} \longmapsto \big(\mathfrak{g} \xrightarrow{0} T_A\big).$$

of the category of dg-Lie algebras over A into the category of dg-Lie algebroids.

Proposition 3.1.17. *Endow the category* $\mathrm{Lie}_A^{\mathrm{dg}}$ *of dg-Lie algebras over A with the model structure transferred from* $\mathrm{Mod}_A^{\mathrm{dg}}$*. Then the above inclusion functor is part of a Quillen adjunction*

$$i\colon \mathrm{Lie}_A^{\mathrm{dg}} \rightleftarrows \mathrm{LieAlgd}_A^{\mathrm{dg}} \colon \ker$$

whose right adjoint sends a dg-Lie-algebroid to the kernel of its anchor map. The induced right adjoint functor of ∞-categories $\ker\colon \mathrm{LieAlgd}_A \longrightarrow \mathrm{Lie}_A$ *detects equivalences and preserves sifted colimits.*

Proof. One easily verifies that the functor ker is right Quillen and fits into a commuting diagram of right Quillen functors

$$\begin{tikzcd} \mathrm{LieAlgd}_A^{\mathrm{dg}} \arrow[r, "\ker"] \arrow[d, "U"'] & \mathrm{Lie}_A^{\mathrm{dg}} \arrow[d, "U"] \\ \mathrm{Mod}_A^{\mathrm{dg}}/T_A \arrow[r, "\ker"'] & \mathrm{Mod}_A^{\mathrm{dg}} \end{tikzcd}$$

Since the vertical forgetful functors detect equivalences and preserve sifted homotopy colimits (see [74, Proposition 7.8] for the case of Lie algebras), it suffices to check the right derived functor of $\ker\colon \mathrm{Mod}_A^{\mathrm{dg}}/T_A \longrightarrow \mathrm{Mod}_A^{\mathrm{dg}}$ has these properties as well. But it follows immediately from the fact that $\mathrm{Mod}_A^{\mathrm{dg}}$ is a stable model category that taking homotopy pullbacks along $0 \longrightarrow T_A$ detects equivalences and preserves all homotopy colimits indexed by contractible categories. □

Remark 3.1.18. The above proposition asserts that $\mathrm{LieAlgd}_A$ is monadic over the ∞-category Lie_A. This point of view is used extensively in [33] (where the monad is not described algebraically, however). In particular, even though the functor $\mathrm{Lie}_A^{\mathrm{dg}} \longrightarrow \mathrm{LieAlgd}_A^{\mathrm{dg}}$ is fully faithful, its derived functor is *not* fully faithful; the derived counit map is given at the level of A-modules by a map $\mathfrak{g} \oplus T_A[-1] \longrightarrow \mathfrak{g}$.

Corollary 3.1.19. *Any dg-Lie algebroid* $\mathfrak{g}$ *is equivalent to the homotopy colimit of a simplicial diagram in* $\mathrm{LieAlgd}_A^{\mathrm{dg}}$ *whose objects are equivalent to Lie algebras over* A*.*

Proof. Such a simplicial diagram is provided by the bar resolution associated to the Quillen adjunction $\mathrm{Lie}_A^{\mathrm{dg}} \leftrightarrows \mathrm{LieAlgd}_A^{\mathrm{dg}}$ [12], or associated to the induced monadic adjunction of ∞-categories [62, Proposition 4.7.4.14]. □

The above results have obvious analogues for L_∞-algebroids and L_∞-algebras over A. Using these, we obtain the following:

Proposition 3.1.20. *The inclusion* $j\colon \mathrm{LieAlgd}_A^{\mathrm{dg}} \longrightarrow \mathrm{L}_\infty\mathrm{Algd}_A^{\mathrm{dg}}$ *is the right adjoint of a Quillen equivalence.*

Proof. The functor j fits into a commuting diagram of right Quillen functors

$$\begin{tikzcd} \mathrm{LieAlgd}_A^{\mathrm{dg}} \arrow[r, "j"] \arrow[d, "\ker"'] & \mathrm{L}_\infty\mathrm{Algd}_A^{\mathrm{dg}} \arrow[d, "\ker"] \\ \mathrm{Lie}_A^{\mathrm{dg}} \arrow[r, "w^*"'] & \mathrm{L}_\infty\mathrm{Alg}_A^{\mathrm{dg}} \end{tikzcd}$$

where w^* is the forgetful functor associated to the map of operads $w\colon L_\infty \longrightarrow \mathrm{Lie}$. This functor is part of a Quillen equivalence because w is a weak equivalence between Σ-cofibrant operads. Since w^* and the vertical functors have right derived functors that detect equivalences and preserve all sifted homotopy colimits, it follows that j has these properties as well.

Let L be the left adjoint to the right Quillen functor j. Because j detects weak equivalences, it suffices to show that the derived unit map $\eta\colon \mathfrak{g} \longrightarrow \mathbb{R}jL(\mathfrak{g})$ is a weak equivalence for each cofibrant L_∞-algebroid $\mathfrak{g}$. Both $\mathbb{R}j$ and L preserve homotopy colimits indexed by $\mathbf{\Delta}^{\mathrm{op}}$ and each L_∞-algebroid is the homotopy colimit of a simplicial diagram of L_∞-algebras (by the variant of Corollary 3.1.19 in the L_∞-case). It therefore suffices to show that η is a weak equivalence when $\mathfrak{g}$ is a cofibrant L_∞-algebra over A. In that case, the (derived) unit map agrees with the derived unit map $\mathfrak{g} \longrightarrow \mathbb{R}w^*\mathbb{L}w_!\mathfrak{g}$, which is a weak equivalence. □

3.1.3 Nonlinear maps and cofibrant resolutions. There is no straightforward way to replace a dg-Lie algebroid or L_∞-algebroid by a fibrant dg-Lie algebroid; this is the main reason for the non-existence of a genuine model structure on dg-Lie algebroids. The purpose of this section is to provide a reasonably concrete *cofibrant* replacement for dg-Lie algebroids and L_∞-algebroids, which is analogous to the cobar resolution for algebras over reduced operads.

Let us start by briefly recalling the relation between L_∞-algebras and cocommutative dg-coalgebras (over $\mathbb{R}$). All cocommutative coalgebras are assumed to be without counit and conilpotent (every element is annihilated by some n-fold composite of the comultiplication). For any cocommutative coalgebra C and an L_∞-algebra $\mathfrak{h}$, the chain complex $\mathrm{Hom}(C, \mathfrak{h})$ has the structure of an L_∞-algebra, with differential given by $\partial\tau = \partial_{\mathfrak{h}} \circ \tau - \tau \circ \partial_C$ and n-ary bracket given by composing the n-ary bracket in $\mathfrak{h}$ with the n-fold comultiplication in C. A *twisting cochain* is a Maurer-Cartan element of this L_∞-algebra, i.e. a map $C \longrightarrow \mathfrak{h}[1]$ which satisfies the Maurer-Cartan equation

$$\partial\tau + \sum_{j\geq 2} \frac{1}{j!}[\tau, ..., \tau]_j = 0. \tag{3.1.21}$$

The infinite sum is well-defined because C is conilpotent. There are natural bijections

$$\mathrm{Hom}_{\mathrm{L}_\infty\mathrm{Alg}^{\mathrm{dg}}_{\mathbb{R}}}(\Omega C, \mathfrak{g}) \cong \mathrm{Twist}(C, \mathfrak{g}) \cong \mathrm{Hom}_{\mathrm{CoAlg}^{\mathrm{dg}}_{\mathbb{R}}}\big(C, \overline{C}_*(\mathfrak{g})\big) \tag{3.1.22}$$

between the set of twisting cochains $C \longrightarrow \mathfrak{g}[1]$, the set of maps of L_∞-algebras $\Omega C \longrightarrow \mathfrak{g}$ from the *cobar* construction of C and the set of maps of cocommutative coalgebras $C \longrightarrow \overline{C}_*(\mathfrak{g})$ to the *reduced (homological) Chevalley-Eilenberg complex* of $\mathfrak{g}$. The latter is the cofree (conilpotent, non-counital) graded-cocommutative coalgebra

$$\overline{C}_*(\mathfrak{g}) := \mathrm{Sym}^{\geq 1}_{\mathbb{R}}\mathfrak{g}[1]$$

endowed with the unique differential extending the map

$$\textstyle\sum_{n\geq 1}[-, \dots, -]_n \colon \mathrm{Sym}^{\geq 1}_{\mathbb{R}}\mathfrak{g}[1] \longrightarrow \mathfrak{g}[1].$$

The natural isomorphisms (3.1.22) realize the cobar functor Ω as a left adjoint to $\overline{C}_*$.

A *nonlinear map* of L_∞-algebras $\mathfrak{g} \rightsquigarrow \mathfrak{h}$ is simply a twisting cochain $\overline{C}_*(\mathfrak{g}) \longrightarrow \mathfrak{h}[1]$, or equivalently, a map of cocommutative dg-coalgebras $\overline{C}_*(\mathfrak{g}) \longrightarrow \overline{C}_*(\mathfrak{h})$.

Definition 3.1.23. Let $\mathfrak{g}$ and $\mathfrak{h}$ be L_∞-algebroids over A. A *nonlinear map* $\mathfrak{g} \rightsquigarrow \mathfrak{h}$ of L_∞-algebroids is a nonlinear map of L_∞-algebras $\mathfrak{g} \rightsquigarrow \mathfrak{h}$, such that

(i) the composite map $\rho_{\mathfrak{h}}(\tau) \colon \overline{C}_*(\mathfrak{g}) \longrightarrow \mathfrak{h}[1] \longrightarrow T_A[1]$ takes the quotient by $\mathrm{Sym}^{\geq 2}_{\mathbb{R}}\mathfrak{g}[1] \subseteq \overline{C}_*(\mathfrak{g})$ and applies the anchor of $\mathfrak{g}$ to the remaining $\mathfrak{g}[1]$.

(ii) the map of graded vector spaces $\tau \colon \mathrm{Sym}^{\geq 1}_{\mathbb{R}}(\mathfrak{g}[1]) \longrightarrow \mathfrak{h}[1]$ descends to a graded A-linear map $\mathrm{Sym}^{\geq 1}_A(\mathfrak{g}[1]) \longrightarrow \mathfrak{h}[1]$.

Let $\mathrm{L}_\infty\mathrm{Algd}^{\mathrm{nonlin}}_A$ be the category of L_∞-algebroids and nonlinear maps between them.

Remark 3.1.24. The category of L_∞-algebras and nonlinear maps between them is a full subcategory of the category of cocommutative dg-coalgebras (on the fibrant objects in the model structure from [40]). We do not know if $\mathrm{L}_\infty\mathrm{Algd}^{\mathrm{nonlin}}$ can be embedded into such a category of coalgebraic objects.

Remark 3.1.25. More generally, let $\overline{C}_{\leq n}(\mathfrak{g}) \subseteq \overline{C}_*(\mathfrak{g})$ be the sub-dg-coalgebra on the polynomials in $\mathfrak{g}[1]$ of order $\leq n$. Let us say that an *n-th order map* $\mathfrak{g} \rightsquigarrow \mathfrak{h}$ of L_∞-algebroids is a twisting cochain $\overline{C}_{\leq n}(\mathfrak{g}) \longrightarrow \mathfrak{h}[1]$ satisfying the obvious analogues of (i) and (ii).

For each L_∞-algebroid $\mathfrak{g}$ and each n, the functor

$$\mathrm{L}_\infty\mathrm{Algd}^{\mathrm{dg}}_A \longrightarrow \mathrm{Set};\ \mathfrak{h} \longmapsto \big\{n - \mathrm{th\ order\ maps\ } \mathfrak{g} \rightsquigarrow \mathfrak{h}\big\}$$

can be corepresented by an L_∞-algebroid $Q^{(n)}(\mathfrak{g})$, depending functorially on $\mathfrak{g}$. The L_∞-algebroid $Q^{(1)}(\mathfrak{g})$ is simply the free L_∞-algebroid generated by the map of A-modules $\rho_\mathfrak{g}\colon \mathfrak{g} \longrightarrow T_A$.

An $(n+1)$-st order map $\mathfrak{g} \rightsquigarrow \mathfrak{h}$ restricts to an n-th order linear map, so that there is a sequence of L_∞-algebroids (depending functorially on $\mathfrak{g}$)

$$Q^{(1)}(\mathfrak{g}) = \mathrm{Free}(\mathfrak{g}) \longrightarrow Q^{(2)}(\mathfrak{g}) \longrightarrow \dots \longrightarrow Q(\mathfrak{g}) \tag{3.1.26}$$

whose colimit $Q(\mathfrak{g})$ corepresents nonlinear maps $\mathfrak{g} \rightsquigarrow \mathfrak{h}$. Our first aim will be to prove that $Q(\mathfrak{g})$ is often a cofibrant L_∞-algebroid.

Definition 3.1.27. An dg-Lie algebroid or L_∞-algebroid $\mathfrak{g}$ is *A-cofibrant* if its underlying dg-A-module is cofibrant.

Lemma 3.1.28. *The maps in Diagram 3.1.26 fit into a pushout square (depending functorially on $\mathfrak{g}$)*

$$\begin{tikzcd} F\Big(\big(\mathrm{Sym}^{n+1}_A \mathfrak{g}[1]\big)[-2]\Big) \arrow[r, "\kappa"] \arrow[d] & Q^{(n)}(\mathfrak{g}) \arrow[d] \\ F\Big(\big(\mathrm{Sym}^{n+1}_A \mathfrak{g}[1]\big)[-2,-1]\Big) \arrow[r] & Q^{(n+1)}(\mathfrak{g}). \end{tikzcd} \tag{3.1.29}$$

Here the left two L_∞-algebroids are freely generated by the twofold desuspension of the dg-A-module $\mathrm{Sym}^{n+1}_A \mathfrak{g}[1]$ and its cone, both equipped with the zero anchor map. Consequently, $Q(\mathfrak{g})$ is cofibrant if $\mathfrak{g} \in \mathrm{L}_\infty\mathrm{Algd}^{\mathrm{dg}}_A$ is A-cofibrant.

Proof. The left map in Diagram (3.1.29) is a cofibration when $\mathfrak{g}$ is cofibrant as a dg-A-module. It follows that the sequence (3.1.26) consists of cofibrations, so that $Q(\mathfrak{g})$ is cofibrant.

To produce the pushout square (3.1.29), observe that without the differential, the map $Q^{(n)}(\mathfrak{g}) \longrightarrow Q^{(n+1)}(\mathfrak{g})$ is given by the obvious map of free graded L_∞-algebroids

$$F\Big(\big(\mathrm{Sym}^{1\leq\bullet\leq n}_A \mathfrak{g}[1]\big)[-1]\Big) \longrightarrow F\Big(\big(\mathrm{Sym}^{1\leq\bullet\leq n+1}_A \mathfrak{g}[1]\big)[-1]\Big).$$

To obtain a pushout of the form (3.1.29), it suffices to check that for every new generator $\tau \in \mathrm{Sym}^{n+1}_A \mathfrak{g}[1]$, its differential $\partial\tau$ is contained in $Q^{(n)}(\mathfrak{g})$. To see this, let

$$\Omega^{(n)}(\mathfrak{g}) := \Omega\big(\overline{C}_{\leq n}(\mathfrak{g})\big)$$

denote the n-th stage of the operadic bar-cobar resolution of $\mathfrak{g}$, thought of as an L_∞-algebra over $\mathbb{R}$. By its universal property, $Q^{(n)}(\mathfrak{g})$ is a quotient of the free L_∞-algebroid $A \otimes \Omega^{(n)}(\mathfrak{g})$ generated by this L_∞-algebra (which has a natural map to T_A). We therefore obtain a commuting diagram of the form

$$\begin{tikzcd} A \otimes \Omega^{(n)}(\mathfrak{g}) \arrow[r, two heads] \arrow[d] & Q^{(n)}(\mathfrak{g}) \arrow[d] \\ A \otimes \Omega^{(n+1)}(\mathfrak{g}) \arrow[r, two heads] & Q^{(n+1)}(\mathfrak{g}) \end{tikzcd}$$

in which the horizontal maps are surjective. Any new generator τ of $Q^{(n+1)}(\mathfrak{g})$ is the image of a new generator $\tilde\tau \in \mathrm{Sym}^{n+1}_\mathbb{R} \mathfrak{g}[1]$ of $\Omega^{(n+1)}(\mathfrak{g})$. It is well-known that $\partial\tilde\tau$ is contained in $\Omega^{(n)}(\mathfrak{g})$ (see e.g. [100, Proposition 2.8 and Section 4.3] for a detailed discussion). Consequently, its image $\partial\tau$ is contained in $Q^{(n)}(\mathfrak{g})$. $\square$

By construction, the functor Q is the left adjoint of the obvious inclusion $\iota\colon \mathrm{L}_\infty\mathrm{Algd}_A^{\mathrm{dg}} \longrightarrow \mathrm{L}_\infty\mathrm{Algd}_A^{\mathrm{nonlin}}$. The induced (linear) map $Q(\mathfrak{g}) \longrightarrow \mathfrak{g}$ of L_∞-algebroids is often an equivalence:

Lemma 3.1.30. *The functor $Q\colon \mathrm{L}_\infty\mathrm{Algd}_A^{\mathrm{nonlin}} \longrightarrow \mathrm{L}_\infty\mathrm{Algd}_A^{\mathrm{dg}}$ enjoys the following properties:*

(a) let $\mathfrak{g} \rightsquigarrow \mathfrak{h}$ be a nonlinear map between A-cofibrant L_∞-algebroids. If the linear part $\mathfrak{g} \longrightarrow \mathfrak{h}$ is a weak equivalence, then $Q(\mathfrak{g}) \longrightarrow Q(\mathfrak{h})$ is a weak equivalence.

(b) the composite functor from the category of A-cofibrant L_∞-algebroids

$$Q\circ\iota\colon \mathrm{L}_\infty\mathrm{Algd}_A^{\mathrm{dg,A-cof}} \longrightarrow \mathrm{L}_\infty\mathrm{Algd}_A^{\mathrm{dg,cof}}$$

is a relative functor that preserves $\mathbf{\Delta}^{\mathrm{op}}$-indexed homotopy colimits.

(c) the counit map $q\colon Q(\mathfrak{g}) \longrightarrow \mathfrak{g}$ is a weak equivalence whenever $\mathfrak{g}$ is A-cofibrant.

Proof. Assertion (a) follows by induction along the filtration (3.1.26), using the pushout square (3.1.29) and the fact that each

$$\mathrm{Free}\Big(\big(\mathrm{Sym}_A^{n+1}\mathfrak{g}[1]\big)[-2]\Big) \longrightarrow \mathrm{Free}\Big(\big(\mathrm{Sym}_A^{n+1}\mathfrak{h}[1]\big)[-2]\Big)$$

is a weak equivalence whenever $\mathfrak{g} \longrightarrow \mathfrak{h}$ is a weak equivalence of cofibrant dg-A-modules (and similarly for the cones).

For (b), note that each functor $Q^{(n)}\colon \mathrm{L}_\infty\mathrm{Algd}_A^{\mathrm{dg}} \longrightarrow \mathrm{L}_\infty\mathrm{Algd}_A^{\mathrm{dg}}$ preserves strict colimits of diagrams indexed by $\mathbf{\Delta}^{\mathrm{op}}$, which can be computed at the level of the underlying chain complexes. Suppose that $\mathfrak{g}_\bullet\colon \mathbf{\Delta}^{\mathrm{op}} \longrightarrow \mathrm{L}_\infty\mathrm{Algd}_A^{\mathrm{dg}}$ is a projectively cofibrant diagram. Then $\mathfrak{g}$ takes values in A-cofibrant L_∞-algebroids (by Theorem 3.1.15) and $\mathrm{colim}(\mathfrak{g}_\bullet)$ is a cofibrant model for the homotopy colimit of the underlying simplicial diagram of dg-A-modules. We have to check that the natural map

$$\mathrm{hocolim}(Q(\mathfrak{g}_\bullet)) \longrightarrow Q(\mathrm{colim}\,\mathfrak{g}_\bullet)$$

is an equivalence. Applying the filtration (3.1.26), we obtain a sequence of $\mathbf{\Delta}^{\mathrm{op}}$-diagrams $Q^{(n)}(\mathfrak{g}_\bullet)$ of dg-Lie-algebroids, which fit into pushout diagrams of the form (3.1.29). We will prove by induction on n that the map

$$\mathrm{hocolim}\,\Big(Q^{(n)}(\mathfrak{g}_\bullet)\Big) \longrightarrow \mathrm{colim}\,\Big(Q^{(n)}(\mathfrak{g}_\bullet)\Big) = Q^{(n)}\big(\,\mathrm{colim}\,\mathfrak{g}_\bullet\big)$$

is a weak equivalence whose codomain is cofibrant. For $n=1$ the statement is trivial. For each $n\geq 1$, we obtain a diagram of L_∞-algebroids

$$\begin{tikzcd}
\mathrm{colim}\,F\Big(\big(\mathrm{Sym}_A^{n+1}\mathfrak{g}_\bullet[1]\big)[-2]\Big) \arrow[r] \arrow[d] & \mathrm{colim}\,\Big(Q^{(n)}(\mathfrak{g}_\bullet)\Big) \arrow[d] \\
\mathrm{colim}\,F\Big(\big(\mathrm{Sym}_A^{n+1}\mathfrak{g}_\bullet[1]\big)[-2,-1]\Big) \arrow[r] & \mathrm{colim}\,\Big(Q^{(n+1)}(\mathfrak{g}_\bullet)\Big).
\end{tikzcd}$$

The free functor and the symmetric power functor commute with sifted colimits and preserve cofibrations between cofibrant objects. Since $\mathrm{colim}(\mathfrak{g}_\bullet)$ is a cofibrant dg-A-module (by Theorem 3.1.15), the left vertical map is a cofibration between cofibrant L_∞-algebroids. Since $\mathrm{colim}\,\big(Q^{(n)}(\mathfrak{g}_\bullet)\big)$ is cofibrant by inductive assumption, the above square is a homotopy pushout square and $\mathrm{colim}\,\big(Q^{(n+1)}(\mathfrak{g}_\bullet)\big)$ is cofibrant.

The above homotopy pushout square is weakly equivalent to the corresponding square of homotopy colimits. Indeed, for the top right colimit this holds by assumption, while the left two colimits are equivalent to the corresponding homotopy colimits because the free functor and the symmetric power functor commute with $\mathbf{\Delta}^{\mathrm{op}}$-indexed homotopy colimits. It follows that the map on homotopy pushouts is a weak equivalence as well.

For part (c), note that by parts (a) and (b), together with Corollary 3.1.19, it suffices to prove this when $\mathfrak{g} = A \otimes_{\mathbb{R}} \mathfrak{h}$ is just the A-linear extension of an ordinary L_∞-algebra $\mathfrak{h}$ over $\mathbb{R}$. In that case, the map $Q(\mathfrak{g}) \longrightarrow \mathfrak{g}$ is just the A-linear extension of the usual map $\Omega(\overline{C}_*(\mathfrak{h})) \longrightarrow \mathfrak{h}$ of L_∞-algebras from the operadic cobar construction of $\mathfrak{h}$. This map is a weak equivalence (see e.g. [58] for a textbook account). $\square$

As usual, the derived mapping space between two L_∞-algebroids can be described by the simplicial set of maps from a cofibrant replacement of the domain to a fibrant simplicial resolution of the codomain. Such a simplicial resolution of fibrant L_∞-algebroids has been described in [101]:

Construction 3.1.31 ([101])**.** Let $\mathfrak{g}$ be an L_∞-algebroid over A and let B be any (possibly unbounded) commutative dg-algebra over $\mathbb{R}$. Then $\mathfrak{g} \otimes B$ has the structure of an A-module and an L_∞-algebra and the anchor map extends to a B-linear map $\mathfrak{g} \otimes B \longrightarrow T_A \otimes B$. Let $\mathfrak{g} \boxtimes B$ be the pullback

$$\begin{tikzcd} \mathfrak{g} \boxtimes B \arrow[r] \arrow[d, "\rho"'] & \mathfrak{g} \otimes B \arrow[d] \\ T_A \arrow[r] & T_A \otimes B. \end{tikzcd}$$

All maps in this diagram are A-linear and preserve L_∞-structures, and one can verify that the induced L_∞-structure on $\mathfrak{g} \boxtimes B$ turns it into an L_∞-algebroid over A (see [101]). We therefore obtain a functor

$$\mathfrak{g} \boxtimes (-)\colon \mathrm{CAlg}^{\mathrm{dg}}_{\mathbb{R}} \longrightarrow \mathrm{L}_\infty\mathrm{Algd}_A$$

which preserves pullbacks and fibrations, and weak equivalences when $\mathfrak{g}$ is fibrant. Furthermore, there are natural isomorphisms $\mathfrak{g} \boxtimes (B \otimes C) \cong \big(\mathfrak{g} \boxtimes B\big) \boxtimes C$.

For any finite simplicial set K, let $\mathfrak{g}^K = \mathfrak{g} \boxtimes \Omega[K]$ be the dg-Lie algebroid obtained by applying this functor to the polynomial differential forms on K.

Lemma 3.1.32. *Let $K \longrightarrow L$ be a cofibration between finite simplicial sets and let $\mathfrak{g} \longrightarrow \mathfrak{h}$ be a fibration. Then $\mathfrak{g}^L \longrightarrow \mathfrak{g}^K \times_{\mathfrak{h}^K} \mathfrak{h}^L$ is a fibration, which is a weak equivalence if $\mathfrak{g} \longrightarrow \mathfrak{h}$ or $K \longrightarrow L$ is a weak equivalence.*

Proof. It is well-known that the map $\mathfrak{g} \otimes \Omega[L] \longrightarrow \mathfrak{g} \otimes \Omega[K] \times_{\mathfrak{h}\otimes\Omega[K]} \mathfrak{h} \otimes \Omega[L]$ is a surjection and a quasi-isomorphism whenever $\mathfrak{g} \longrightarrow \mathfrak{h}$ or $K \longrightarrow L$ is a weak equivalence (cf. [15]). The assertion now follows by considering the two pullback squares

$$\begin{tikzcd} \mathfrak{g}^L \arrow[r] \arrow[d] & \mathfrak{g}^K \times_{\mathfrak{h}^K} \mathfrak{h}^L \arrow[r] \arrow[d] & T_A \arrow[d] \\ \mathfrak{g} \otimes \Omega[L] \arrow[r] & \mathfrak{g} \otimes \Omega[K] \times_{\mathfrak{h}\otimes\Omega[K]} \mathfrak{h} \otimes \Omega[L] \arrow[r] & T_A \otimes \Omega[L] \end{tikzcd}$$

and using that (acyclic) fibrations are stable under base change. $\square$

Corollary 3.1.33. *Let $\mathfrak{g}$ be an A-cofibrant L_∞-algebroid and let $\mathfrak{h}$ be fibrant. Then a model for the derived mapping space $\mathrm{Map}^{\mathbb{R}}(\mathfrak{g}, \mathfrak{h})$ is the simplicial set given in degree n by the nonlinear maps $\mathfrak{g} \rightsquigarrow \mathfrak{h}^{\Delta[n]}$.*

Remark 3.1.34. When the L_∞-algebroids $\mathfrak{g}$ and $\mathfrak{h}$ are concentrated in nonnegative degrees, one can also compute the mapping space using a semi-model structure on connective dg-Lie algebroids. In this case, one just has to assume that the map $\mathfrak{h} \longrightarrow \tau_{\geq 0} T_A$ is a surjection in degrees > 0; this becomes particularly easy when A is discrete (so that T_A is concentrated in degree 0).

Remark 3.1.35. The simplicial sets of maps $\mathrm{Hom}_{\mathrm{L}_\infty\mathrm{Algd}^{\mathrm{nonlin}}}(\mathfrak{g}, \mathfrak{h}^{\Delta[\bullet]})$ equip the category $\mathrm{L}_\infty\mathrm{Algd}^{\mathrm{nonlin}}$ with an enrichment over simplicial sets. Corollary 3.1.33 shows that the ∞-categorical localization of the semi-model category of L_∞-algebroids can be modeled by the full simplicial subcategory of $\mathrm{L}_\infty\mathrm{Algd}^{\mathrm{nonlin}}$ on the L_∞-algebroids which are fibrant and A-cofibrant.

Remark 3.1.36. Nonlinear maps of L_∞-algebroids are frequently used in the literature, where (in the finite rank case) they are often identified with maps of so-called NQ-supermanifolds (originating in [1]). For example, they naturally arise when studying homotopy transfer of L_∞-algebroid structures (see [79]).

3.1.4 Naturality.

The semi-model category of dg-Lie algebroids over A depends on A in a somewhat delicate way. Indeed, a map of dg-$\mathcal{C}^\infty$-rings $f\colon A \longrightarrow B$ does not induce a Quillen pair between the categories of dg-Lie algebroids over A and B because there is no direct way to compare the tangent modules T_A and T_B. To adress this issue, let us make the following definition:

Definition 3.1.37. Let $f\colon A \longrightarrow B$ be a map of dg-$\mathcal{C}^\infty$-rings, $\mathfrak{g}$ be a dg-Lie algebroid over A and $\mathfrak{h}$ be a dg-Lie algebroid over B. A *map of dg-Lie algebroids over f* is an A-linear map

$$\phi\colon \mathfrak{g} \longrightarrow \mathfrak{h}$$

which preserves the Lie bracket and intertwines the actions on A and B, in the sense that

$$\rho_{\mathfrak{h}}\big(\phi(X)\big)(f(a)\big) = f\big(\rho_{\mathfrak{g}}(X)(a)\big).$$

Example 3.1.38. Suppose that $f\colon A = \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(U) = B$ is induced by restriction along an open inclusion $U \subseteq M$ of smooth manifolds. Then there is a map $T_A \longrightarrow T_B$ over f which restricts a vector field on M to the open U.

Example 3.1.39. Let $f\colon A \longrightarrow B$ be any map of dg-$\mathcal{C}^\infty$-rings and let $\mathfrak{h}$ be a dg-Lie algebroid over B. Consider the dg-A-module

$$f_\sharp(\mathfrak{h}) := f_*(\mathfrak{h}) \times_{\mathrm{Der}(A,B)} T_A$$

consisting of tuples (X, v) with $X \in \mathfrak{h}$ and $v \in T_A$ such that

$$\rho_{\mathfrak{h}}(X)(f(a)) = f(v(a)).$$

Then $f_\sharp(\mathfrak{h})$ has the structure of a dg-Lie algebroid over A, with bracket given by the bracket on $\mathfrak{h}$ and T_A and the anchor given by the projection to T_A. The projection $f_\sharp(\mathfrak{h}) \longrightarrow \mathfrak{h}$ is a map of dg-Lie algebroids over f.

Lemma 3.1.40. *Let $f\colon A \longrightarrow B$ be a map of dg-$\mathcal{C}^\infty$-rings and let $\phi\colon \mathfrak{g} \longrightarrow \mathfrak{h}$ be a map of dg-Lie algebroids over f. Then the following holds:*

(1) If $\mathfrak{g}' \longrightarrow \mathfrak{g}$ is a map of dg-Lie algebroids over A, then the B-linear map

$$p\colon f^*(\mathfrak{g}') = B \otimes_A \mathfrak{g}' \longrightarrow \mathfrak{h}$$

is a map of dg-Lie algebroids over B.

(2) If $\mathfrak{h}' \longrightarrow \mathfrak{h}$ is a map of dg-Lie algebroids over B, then the A-linear projection map

$$q\colon f_\sharp(\mathfrak{h}') := f_*(\mathfrak{h}') \times_{f_*(\mathfrak{h})} \mathfrak{g} \longrightarrow \mathfrak{g}$$

is a map of dg-Lie algebroids over A.

Proof. For (1), let us denote the representation of $X \in \mathfrak{h}$ on $b \in B$ by $\nabla_X(b)$. We can endow $f^*(\mathfrak{g}') = B \otimes_A \mathfrak{g}'$ with the bracket (without Koszul signs)

$$\begin{aligned}[b_1 \otimes X_1, b_2 \otimes X_2] = b_1 b_2 \otimes [X_1, X_2]& \\ + \nabla_{p(b_1\otimes X_1)}(b_2) \otimes X_2 - \nabla_{p(b_2\otimes X_2)}&(b_1) \otimes X_1.\end{aligned}$$

This is well defined because the composite map $\mathfrak{g}' \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{h}$ is a map of dg-Lie algebroids over f. Furthermore, the B-linear map $p\colon f^*(\mathfrak{g}') \longrightarrow \mathfrak{h}$ preserves the bracket by construction.

It remains to verify that $f^*(\mathfrak{g}')$ is indeed a dg-Lie algebroid over B. Note that the Leibniz rule holds by construction and that the Jacobi identity holds for all elements of the form $1 \otimes X$. More generally, if the Jacobi identity holds for elements X_1, X_2, X_3 in $f^*(\mathfrak{g}')$ and $b \in B$, then

$$\begin{aligned}\Big[[X_1, X_2], bX_3\Big] &= b \cdot \Big[[X_1, X_2], X_3\Big] + \nabla_{p([X_1,X_2])}(b) \otimes X_3 \\ &= b \cdot \Big[X_1, [X_2, X_3]\Big] + \Big(\nabla_{p(X_1)}\nabla_{p(X_2)}(b)\Big)X_3 \\ &\quad - b \cdot \Big[X_2, [X_1, X_3]\Big] - \Big(\nabla_{p(X_2)}\nabla_{p(X_1)}(b)\Big)X_3 \\ &= \big[X_1, [X_2, bX_2]\big] - \big[X_2, [X_1, bX_3]\big].\end{aligned}$$

Here the first and last equation follow from the Leibniz rule. The second equation follows from the Jacobi identity for X_1, X_2, X_3 and the fact that p preserves the bracket. It follows that the Jacobi identity holds for X_1, X_2 and bX_3. By symmetry and additivity we deduce that the Jacobi identity holds for all elements in $f^*(\mathfrak{g}')$, so that $f^*(\mathfrak{g}')$ is indeed a dg-Lie algebroid.

For (2), we can endow $f_\sharp(\mathfrak{h}')$ with the bracket

$$\Big[(X, v), (Y, w)\Big] = \Big([X, Y]_{\mathfrak{h}'}, [v, w]_{\mathfrak{g}}\Big).$$

This is well-defined because $\mathfrak{h}' \longrightarrow \mathfrak{h}$ is a map of dg-Lie algebroids over B and $\mathfrak{g} \longrightarrow \mathfrak{h}$ is a map of dg-Lie algebroids over f. One easily verifies that this makes the projection map $f_\sharp(\mathfrak{h}') \longrightarrow \mathfrak{g}$ a map of dg-Lie algebroids over A. ☐

Lemma 3.1.41. *Let $f\colon A \longrightarrow B$ be a map of dg-$\mathcal{C}^\infty$-rings and let $\phi\colon \mathfrak{g} \longrightarrow \mathfrak{h}$ be a map of dg-Lie algebroids over f. Then there is a Quillen pair (depending on the map ϕ)*

$$f^*\colon \mathrm{LieAlgd}^{\mathrm{dg}}_A/\mathfrak{g} \rightleftarrows \mathrm{LieAlgd}^{\mathrm{dg}}_B/\mathfrak{h} \colon f_\sharp.$$

This Quillen pair is a Quillen equivalence if f and ϕ are both quasi-isomorphisms.

Proof. One can easily verify that the functors f^* and $f_\sharp$ from Lemma 3.1.40 are adjoints. The functor $f_\sharp$ preserves fibrations and trivial fibrations since it is given at the level of chain complexes by

$$\big(\mathfrak{h}' \longrightarrow \mathfrak{h}\big) \longmapsto \mathfrak{h}' \times_{\mathfrak{h}} \mathfrak{g}.$$

Given a cofibrant object $\mathfrak{g}'$ in $\mathrm{LieAlgd}^{\mathrm{dg}}_A/\mathfrak{g}$, the derived unit map can be identified at the level of chain complexes with the natural map

$$\mathfrak{g}' \longrightarrow B \otimes_A \mathfrak{g}' \times^h_{\mathfrak{h}} \mathfrak{g}$$

into the homotopy pullback. On the other hand, for any fibrant object $\mathfrak{h}'$ in $\mathrm{LieAlgd}_A^{\mathrm{dg}}/\mathfrak{h}$, the derived counit map can be identified at the level of chain complexes with the natural map

$$B \otimes_A \left(\mathfrak{g} \times^h_{\mathfrak{h}} \mathfrak{h}'\right) \longrightarrow \mathfrak{h}'.$$

When $f\colon A \longrightarrow B$ and $\phi\colon \mathfrak{g} \longrightarrow \mathfrak{h}$ are quasi-isomorphism, each of these two maps is a quasi-isomorphism. It follows that $(f^*, f_\sharp)$ is a Quillen equivalence. □

Example 3.1.42. Given a map $f\colon A \longrightarrow B$ of dg-$\mathcal{C}^\infty$-rings, Example 3.1.39 provides a map $f_\sharp(T_B) \longrightarrow T_B$ of dg-Lie algebroids over f, where $f_\sharp(T_B)$ is the dg-Lie algebroid over A of compatible $\mathcal{C}^\infty$-derivations $v\colon A \longrightarrow A$ and $w\colon B \longrightarrow B$. There is a zig-zag of right Quillen functors

$$\mathrm{LieAlgd}_A^{\mathrm{dg}} \xrightarrow{(-)\times_{T_A} f_\sharp(T_B)} \mathrm{LieAlgd}_A^{\mathrm{dg}}/f_\sharp(T_B) \xleftarrow{f_\sharp} \mathrm{LieAlgd}_B^{\mathrm{dg}}.$$

Both of the maps $f_\sharp(T_B) \longrightarrow T_A$ and $f_\sharp(T_B) \longrightarrow T_B$ are quasi-isomorphisms as soon as f is either a trivial cofibration or a trivial fibration between cofibrant dg-$\mathcal{C}^\infty$-rings. In this case, Lemma 3.1.41 implies that the above functors are right Quillen equivalences. In particular, when A and B are weakly equivalent cofibrant dg-$\mathcal{C}^\infty$-rings, there is a zig-zag of Quillen equivalences

$$\mathrm{LieAlgd}_A^{\mathrm{dg}} \xrightarrow{\sim} \dots \xleftarrow{\sim} \mathrm{LieAlgd}_B^{\mathrm{dg}}.$$

In fact, the same argument applies as soon as Ω_A is a cofibrant dg-A-module and $L_A \longrightarrow \Omega_A$ is a weak equivalence, and similarly for B: in this case, the module T_A is a model for the derived A-linear dual of L_A. For instance, the function ring $\mathcal{C}^\infty(M)$ of a smooth manifold has this property, by Example 2.2.18. It follows that dg-Lie algebroids over $\mathcal{C}^\infty(M)$ provide a model for the ∞-category $\mathrm{LieAlgd}_{\mathcal{C}^\infty(M)}$.

3.2 Technicalities on the semi-model structure

This section is devoted to the proofs of Theorem 3.1.10 and Theorem 3.1.15. Just like for algebras over an operad, these proofs rely on an analysis of the pushout of a diagram of dg-Lie algebroids (and L_∞-algebroids) of the form

$$\begin{array}{ccc} F(V) & \xrightarrow{F(i)} & F(W) \\ \downarrow & & \downarrow \\ \mathfrak{g} & \longrightarrow & \mathfrak{h} \end{array} \tag{3.2.1}$$

where $F\colon \mathrm{Mod}_{\mathbb{R}}^{\mathrm{dg}}/T_A \longrightarrow \mathrm{L}_\infty\mathrm{Algd}_A^{\mathrm{dg}}$ denotes the free functor. We will show that the map $\mathfrak{g} \longrightarrow \mathfrak{h}$ can be decomposed into a sequence of maps $\mathfrak{g}^{(p)} \longrightarrow \mathfrak{g}^{(p+1)}$, whose associated graded is controlled by the *reduced enveloping operad* of the dg-Lie algebroid $\mathfrak{g}$. The difference from the case of operadic algebras is that the maps $\mathfrak{g}^{(p)} \longrightarrow \mathfrak{g}^{(p+1)}$ need not be injective in general.

Throughout, we will only treat L_∞-algebroids; the case of dg-Lie algebroids proceeds in exactly the same manner, replacing all appearances of the L_∞-operad by the Lie operad.

3.2.1 Filtrations. Let $\mathrm{Mod}_{\mathbb{R}}^{\mathbb{N},\mathrm{dg}}$ be the category of sequences of chain complexes

$$V^{(0)} \longrightarrow V^{(1)} \longrightarrow V^{(2)} \longrightarrow \dots \tag{3.2.2}$$

endowed with the Reedy model structure. We will refer to an object V of $\mathrm{Mod}_{\mathbb{R}}^{\mathbb{N},\mathrm{dg}}$ as a *weakly filtered* chain complex. An object is Reedy cofibrant if and only if (3.2.2) consists of monomorphisms, in which case it can be interpreted as a genuine filtration on $\operatorname{colim} V$. We will say that an element in $V^{(p)}$ is of *weight* $\leq p$ and an element of $V^{(p)}/V^{(p-1)}$ is of *weight* p, so that degrees always indicate homological degrees.

The category of weakly filtered chain complexes has a closed symmetric monoidal structure, given by

$$(V \otimes W)^{(n)} = \operatorname{colim} V^{(p)} \otimes W^{(q)}.$$

The colimit is taken over the full subcategory of $(p, q) \in \mathbb{N} \times \mathbb{N}$ for which $p + q \leq n$. The symmetry isomorphism given by the symmetry isomorphisms of chain complexes $V^{(p)} \otimes W^{(q)} \longrightarrow W^{(q)} \otimes V^{(p)}$, i.e. there are no extra signs depending on p and q.

There are two Quillen pairs

$$\operatorname{colim}\colon \mathrm{Mod}_{\mathbb{R}}^{\mathbb{N},\mathrm{dg}} \rightleftarrows \mathrm{Mod}_{\mathbb{R}}^{\mathrm{dg}} : i \qquad \mathrm{gr}\colon \mathrm{Mod}_{\mathbb{R}}^{\mathbb{N},\mathrm{dg}} \rightleftarrows \mathrm{Mod}_{\mathbb{R}}^{\mathrm{gr},\mathrm{dg}} : j.$$

Here i sends a chain complex to the constant diagram on V (and will be omitted from the notation) and 'gr' sends a sequence V to the $\mathbb{N}$-graded chain complex $V^{(\bullet)}/V^{(\bullet-1)}$, with right adjoint sending a graded chain complex W to the sequence consisting of zero maps. Each of the above functors is symmetric monoidal.

Remark 3.2.3. The functor $\mathrm{gr}\colon \mathrm{Mod}_{\mathbb{R}}^{\mathbb{N},\mathrm{dg}} \longrightarrow \mathrm{Mod}_{\mathbb{R}}^{\mathrm{gr},\mathrm{dg}}$ detects weak equivalences between cofibrant objects: this is just the well-known fact that weak equivalences of filtered chain complexes are detected on the associated graded.

The notions of L_∞-algebras and L_∞-algebroids over A have obvious weakly filtered and graded analogues (A is always of weight ≤ 0). For example, a weakly filtered L_∞-algebroid over A is an object $\mathfrak{g}$ in $\mathrm{Mod}_{\mathbb{R}}^{\mathbb{N},\mathrm{dg}}$ together with

(1) the structure of an A-module, i.e. natural chain maps $A \otimes \mathfrak{g}^{(i)} \longrightarrow \mathfrak{g}^{(i)}$.

(2) an L_∞-algebra structure in $\mathrm{Mod}_{\mathbb{R}}^{\mathbb{N},\mathrm{dg}}$, i.e. for each $p \geq 0$ a matching family of n-ary maps $[-, \cdots, -]\colon \mathfrak{g}^{(i_1)} \otimes \cdots \otimes \mathfrak{g}^{(i_n)} \longrightarrow \mathfrak{g}^{(p)}$, for all $i_1 + \cdots + i_n \leq p$.

(3) a map $\mathfrak{g} \longrightarrow T_A$ of L_∞-algebras and A-modules in $\mathrm{Mod}_{\mathbb{R}}^{\mathbb{N},\mathrm{dg}}$, where T_A has weight ≤ 0.

When $\mathfrak{g}$ is Reedy cofibrant (i.e. a filtered chain complex), this is simply the structure of an L_∞-algebroid on $\operatorname{colim}(\mathfrak{g})$ whose entire structure respects the filtration. Let us denote the categories of weakly filtered and graded L_∞-algebroids over A by

$$\mathrm{L}_\infty\mathrm{Algd}_A^{\mathbb{N},\mathrm{dg}} \qquad \text{and} \qquad \mathrm{L}_\infty\mathrm{Algd}_A^{\mathrm{gr},\mathrm{dg}}.$$

The description of the free L_∞-algebroid on a chain complex over T_A also applies to the weakly filtered and graded settings: one first takes the free (weakly filtered, graded) L_∞-algebra over T_A and then takes the associated action L_∞-algebroid (Example 3.1.9). This yields a commuting diagram of left adjoints

$$\begin{tikzcd}
\mathrm{Mod}_{\mathbb{R}}^{\mathbb{N},\mathrm{dg}}/T_A \arrow[d, "\mathrm{Free}"'] & \mathrm{Mod}_{\mathbb{R}}^{\mathrm{dg}}/T_A \arrow[l, "i"'] \arrow[d, "\mathrm{Free}"'] & \mathrm{Mod}_{\mathbb{R}}^{\mathbb{N},\mathrm{dg}}/T_A \arrow[l, "\mathrm{colim}"'] \arrow[r, "\mathrm{gr}"] \arrow[d, "\mathrm{Free}"] & \mathrm{Mod}_{\mathbb{R}}^{\mathrm{gr},\mathrm{dg}}/T_A \arrow[d, "\mathrm{Free}"] \\
\mathrm{L}_\infty\mathrm{Algd}_A^{\mathbb{N},\mathrm{dg}} & \mathrm{L}_\infty\mathrm{Algd}_A^{\mathrm{dg}} \arrow[l, "i"] & \mathrm{L}_\infty\mathrm{Algd}_A^{\mathbb{N},\mathrm{dg}} \arrow[l, "\mathrm{colim}"] \arrow[r, "\mathrm{gr}"'] & \mathrm{L}_\infty\mathrm{Algd}_A^{\mathrm{gr},\mathrm{dg}}
\end{tikzcd}$$

where the vertical functors are the free functors, sending a (weakly filtered, graded) chain complex V over T_A to $A \otimes L_\infty(V)$. All horizontal functors can be computed at the level of chain complexes. For example, the colimit of a weakly filtered L_∞-algebroid is simply the colimit of the underlying sequence of chain complexes, together with a certain L_∞-algebroid structure on it.

3.2.2 Coproducts with L_∞-algebras. In this section we will study the simplest type of pushout diagram (3.2.1): the case of a coproduct of a (weakly filtered) L_∞-algebroid $\mathfrak{g}$ over A with the free L_∞-algebroid generated by a (weakly filtered) chain complex V, equipped with the *zero map* to T_A.

Such coproducts are much easier to describe than coproducts for nonzero maps $V \longrightarrow T_A$. Indeed, the coproduct $\mathfrak{g} \amalg F\big(V \xrightarrow{0} T_A\big)$ fits into a retract diagram

$$\mathfrak{g} = \mathfrak{g} \amalg F(0) \longrightarrow \mathfrak{g} \amalg F\big(V \xrightarrow{0} T_A\big) \longrightarrow \mathfrak{g}.$$

This construction is the left adjoint in an adjunction

$$\mathfrak{g} \amalg F\big((-) \xrightarrow{0} T_A\big) \colon \mathrm{Mod}_{\mathbb{R}}^{\mathbb{N},\mathrm{dg}} \rightleftarrows \mathfrak{g}/\mathrm{L}_\infty\mathrm{Algd}_A^{\mathbb{N},\mathrm{dg}}/\mathfrak{g} \colon \ker$$

where the right adjoint sends a retract diagram $\mathfrak{g} \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{g}$ to the kernel of $\mathfrak{h} \longrightarrow \mathfrak{g}$. The category of retract diagrams of (weakly filtered) L_∞-algebroids

$$\mathfrak{g} \longrightarrow \mathfrak{h} = \mathfrak{g} \oplus \mathfrak{m} \longrightarrow \mathfrak{g}$$

can be identified with the category of algebras over an operad in (weakly filtered) chain complexes over $\mathbb{R}$. Indeed, such a retract diagram can equivalently be encoded by the following kind of algebraic structure on $\mathfrak{m}$:

- $\mathfrak{m}$ has the structure of a (weakly filtered) A-module.
- $\mathfrak{m}$ comes equipped with an A-linear L_∞-structure, since the anchor map vanishes on $\mathfrak{m}$.
- for each set of elements $\xi_1, \dots, \xi_n \in \mathfrak{g}$, the $(n+k)$-ary bracket on $\mathfrak{g} \oplus \mathfrak{m}$ determines a k-ary operation $[\xi_1, \dots, \xi_n, (-)] \colon \mathfrak{m}^{\otimes k} \longrightarrow \mathfrak{m}$ of degree $k-2$, for each $k \geq 1$.

These operations have to satisfy equations stating that certain sums of their composites are zero. This type of algebraic structure can precisely be encoded by means of an operad, which has no nullary operations (as one sees from the above description).

Definition 3.2.4. The *reduced enveloping operad* $\overline{\mathrm{Env}}_{\mathfrak{g}}$ of a weakly filtered L_∞-algebroid $\mathfrak{g}$ is the (reduced) weakly filtered dg-operad over $\mathbb{R}$ whose algebras $\mathfrak{m}$ are retract diagrams of L_∞-algebroids $\mathfrak{g} \longrightarrow \mathfrak{h} = \mathfrak{g} \oplus \mathfrak{m} \longrightarrow \mathfrak{g}$.

Remark 3.2.5. The above definition is somewhat imprecise. More accurately, one can construct the operad $\overline{\mathrm{Env}}_{\mathfrak{g}}$ in terms of generators of the form

- μ_a for $a \in A$ (left multiplication by a)
- $[-, \dots, -]$ (the L_∞-structure on $\mathfrak{m}$)
- $[\xi_1, \dots, \xi_n, -, \dots, -]$ for elements $\xi_1, \dots, \xi_n$ in $\mathfrak{g}$.

These generators have to satisfy an obvious list of equations. For example, there are equations expressing the anti-symmetry and Jacobi identities for the various brackets. Furthermore, the brackets $[\xi_1, \dots, \xi_n, -, \dots, -]$ depend A-multilinearly on the elements ξ_i and are mostly A-multilinear operations themselves, i.e.

$$\begin{aligned}
[a \cdot \xi_1, \dots, \xi_n, -, \dots, -] &= \mu_a \circ [\xi_1, \dots, \xi_n, -, \dots, -] \\
[\xi, -] \circ \mu_a &= \mu_a \circ [\xi, -] + \mu_{\xi(a)} \\
[\xi_1, \dots, \xi_n, -, \dots, -] \circ_i \mu_a &= \mu_a \circ [\xi_1, \dots, \xi_n, -, \dots, -].
\end{aligned}$$

Example 3.2.6. Suppose that $\mathfrak{g}$ is an A-linear L_∞-algebra. Then the reduced enveloping operad of $\mathfrak{g}$ is simply the arity ≥ 1 part of the usual enveloping operad of $\mathfrak{g}$.

Remark 3.2.7. A map of L_∞-algebroids $f\colon \mathfrak{g} \longrightarrow \mathfrak{h}$ induces a map of reduced enveloping operads $f\colon \overline{\mathrm{Env}}_{\mathfrak{g}} \longrightarrow \overline{\mathrm{Env}}_{\mathfrak{h}}$, which sends a generator $[\xi_1, \dots, \xi_n, -, \dots, -]$ to the generator $[f(\xi_1), \dots, f(\xi_n), -, \dots, -]$. The corresponding restriction functor between categories of algebras can be identified with the functor

$$f^*\colon \mathfrak{h}/\mathrm{L}_\infty\mathrm{Algd}_A^{\mathrm{dg}}/\mathfrak{h} \longrightarrow \mathfrak{g}/\mathrm{L}_\infty\mathrm{Algd}_A^{\mathrm{dg}}/\mathfrak{g}$$

sending $\mathfrak{h} \longrightarrow \mathfrak{h} \oplus \mathfrak{m} \longrightarrow \mathfrak{h}$ to the pullback $\mathfrak{g} \longrightarrow (\mathfrak{h} \oplus \mathfrak{m}) \times_{\mathfrak{h}} \mathfrak{g} \longrightarrow \mathfrak{g}$.

The operad structure on $\overline{\mathrm{Env}}_{\mathfrak{g}}$ is not A-linear, but there is a canonical map of operads $\mu\colon A \longrightarrow \overline{\mathrm{Env}}_{\mathfrak{g}}$. Here we consider A as an operad with only unary operations. In particular, for every (weakly filtered) chain complex V, there is an isomorphism of left A-modules

$$\begin{aligned} \mathfrak{g} \coprod F\big(V \xrightarrow{0} T_A\big) &\cong \mathfrak{g} \oplus \Big(\overline{\mathrm{Env}}_{\mathfrak{g}} \circ V\Big) \\ &= \mathfrak{g} \oplus \bigoplus_{p\geq 1} \overline{\mathrm{Env}}_{\mathfrak{g}}(p) \otimes_{\Sigma_p} V^{\otimes p} \end{aligned}$$

Here $\circ$ denotes the usual composition product of symmetric sequences. To simplify the above formulas, let us make the following definition:

Definition 3.2.8. For any (weakly filtered) L_∞-algebroid $\mathfrak{g}$, let $\mathrm{Env}_{\mathfrak{g}}$ be the symmetric sequence of (weakly filtered) chain complexes given by $\mathrm{Env}_{\mathfrak{g}}(0) = \mathfrak{g}$ and $\mathrm{Env}_{\mathfrak{g}}(p) = \overline{\mathrm{Env}}_{\mathfrak{g}}(p)$ for $p \geq 1$. This determines a functor

$$\mathrm{Env}\colon \mathrm{L}_\infty\mathrm{Algd}_A^{\mathrm{dg},\mathbb{N}} \longrightarrow \big(\mathrm{Mod}_A^{\mathrm{dg},\mathbb{N}}\big)^{\Sigma}$$

to the category of (weakly filtered) symmetric sequences of left A-modules.

Remark 3.2.9. The symmetric sequence $\mathrm{Env}_{\mathfrak{g}}$ has no natural operad structure.

Remark 3.2.10. Let $\mathfrak{g}$ be a weakly filtered L_∞-algebroid of weight ≤ 0, i.e. an ordinary L_∞-algebroid. Then $\mathrm{Env}_{\mathfrak{g}}$ is of weight ≤ 0 as well. Similarly, if $\mathfrak{g}$ is a graded L_∞-algebroid, then $\mathrm{Env}_{\mathfrak{g}}$ is a symmetric sequence of graded complexes. In other words, there is a commuting diagram

$$\begin{tikzcd} \mathrm{L}_\infty\mathrm{Algd}_A^{\mathrm{dg}} \arrow[r] \arrow[d, "\mathrm{Env}_{(-)}"'] & \mathrm{L}_\infty\mathrm{Algd}_A^{\mathbb{N},\mathrm{dg}} \arrow[d, "\mathrm{Env}_{(-)}"'] & \mathrm{L}_\infty\mathrm{Algd}_A^{\mathrm{gr},\mathrm{dg}} \arrow[l] \arrow[d, "\mathrm{Env}_{(-)}"] \\ \mathrm{Mod}_A^{\Sigma,\mathrm{dg}} \arrow[r] & \mathrm{Mod}_A^{\Sigma,\mathbb{N},\mathrm{dg}} & \mathrm{Mod}_A^{\Sigma,\mathrm{gr},\mathrm{dg}} \arrow[l] \end{tikzcd}$$

where the horizontal functors are the obvious inclusions.

Lemma 3.2.11. *The functor* $\mathrm{Env}\colon \mathrm{L}_\infty\mathrm{Algd}_A^{\mathbb{N},\mathrm{dg}} \longrightarrow \mathrm{Mod}_A^{\Sigma,\mathbb{N},\mathrm{dg}}$ *has the following properties:*

(1) *It preserves all filtered colimits and reflexive coequalizers.*
(2) *Suppose that* $\mathfrak{g} \longrightarrow T_A$ *is a map of weakly filtered* $\mathbb{R}$*-linear* L_∞*-algebras and let* $A\otimes\mathfrak{g} \longrightarrow T_A$ *be the associated action* L_∞*-algebroid. Then there is a natural isomorphism of symmetric* A*-bimodules*

$$\mathrm{Env}_{A\otimes\mathfrak{g}} \cong A \otimes (L_\infty)_{\mathfrak{g}}$$

where $(L_\infty)_{\mathfrak{g}}$ *denotes the enveloping operad of the* L_∞*-algebra* $\mathfrak{g}$ *(see e.g. [10]).*
(3) *If* $\mathfrak{g}$ *is a weakly filtered* L_∞*-algebroid and* V *is a weakly filtered chain complex, then there is an isomorphism of symmetric* A*-modules*

$$\mathrm{Env}_{\mathfrak{g}\coprod F\left(V\xrightarrow{0}T_A\right)}(p) \cong \mathrm{Env}_{\mathfrak{g}}((-)+p)\circ V.$$

(4) The functor $\mathrm{Env}_{(-)}$ *commutes with taking the colimit and associated graded of a weakly filtered* L_∞*-algebroid. In other words, there is a commuting diagram*

$$\begin{tikzcd}
\mathrm{L}_\infty\mathrm{Algd}_A^{\mathrm{dg}} \arrow[d, "\mathrm{Env}_{(-)}"'] & \mathrm{L}_\infty\mathrm{Algd}_A^{\mathbb{N},\mathrm{dg}} \arrow[l, "\mathrm{colim}"'] \arrow[r, "\mathrm{gr}"] \arrow[d, "\mathrm{Env}_{(-)}"'] & \mathrm{L}_\infty\mathrm{Algd}_A^{\mathrm{gr},\mathrm{dg}} \arrow[d, "\mathrm{Env}_{(-)}"] \\
\mathrm{Mod}_A^{\Sigma,\mathrm{dg}} & \mathrm{Mod}_A^{\Sigma,\mathbb{N},\mathrm{dg}} \arrow[l, "\mathrm{colim}"] \arrow[r, "\mathrm{gr}"'] & \mathrm{Mod}_A^{\Sigma,\mathrm{gr},\mathrm{dg}}.
\end{tikzcd}$$

Proof. Since filtered colimits and reflexive coequalizers of L_∞-algebroids are computed at the level of the underlying complexes, part (1) follows either from the explicit description of $\mathrm{Env}_{\mathfrak{g}}$ in terms of generators and relations (Remark 3.2.5) or from the fact that for any such diagram $\mathfrak{g}_\bullet$, there is an isomorphism

$$\begin{aligned}
\mathrm{Env}_{\mathrm{colim}(\mathfrak{g}_\bullet)} \circ V &\cong \big(\mathrm{colim}\,\mathfrak{g}_\bullet\big) \amalg F\big(V \xrightarrow{0} T_A\big) \\
&\cong \mathrm{colim}\Big(\mathfrak{g}_\bullet \amalg F\big(V \xrightarrow{0} T_A\big)\Big) \cong \mathrm{colim}\,\big(\mathrm{Env}_{\mathfrak{g}_\bullet}\big) \circ V.
\end{aligned}$$

For (2), consider an action L_∞-algebroid $A \otimes \mathfrak{g}$ and let V be a chain complex. The free L_∞-algebroid on $0\colon V \longrightarrow T_A$ is the A-linear extension of the free L_∞-algebra $L_\infty(V)$ generated by V. It follows that

$$\big(A \otimes \mathfrak{g}\big) \amalg F\big(V \xrightarrow{0} T_A\big) \cong A \otimes \Big(\mathfrak{g} \cup L_\infty(V)\Big)$$

is the A-linear extension of the coproduct of L_∞-algebras $\mathfrak{g} \cup L_\infty(V)$. Using the description of this coproduct of L_∞-algebras in terms of the enveloping operad $(L_\infty)_{\mathfrak{g}}$, one obtains a natural isomorphism

$$\mathrm{Env}_{A\otimes\mathfrak{g}} \circ V \cong \big(A \otimes (L_\infty)_{\mathfrak{g}}\big) \circ V.$$

This induces an isomorphism of symmetric sequences $\mathrm{Env}_{A\otimes\mathfrak{g}} \cong A \otimes (L_\infty)_{\mathfrak{g}}$.

For (3), observe that for any chain complex W, there are natural isomorphisms

$$\begin{aligned}
\mathrm{Env}_{\mathfrak{g}\amalg F(V \xrightarrow{0} T_A)} \circ W &\cong \mathfrak{g} \amalg F(V \xrightarrow{0} T_A) \amalg F(W \xrightarrow{0} T_A) \\
&\cong \mathfrak{g} \amalg F(V \oplus W \xrightarrow{0} T_A) \\
&\cong \mathrm{Env}_{\mathfrak{g}} \circ (V \oplus W) \\
&\cong \bigoplus_{p\geq 0} \left(\bigoplus_{q\geq 0} \mathrm{Env}_{\mathfrak{g}}(q+p) \otimes_{\Sigma_q} V^{\otimes q}\right) \otimes_{\Sigma_p} W^{\otimes p}.
\end{aligned}$$

This induces the desired isomorphism of symmetric sequences.

For (4), recall that Env commutes with the inclusions of objects of weight ≤ 0 (resp. graded objects) into weakly filtered objects (Remark 3.2.10). This implies that there is a natural transformation

$$\nu\colon\ \mathrm{colim} \circ \mathrm{Env} \longrightarrow \mathrm{Env} \circ \mathrm{colim}$$

and similarly for the functor taking the associated graded. Any weakly filtered L_∞-algebroid can be obtained as a reflexive coequalizer of free L_∞-algebroids generated by weakly filtered chain complexes over T_A. Since the functors colim and Env preserve reflexive coequalizers, it suffices to check that ν induces an isomorphism for such a free L_∞-algebroid.

For a free L_∞-algebroid $F(\rho\colon V \longrightarrow T_A)$, we can use part (2) and the description of the enveloping operad of a free L_∞-algebra to see that the map

$$\nu\colon\ \mathrm{colim}\Big(\mathrm{Env}_{F(V)}(p)\Big) \longrightarrow \mathrm{Env}_{\mathrm{colim}(F(V))}(p)$$

is the A-linear extension of the map

$$\operatorname{colim}\Big(\bigoplus_{q\geq 0} L_\infty(p+-)\otimes_{\Sigma_q} V^{\otimes q}\Big) \longrightarrow \bigoplus_{q\geq 0} L_\infty(p+q)\otimes_{\Sigma_q}\big(\operatorname{colim} V\big)^{\otimes q}.$$

This map is an isomorphism for any $\mathbb{N}$-diagram V. The same argument applies to the functor taking the associated graded. $\square$

3.2.3 Pushouts along free maps. Let us now consider more general pushout diagrams of L_∞-algebroids of the form

$$\begin{tikzcd} F(W \longrightarrow T_A) \arrow[r, "F(i)"] \arrow[d] & F(V \longrightarrow T_A) \arrow[d] \\ \mathfrak{g} \arrow[r] & \mathfrak{g}\coprod_{F(W)} F(V). \end{tikzcd} \tag{3.2.12}$$

Here $i\colon V \longrightarrow W$ is any monomorphism of chain complexes over T_A; the maps $V \longrightarrow T_A$ and $W \longrightarrow T_A$ need not be zero.

We can realize Diagram (3.2.12) as the colimit of a pushout diagram of *weakly filtered* L_∞-algebroids. More precisely, we endow the objects appearing in the above square with the following filtrations:

- give $\mathfrak{g}$ and W the filtration where everything has weight ≤ 0, i.e. take the constant $\mathbb{N}$-diagrams on $\mathfrak{g}$ and W.
- let $\widetilde{V}$ be the filtered complex

  $$W \xrightarrow{f} V \xrightarrow{=} V \longrightarrow V \longrightarrow \cdots .$$

  together with the obvious map to T_A (which has weight ≤ 0).

Diagram (3.2.12) is the colimit over $\mathbb{N}$ of the pushout square of weakly filtered L_∞-algebroids

$$\begin{tikzcd} F(W) \arrow[r, "F(f)"] \arrow[d] & F(\widetilde{V}) \arrow[d] \\ \mathfrak{g} \arrow[r] & \mathfrak{g}\coprod_{F(W)} F(\widetilde{V}) =: \mathfrak{h}. \end{tikzcd} \tag{3.2.13}$$

Indeed, the colimit of $\widetilde{V}$ is simply V and taking colimits over $\mathbb{N}$ commutes with all colimits and free functors. On the other hand, the associated graded of (3.2.13) is given by

$$\operatorname{gr}\Big(\mathfrak{g}\coprod_{F(W)} F(\widetilde{V})\Big) \cong \mathfrak{g}\coprod_{F(W)} F\big(\operatorname{gr}(\widetilde{V})\big) \cong \mathfrak{g}\amalg F\big(V/W\big). \tag{3.2.14}$$

The last isomorphism uses that the associated graded of $\widetilde{V}$ is $W\oplus V/W$, with W of weight 0 and V/W of weight 1. In particular, the map $V/W \longrightarrow T_A$ is the *zero map*, since T_A has weight 0.

Proposition 3.2.15. *Let $i\colon W \longrightarrow V$ be a monomorphism of chain complexes over T_A and suppose that $\mathfrak{g}$ is a cofibrant L_∞-algebroid, i.e. the map $0 \longrightarrow \mathfrak{g}$ is contained in the weakly saturated class generated by the maps $F(j)$, where j is a monomorphism of chain complexes.*

Then the weakly filtered L_∞-algebroid $\mathfrak{h} := \mathfrak{g}\coprod_{F(W)} F(\widetilde{V})$ from (3.2.13) has the following two properties:

(1) the symmetric sequence of weakly filtered dg-A-modules $\mathrm{Env}_{\mathfrak{h}}$ *is* filtered*: in other words, for each p, there is a sequence of* injections

$$\mathrm{Env}_{\mathfrak{h}}(p)^{(0)} \longrightarrow \mathrm{Env}_{\mathfrak{h}}(p)^{(1)} \longrightarrow \mathrm{Env}_{\mathfrak{h}}(p)^{(2)} \longrightarrow \cdots .$$

(2) the filtration on $\mathrm{Env}_{\mathfrak{h}}$ *has associated graded*

$$\mathrm{gr}\big(\mathrm{Env}_{\mathfrak{h}}(p)\big) \cong \mathrm{Env}_{\mathfrak{g}}((-)+p)\circ(W/V)$$

where W/V *has weight* 1*.*

Proof. Let us start by verifying part (2): recall from Lemma 3.2.11 that Env commutes with taking the associated graded. We then find that

$$\mathrm{gr}(\mathrm{Env}_{\mathfrak{h}}(p)) \cong \mathrm{Env}_{\mathfrak{g}\coprod F(V/W)}(p) \cong \mathrm{Env}_{\mathfrak{g}}((-)+p)\circ(W/V)$$

using the isomorphism (3.2.14) and Lemma 3.2.11.

To verify assertion (1), we can forget about all differentials. Since $\mathfrak{g}$ is cofibrant, it is the retract of an L_∞-algebroid which is freely generated (without differentials) by a certain map $M \longrightarrow T_A$ of graded vector spaces. Similarly, without differentials we can split $\widetilde{V}$ as a direct sum

$$\widetilde{V} \cong W \oplus W^{\perp} \longrightarrow T_A.$$

Here $W^{\perp}$ has weight 1 and comes equipped with a nontrivial map to T_A. The map $F(W) \longrightarrow F(\widetilde{V})$ can then be identified with the inclusion

$$F(W) \longrightarrow F(W) \amalg F(W^{\perp})$$

into the coproduct with the free L_∞-algebroid on the filtered $\mathbb{R}$-module $W^{\perp}$. It follows that the weakly filtered L_∞-algebroid $\mathfrak{h}$ is a retract of the free L_∞-algebroid (without differentials)

$$F(M) \coprod_{F(W)} F(\widetilde{V}) \cong F(M \oplus W^{\perp} \longrightarrow T_A).$$

Here M has weight 0 and $W^{\perp}$ has weight 1. By Lemma 3.2.11, each $\mathrm{Env}_{\mathfrak{h}}(p)$ is (without differentials) the retract of

$$\mathrm{Env}_{\mathrm{F}(M\oplus W^{\perp})}(p) \cong \bigoplus_{q\geq 0} A \otimes L_\infty(p+q) \otimes_{\Sigma_q} (M \oplus W^{\perp})^{\otimes q}.$$

Since $M \oplus W^{\perp}$ is filtered, the above symmetric sequence is filtered and the retract $\mathrm{Env}_{\mathfrak{h}}$ is filtered as well. ☐

3.2.4 Proof of Theorem 3.1.10. We have to prove that the free-forgetful adjunction

$$F\colon \mathrm{Mod}^{\mathrm{dg}}_{\mathbb{R}}/T_A \rightleftarrows \mathrm{L}_\infty\mathrm{Algd}^{\mathrm{dg}}_A : U.$$

satisfies the conditions of Lemma 2.1.19, guaranteeing the existence of a transferred semi-model structure on L_∞-algebroids. Since the forgetful functor U preserves filtered colimits, it suffices to show that for any cofibrant L_∞-algebroid $\mathfrak{g}$ and any generating cofibration $0 \longrightarrow \mathbb{R}[n, n+1]$ in $\mathrm{Mod}^{\mathrm{dg}}_{\mathbb{R}}/T_A$, the map

$$\mathfrak{g} \longrightarrow \mathfrak{g} \amalg F(\mathbb{R}[n, n+1] \longrightarrow T_A)$$

is a trivial cofibration of chain complexes.

By Proposition 3.2.15, this map is the inclusion of the weight 0 part of a filtration on $F(\mathbb{R}[n, n+1])$ whose associated graded is

$$\mathrm{Env}_{\mathfrak{g}} \circ \mathbb{R}[n, n+1].$$

Since $\mathbb{R}[n, n+1]$ is acyclic, the associated graded is acyclic as well, so that $\mathfrak{g} \longrightarrow \mathfrak{g} \amalg F(\mathbb{R}[n, n+1])$ is a trivial cofibration of chain complexes.

Remark 3.2.16. The same argument shows that the map of symmetric sequences $\mathrm{Env}_{\mathfrak{g}} \longrightarrow \mathrm{Env}_{\mathfrak{g} \coprod F(\mathbb{R}[n,n+1])}$ is a trivial cofibration in each degree. This implies that the functor Env preserves weak equivalences between cofibrant L_∞-algebroids.

3.2.5 Proof of Theorem 3.1.15.

We have to prove that the forgetful functor

$$U : \mathrm{L}_\infty\mathrm{Algd}_A^{\mathrm{dg}} \longrightarrow \mathrm{Mod}_A^{\mathrm{dg}}/T_A$$

preserves cofibrant objects and sifted homotopy colimits. Our proof will follow the lines of e.g. [62, Lemma 4.5.4.12]:

Definition 3.2.17. Let $\mathcal{M}$ be a (semi-) model category and let $X : \mathcal{J} \longrightarrow \mathcal{M}$ be a diagram. We will say that X is *good* if it satisfies the following conditions:

- each object $X(j)$ is cofibrant.
- the colimit $\mathrm{colim}\, X$ is cofibrant.
- the map $\mathrm{hocolim}\, X \longrightarrow \mathrm{colim}\, X$ is a weak equivalence.

More generally, we say that a map of $\mathcal{J}$-diagrams $X \longrightarrow Y$ in $\mathcal{M}$ is good if

(i) X and Y are both good diagrams.
(ii) each $X(j) \longrightarrow Y(j)$ is a cofibration in $\mathcal{M}$.
(iii) $\mathrm{colim}\, X \longrightarrow \mathrm{colim}\, Y$ is a cofibration in $\mathcal{M}$.

With these definitions, observe that Theorem 3.1.15 follows from the following assertion by taking $p = 0$:

Theorem 3.2.18. *Let $\mathcal{J}$ be a homotopy sifted category. If $\mathfrak{g} : \mathcal{J} \longrightarrow \mathrm{L}_\infty\mathrm{Algd}_A^{\mathrm{dg}}$ is a projectively cofibrant diagram, then each diagram $\mathrm{Env}_{\mathfrak{g}}(p) : \mathcal{J} \longrightarrow \mathrm{Mod}_A^{\mathrm{dg}}$ is a good diagram of dg-A-modules.*

We will prove Theorem 3.2.18 by 'induction on cells', using the following simple stability properties of good maps:

Lemma 3.2.19. *Let $\mathcal{J}$ be a homotopy sifted category. We have the following properties of good maps between $\mathcal{J}$-indexed diagrams in $\mathcal{M}$:*

(1) every projectively cofibrant diagram $X : \mathcal{J} \longrightarrow \mathcal{M}$ is good and every projective cofibration between projectively cofibrant diagrams in $\mathcal{M}$ is good.

(2) good morphisms are closed under transfinite composition and retracts.

(3) given a pushout diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X' \arrow[r, "f'"'] & Y' \end{tikzcd}$$

in which f is good and X' is good, we have that f' is good.

When $\mathcal{M} = \mathrm{Mod}_A^{\mathrm{dg}}$ is the model category of dg-A-modules, we furthermore have:

(4) if X and Y are good, then $X \otimes_A Y$ is good.

(5) if X is a good diagram with a Σ_n-action, then X/Σ_n is good.

(6) let $f\colon X \longrightarrow Y$ be a natural monomorphism with a good domain and a good cokernel, such that the map $\operatorname{colim} X \longrightarrow \operatorname{colim} Y$ is a monomorphism. Then f is good.

Proof. The first three properties are easily verified. For (4), it is clear that each $X \otimes_A Y(j)$ is cofibrant. Since $\mathcal{J}$ is sifted and $\otimes_A$ is a Quillen bifunctor, there is a commuting diagram of (homotopy) colimits

$$\begin{tikzcd}
\operatorname*{hocolim}_n X_n \otimes Y_n \arrow[r, "\sim"] \arrow[d] & \operatorname*{hocolim}_{i,j} X_i \otimes Y_j \arrow[r, "\sim"] \arrow[d] & \big(\operatorname*{hocolim}_i X_i\big) \otimes \big(\operatorname*{hocolim}_j Y_j\big) \arrow[d, "\sim"] \\
\operatorname*{colim}_n X_n \otimes Y_n \arrow[r, "\cong"'] & \operatorname*{colim}_{i,j} X_i \otimes Y_j \arrow[r, "\cong"'] & \big(\operatorname*{colim}_i X_i\big) \otimes \big(\operatorname*{colim}_j Y_j\big)
\end{tikzcd}$$

It follows that $X \otimes_A Y$ is good as well.

Assertion (5) follows from the fact that we are working in characteristic zero, so that $\operatorname{colim}\colon \mathrm{Mod}_A^{\mathrm{dg},\Sigma_n} \longrightarrow \mathrm{Mod}_A^{\mathrm{dg}}$ is left Quillen for the *injective* model structure.

Finally, for (6) we use that a map of dg-A-modules is a cofibration if and only if it is a monomorphism with cofibrant cokernel. This implies that the maps $X(i) \longrightarrow Y(i)$ and $\operatorname{colim} X \longrightarrow \operatorname{colim} Y$ are cofibrations, so that all $Y(i)$ and $\operatorname{colim} Y$ are cofibrant. Furthermore, we obtain a commuting diagram

$$\begin{tikzcd}
\operatorname{hocolim} X \arrow[r] \arrow[d, "\sim"'] & \operatorname{hocolim} Y \arrow[r] \arrow[d] & \operatorname{hocolim} Y/X \arrow[d, "\sim"] \\
\operatorname{colim} X \arrow[r] & \operatorname{colim} Y \arrow[r] & \operatorname{colim} Y/X
\end{tikzcd}$$

Both horizontal sequences are cofiber sequences of dg-A-modules, so the map $\operatorname{hocolim} Y \longrightarrow \operatorname{colim} Y$ is a weak equivalence. ☐

Lemma 3.2.20. *Let $\mathcal{J}$ be a homotopy sifted category and consider a pushout square*

$$\begin{tikzcd}
F(W) \arrow[r, "\mathrm{Free}(i)"] \arrow[d] & F(V) \arrow[d] \\
\mathfrak{g} \arrow[r] & \mathfrak{h}
\end{tikzcd}$$

where i is a projective cofibration of $\mathcal{J}$-diagrams of chain complexes and $\mathfrak{g}$ is a projectively cofibrant diagram of L_∞-algebroids. If each $\mathrm{Env}_{\mathfrak{g}}(p)$ is good, then each $\mathrm{Env}_{\mathfrak{g}}(p) \longrightarrow \mathrm{Env}_{\mathfrak{h}}(p)$ is good.

Proof. Since each L_∞-algebroid $\mathfrak{g}(j)$ is cofibrant, Proposition 3.2.15 provides a natural filtration

$$\mathrm{Env}_{\mathfrak{g}}(p) = \mathrm{Env}_{\mathfrak{h}}(p)^{(0)} \longrightarrow \mathrm{Env}_{\mathfrak{h}}(p)^{(1)} \longrightarrow \cdots \longrightarrow \mathrm{Env}_{\mathfrak{h}}(p)$$

on $\mathrm{Env}_{\mathfrak{h}}(p)$. There is a similar filtration on the colimit, because $\operatorname{colim}(\mathfrak{g})$ is a cofibrant L_∞-algebroid as well. By parts (2) and (6) of Lemma 3.2.19, it then suffices to verify that the associated graded

$$\bigoplus_{q\geq 0} \mathrm{Env}_{\mathfrak{g}}(p+q) \otimes_{\Sigma_q} (V/W)^{\otimes q}$$

consists of good $\mathcal{J}$-diagrams of dg-A-modules. This follows from parts (4) and (5) of Lemma 3.2.19. □

Proof (of Theorem 3.2.18). Let $\mathcal{K}$ be the subcategory of $\mathfrak{g}\colon \mathcal{J} \longrightarrow \mathrm{L}_\infty\mathrm{Algd}_A^{\mathrm{dg}}$ for which each diagram of dg-A-modules $\mathrm{Env}_\mathfrak{g}(p)$ is good. By part (2) of Lemma 3.2.19, $\mathcal{K}$ is closed under retracts and contains the colimit of a transfinite sequence $\mathfrak{g}_\bullet$ for which each $\mathrm{Env}_{\mathfrak{g}_\alpha}(p) \longrightarrow \mathrm{Env}_{\mathfrak{g}_\beta}(p)$ is good. It therefore suffices to show that $\mathcal{K}$ is closed under pushouts along generating cofibrations, which is Lemma 3.2.20. □

3.3 Representations

In this section we will recall the basic theory of (left) representations of dg-Lie algebroids.

Definition 3.3.1. Let $\mathfrak{g}$ be a dg-Lie algebroid over A. A $\mathfrak{g}$-*representation* is a dg-A-module E, together with a Lie algebra representation

$$\nabla\colon \mathfrak{g} \otimes_\mathbb{R} E \longrightarrow E$$

such that $\nabla_{aX}s = a\nabla_X s$ and $\nabla_X(as) = X(a)s + (-1)^{aX}a\nabla_X s$ for all $a \in A$, $X \in \mathfrak{g}$, and $s \in E$. Let $\mathrm{Rep}_\mathfrak{g}^{\mathrm{dg}}$ be the category of $\mathfrak{g}$-representations, whose maps are maps of chain complexes that preserve the actions of A and $\mathfrak{g}$.

Example 3.3.2. Every dg-Lie algebroid has a natural representation on A (via the anchor) and on the kernel of its anchor map (via the Lie bracket).

Example 3.3.3. The category $\mathrm{Rep}_\mathfrak{g}^{\mathrm{dg}}$ has a closed symmetric monoidal structure, given by $E \otimes_A F$ endowed with the $\mathfrak{g}$-representation

$$\nabla_X(e \otimes f) = \nabla_X(e) \otimes f + (-1)^{Xe} e \otimes \nabla_X(f).$$

The internal hom is given by the mapping complex $\mathrm{Hom}_A(E, F)$, equipped with the conjugate representation of $\mathfrak{g}$.

There are at least two other equivalent ways of describing a $\mathfrak{g}$-representation on a dg-A-module E:

(1) Let $\mathrm{At}(E)$ be the Atiyah Lie algebroid of E, as in Example 3.1.2. Then a representation of $\mathfrak{g}$ is just a map of dg-Lie algebroids

$$\mathfrak{g} \longrightarrow \mathrm{At}(E).$$

(2) If E is a $\mathfrak{g}$-representation, then $\mathfrak{g} \oplus E$ has the structure of a dg-Lie algebroid, with anchor map $(\rho, 0)\colon \mathfrak{g} \oplus E \longrightarrow T_A$ and bracket

$$[(X, s), (Y, t)] = ([X, Y], \nabla_X t - \nabla_Y s).$$

There is an obvious inclusion and retraction $\mathfrak{g} \longrightarrow \mathfrak{g} \oplus E \longrightarrow \mathfrak{g}$. Using these maps, one can realize the category of $\mathfrak{g}$-representations as the full subcategory of $\mathfrak{g}/\mathrm{LieAlgd}_A^{\mathrm{dg}}/\mathfrak{g}$ on those retract diagrams $\mathfrak{g} \longrightarrow \mathfrak{g} \oplus \mathfrak{m} \longrightarrow \mathfrak{g}$ for which the Lie bracket vanishes on $\mathfrak{m} \otimes \mathfrak{m}$.

Remark 3.3.4. Recall that the category $\mathfrak{g}/\mathrm{LieAlgd}_A^{\mathrm{dg}}/\mathfrak{g}$ can be identified with the category of algebras over the reduced enveloping operad $\overline{\mathrm{Env}}_\mathfrak{g}$ of $\mathfrak{g}$ (Definition 3.2.4). It is well-known (see e.g. [10]) that the following categories are equivalent:

- the category of *abelian group objects* in $\mathrm{LieAlgd}_A^{\mathrm{dg}}/\mathfrak{g}$.

- the full subcategory of $\mathfrak{g}/\mathrm{LieAlgd}_A^{\mathrm{dg}}/\mathfrak{g}$ on the $\mathfrak{g}\oplus\mathfrak{m}$ such that the Lie bracket vanishes on $\mathfrak{m}\otimes\mathfrak{m}$.
- the category of left modules over $\overline{\mathrm{Env}}_{\mathfrak{g}}(1)$, the dg-algebra of unary operations in the reduced enveloping operad (the analogue of Definition 3.2.4 for dg-Lie algebroids).

By Remark 3.3.4, the category of $\mathfrak{g}$-representations is equivalent to the category of (left) modules over the associative dg-algebra $\overline{\mathrm{Env}}_{\mathfrak{g}}(1)$. Unwinding the definitions, this associative dg-algebra can be identified with the *enveloping algebra* $\mathcal{U}(\mathfrak{g})$ of the dg-Lie algebroid $\mathfrak{g}$, as described originally in [82]. Recall that $\mathcal{U}(\mathfrak{g})$ is the initial associative dg-algebra over $\mathbb{R}$ equipped with the data of

- a map of dg-algebras $\kappa_A\colon A\longrightarrow\mathcal{B}$
- a map of dg-Lie algebras $\kappa_{\mathfrak{g}}\colon\mathfrak{g}\longrightarrow\mathcal{B}$

such that $\kappa_A(a)\kappa_{\mathfrak{g}}(X)=\kappa_{\mathfrak{g}}(aX)$ and $\kappa_{\mathfrak{g}}(X)\kappa_A(a)=\kappa_A(a)\kappa_{\mathfrak{g}}(X)+\kappa_A(\rho(X)a)$ without Koszul signs. Explicitly, $\mathcal{U}(\mathfrak{g})$ is generated freely by elements from A and $\mathfrak{g}$, subject to conditions as in Remark 3.2.5.

Remark 3.3.5. Geometrically, $\mathcal{U}(\mathfrak{g})$ can be thought of as the algebra of differential operators generated by $\mathfrak{g}$ (where elements in A are differential operators of degree zero and elements in $\mathfrak{g}$ are differential operators of degree 1).

Example 3.3.6. When $\mathfrak{g}$ is a Lie algebra over A, the map $A\longrightarrow\mathcal{U}(\mathfrak{g})$ realizes $\mathcal{U}(\mathfrak{g})$ as an algebra in $\mathrm{Mod}_A^{\mathrm{dg}}$, which is simply the enveloping algebra of the Lie algebra $\mathfrak{g}$ over A.

Recall that the behaviour of the enveloping algebra is described concisely by the Poincaré-Birkhoff-Witt theorem (as described in [82]). More precisely, the map of algebras $A\longrightarrow\mathcal{U}(\mathfrak{g})$ realizes $\mathcal{U}(\mathfrak{g})$ as a bimodule over A. There is a natural filtration of A-bimodules

$$\mathcal{U}^{(0)}(\mathfrak{g})\subseteq\mathcal{U}^{(1)}(\mathfrak{g})\subseteq\cdots\subseteq\mathcal{U}(\mathfrak{g})$$

on $\mathcal{U}(\mathfrak{g})$ by declaring generators from A to be of filtration weight 0 and generators from $\mathfrak{g}$ to be of weight 1. Each $\mathcal{U}^{(n)}(\mathfrak{g})$ is the left dg-A-submodule of $\mathcal{U}(\mathfrak{g})$ generated by the unit and all products of $\leq n$ elements in $\mathfrak{g}$. In particular, $\mathcal{U}^{(0)}(\mathfrak{g})=A$ and the associated graded

$$\mathrm{gr}\big(\mathcal{U}(\mathfrak{g})\big)=\bigoplus_n\mathcal{U}^{(n)}(\mathfrak{g})/\mathcal{U}^{(n-1)}(\mathfrak{g})$$

has the structure of a dg-bimodule over A. For all $u\in\mathcal{U}^{(n)}(\mathfrak{g})$ and $a\in A$, we have that $au-(-1)^{ua}ua\in\mathcal{U}^{(n-1)}(\mathfrak{g})$, so that the left and right A-module structure on $\mathrm{gr}\big(\mathcal{U}(\mathfrak{g})\big)$ agree. In fact, $\mathrm{gr}\big(\mathcal{U}(\mathfrak{g})\big)$ has the natural structure of a (graded) dg-algebra over A, which is commutative because for all $u\in\mathcal{U}^{(n)}(\mathfrak{g})$ and $v\in\mathcal{U}^{(m)}(\mathfrak{g})$, the graded commutator lies in $\mathcal{U}^{(n+m-1)}(\mathfrak{g})$.

The canonical map of dg-A-modules $\mathfrak{g}\longrightarrow\mathcal{U}(\mathfrak{g})$ therefore induces a map of (graded) commutative dg-algebras over A

$$\mathrm{Sym}_A(\mathfrak{g})\longrightarrow\mathrm{gr}\big(\mathcal{U}(\mathfrak{g})\big).$$

This map is always surjective, since every object in $\mathcal{U}(\mathfrak{g})$ can be written as a finite sum of products of elements in A and $\mathfrak{g}$. The PBW-theorem asserts that it is injective for sufficiently nice dg-Lie algebroids $\mathfrak{g}$:

Proposition 3.3.7 (PBW, [82])**.** *Let A be a non-negatively graded commutative dg-algebra and let $\mathfrak{g}$ be an A-cofibrant dg-Lie algebroid over A (Definition 3.1.27). Then the map* $\mathrm{Sym}_A(\mathfrak{g})\longrightarrow\mathrm{gr}(\mathcal{U}(\mathfrak{g}))$ *is an isomorphism.*

Proof. We can forget about the differential, in which case $\mathfrak{g}$ is a projective graded module over the graded-commutative algebra underlying A. The result now follows from (a graded analogue of) [82]. □

Corollary 3.3.8. *The category* $\mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}}$ *of representations of a dg-Lie algebroid* $\mathfrak{g}$ *carries the* projective *model structure, in which a map is a weak equivalence (fibration) if it is a quasi-isomorphism (surjection). Each map of dg-Lie algebroids* $f\colon \mathfrak{g} \longrightarrow \mathfrak{h}$ *induces a Quillen adjunction*

$$f_* = \mathcal{U}(\mathfrak{h}) \otimes_{\mathcal{U}(\mathfrak{g})} (-)\colon \mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}} \rightleftarrows \mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{h}} \colon f^!$$

whose right adjoint $f^!$ *restricts a representation of* $\mathfrak{h}$ *along* f*. When* f *is a weak equivalence between* A*-cofibrant dg-Lie algebroids, this is a Quillen equivalence.*

Proof. The category $\mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}}$ can be identified with the category of left dg-$\mathcal{U}(\mathfrak{g})$-modules, which clearly admits the projective model structure. Every map $f\colon \mathfrak{g} \longrightarrow \mathfrak{h}$ of dg-Lie algebroids induces a map $\mathcal{U}(f)\colon \mathcal{U}(\mathfrak{g}) \longrightarrow \mathcal{U}(\mathfrak{h})$ of enveloping algebras. The Quillen pair $(f_*, f^!)$ is then identified with the Quillen pair given by restriction and induction along $\mathcal{U}(f)$.

When $\mathfrak{g}$ and $\mathfrak{h}$ are A-cofibrant dg-Lie algebroids, their enveloping algebras carry the PBW-filtration, which is preserved by $\mathcal{U}(f)$. The associated map on the associated graded agrees with $\mathrm{Sym}_A(f)\colon \mathrm{Sym}_A(\mathfrak{g}) \longrightarrow \mathrm{Sym}_A(\mathfrak{h})$ by the PBW theorem. This map is a quasi-isomorphism if f is a quasi-isomorphism, so that $\mathcal{U}(f)$ is a quasi-isomorphism as well. This implies that the Quillen pair $(f_*, f^!)$ is a Quillen equivalence. □

Variant 3.3.9. Let $\mathfrak{g}$ be an A-cofibrant dg-Lie algebroid. Then the category $\mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}}$ carries a second model structure, which we will refer to as the *A-model structure*, in which a map is a weak equivalence (cofibration) if and only if the underlying map of dg-A-modules is a weak equivalence (cofibration) in the projective model structure.

Indeed, the forgetful functor $\mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}} \longrightarrow \mathrm{Mod}^{\mathrm{dg}}_A$ is a left adjoint and the projective model structure on $\mathrm{Mod}^{\mathrm{dg}}_A$ transfers along this left adjoint once the following condition is satisfied (see [7]): if a map $p\colon F \longrightarrow E$ in $\mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}}$ has the right lifting property against all A-cofibrations, then its underlying map of dg-A-modules is a weak equivalence. Because $\mathfrak{g}$ is A-cofibrant, the generating cofibrations of the projective model structure

$$\mathcal{U}(\mathfrak{g})[n] \longrightarrow \mathcal{U}(\mathfrak{g})[n, n+1]$$

are all A-cofibrations by the PBW theorem 3.3.7. Any p with the right lifting property against the A-cofibrations is therefore a projective trivial fibration, so in particular a weak equivalence. It follows that the A-model structure on dg-representations exists and that the identity functor

$$\mathrm{id}\colon \left(\mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}}\right)_{\mathrm{proj}} \longrightarrow \left(\mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}}\right)_A$$

is a left Quillen equivalence from the projective to the A-model structure. For every map $f\colon \mathfrak{g} \longrightarrow \mathfrak{h}$ of A-cofibrant dg-Lie algebroids, the restriction functor $f^!$ is the *left adjoint* in a Quillen pair

$$f^!\colon \left(\mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}}\right)_A \rightleftarrows \left(\mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}}\right) \colon f_!$$

with right adjoint $f_!$ given the coinduction functor $\mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(\mathcal{U}(\mathfrak{h}), -)$.

Variant 3.3.10. Let $\mathfrak{g}$ be an A-cofibrant dg-Lie algebroid. By the same argument, the category $\mathrm{Rep}^{\geq 0,\mathrm{dg}}_{\mathfrak{g}}$ of connective $\mathfrak{g}$-representations carries a model structure, in which a map is a weak equivalence (cofibration) if the underlying map in $\mathrm{Mod}^{\geq 0,\mathrm{dg}}_A$ is a weak equivalence (cofibration) in the projective model structure. Any map $f\colon \mathfrak{g} \longrightarrow \mathfrak{h}$ between A-cofibrant dg-Lie algebroids induces a Quillen adjunction

$$f^!\colon \big(\mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}}\big)_A \rightleftarrows \big(\mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}}\big)\colon \tau_{\geq 0} f_!$$

where the right adjoint takes the connective cover of the coinduced representation. Unless $\mathfrak{g}$ is connective, there is no analogue of the projective model structure on connective dg-representations

Definition 3.3.11. For any A-cofibrant dg-Lie algebroid $\mathfrak{g}$, let

$$\mathrm{Rep}_{\mathfrak{g}} = \mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}}[W^{-1}]$$

be the (locally presentable) ∞-category of $\mathfrak{g}$-representations. If $\mathfrak{g}$ is not A-cofibrant, then one should take representations over some cofibrant replacement.

Lemma 3.3.12. *When $\mathfrak{g}$ is A-cofibrant, the closed symmetric monoidal structure on* $\mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}}$ *from Example 3.3.3 endows the ∞-category* $\mathrm{Rep}_{\mathfrak{g}}$ *with a closed symmetric monoidal structure.*

Proof. The tensor product from Example 3.3.3 makes $\mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}}$ a monoidal model category with respect to the A-model structure of Variant 3.3.9. The localization $\mathrm{Rep}_{\mathfrak{g}}$ is a closed symmetric monoidal ∞-category by [62, Example 4.1.7.6, Lemma 4.1.8.8]. □

Remark 3.3.13. When $\mathfrak{g}$ is A-cofibrant, the inclusion $\mathrm{Rep}^{\geq 0,\mathrm{dg}}_{\mathfrak{g}} \longrightarrow \mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}}$ is a symmetric monoidal left Quillen functor, which induces a fully faithful, symmetric monoidal functor of ∞-categories $\mathrm{Rep}^{\geq 0}_{\mathfrak{g}} \longrightarrow \mathrm{Rep}_{\mathfrak{g}}$.

Remark 3.3.14. Let $\mathcal{P}$ be an operad enriched over chain complexes. Then a $\mathcal{P}$-algebra in $\mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}}$ is simply given by an $\mathcal{P}$-algebra in $\mathrm{Mod}^{\mathrm{dg}}_A$ which carries a representation of $\mathfrak{g}$ such that each map $\nabla = [X_1, \dots, X_k, -]\colon B \longrightarrow B$ is a derivation for the $\mathcal{P}$-algebra structure on B, in the sense that

$$\begin{aligned}
\nabla(\phi(b_1, \dots, b_n)) &= \sum_{i=1}^{n} \phi(b_1, \dots, \nabla(b_i), \dots, b_n) && n \geq 1\\
\nabla(\phi(a)) &= 0 && n = 0, k \geq 2\\
\nabla(\phi(a)) &= X_1(a) \cdot \phi(1) && n = 0, k = 1
\end{aligned}$$

for all $b_i \in B$, $a \in A$ and $\phi \in \mathcal{P}$ (the last two equations describe the behaviour on nullary operations).

Remark 3.3.15. Similarly, let $A \longrightarrow B$ be a map of dg-$\mathcal{C}^\infty$-rings. We will say that $\mathfrak{g}$ *acts on B by $\mathcal{C}^\infty$-derivations* if

$$\nabla(\phi(b_1, \dots, b_n)) = \sum \frac{\partial \phi}{\partial x_i}(b_1, \dots, b_n) \cdot \nabla(b_i)$$

for every $\phi \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ and $\nabla = [X_1, \dots, X_k, -]\colon B \longrightarrow B$. For example, $\mathfrak{g}$ acts by $\mathcal{C}^\infty$-derivations on A itself via the anchor map.

Variant 3.3.16. Similar observations can be made about representations of L_∞-algebroids. More precisely, if E is a dg-A-module, then a *representation* of an L_∞-algebroid $\mathfrak{g}$ on E is one of the following equivalent pieces of data (see [71] for more details):

(1) an abelian group in the category $\mathfrak{g}/\mathrm{L}_\infty\mathrm{Algd}^{\mathrm{dg}}_A/\mathfrak{g}$, whose underlying dg-$A$-module is $\mathfrak{g} \oplus E$.

(2) an object of the form $A \oplus E$ in $\mathfrak{g}/\mathrm{L}_\infty\mathrm{Algd}_A^{\mathrm{dg}}/\mathfrak{g}$ which is square zero, i.e. all brackets vanish when evaluated on at least two element from E.

(3) the compatible structure of a left $\overline{\mathrm{Env}}_{\mathfrak{g}}(1)$-module on E, where $\overline{\mathrm{Env}}_{\mathfrak{g}}(1)$ is the associative dg-algebra of unary operations in the reduced enveloping operad of $\mathfrak{g}$ (Definition 3.2.4).

(4) a collection of operations $[X_1, \ldots, X_n, -]\colon E \longrightarrow E$ of degree $|X_1| + \ldots + |X_n| + n - 2$ for every $X_1, \ldots, X_n \in \mathfrak{g}$, such that (ignoring all Koszul signs due to permutations of variables)

$$\begin{aligned} [X_{\sigma(1)}, ..., X_{\sigma(n)}, v] &= (-1)^\sigma [X_1, ..., X_n, v] && \sigma \in \Sigma_n \\ [a \cdot X_1, ..., X_n, v] &= (-1)^{(n-1)a} a \cdot [X_1, ..., X_n, v] \\ [X_1, ..., X_n, a \cdot v] &= (-1)^{(n-1)a} a \cdot [X_1, ..., X_n, v] && n \geq 2 \\ [X_1, a \cdot v] &= a \cdot [X_1, v] + X_1(a) \cdot v \end{aligned}$$

and such that the brackets determine an the structure of a module over the L_∞-algebra $\mathfrak{g}$, i.e.

$$J^{n+1}(X_1, \ldots, X_n, v) = 0$$

for all $n \geq 0$, where J^{n+1} is the Jacobiator from 3.1.5, $X_i \in \mathfrak{g}$ and $v \in E$.

(5) a nonlinear map of L_∞-algebroids $\mathfrak{g} \rightsquigarrow \mathrm{At}(E)$.

The equivalence between (1), (2) and (3) follows as in Remark 3.3.4. Given an L_∞-algebroid $\mathfrak{g} \oplus E$ as in (2), the brackets of elements in $\mathfrak{g}$ with an element in E exactly determine the data of (4). The equivalence of (4) and (5) follows from the description of L_∞-algebra representations given in [43] (see [71] for more details).

Let $\mathfrak{g}$ be an L_∞-algebroid and let $\tilde{\mathfrak{g}}$ be the dg-Lie algebroid obtained by applying the left Quillen equivalence of Proposition 3.1.20 to its cobar resolution $Q(\mathfrak{g})$ (3.1.26). In light of (5), the category of representations of $\mathfrak{g}$ is equivalent to the category of representations of the dg-Lie algebroid $\tilde{\mathfrak{g}}$ in the sense of Definition 3.3.1.

In particular, the category of L_∞-algebroid representations of $\mathfrak{g}$ is symmetric monoidal, with tensor product given by $E \otimes_A F$, and can be endowed with the projective and A-model structure. A weak equivalence between A-cofibrant L_∞-algebroids induces a Quillen equivalence between the model categories of representations.

Remark 3.3.17. A dg-Lie algebroid $\mathfrak{g}$ now gives rise to two (model) categories of 'representations':

(1) the model category $\mathrm{Rep}_{\mathfrak{g}}^{\mathrm{dg}}$ from Definition 3.3.1.

(2) a model category $\mathrm{Rep}_{\mathfrak{g}}^{L_\infty}$ of L_∞-algebroid representations of $\mathfrak{g}$.

The objects of the latter category are exactly the representations up to homotopy from [3]. There is an obvious fully faithful inclusion

$$\mathrm{Rep}_{\mathfrak{g}}^{\mathrm{dg}} \longrightarrow \mathrm{Rep}_{\mathfrak{g}}^{L_\infty} \tag{3.3.18}$$

of the category of 'strict' representations of $\mathfrak{g}$ into the category of L_∞-algebroid representations of $\mathfrak{g}$. Unwinding the definitions (see [71]), one can identify this inclusion with the right Quillen functor that restricts a representation of $\mathfrak{g}$ along the canonical map $q\colon Q(\mathfrak{g}) \longrightarrow \mathfrak{g}$ from its cobar resolution (taken in the category of dg-Lie algebroids). When $\mathfrak{g}$ is A-cofibrant, the map q is a quasi-isomorphism (Lemma 3.1.30). Corollary 3.3.8 then implies that the inclusion (3.3.18) is the right adjoint of a Quillen equivalence.

Chapter 4

Formal moduli problems and Lie algebroids

The aim of this section is to relate the theory of Lie algebroids over a $\mathcal{C}^\infty$-ring A to formal deformation theory over A, in the sense of Section 2.3.3. More precisely, our aim will be to prove the following:

Theorem 4.2.1. *Let A be a $\mathcal{C}^\infty$-ring which is eventually coconnective, i.e. such that $\pi_n(A) = 0$ for $n \gg 0$. Then there is an equivalence of ∞-categories*

$$\mathrm{FMP}_A \simeq \mathrm{LieAlgd}_A$$

between the ∞-category of Lie algebroids over A (Definition 3.1.13) and the ∞-category of formal moduli problems under A (Definition 2.3.34).

Remark 4.0.1. The idea that Lie algebroids can equivalently be considered as formal moduli problems also appears in [33], where Lie algebroids are essentially defined to be formal moduli problems (indexed by a slightly larger category of small extensions). Theorem 4.2.1 can be viewed as a rectification result supporting this idea. A similar result was obtained independently by Calaque and Grivaux [17].

Remark 4.0.2. As we asserted in the introduction (see Theorem I), there is a similar equivalence when A is a connective commutative algebra over a field of characteristic zero. Our proof of Theorem 4.2.1 applies essentially verbatim in this case; we refer to [72] for details.

Geometrically, Theorem 4.2.1 asserts that the deformation theory of a map from $\mathrm{Spec}(A)$ to some moduli space is controlled by a Lie algebroid over A. When $\mathrm{Spec}(A)$ is a point, i.e. when $A = \mathbb{R}$, this reproduces the well-known relation between Lie algebras and deformation problems, originating in the work of Deligne and Drinfeld, developed by many others (among which Kontsevich, Soibelman, Hinich and Manetti) and culminating in the recent work of Pridham [75] and Lurie [61]. Our proof of the above result is follows the method described in [61] and its exposition in the work of Hennion [38], which discusses Lie algebras over a nontrivial base.

The core of the proof of Theorem 4.2.1 consists of an identification of the ∞-category of small extensions $A' \longrightarrow A$ with a certain category of Lie algebroids that are obtained from finitely many 'cell attachments'. This identification is treated in Section 4.1 and the proof of Theorem 4.2.1 is then completed in Section 4.2.

In Section 4.3 we consider the behaviour of representations of Lie algebroids under this equivalence. We show (Theorem 4.3.1) that connective representations of a Lie algebroid $\mathfrak{g}$ can equivalently be described as quasi-coherent modules on the associated formal moduli problem $\mathrm{MC}_{\mathfrak{g}}$. Finally, Section 4.4 considers the formal moduli problems arising from deformations of connective algebras (as in Example 2.3.35). We show that these formal moduli problems are classified by the explicit 'Atiyah Lie algebroids' from Example 3.1.3.

Assumption 4.0.3. Throughout, we assume that A is a cofibrant dg-$\mathcal{C}^\infty$-ring and only use the presentation of the ∞-category $\mathrm{LieAlgd}_A$ presented by the semi-model category $\mathrm{LieAlgd}_A^{\mathrm{dg}}$ of dg-Lie algebroids over A.

4.1 Lie algebroid cohomology

The aim of this section is to prove the following result:

Proposition 4.1.1. *Let A be a $\mathcal{C}^\infty$-ring A. Then there is an adjunction of ∞-categories*

$$\mathrm{c}^*\colon \mathrm{LieAlgd}_A \rightleftarrows \left(\mathcal{C}^\infty\mathrm{Alg}/A\right)^{\mathrm{op}} \colon \mathfrak{D}.$$

Here $\mathrm{c}^*(\mathfrak{g})$ is the connective cover of the Lie algebroid cohomology of $\mathfrak{g}$ and the anchor map of $\mathfrak{D}(B \longrightarrow A)$ is the A-linear dual of the map between cotangent complexes $L_A \longrightarrow L_{A/B}$.

Proposition 4.1.2. *Suppose that A is eventually coconnective. Then the unit map $\mathfrak{g} \longrightarrow \mathfrak{D}\mathrm{c}^*(\mathfrak{g})$ is an equivalence for every dg-Lie algebroid $\mathfrak{g}$ satisfying the following two conditions:*

(i) $\mathfrak{g}$ is cofibrant as a dg-A-module and is freely generated as a graded A-module by a set $\{x_i\}$.

(ii) There are finitely many x_i in each single degree, and no generators of (homological) degree ≥ 0.

As we will see (Remark 4.1.19), the functor c^* sends every Lie algebroid $\mathfrak{g}$ to the limit of a tower of nilpotent extensions of A. Proposition 4.1.2 asserts that this operation is not too far from being an equivalence: sufficiently finite-dimensional Lie algebroids can be recovered from the corresponding pro-nilpotent extensions of A.

In Section 4.1.1, we will begin our discussion of the above results by describing the functor c^* in more detail. In particular, we show that $\mathrm{c}^*(\mathfrak{g})$ essentially computes (the connective cover of) the dual of the *cotangent complex* of the Lie algebroid $\mathfrak{g}$, just like its right adjoint $\mathfrak{D}$ takes the dual of the $\mathcal{C}^\infty$-algebraic cotangent complex. We will use this in Section 4.1.2 to prove Proposition 4.1.1. Section 4.1.3 is devoted to a proof of Proposition 4.1.2.

4.1.1 The cotangent complex of a Lie algebroid. Recall from Section 3.3 that for each dg-Lie algebroid $\mathfrak{g}$, there is a right Quillen functor

$$\mathfrak{g} \oplus (-)\colon \mathrm{Rep}_{\mathfrak{g}}^{\mathrm{dg}} \longrightarrow \mathrm{LieAlgd}_A^{\mathrm{dg}}/\mathfrak{g}$$

taking the split square zero extension of $\mathfrak{g}$ by a (strict) $\mathfrak{g}$-representation. Using that that this right adjoint functor depends naturally on $\mathfrak{g}$, one sees that that its left adjoint is given by

$$\left(\mathfrak{h} \xrightarrow{f} \mathfrak{g}\right) \longmapsto f_*\Upsilon_{\mathfrak{h}} = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} \Upsilon_{\mathfrak{h}}. \tag{4.1.3}$$

Here $\Upsilon_{\mathfrak{h}}$ is the value of the left adjoint to $\mathfrak{h} \oplus (-)$ on $\mathfrak{h}$ itself.

Definition 4.1.4. Let $\mathfrak{g}$ be an A-cofibrant dg-Lie algebroid (Definition 3.1.27). The cotangent complex $L_{\mathfrak{g}}$ of $\mathfrak{g}$ is the value of the left derived functor of (4.1.3) on the identity map of $\mathfrak{g}$.

Example 4.1.5. Let $\rho\colon V \longrightarrow T_A$ be a map of dg-A-modules and let $\mathfrak{g} = F(V)$ be the free dg-Lie algebroid associated to it. If E is a $\mathfrak{g}$-representation, then maps of dg-Lie algebroids $\mathfrak{g} = F(V) \longrightarrow \mathfrak{g} \oplus E$ over $\mathfrak{g}$ are in natural bijection with maps of dg-A-modules $V \longrightarrow E$. It follows that $\Upsilon_{F(V)} = \mathcal{U}(\mathfrak{g}) \otimes_A V$ is the free $\mathfrak{g}$-representation generated by the dg-A-module V.

One can compute the cotangent complex $L_{\mathfrak{g}}$ using the explicit cofibrant replacement $q\colon Q(\mathfrak{g}) \longrightarrow \mathfrak{g}$ from Section 3.1.3. The datum of a map

$$(q, \alpha)\colon Q(\mathfrak{g}) \longrightarrow \mathfrak{g} \oplus E$$

is equivalent to the datum of a graded A-linear map

$$(q, \alpha)\colon \mathrm{Sym}_A^{\geq 1} \mathfrak{g}[1] \longrightarrow \mathfrak{g}[1] \oplus E[1]$$

satisfying the Maurer-Cartan equation (3.1.21), where q is the obvious projection onto $\mathfrak{g}[1]$. Since q already satisfies the Maurer-Cartan equation and there are no nontrivial brackets between elements in E, the Maurer-Cartan equation reduces to the following linear equation for α:

$$\partial_{\mathrm{CE}}\alpha := \partial_E \circ \alpha - \alpha \circ \partial_{\overline{C}_*(\mathfrak{g})} + [q, \alpha] = 0. \tag{4.1.6}$$

Remark 4.1.7. Formula (4.1.6) determines a graded $\mathbb{R}$-linear map

$$\partial_{\mathrm{CE}}\colon \mathrm{Hom}_A(\mathrm{Sym}_A^{\geq 1}\mathfrak{g}[1], E[1]) \longrightarrow \mathrm{Hom}_{\mathbb{R}}(\mathrm{Sym}_{\mathbb{R}}^{\geq 1}\mathfrak{g}[1], E[1])$$

of degree -1. To analyze this map ∂_{CE} a bit further, let us replace E by its cone $E[0,1]$. Unraveling the above definitions, one can see that a map $Q(\mathfrak{g}) \longrightarrow \mathfrak{g} \oplus E[0,1]$ over $\mathfrak{g}$ is determined uniquely by a pair of maps

$$\alpha\colon \mathrm{Sym}_A^{\geq 1}\mathfrak{g}[1] \longrightarrow E[1] \qquad \beta\colon \mathrm{Sym}_A^{\geq 1}\mathfrak{g}[1] \longrightarrow E[2]$$

subject to the condition that $\partial_{\mathrm{CE}}\alpha = \beta$ and $\partial_{\mathrm{CE}}\beta = 0$.

On the other hand, a map $Q(\mathfrak{g}) \longrightarrow \mathfrak{g} \oplus E[0,1]$ over $\mathfrak{g}$ is determined uniquely by a map $Q(\mathfrak{g}) \longrightarrow \mathfrak{g} \oplus E$ of dg-Lie algebroids without differential over $\mathfrak{g}$. But without the differential, $Q(\mathfrak{g})$ is just freely generated by the graded A-module $\big(\mathrm{Sym}_A^{\geq 1}\mathfrak{g}[1]\big)[-1]$. This means that every map $\alpha\colon \mathrm{Sym}_A^{\geq 1}\mathfrak{g}[1] \longrightarrow E[1]$ determines a map $Q(\mathfrak{g}) \longrightarrow \mathfrak{g} \oplus E$.

In other words, it follows that for any α there exists a map β satisfying the above two conditions. This means that the map ∂_{CE} preserves A-multilinear maps and squares to zero.

Definition 4.1.8. The *reduced Chevalley-Eilenberg complex* of $\mathfrak{g}$ with coefficients in E is the chain complex

$$\overline{C}^*(\mathfrak{g}, E) = \mathrm{Hom}_A(\mathrm{Sym}_A^{\geq 1}\mathfrak{g}[1], E)$$

equipped with the differential ∂_{CE} given by formula 4.1.6.

Remark 4.1.9. The differential ∂_{CE} can be computed explicitly (without Koszul signs due to the ordering of variables) as the usual Chevalley-Eilenberg differential

$$\begin{aligned}(\partial_{\mathrm{CE}}\alpha)(X_1, ..., X_n) = \partial_E\big(\alpha(X_1, ..., X_n)\big) + \sum_i \nabla_{X_i}\alpha(\dots, \hat{X}_i, \dots) \\ - \sum_i \alpha(\dots, \partial X_i, \dots) - \sum_{i<j} \alpha\Big([X_i, X_j], X_1, \dots, X_n\Big)\end{aligned} \tag{4.1.10}$$

It follows that $\mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(L_{\mathfrak{g}}, E)$ can be modeled by $\overline{C}^*(\mathfrak{g}, E[-1])$. Dually, this given the following description of the cotangent complex $L_{\mathfrak{g}}$ itself:

Corollary 4.1.11. *Let $\mathfrak{g}$ be a dg-Lie algebroid whose underlying dg-A-module is cofibrant. Then*

$$L_{\mathfrak{g}} = \mathcal{U}(\mathfrak{g}) \otimes_A \big(\mathrm{Sym}_A^{\geq 1}\mathfrak{g}[1]\big)[-1] \tag{4.1.12}$$

with differential given (modulo Koszul signs) by

$$\begin{aligned}\partial(u\otimes X_1\dots X_n) &= (\partial u)\otimes X_1\dots X_n + \sum_i u\otimes X_1\dots\partial(X_i)\dots X_n \\ &\overset{(n>1)}{+} \sum_i u\cdot X_k\otimes X_1\dots X_n + \sum_{i<j} u\otimes [X_i,X_j]X_1\dots X_n.\end{aligned} \tag{4.1.13}$$

The first term in the second row only applies when $n>1$.

Definition 4.1.14. The cotangent complex (4.1.12) comes with a $\mathcal{U}(\mathfrak{g})$-linear map

$$L_{\mathfrak{g}} = \mathcal{U}(\mathfrak{g})\otimes_A \big(\mathrm{Sym}_A^{\geq 1}\mathfrak{g}[1]\big)[-1] \longrightarrow \mathcal{U}(\mathfrak{g})$$

sending $u\otimes X_1\dots X_n$ to zero when $n>1$ and to $u\cdot X_1$ when $n=1$. The *Koszul complex* $K(\mathfrak{g})$ of $\mathfrak{g}$ is the cofiber

$$L_{\mathfrak{g}} \longrightarrow \mathcal{U}(\mathfrak{g}) \longrightarrow K(\mathfrak{g}).$$

Explicitly, $K(\mathfrak{g})$ can be identified with $\mathcal{U}(\mathfrak{g})\otimes_A \mathrm{Sym}_A\mathfrak{g}$ with the same differential as in (4.1.13), where the first term in the second row is also included when $n=1$.

Remark 4.1.15. The composite map

$$L_{\mathfrak{g}} = \mathcal{U}(\mathfrak{g})\otimes_A \big(\mathrm{Sym}_A^{\geq 1}\mathfrak{g}[1]\big)[-1] \longrightarrow \mathcal{U}(\mathfrak{g}) \xrightarrow{u\mapsto u\cdot 1} A$$

is equal to zero, so that there is a $\mathcal{U}(\mathfrak{g})$-linear map $K(\mathfrak{g})\longrightarrow A$. When $\mathfrak{g}$ is A-cofibrant, this map is a weak equivalence. Indeed, the PBW filtration on $\mathcal{U}(\mathfrak{g})$ (Proposition 3.3.7) and the filtration on $\mathrm{Sym}_A\mathfrak{g}[1]$ by polynomial degree determine a total filtration on the Koszul complex $\mathcal{K}(\mathfrak{g})$. The map on the associated graded is the obvious projection

$$\mathrm{Sym}_A(\mathfrak{g}[0,1]) = \mathrm{Sym}_A\mathfrak{g}\otimes_A \mathrm{Sym}_A\mathfrak{g}[1] \longrightarrow A$$

from the symmetric algebra on the cone $\mathfrak{g}[0,1]$, which is a weak equivalence.

Definition 4.1.16. The *Chevalley-Eilenberg complex* is the mapping complex

$$C^*(\mathfrak{g},E) := \mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(K(\mathfrak{g}),E) \cong \mathrm{Hom}_A(\mathrm{Sym}_A\mathfrak{g}[1],E)$$

where the latter is equipped with the Chevalley-Eilenberg differential (4.1.10) . When $E=A$ is the canonical $\mathfrak{g}$-representation, we will write $C^*(\mathfrak{g}) = C^*(\mathfrak{g},A)$.

The upshot of our discussion is that A-cofibrant dg-Lie algebroids have a fiber sequence

$$\overline{C}^*(\mathfrak{g},E) = \mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(L_{\mathfrak{g}},E)[-1] \longrightarrow C^*(\mathfrak{g},E) = \mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(A,E) \longrightarrow E. \tag{4.1.17}$$

We will refer to the canonical map $C^*(\mathfrak{g},E)\longrightarrow E$ as the *augmentation* of the Chevalley-Eilenberg complex.

Remark 4.1.18. Since A is the unit of the symmetric monoidal structure on $\mathrm{Rep}_{\mathfrak{g}}$ provided by Example 3.3.3, it has a canonical cocommutative coalgebra structure. This coalgebra structure on A induces a lax symmetric monoidal structure on the functor $\mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(A,-)$, which can be described by means of the shuffle product of forms.

Recall that for $\alpha\in C^*(\mathfrak{g},E)$ and $\beta\in C^*(\mathfrak{g},F)$, their shuffle product in $C^*(\mathfrak{g},E\otimes_A F)$ is given by

$$(\alpha\times\beta)(X_1,...,X_n) = \sum_{k,\sigma\in\mathrm{Sh}(k,n-k)} \alpha(X_{\sigma(1)},\dots,X_{\sigma(k)})\otimes\beta(X_{\sigma(k+1)},\dots,X_{\sigma(n)}).$$

This endows $C^*(\mathfrak{g},-)\colon \mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}} \longrightarrow \mathrm{Mod}^{\mathrm{dg}}_{\mathbb{R}}$ with a lax symmetric monoidal structure, which is compatible with the augmentation in the sense that there is a commuting square

$$\begin{tikzcd} C^*(\mathfrak{g},E)\otimes_{\mathbb{R}} C^*(\mathfrak{g},F) \arrow[r, "\times"] \arrow[d] & C^*(\mathfrak{g},E\otimes_A F) \arrow[d] \\ E\otimes_k F \arrow[r] & E\otimes_A F. \end{tikzcd}$$

In particular, taking coefficients with values in the commutative algebra A, one obtains a functor with values in commutative dg-algebras over A

$$C^*\colon \mathrm{LieAlgd}^{\mathrm{dg}}_A \longrightarrow \left(\mathrm{CAlg}^{\mathrm{dg}}_{\mathbb{R}}/A\right)^{\mathrm{op}};\ \mathfrak{g}\longmapsto C^*(\mathfrak{g}) := C^*(\mathfrak{g},A).$$

Furthermore, for every dg-Lie algebroid $\mathfrak{g}$, there is a lax symmetric monoidal functor $C^*(\mathfrak{g},-)\colon \mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}} \longrightarrow \mathrm{Mod}^{\mathrm{dg}}_{C^*(\mathfrak{g})}$. In particular, this functor preserves algebras over dg-operads.

Remark 4.1.19. Each $C^*(\mathfrak{g},E)$ arises naturally as the limit of a tower

$$\dots \longrightarrow C^{\leq 2}(\mathfrak{g},E) \longrightarrow C^{\leq 1}(\mathfrak{g},E) \longrightarrow E$$

where $C^{\leq n}(\mathfrak{g},E)$ is the quotient of $C^*(\mathfrak{g},E)$ given by $\mathrm{Hom}(\mathrm{Sym}^{\geq n+1}_A\mathfrak{g}[1],E)$. This filtration is simply induced by the filtration of Lemma 3.1.28 on the cobar resolution of $\mathfrak{g}$. There are natural multiplication maps $C^{\leq m}(\mathfrak{g},A)\otimes C^{\leq n}(\mathfrak{g},E) \longrightarrow C^{\leq m+n}(\mathfrak{g},E)$. In particular, $C^*(\mathfrak{g})$ arises as the limit of a tower of (strict) square zero extensions of commutative dg-algebras.

Lemma 4.1.20. *Suppose that $A \longrightarrow B$ is a map of dg-$\mathcal{C}^\infty$-rings and that the dg-Lie algebroid $\mathfrak{g}$ acts on B by $\mathcal{C}^\infty$-derivations (Remark 3.3.15). Then the connective cover $\tau_{\geq 0}C^*(\mathfrak{g},B)$ has the natural structure of a dg-$\mathcal{C}^\infty$-ring and the map $\tau_{\geq 0}C^*(\mathfrak{g}) \longrightarrow B$ is a map of dg-$\mathcal{C}^\infty$-rings.*

Proof. Note that $C^*(\mathfrak{g})_0$ arises as the limit of a tower of nilpotent extensions $C^{\leq n}(\mathfrak{g})_0 \longrightarrow B_0$ and admits a natural section $B_0 \longrightarrow C^*(\mathfrak{g})$. It follows from [19, Proposition 3.22] that each $C^{\leq n}(\mathfrak{g},B)_0$ carries a unique $\mathcal{C}^\infty$-ring structure compatible with the projection to B_0 and this splitting, which induces a $\mathcal{C}^\infty$-ring structure on the limit $C^*(\mathfrak{g})_0$.

We have to show that these $\mathcal{C}^\infty$-ring structures restrict to zero cycles. For this, it suffices to verify that the $\mathbb{R}$-algebra derivation $\partial_{\mathrm{CE}}\colon C^{\leq n}(\mathfrak{g},B)_0 \longrightarrow C^{\leq n}(\mathfrak{g},B)_{-1}$ is a $\mathcal{C}^\infty$-derivation. Invoking [19, Proposition 3.22] once more, it suffices to show that the composite map

$$B_0 \longrightarrow C^{\leq n}(\mathfrak{g},B)_0 \xrightarrow{\partial_{\mathrm{CE}}} C^{\leq n}(\mathfrak{g},B)_{-1}$$

is $\mathcal{C}^\infty$-derivation. For every $b\in B_0$, $\partial_{\mathrm{CE}}(b)$ can be identified with the map

$$\mathfrak{g}_0[1] \longrightarrow B_0;\ X \longmapsto \nabla_X b.$$

Since $\mathfrak{g}$ acts on B by $\mathcal{C}^\infty$-derivations, the map $b\mapsto \partial_{\mathrm{CE}}(b)$ is a $\mathcal{C}^\infty$-derivation. We conclude that $\tau_{\geq 0}C^*(\mathfrak{g},B)$ has the natural structure of a dg-$\mathcal{C}^\infty$-ring. □

Definition 4.1.21. The connective cover of the Chevalley-Eilenberg complex $C^*(\mathfrak{g},E)$ will be denoted by

$$\mathrm{c}^*(\mathfrak{g},E) := \tau_{\geq 0}C^*(\mathfrak{g},E)$$

Each $\mathrm{c}^*(\mathfrak{g},E)$ is a module over the dg-$\mathcal{C}^\infty$-ring $\mathrm{c}^*(\mathfrak{g}) = \mathrm{c}^*(\mathfrak{g},A)$.

With these definitions, we can already address half of Proposition 4.1.1:

Lemma 4.1.22. *Let A be a dg-$\mathcal{C}^\infty$-ring. Then the functor* c^* *induces an adjunction of ∞-categories*

$$\mathrm{c}^*\colon \mathrm{LieAlgd}_A \rightleftarrows \big(\mathcal{C}^\infty\mathrm{Alg}/A\big)^{\mathrm{op}}\colon \mathfrak{D}.$$

Proof. By Lemma 4.1.20, there is a functor

$$\mathrm{c}^*\colon \mathrm{LieAlgd}_A^{\mathrm{dg}} \longrightarrow \big(\mathcal{C}^\infty\mathrm{Alg}^{\mathrm{dg}}/A\big)^{\mathrm{op}}.$$

The fiber sequence (4.1.17) shows that c^* preserves weak equivalences between A-cofibrant dg-Lie algebroids. Every cofibrant dg-Lie algebroid is A-cofibrant (Theorem 3.1.15), so that c^* induces a functor between ∞-categories. Since $\mathrm{LieAlgd}_A$ is a locally presentable ∞-category (because it arises from a tractable semi-model category, by Proposition 2.1.24), c^* admits a right adjoint as soon as it preserves all (homotopy) colimits.

Limits of $\mathcal{C}^\infty$-rings are computed at the level of the underlying chain complexes, so it suffices to check that the functor

$$\mathrm{c}^*\colon \mathrm{LieAlgd}_A \xrightarrow{C^*} \big(\mathrm{Mod}_{\mathbb{R}}/A\big)^{\mathrm{op}} \xrightarrow{\tau_{\geq 0}} \big(\mathrm{Mod}_{\mathbb{R}}^{\geq 0}/A\big)^{\mathrm{op}}$$

preserves all colimits. Taking connective covers preserves limits, so it suffices to show that C^* preserves colimits. Furthermore, taking kernels provides a functor $\ker\colon \mathrm{Mod}_{\mathbb{R}}/A \longrightarrow \mathrm{Mod}_{\mathbb{R}}$ that preserves and detects all limits. It therefore suffices to show that $\ker\circ C^*$ sends colimits of dg-Lie algebroids to limits of chain complexes. By the fiber sequence (4.1.17), this functor just sends $\mathfrak{g}$ to the mapping complex $\mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(L_{\mathfrak{g}}, A[-1])$. Since the cotangent complex functor preserves colimits, the result follows. $\square$

4.1.2 The cotangent complex for free Lie algebroids. When $\mathfrak{g} = F(V)$ is the free dg-Lie algebroid generated by a cofibrant dg-A-module over T_A, we now have two different (but weakly equivalent) descriptions of the cotangent complex $L_{\mathfrak{g}}$: Example 4.1.5 simply evaluates the left Quillen functor (4.1.3) on $\mathfrak{g}$ itself, while Corollary 4.1.11 computes the value of (4.1.3) on the 'cobar' resolution $Q(\mathfrak{g})$. Of course, the first description is significantly smaller than the second.

When $\mathfrak{g} = F(V)$ is a free Lie algebroid, there is a canonical section of the map $Q(\mathfrak{g}) \xrightarrow{\sim} \mathfrak{g}$, induced by the canonical inclusion

$$V \hookrightarrow \mathfrak{g} \longrightarrow \Big(\mathrm{Sym}_A \mathfrak{g}[1]\Big)[-1] \subseteq Q(\mathfrak{g}).$$

Applying (4.1.3) to this section produces a weak equivalence between the two models for the cotangent complex $L_{\mathfrak{g}}$, given by the $\mathcal{U}(\mathfrak{g})$-linear extension of the above inclusion

$$\mathcal{U}(\mathfrak{g})\otimes_A V \xrightarrow{\sim} \mathcal{U}(\mathfrak{g})\otimes_A \Big(\mathrm{Sym}_A^{\geq 1}\mathfrak{g}[1]\Big)[-1].$$

Mapping into A and taking connective covers, one obtains a weak equivalence to a significantly smaller complex

$$\kappa\colon \mathrm{c}^*(\mathfrak{g}, A) \xrightarrow{\simeq} A^V = A\oplus_{\rho^\vee} \tau_{\geq 0}\mathrm{Hom}_A\big(V[1], A\big);\;\; \alpha \longmapsto \Big(\alpha(1), \alpha\big|_{V[1]}\Big).$$

This map is a map of dg-$\mathcal{C}^\infty$-rings, where the codomain A^V almost has the structure of a square zero extension of A: it fits into a (homotopy) pullback square of dg-$\mathcal{C}^\infty$-rings

$$\begin{tikzcd} A^V = A\oplus_{\rho^\vee}\tau_{\geq 0}(V[1]^\vee) \arrow[r] \arrow[d] & A\oplus \tau_{\geq 0}(V[0,1]^\vee) \arrow[d] \\ A \arrow[r, "\rho^\vee"'] & A\oplus\tau_{\geq 0}(V^\vee). \end{tikzcd} \tag{4.1.23}$$

where the bottom map is classified by the map

$$\rho^\vee : \Omega_A \longrightarrow V^\vee;\ d_{\mathrm{dR}}a \longmapsto \Big(v \mapsto \rho(v)(a)\Big).$$

Using this pullback square, one can easily check that the functor

$$A^{(-)} : \mathrm{Mod}_A^{\mathrm{dg}}/T_A \longrightarrow \big(\mathcal{C}^\infty\mathrm{Alg}^{\mathrm{dg}}/A\big)^{\mathrm{op}}; V \longmapsto A^V = A \oplus_{\rho^\vee} \tau_{\geq 0}\big(V[1]^\vee\big).$$

is a left Quillen functor, whose right adjoint sends $B \longrightarrow A$ to the mapping fiber $T_{A/B}$ of

$$T_A = \mathrm{Der}(A, A) \longrightarrow \mathrm{Der}(B, A),$$

together with its natural projection to T_A. We may therefore summarize the previous discussion by the following result:

Corollary 4.1.24. *There is a natural transformation to a left Quillen functor*

$$\begin{tikzcd} \mathrm{Mod}_A^{\mathrm{dg}}/T_A \arrow[rr, bend left, "\mathrm{c}^*\circ F"] \arrow[rr, bend right, "A^{(-)}"'] \arrow[rr, phantom, "\Downarrow\kappa"] & & \big(\mathcal{C}^\infty\mathrm{Alg}^{\mathrm{dg}}/A\big)^{\mathrm{op}} \end{tikzcd}$$

which is a weak equivalence when restricted to cofibrant dg-A-modules over T_A.

Let us finally turn to the proof of Proposition 4.1.1:

Proof (of Proposition 4.1.1). We have already seen in Lemma 4.1.22 that the functor c^* admits a right adjoint $\mathfrak{D}$. It remains to show that the underlying anchor map of the Lie algebroid is indeed the A-linear dual of the map $L_A \longrightarrow L_{B/A}$. To see this, note that the composition of $\mathfrak{D}$ with the forgetful functor $\mathrm{LieAlgd}_A \longrightarrow \mathrm{Mod}_A/T_A$ is the right adjoint of the composite

$$\mathrm{Mod}_A/T_A \xrightarrow{F} \mathrm{LieAlgd}_A \xrightarrow{\mathrm{c}^*} \big(\mathcal{C}^\infty\mathrm{Alg}/A\big)^{\mathrm{op}}.$$

The natural equivalence κ from (4.1.24) identifies this functor with the left derived functor of the left Quillen functor $A^{(-)}$. It follows that the underlying anchor map of $\mathfrak{D}(B \longrightarrow A)$ can be computed by the right derived functor of the right adjoint to $A^{(-)}$. By the above discussion and the fact that A is cofibrant, this is exactly the functor taking the A-linear dual of $L_A \longrightarrow L_{A/B}$. □

Remark 4.1.25. Suppose that $\phi\colon B \longrightarrow A$ is a cofibration between cofibrant dg-$\mathcal{C}^\infty$-rings. The map of A-modules $\mathrm{Der}(A, A) \longrightarrow \mathrm{Der}(B, A)$ is already a fibration and its kernel $\mathfrak{g} \subseteq T_A = \mathrm{Der}(A, A)$ is a model for the A-linear dual of $L_{A/B}$. The latter is clearly closed under the commutator bracket and forms a dg-Lie algebroid over A. This dg-Lie algebroid represents $\mathfrak{D}(B \longrightarrow A)$.

Indeed, there is a natural diagram of dg-$\mathcal{C}^\infty$-rings

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[dr, "\phi"'] & \mathrm{c}^*(\mathfrak{g}) \arrow[r] \arrow[d] & \mathrm{c}^*(\tilde{\mathfrak{g}}) \arrow[dl] \\ & A & \end{tikzcd}$$

where $\tilde{\mathfrak{g}} \longrightarrow \mathfrak{g}$ is a cofibrant replacement of $\mathfrak{g}$ and the map f sends $b \in B$ to the map $\mathrm{Sym}_A\mathfrak{g}[1] \longrightarrow A$ whose value on 1 is $\phi(b)$ and which vanishes on powers of $\mathfrak{g}[1]$. This is a map of dg-$\mathcal{C}^\infty$-rings since all derivations in $\mathfrak{g}$ annihilate $\phi(b)$. The composite map $B \longrightarrow \mathrm{c}^*(\tilde{\mathfrak{g}})$ is adjoint to a map $\tilde{\mathfrak{g}} \longrightarrow \mathfrak{D}(B \longrightarrow A)$. This map induces an equivalence on the underlying anchor map by Proposition 4.1.1, so that $\tilde{\mathfrak{g}}$ and $\mathfrak{g}$ are equivalent to $\mathfrak{D}(B \longrightarrow A)$.

4.1.3 Koszul duality. The goal of this section is to give a proof of Proposition 4.1.2. In fact, Proposition 4.1.2 follows immediately from the following, slightly simpler, assertion:

Proposition 4.1.26. *Let A be a cofibrant dg-$\mathcal{C}^\infty$-ring and suppose that $\mathfrak{g}$ is a dg-Lie algebroid satisfying the following conditions:*

(i) $\mathfrak{g}$ is cofibrant as a dg-A-module and is freely generated as a graded A-module by a set $\{x_i\}$.

(ii) There are finitely many x_i in each single degree, and no generators of (homological) degree ≥ 0.

Then the (derived) unit map $\mathfrak{g} \longrightarrow \mathfrak{D}\mathrm{c}^*(\mathfrak{g})$ can be identified at the level of dg-A-modules with the canonical map $\mathfrak{g} \longrightarrow \mathfrak{g}^{\vee\vee}$ from $\mathfrak{g}$ into its A-linear bidual.

Proof (of Proposition 4.1.2). By Proposition 4.1.26, the unit map $\mathfrak{g} \longrightarrow \mathfrak{D}\mathrm{c}^*(\mathfrak{g})$ is equivalent to the canonical map $\mathfrak{g} \longrightarrow \mathfrak{g}^{\vee\vee}$ from $\mathfrak{g}$ into its bidual. By the conditions on $\mathfrak{g}$, both $\mathfrak{g}$ and $\mathfrak{g}^\vee$ are cofibrant dg-A-modules (the latter is free on a set of generators of degree ≥ 0). It follows that $\mathfrak{g}^{\vee\vee}$ is a model for the *derived* bidual of $\mathfrak{g}$. To see that this map is an equivalence, we may therefore replace A by a weakly equivalent dg-algebra, so that we can assume that A is concentrated in degrees $[0, n]$.

By the conditions on $\mathfrak{g}$, we can write $\mathfrak{g} = \bigoplus_{i<0} A^{\oplus k_i}[i]$ at the level of graded A-modules. The map $\mathfrak{g} \longrightarrow \mathfrak{g}^{\vee\vee}$ can then be identified with the natural map of graded A-modules

$$\textstyle\bigoplus_{i<0} A^{\oplus k_i}[i] \longrightarrow \prod_{i<0} A^{\times k_i}[i].$$

Since A is bounded, this map is an isomorphism. □

Remark 4.1.27. Let $\mathfrak{g}$ be a dg-Lie algebroid satisfying conditions (i) and (ii) of Proposition 4.1.26. Then $C^*(\mathfrak{g})$ is concentrated in nonnegative degrees so that $\mathrm{c}^*(\mathfrak{g}) = C^*(\mathfrak{g})$. We will therefore simply write $C^*(\mathfrak{g})$ instead of $\mathrm{c}^*(\mathfrak{g})$ throughout this section.

The remainder of this section is devoted to a proof of Proposition 4.1.26. let us start by considering the map of commutative dg-algebras over A

$$c\colon C^*(\mathfrak{g}) \longrightarrow A^{\mathfrak{g}} = A \oplus_{\rho^\vee} \mathfrak{g}[1]^\vee \tag{4.1.28}$$

which sends $\alpha\colon \mathrm{Sym}_A \mathfrak{g}[1] \longrightarrow A$ to $\big(\alpha(1), \alpha\big|_{\mathfrak{g}[1]}\big)$. In the proof of Proposition 4.1.1, we have seen that the functor $V \mapsto A^V$ was a right Quillen functor, whose derived left adjoint sent

$$\big(B \longrightarrow A\big) \longmapsto \big(L^\vee_{A/B} \longrightarrow L^\vee_A = T_A\big).$$

Lemma 4.1.29. *Let $\mathfrak{g}$ be a cofibrant dg-Lie algebroid over A satisfying conditions (i) and (ii) from Proposition 4.1.26. Then the A-linear map $\mathfrak{g} \longrightarrow L^\vee_{A/C^*(\mathfrak{g})}$ adjoint to (4.1.28) is equivalent to the A-linear map underlying the unit map $\mathfrak{g} \longrightarrow \mathfrak{D}C^*(\mathfrak{g})$.*

Proof. Let $F\colon \mathrm{Mod}_A/T_A \leftrightarrows \mathrm{LieAlgd}_A\colon U$ be the free-forgetful adjunction. The A-linear map $U(\mathfrak{g}) \longrightarrow U\mathfrak{D}C^*(\mathfrak{g})$ in Mod_A/T_A corresponds by adjunction to the map

$$C^*(\mathfrak{g}) \longrightarrow C^*(F(\mathfrak{g}))$$

in $\mathrm{CAlg}^{\geq 0}/A$. The composition of this map with the equivalence $\kappa\colon C^*(F(\mathfrak{g})) \longrightarrow A^{\mathfrak{g}}$ from Corollary 4.1.24 is exactly the map (4.1.28). This means that the maps

$$\mathfrak{g} \longrightarrow L^\vee_{A/C^*(\mathfrak{g})} \quad \text{and} \quad U(\mathfrak{g}) \longrightarrow U\mathfrak{D}C^*(\mathfrak{g})$$

are identified under the adjoint equivalence between $U\mathfrak{D}$ and the functor sending $B \longrightarrow A$ to $L^\vee_{A/B}$. □

To use Lemma 4.1.29, we will have to compute the relative cotangent complex of the map $C^*(\mathfrak{g}) \longrightarrow A$. Unfortunately, $C^*(\mathfrak{g})$ has the structure of a power series algebra, which means that $C^*(\mathfrak{g})$ is not cofibrant and computing its cotangent complex requires some effort. Let us therefore introduce the following 'global' variant of the Chevalley-Eilenberg complex:

Construction 4.1.30. Let $\mathfrak{g}$ be a dg-Lie algebroid over a cofibrant dg-$\mathcal{C}^\infty$-ring A satisfying conditions (i) and (ii) from Proposition 4.1.26. Consider the graded-commutative algebra

$$C^*_{\mathrm{poly}}(\mathfrak{g}) := \mathrm{Sym}_A\big(\mathfrak{g}[1]^\vee\big) \subseteq C^*(\mathfrak{g})$$

consisting of graded A-linear maps $\mathrm{Sym}_A\mathfrak{g}[1] \longrightarrow A$ that vanish on some power $\mathrm{Sym}^{\geq n}_A\mathfrak{g}[1]$. This graded subalgebra of $C^*(\mathfrak{g})$ is closed under the differential of $C^*(\mathfrak{g})$, since this differential sends a function vanishing on $\mathrm{Sym}^{\geq n}_A\mathfrak{g}[1]$ to a function vanishing on $\mathrm{Sym}^{\geq n+1}_A\mathfrak{g}[1]$.

The degree zero part of $C^*_{\mathrm{poly}}(\mathfrak{g})$ is given by a polynomial algebra $A_0[x_i]$, where the x_i are the duals of the degree -1 generators of $\mathfrak{g}$. Consider the dg-$\mathcal{C}^\infty$-ring

$$C^*_{\mathrm{sm}}(\mathfrak{g}) = A_0\{x_i\} \otimes_{A_0[x_i]} C^*_{\mathrm{poly}}(\mathfrak{g})$$

where $A_0\{x_i\}$ is the free discrete $\mathcal{C}^\infty$-ring on A_0 and the generators x_i. There is a natural map of dg-$\mathcal{C}^\infty$-rings $C^*_{\mathrm{sm}}(\mathfrak{g}) \longrightarrow C^*(\mathfrak{g})$ obtained by sending the generators x_i in degree zero to their image in $C^*(\mathfrak{g})$.

Example 4.1.31. Let $\mathfrak{g} = A^{\oplus n}[-1]$ be the trivial dg-Lie algebroid on n generators of degree -1. Then $C^*(\mathfrak{g})$ is isomorphic to the ring of power-series $A[[x_1, ..., x_n]]$ and the inclusion $C^*_{\mathrm{poly}}(\mathfrak{g}) \subseteq C^*(\mathfrak{g})$ is the inclusion of the polynomial algebra $A[x_1, ..., x_n] \subseteq A[[x_1, ..., x_n]]$. The map $C^*_{\mathrm{sm}}(\mathfrak{g}) \longrightarrow C^*(\mathfrak{g})$ is the map $A\{x_1, \dots, x_n\} \longrightarrow A[[x_1, \dots, x_n]]$ taking Taylor expansions at $x_i = 0$.

Warning 4.1.32. The commutative dg-algebras $C^*_{\mathrm{poly}}(\mathfrak{g})$ and $C^*_{\mathrm{sm}}(\mathfrak{g})$ are not homotopy invariant.

Lemma 4.1.33. *Let $A \in \mathcal{C}^\infty\mathrm{Alg}^{\mathrm{dg}}$ be cofibrant and let $\mathfrak{g}$ be a dg-Lie algebroid over A such that $\mathfrak{g} \cong \bigoplus_{i<0} A^{\oplus k_i}[i]$ as a graded A-module. Then the following hold:*

(1) the dg-$\mathcal{C}^\infty$-ring $C^*_{\mathrm{sm}}(\mathfrak{g})$ is cofibrant.

(2) the map $\Omega_{C^*_{\mathrm{sm}}(\mathfrak{g})} \otimes_{C^*_{\mathrm{sm}}(\mathfrak{g})} A \longrightarrow \Omega_A$ of $\mathcal{C}^\infty$-algebraic Kähler differentials can be identified with the projection map $\Omega_A \oplus \mathfrak{g}[1]^\vee \longrightarrow \Omega_A$. Here $\Omega_A \oplus \mathfrak{g}[1]^\vee$ has differential given by

$$\partial(d_{\mathrm{dR}}(a), \alpha) = \big(d_{\mathrm{dR}}(\partial_A a), \partial_{\mathfrak{g}[1]^\vee}(\alpha) + \rho^\vee(d_{\mathrm{dR}}a)\big)$$

where $\rho^\vee\colon \Omega_A \longrightarrow \mathfrak{g}^\vee$ is the adjoint of the anchor map $\mathfrak{g} \longrightarrow T_A$.

Proof. Since $\mathfrak{g}$ is given as a graded A-module by $\bigoplus_{i<0} A^{\oplus k_i}[i]$, it follows that $C^*_{\mathrm{poly}}(\mathfrak{g})$ is given without the differential by a polynomial algebra over A, generated by the free module $\mathfrak{g}[1]^\vee$. Consequently, $C^*_{\mathrm{sm}}(\mathfrak{g})$ is freely generated over A if we forget the differential. In addition, A is cofibrant, so that it is the retract of a dg-$\mathcal{C}^\infty$-ring which is freely generated without the differential. This implies that $C^*_{\mathrm{sm}}(\mathfrak{g})$ is also a retract of a dg-$\mathcal{C}^\infty$-ring which is freely generated without differential, so that $C^*_{\mathrm{sm}}(\mathfrak{g})$ is cofibrant.

For the second assertion, observe that without differentials, $C^*_{\mathrm{sm}}(\mathfrak{g})$ is freely generated over A by the A-module $\mathfrak{g}[1]^\vee$. It follows that

$$\Omega_{C^*_{\mathrm{sm}}(\mathfrak{g})} \otimes_{C^*_{\mathrm{sm}}(\mathfrak{g})} A \cong \Omega_A \oplus \mathfrak{g}[1]^\vee$$

as a graded A-module. To identify the differential, observe that the Chevalley-Eilenberg differential sends an element $a \in A \subseteq C^*_{\mathrm{sm}}(\mathfrak{g})$ to the element $\partial_A a + \rho^\vee(d_{\mathrm{dR}}a)$ and an element $\alpha \in \mathfrak{g}[1]^\vee \subseteq C^*_{\mathrm{sm}}(\mathfrak{g})$ to $\partial_{\mathfrak{g}[1]^\vee}\alpha$, modulo higher order terms in the generators $\mathfrak{g}[1]^\vee$. $\square$

Lemma 4.1.34. *Let A be a nonnegatively graded commutative algebra over $\mathbb{R}$, let V be a finite-dimensional $\mathbb{R}$-vector space and let W be a degreewise finite-dimensional graded $\mathbb{R}$-vector space, in strictly negative degrees. Then there is a natural isomorphism of graded-commutative A-algebras*

$$\mathrm{Hom}\big(\mathrm{Sym}_{\mathbb{R}}V, A\big) \otimes_{\mathbb{R}} \mathrm{Sym}_{\mathbb{R}}(W^{\vee}) \longrightarrow \mathrm{Hom}\big(\mathrm{Sym}_{\mathbb{R}}(V \oplus W), A\big)$$

where Hom *is the internal hom in graded vector spaces and* $\mathrm{Sym}_{\mathbb{R}}W^{\vee}$ *is the graded polynomial algebra on the dual vector space of W.*

Proof. Observe that there is an isomorphism of graded cocommutative coalgebras $\mathrm{Sym}_{\mathbb{R}}(V \oplus W) \cong \mathrm{Sym}_{\mathbb{R}}V \otimes_{\mathbb{R}} \mathrm{Sym}_{\mathbb{R}}W$. There is a natural map of graded-commutative algebras

$$\mathrm{Hom}\big(\mathrm{Sym}_{\mathbb{R}}V, A\big) \otimes \mathrm{Hom}(\mathrm{Sym}_{\mathbb{R}}W, \mathbb{R}) \xrightarrow{\ \mu\ } \mathrm{Hom}\big(\mathrm{Sym}_{\mathbb{R}}V \otimes_{\mathbb{R}} \mathrm{Sym}_{\mathbb{R}}W, A\big).$$

sending two maps $\alpha\colon \mathrm{Sym}_{\mathbb{R}}V \longrightarrow A$ and $\beta\colon \mathrm{Sym}_{\mathbb{R}}W \longrightarrow \mathbb{R}$ to $\alpha \otimes \beta$. Using that $\mathrm{Sym}_{\mathbb{R}}W$ is degreewise finite dimensional, one can identify $\mathrm{Sym}_{\mathbb{R}}(W^{\vee}) \simeq \mathrm{Hom}(\mathrm{Sym}_{\mathbb{R}}W, \mathbb{R})$ and see that the map μ is an isomorphism. ☐

Lemma 4.1.35. *Let $A \in \mathcal{C}^{\infty}\mathrm{Alg}^{\mathrm{dg}}$ be cofibrant and let $\mathfrak{g}$ be as in Proposition 4.1.26. Then the map $C^*_{\mathrm{sm}}(\mathfrak{g}) \longrightarrow C^*(\mathfrak{g})$ of commutative dg-algebras over A induces an equivalence*

$$L_{C^*_{\mathrm{sm}}(\mathfrak{g})} \otimes_{C^*_{\mathrm{sm}}(\mathfrak{g})} A \xrightarrow{\ \simeq\ } L_{C^*(\mathfrak{g})} \otimes_{C^*(\mathfrak{g})} A.$$

Proof. Consider the trivial cofibration, followed by a fibration

$$0 \xrightarrow{\ \sim\ } \mathfrak{h} = F(\mathfrak{g}[0,-1]) \longrightarrow\!\!\!\!\!\rightarrow \mathfrak{g}$$

where $\mathfrak{h}$ is the free dg-Lie algebroid on the map $\mathfrak{g}[0,-1] \longrightarrow \mathfrak{g} \longrightarrow T_A$ from the path space of $\mathfrak{g}$. Let V be the free graded $\mathbb{R}$-vector space spanned by the generators x_i of $\mathfrak{g}$, so that $\mathfrak{g} = A \otimes_{\mathbb{R}} V$. As a graded Lie algebroid, $\mathfrak{h}$ is then freely generated by the graded $\mathbb{R}$-vector space $V[0,-1]$. Consequently, the map $\mathfrak{h} \longrightarrow \mathfrak{g}$ is given without differentials by the A-linear extenion of a map from the free Lie algebra

$$\mathfrak{h} = A \otimes \mathrm{Lie}\big(V[0,-1]\big) \longrightarrow A \otimes V = \mathfrak{g}$$

which sends V to the generators of $\mathfrak{g}$ and $V[-1]$ to zero. This map has a splitting, induced by the inclusion $V \longrightarrow \mathrm{Lie}(V[0,-1])$, so that $\mathfrak{h} \longrightarrow \mathfrak{g}$ can be identified with

$$\mathfrak{h} = A \otimes (V \oplus W) \xrightarrow{(\mathrm{id},0)} A \otimes V = \mathfrak{g}.$$

Here W is a graded $\mathbb{R}$-vector space isomorphic to $\mathrm{Lie}(V[0,-1])/V$, which is degreewise finite dimensional and concentrated in degrees < -1.

Let us now consider the commutative diagram of cdgas associated to $\mathfrak{h} \longrightarrow \mathfrak{g}$

$$\begin{tikzcd} C^*_{\mathrm{poly}}(\mathfrak{g}) \arrow[r] \arrow[d] & C^*_{\mathrm{poly}}(\mathfrak{h}) \arrow[r] \arrow[d] & A \arrow[d] \\ C^*(\mathfrak{g}) \arrow[r] & C^*(\mathfrak{h}) \arrow[r, "\sim"'] & A. \end{tikzcd} \tag{4.1.36}$$

The right bottom map is a weak equivalence since $\mathfrak{h}$ is cofibrant and weakly contractible. The map $C^*_{\mathrm{poly}}(\mathfrak{g}) \longrightarrow C^*_{\mathrm{poly}}(\mathfrak{h})$ can be identified with a map of polynomial algebras

$$A \otimes_{\mathbb{R}} \mathrm{Sym}_{\mathbb{R}}(V[1]^{\vee}) \longrightarrow A \otimes_{\mathbb{R}} \mathrm{Sym}_{\mathbb{R}}\big((V \oplus W)[1]^{\vee}\big).$$

It follows that $C^*_{\mathrm{poly}}(\mathfrak{h})$ is freely generated over $C^*_{\mathrm{poly}}(\mathfrak{g})$ by $W[1]^\vee$, which is degreewise finite dimensional and concentrated in degrees ≥ 1. In particular, $C^*_{\mathrm{poly}}(\mathfrak{g}) \longrightarrow C^*_{\mathrm{poly}}(\mathfrak{h})$ is a cofibration of cdgas.

On the other hand, the map $C^*(\mathfrak{g}) \longrightarrow C^*(\mathfrak{h})$ is given without differentials by the natural map

$$\mathrm{Hom}_{\mathbb{R}}\Big(\mathrm{Sym}_{\mathbb{R}} V[1], A\Big) \longrightarrow \mathrm{Hom}_{\mathbb{R}}\Big(\mathrm{Sym}_{\mathbb{R}}(V[1] \oplus W[1]), A\Big).$$

It now follows from Lemma 4.1.34 that the left square in (4.1.36) is a (homotopy) pushout square of cdgas. Its image under $L_{(-)} \otimes_{(-)} A$

$$\begin{tikzcd} L_{C^*_{\mathrm{poly}}(\mathfrak{g})} \otimes_{C^*_{\mathrm{poly}}(\mathfrak{g})} A \arrow[r] \arrow[d] & L_{C^*_{\mathrm{poly}}(\mathfrak{h})} \otimes_{C^*_{\mathrm{poly}}(\mathfrak{h})} A \arrow[d] \\ L_{C^*(\mathfrak{g})} \otimes_{C^*(\mathfrak{g})} A \arrow[r] & L_{C^*(\mathfrak{h})} \otimes_{C^*(\mathfrak{h})} A \end{tikzcd} \tag{4.1.37}$$

is a homotopy pushout square as well. Since $C^*(\mathfrak{h}) \longrightarrow A$ is a weak equivalence, the map $L_{C^*(\mathfrak{h})} \longrightarrow L_A$ is a weak equivalence. On the other hand, the map $L_{C^*_{\mathrm{poly}}(\mathfrak{h})} \otimes_{C^*_{\mathrm{poly}}(\mathfrak{h})} A \longrightarrow L_A$ is identified with the projection map

$$\Omega_A \oplus \mathfrak{h}[1]^\vee \longrightarrow \Omega_A$$

by Lemma 4.1.33. The kernel of this map is contractible, since $\mathfrak{h}$ is a cofibrant contractible dg-A-module. It follows that the right vertical map in Diagram (4.1.37) is an equivalence, so that the left map is an equivalence as well. □

Proof (of Proposition 4.1.26). By Lemma 4.1.29, it suffices to show that the map $\mathfrak{g} \longrightarrow L^\vee_{A/C^*}$ is adjoint to a weak equivalence $L_{A/C^*(\mathfrak{g})} \longrightarrow \mathfrak{g}^\vee$. This map fits into a sequence of maps

$$L_{A/C^*_{\mathrm{sm}}(\mathfrak{g})} \longrightarrow L_{A/C^*(\mathfrak{g})} \longrightarrow \mathfrak{g}^\vee$$

classifying the composite map of commutative dg-algebras over A

$$C^*_{\mathrm{sm}}(\mathfrak{g}) \longrightarrow C^*(\mathfrak{g}) \xrightarrow{c} A^{\mathfrak{g}}$$

where c is as in (4.1.28). The map $L_{A/C^*_{\mathrm{sm}}(\mathfrak{g})} \longrightarrow L_{A/C^*(\mathfrak{g})}$ is an equivalence by Lemma 4.1.35, so it suffices to show that $L_{A/C^*_{\mathrm{sm}}(\mathfrak{g})} \longrightarrow \mathfrak{g}^\vee$ is an equivalence. This map can be computed explicitly: the map

$$C^*_{\mathrm{sm}}(\mathfrak{g}) \longrightarrow A^{\mathfrak{g}} = A \oplus_{\rho^\vee} \mathfrak{g}[1]^\vee$$

is simply the quotient of $C^*_{\mathrm{sm}}(\mathfrak{g})$ by the augmentation ideal $(\mathfrak{g}[1]^\vee)^2$. Unwinding the definitions, e.g. using the pullback square (4.1.23), one finds the following description of the classifying map $L_{A/C^*_{\mathrm{sm}}(\mathfrak{g})} \longrightarrow \mathfrak{g}^\vee$: it is the canonical map from the mapping cone of

$$\Omega_{C^*_{\mathrm{sm}}(\mathfrak{g})} \otimes_{C^*_{\mathrm{sm}}(\mathfrak{g})} A \cong \Omega_A \oplus \mathfrak{g}[1]^\vee \longrightarrow \Omega_A$$

to $\mathfrak{g}^\vee$. This map is a weak equivalence, which concludes the proof. □

4.2 Koszul duality

In this section we will use the results of Section 4.1 to establish equivalences between Lie algebroids and formal moduli problems (Definition 2.3.34):

Theorem 4.2.1. *For any $\mathcal{C}^\infty$-ring A, there is an adjunction of ∞-categories*

$$\mathrm{MC}\colon \mathrm{LieAlgd}_A \rightleftarrows \mathrm{FMP}_A \colon T_{A/}.$$

This adjunction is an equivalence when A is eventually coconnective.

Remark 4.2.2. The above adjunction induces an adjunction

$$\mathrm{MC}\colon \mathrm{Lie}_A \rightleftarrows \mathrm{FMP}_A^{\mathrm{aug}} \colon T_{A/} \tag{4.2.3}$$

between the ∞-category of A-linear Lie algebras (i.e. Lie algebroids over the zero Lie algebroid) and the category $\mathrm{FMP}_A^{\mathrm{aug}}$ of *augmented* formal moduli problems, i.e. objects over the functor $\mathrm{Spf}(A)$ corepresented by A. This category can also be identified with the ∞-category of functors $F\colon A/\mathcal{C}^\infty\mathrm{Alg}^{\mathrm{sm}}/A \longrightarrow \mathcal{S}$ from the ∞-category of 'split' small extensions $A \longrightarrow A' \longrightarrow A$, such that $F(A) \simeq *$ and such that F preserves pullbacks along the maps $A \longrightarrow A[\epsilon_n]$. In these terms, the equivalence (4.2.3) is established in [38].

These theorems follow formally from Proposition 4.1.2, by means of a general procedure due to Lurie [61] that we will briefly recall. We will then prove Theorem 4.2.1 in Section 4.2.2.

4.2.1 Generating the category of Lie algebroids. Categories of chain complexes or spectra endowed with a certain algebraic structure often admit a presentation in terms of generators and relations.

Definition 4.2.4. Let Ξ be a locally presentable ∞-category equipped with a collection of right adjoint functors

$$e_\alpha\colon \Xi \longrightarrow \mathrm{Sp}$$

to the ∞-category of spectra. The left adjoint to e_α sends the canonical map $\mathbb{S}^n \longrightarrow 0$ in Sp to a map in Ξ that we will denote by $K_{\alpha,n} \longrightarrow \emptyset$. We will say that an object $V \in \Xi$ is *good* if it admits a finite filtration

$$\emptyset = V^{(0)} \longrightarrow V^{(1)} \longrightarrow \dots \longrightarrow V^{(n)} \tag{4.2.5}$$

where each i admits an α and $n \leq -2$ such that $V^{(i-1)} \longrightarrow V^{(i)}$ fits into a pushout square

$$\begin{tikzcd} K_{\alpha,n} \arrow[r] \arrow[d] & V^{(i-1)} \arrow[d] \\ \emptyset \arrow[r] & V^{(i)}. \end{tikzcd} \tag{4.2.6}$$

Let $\Xi^{\mathrm{good}} \subseteq \Xi$ be the full subcategory on the good objects; it is the smallest subcategory of Ξ which contains $\emptyset$ and is closed under pushouts along the maps $K_{\alpha,n} \longrightarrow \emptyset$ with $n \leq -2$.

Proposition 4.2.7 ([61, Theorem 1.3.12])**.** *Let (Ξ, e_α) be as in Definition 4.2.4. Suppose that each e_α preserves small sifted homotopy colimits and that a map f in Ξ is an equivalence if and only if each $e_\alpha(f)$ is an equivalence of spectra. Then the right adjoint functor*

$$j^*\colon \Xi \longrightarrow \mathrm{PSh}(\Xi^{\mathrm{good}});\ V \longmapsto \mathrm{Map}_\Xi(-, V) \tag{4.2.8}$$

is fully faithful, with essential image consisting of those (space-valued) presheaves F satisfying the following two conditions:

(a) $F(\emptyset)$ is contractible.

(b) For any α and $n < 0$, F sends a pushout square of the form (4.2.6) to a pullback square of spaces.

Remark 4.2.9. One can think of Proposition (4.2.7) as a version of Brown representability: the pushout diagram (4.2.6) describes the process of attaching a cell and the proposition asserts that a functor on finite cell complexes is representable, as soon as it respects all cell attachments.

Example 4.2.10. Let A be a connective commutative dg-algebra (over a field of characteristic zero) and let Mod_A be the ∞-category of A-modules. The single functor $e\colon \mathrm{Mod}_A \longrightarrow \mathrm{Sp}$, forgetting the A-module structure, satisfies the conditions of Proposition 4.2.7. In this case, the good A-modules can be presented by the dg-A-modules whose underlying graded A-module is free on finitely many generators x_i of degree < 0. There is a natural equivalence of ∞-categories

$$\mathrm{Mod}_A^{\mathrm{good,op}} \longrightarrow \mathrm{Mod}_A^{\mathrm{f.p.},\geq 0};\ \ E \longrightarrow E[1]^\vee$$

to the ∞-category of finitely presented connective A-modules, i.e. dg-A-modules generated by finitely many generators of degree ≥ 0. Combining this equivalence with Proposition 4.2.7, one finds that Mod_A is equivalent to the ∞-category of functors

$$F\colon \mathrm{Mod}_A^{\mathrm{f.p.},\geq 0} \longrightarrow \mathcal{S}$$

that send 0 to a contractible space and preserve pullbacks along the maps $0 \longrightarrow A[n]$ with $n \geq 1$.

Proposition 4.2.7 is exactly [61, Theorem 1.3.12], replacing the category Υ^{sm} from loc. cit. by the opposite of Ξ^{good}. For completeness, we have included a brief recollection of the main ingredients of the proof (as given in [61]) at the end of this section, in Section 4.2.3.

Let us first discuss how Proposition 4.2.7 can be applied when $\Xi = \mathrm{LieAlgd}_A$ is the ∞-category of Lie algebroids over a $\mathcal{C}^\infty$-ring A. In this case, there is a composite forgetful functor

$$\mathrm{LieAlgd}_A \xrightarrow{U} \mathrm{Mod}_A/T_A \xrightarrow{\ker} \mathrm{Mod}_A \xrightarrow{e} \mathrm{Sp}$$

which preserves small limits and detects equivalences. The corresponding notion of a good Lie algebroid then unwinds as follows:

Definition 4.2.11. Let $\mathfrak{s}_n := F(0\colon A[n] \longrightarrow T_A)$ be the free Lie algebroid generated by a null-homotopic map from $A[n]$ to T_A.

Definition 4.2.12. A Lie algebroid $\mathfrak{g} \in \mathrm{LieAlgd}_A$ over A is called *good* if there exists a finite sequence of maps

$$0 = \mathfrak{g}^{(0)} \longrightarrow \mathfrak{g}^{(1)} \longrightarrow \cdots \longrightarrow \mathfrak{g}^{(n)} = \mathfrak{g}$$

in $\mathrm{LieAlgd}_A$ such that each map $\mathfrak{g}^{(i)} \longrightarrow \mathfrak{g}^{(i+1)}$ fits into a pushout square

$$\begin{tikzcd} \mathfrak{s}_{n_i} \arrow[r] \arrow[d] & \mathfrak{g}^{(i)} \arrow[d] \\ 0 \arrow[r] & \mathfrak{g}^{(i+1)} \end{tikzcd}$$

where $n_i \leq -2$. Let $\mathrm{LieAlgd}_A^{\mathrm{good}} \subseteq \mathrm{LieAlgd}_A$ be the full sub-∞-category on the good Lie algebroids.

Remark 4.2.13. The good Lie algebroids form the smallest subcategory of $\mathrm{LieAlgd}_A$ which contains the zero Lie algebroid 0 and which is closed under pushouts along all possible map $\mathfrak{s}_n \longrightarrow 0$ where $n \leq -2$. Note that there are many inequivalent maps $\mathfrak{s}_n \longrightarrow 0$, corresponding to the various null-homotopies of the map $0\colon A[n] \longrightarrow T_A$ (which is null-homotopic by assumption). In particular, an object of $\mathrm{LieAlgd}_A^{\mathrm{good}}$ need not admit a map to the zero Lie algebroid.

Remark 4.2.14. The good Lie algebroids admit the following description in terms of the model structure on dg-Lie algebroids: let us say that a dg-Lie algebroid $\mathfrak{g}$ is *very good* if it allows for a finite sequence of cofibrations

$$0 = \mathfrak{g}^{(0)} \longrightarrow \cdots \longrightarrow \mathfrak{g}^{(n)} = \mathfrak{g},$$

each of which is the pushout of a (generating) cofibration with $n_i \leq -2$

$$\mathrm{Free}\big(\partial\phi\colon A[n_i] \longrightarrow T_A\big) \longrightarrow \mathrm{Free}\big(\phi\colon A[n_i, n_i+1] \longrightarrow T_A\big). \tag{4.2.15}$$

Here ϕ is a map from the cone of $A[n_i]$ to T_A, which is determined uniquely by a degree $(n_i + 1)$ element of T_A. Then the good Lie algebroids can be presented by the very good dg-Lie algebroids over A.

Corollary 4.2.16. *Let A be a $\mathcal{C}^\infty$-ring. Then there is an equivalence between the ∞-category $\mathrm{LieAlgd}_A$ of Lie algebroids over A and the ∞-category of functors*

$$X\colon \mathrm{LieAlgd}_A^{\mathrm{good,op}} \longrightarrow \mathcal{S}$$

satisfying condition (a) and (b) of Proposition 4.2.7.

4.2.2 Proof of Theorem 4.2.1. We will now deduce Theorem 4.2.1 from Proposition 4.1.2 and Corollary 4.2.16.

Lemma 4.2.17. *Let $\mathfrak{g}$ be a very good dg-Lie algebroid over A. Then the following hold:*

(1) $\mathfrak{g}$ has a cofibrant underlying dg-A-module.

(2) Without the differential, $\mathfrak{g}$ is freely generated by a negatively graded finite-dimensional vector space over T_A.

(3) $\mathfrak{g}$ is isomorphic as a graded A-module to $\bigoplus_{n<0} A^{\oplus k_n}[n]$ for some sequence of $k_n \in \mathbb{N}_{\geq 0}$.

Proof. Assertion (1) is obvious and (3) follows immediately from (2). For (2), note that each pushout along a map (4.2.15) freely adds a single generator of degree < 0 at the level of graded Lie algebroids. □

Corollary 4.2.18. *The functor* $\mathrm{c}\colon \mathrm{LieAlgd}_A \longrightarrow \big(\mathcal{C}^\infty\mathrm{Alg}/A\big)^{\mathrm{op}}$ *restricts to a functor*

$$\mathrm{c}^*\colon \mathrm{LieAlgd}_A^{\mathrm{good}} \longrightarrow \big(\mathcal{C}^\infty\mathrm{Alg}^{\mathrm{sm}}/A\big)^{\mathrm{op}} \tag{4.2.19}$$

between the full subcategories of good Lie algebroids over A and small extensions of A. When A is an eventually coconnective $\mathcal{C}^\infty$-ring, this functor is an equivalence.

Proof. For any $n \leq -1$, Corollary 4.1.24 identifies the image of the free Lie algebroid $\mathfrak{s}_n = F(A[n])$ with

$$\mathrm{c}^*(\mathfrak{s}_n) \simeq A \oplus A[-n-1].$$

The ∞-category $\mathrm{LieAlgd}_A^{\mathrm{good}}$ is generated by pushouts along maps $\mathfrak{s}_n \longrightarrow 0$ for $n \leq -2$, while the ∞-category $\mathcal{C}^\infty\mathrm{Alg}^{\mathrm{sm}}/A$ is generated by pullbacks along $A \longrightarrow A \oplus A[n]$ with $n \geq 1$. Since c is a left adjoint, good Lie algebroids are sent to small extensions of A.

Now suppose that A is eventually coconnective. By Lemma 4.2.17, every good Lie algebroid satisfies the conditions of Proposition 4.1.2, so that (4.2.19) is fully faithful. It remains to show that it is essentially surjective.

To see this, note that the essential image of (4.2.19) is closed under pullbacks along maps $0\colon A \longrightarrow A \oplus A[n]$ with $n \geq 1$. Indeed, these maps are contained in the essential image of c^*, so that the pullback along them can equivalently be computed as the pushout along $\mathfrak{s}_{-n-1} \longrightarrow 0$ in Lie algebroids. Since $\mathrm{LieAlgd}_A^{\mathrm{good}}$ was designed to be closed under such pushouts, its image under c^* is closed under pullbacks along $A \longrightarrow A \oplus A[n]$. The small extensions form the smallest subcategory of $\mathcal{C}^\infty\mathrm{Alg}/A$ which is closed under pullbacks along these maps, so that the result follows. $\square$

Proof (of Theorem 4.2.1). Let us denote the ∞-category of presheaves satisfying conditions (a) and (b) of Proposition 4.2.7 by

$$\mathcal{E} \subseteq \mathrm{PSh}(\mathrm{LieAlgd}_A^{\mathrm{good}}).$$

Corollary 4.2.16 provides an equivalence $\mathrm{LieAlgd}_A \simeq \mathcal{E}$, so that it suffices to produce the required adjunction (equivalence) between $\mathcal{E}$ and the ∞-category of formal moduli problems. By Corollary 4.2.18, the functor c^* induces a functor (4.2.19) from good Lie algebroids to small extensions. The restriction of a formal moduli problem along c^* is a presheaf contained in $\mathcal{E}$. We therefore obtain a right adjoint functor

$$T_{A/}\colon \mathrm{FMP}_A \xrightarrow{(\mathrm{c}^*)^*} \mathcal{E} \xrightarrow{\simeq} \mathrm{LieAlgd}_A.$$

By Corollary 4.2.18, this right adjoint is an equivalence whenever A is eventually connective. $\square$

Let us conclude with some remarks about the equivalence of Theorem 4.2.1:

Remark 4.2.20. If A is eventually coconnective, the functor c^* has inverse given by the functor

$$\mathfrak{D}\colon \mathcal{C}^\infty\mathrm{Alg}^{\mathrm{sm}}/A \longrightarrow \mathrm{LieAlgd}_A^{\mathrm{good},\mathrm{op}}. \tag{4.2.21}$$

It follows that for every Lie algebroid $\mathfrak{g}$, the functor $\mathrm{MC}_\mathfrak{g}$ can be identified with the functor

$$\mathrm{MC}_\mathfrak{g}(A') = \mathrm{Map}(\mathfrak{D}(A'), \mathfrak{g}).$$

By Remark 4.1.25, one can identify $\mathfrak{D}(A')$ with the Lie algebroid of derivations of A along the fibers of $\mathrm{Spec}(A) \longrightarrow \mathrm{Spec}(A')$. One can therefore think of $\mathrm{MC}_\mathfrak{g}$ as sending every small extension A' of A to the space of flat $\mathfrak{g}$-valued connections along the fibers of $\mathrm{Spec}(A) \longrightarrow \mathrm{Spec}(A')$.

Remark 4.2.22. Recall from Example 2.3.36 that there is an adjoint pair

$$\mathcal{O}\colon \mathrm{FMP}_A \rightleftarrows \big(\mathrm{CAlg}/A\big)^{\mathrm{op}} : \mathrm{Spf}.$$

The composite functor $\mathcal{O} \circ \mathrm{MC}\colon \mathrm{LieAlgd}_A \longrightarrow \big(\mathrm{CAlg}/A\big)^{\mathrm{op}}$ preserves colimits and sends each good Lie algebroid to the commutative algebra $C^*(\mathfrak{g})$. It follows that for every Lie algebroid $\mathfrak{g}$, there is a natural equivalence of commutative algebras

$$C^*(\mathfrak{g}) \simeq \mathcal{O}(\mathrm{MC}_\mathfrak{g})$$

between the Chevalley-Eilenberg complex of $\mathfrak{g}$ and the function algebra of the formal moduli problem $\mathrm{MC}_\mathfrak{g}$. On connective covers, this equivalence identifies the $\mathcal{C}^\infty$-ring structures on both sides.

Example 4.2.23. The previous remark shows by adjunction that for any $\mathcal{C}^\infty$-ring B over A, the Lie algebroid associated to $\mathrm{Spf}(B)$ (Example 2.3.36) is given by the Lie algebroid $\mathcal{D}(B \longrightarrow A)$, i.e. the linear dual $T_{A/B} = \mathrm{Map}_A(L_{A/B}, A)$ with the Lie algebroid structure presented by Remark 4.1.25.

Example 4.2.24. If $X\colon \mathcal{C}^\infty\mathrm{Alg}^{\mathrm{sm}}/A \longrightarrow \mathcal{S}$ is a formal moduli problem, the (unique) base-point $* \in X(A)$ corresponds to a map of formal moduli problems

$$x\colon \mathrm{Spf}(A) \longrightarrow X.$$

This induces a map of tangent complexes (see Example 2.3.38) $T_{\mathrm{Spf}(A)} \longrightarrow T_X$, whose domain $T_{\mathrm{Spf}(A)}$ can be identified with (the underlying spectrum of) the tangent module $T_A = \mathrm{Hom}(L_A, A)$. Using that $\mathrm{MC}_{\mathfrak{s}_n} \simeq \mathrm{Spf}(A \oplus A[-n-1])$, one finds that the anchor map of $T_{A/X}$ fits into a fiber sequence

$$T_{A/X} \longrightarrow T_A = T_{\mathrm{Spf}(A)} \xrightarrow{\;x\;} T_X.$$

Alternatively, the anchor map can be described by the functor it represents. More precisely, Example 4.2.10 identifies the ∞-category Mod_A/T_A with the ∞-category of reduced excisive functors

$$L_A/\mathrm{Mod}_A^{\mathrm{f.p.},\geq 1} \longrightarrow \mathcal{S}$$

from the ∞-category of finitely presented, 1-connective A-modules under L_A. Under this equivalence, an object $V \longrightarrow T_A$ of Mod_A/T_A represents the functor

$$\big(L_A \longrightarrow E\big) \longmapsto \mathrm{Map}_{/T_A}(E^\vee, V).$$

The anchor map $T_{A/X} \longrightarrow T_A$ associated to the formal moduli problem X then represents the functor

$$L_A/\mathrm{Mod}_A^{\mathrm{f.p.},\geq 1} \longrightarrow \mathcal{S};\ \ \big(L_A \xrightarrow{\eta} E\big) \longmapsto X(A_\eta).$$

Example 4.2.25. Recall from Example 2.3.40 that for each $m, n \geq 0$, there is a map

$$\eta\colon A[\epsilon_m] \times_A A[\epsilon_n] \longrightarrow A[\epsilon_{m+n+1}]$$

which classifies the square zero extension $A[\epsilon_m, \epsilon_n]/(\epsilon_m^2, \epsilon_n^2)$ of $A[\epsilon_m] \times_A A[\epsilon_n]$. This map is the image under c^* of a certain map between free A-linear Lie algebras

$$f\colon F(w) \longrightarrow F(u, v).$$

The generators u, v, w have degrees $-(n+1), -(m+2)$ and $-(n+m+2)$, respectively. Unwinding the definitions, one finds that $f(w) = [u, v]$. The map $X(\eta)$ of Example 2.3.40 can therefore be identified with the Lie bracket

$$[-,-]\colon \pi_{-m-1}\ker(T_{A/X}) \times \pi_{-n-1}\ker(T_{A/X}) \longrightarrow \pi_{-m-n-2}\ker(T_{A/X})$$

restricted to the kernel of the anchor map $T_{A/X} \longrightarrow T_A$. Note that this kernel coincides with the desuspension of the tangent complex T_X from Example 2.3.38. In other words, the Lie bracket of $T_{A/X}$ controls the obstructions to extending a deformation from the wedge of two infinitesimal lines to the infinitesimal square spanned by them.

4.2.3 Proof of Proposition 4.2.7.

This section gives a brief recollection of the ideas going into the ingredients of the proof of Proposition 4.2.7, mainly so that we can refer to some of them later; all results can be found in [61, Section 1].

Lemma 4.2.26 ([61, Proposition 1.2.10, 1.5.5])**.** *Let X and Y be two functors $\Xi^{\mathrm{good,op}} \longrightarrow \mathcal{S}$ satisfying conditions (a) and (b) of Proposition 4.2.7 and let $f\colon X \longrightarrow Y$ be a natural transformation between them. Then the following assertions hold:*

(1) Suppose that f induces equivalences $X(K_{\alpha,n}) \longrightarrow Y(K_{\alpha,n})$ for all α and $n \leq -2$. Then f is a natural equivalence.

(2) Suppose that the fiber of the map $f\colon X(K_{\alpha,n}) \longrightarrow Y(K_{\alpha,n})$ over the point $Y(0) \longrightarrow Y(K_{\alpha,n})$ is connected for each α and $n \leq -2$. Then each map $X(\mathfrak{h}) \longrightarrow Y(\mathfrak{h})$ induces a surjection on π_0.

Proof. Induction along the filtration (4.2.5). □

Remark 4.2.27. For each α, the spaces $X(K_{n,\alpha})$ can be organized into a spectrum $T_\alpha(X)$, the *tangent complex* at $K_{-1,\alpha}$. In terms of reduced excisive functors, the spectrum $T_\alpha(X)$ classifies the functor

$$\mathcal{S}_*^{\mathrm{fin}} \xrightarrow{S \mapsto (K_{\alpha,-1})^S} \Xi^{\mathrm{good,op}} \xrightarrow{X} \mathcal{S}.$$

Conditions (a) and (b) of Proposition 4.2.7 guarantee that this is indeed a reduced excisive functor. The conditions of (1) and (2) of Lemma 4.2.26 are the equivalent to the conditions

(1') f induces an equivalence of spectra $T_\alpha(X) \longrightarrow T_\alpha(Y)$ for each α.
(2') the map f induces a map of spectra $T_\alpha(X) \longrightarrow T_\alpha(Y)$ whose fiber is connective.

When $X = j^*(V)$ is representable by an object $V \in \Xi$, the spectrum $T_\alpha(X)$ is equivalent to the suspension $e_\alpha(V)[1]$.

Lemma 4.2.28 ([61, Remark 1.5.4])**.** *Let $j_!$ denote the left adjoint of the functor j^* (4.2.8). If $X\colon \Xi^{\mathrm{good,op}} \longrightarrow \mathcal{S}$ is an ind-representable presheaf, then the unit map $X \longrightarrow j^*j_!X$ is an equivalence.*

Proof. Each object $K_{\alpha,n} \in \Xi$ is compact, so that Ξ^{good} consists of compact objects. It follows that j^* preserves filtered colimits, so that the result follows from the case where X is representable. □

Lemma 4.2.29 ([61, Proposition 1.5.8])**.** *Let $X\colon \Xi^{\mathrm{good,op}} \longrightarrow \mathcal{S}$ be a presheaf satisfying conditions (a) and (b) of Proposition 4.2.7. Then there exists a simplicial object $U_\bullet\colon \mathbf{\Delta}^{\mathrm{op}} \longrightarrow \mathrm{PSh}(\Xi^{\mathrm{good}})/X$ such that*

(a) each U_k with $k \geq 0$ is ind-representable.

(b) for each matching map $U_k \longrightarrow M_k(U)$, computed in $\mathrm{PSh}(\Xi^{\mathrm{good}})/X$, the induced map on tangent complexes has a connective fiber.

In particular, X is equivalent to the colimit $\operatorname{colim} U_\bullet$ in $\mathrm{PSh}(\Xi^{\mathrm{good}})$.

Proof. This follows from an application of the small object argument, see [61, Section 1.4]. □

Proof sketch of Proposition 4.2.7. It is clear that for each $V \in \Xi$, the presheaf $j^*(V) = \mathrm{Map}_\Xi(-, V)$ satisfies conditions (a) and (b) of Proposition 4.2.7. Furthermore, the functor j^* detects equivalences by Remark 4.2.27.

It therefore suffices to prove that for any presheaf X satisfying conditions (a) and (b), the associated unit map $\eta\colon X \longrightarrow j^*j_!X$ is an equivalence. When X is ind-representable, this is Lemma 4.2.28. For a general X, let $U_\bullet \longrightarrow X$ be a resolution of X by ind-representable presheaves as in Lemma 4.2.29. The unit map η determines a diagram of presheaves

$$\begin{tikzcd}
\operatorname{colim} U_\bullet \arrow[r, "\operatorname{colim}\eta", "\sim"'] \arrow[d, "\sim"'] & \operatorname{colim}(j^*j_!U_\bullet) \arrow[d] \\
X \arrow[r, "\eta"'] & j^*j_!X
\end{tikzcd}$$

The top map is the colimit of a natural equivalence and therefore an equivalence. In particular, it follows that the functor $\operatorname{colim}(j^*j_!U_\bullet)$ satisfies conditions (a) and (b). To see that the right vertical map is an equivalence, it suffices to show that it induces an equivalence on tangent complexes. The description of the tangent complex from Remark 4.2.27 shows that there is an equivalence of spectra

$$\operatorname{colim} T_\alpha(j^*j_!U_\bullet) \xrightarrow{\sim} T_\alpha\big(\operatorname{colim} j^*j_!U_\bullet\big).$$

In fact, both colimits can be computed pointwise in the category of functors $\mathcal{S}^{\mathrm{fin}}_* \longrightarrow \mathcal{S}$; the result is automatically reduced excisive. Using that $T_\alpha(j^*V) \simeq e_\alpha(V)[1]$, it then suffices to show that the natural map

$$\operatorname{colim}\Big(e_\alpha(j_!U_\bullet)[1]\Big) \longrightarrow e_\alpha(j_!X)$$

is an equivalence. This follows from the fact that $j_!$ preserves sifted colimits, since it is a left adjoint, while e_α preserves sifted colimits by assumption. □

4.3 Quasi-coherent modules

Let $X\colon \mathcal{C}^\infty\mathrm{Alg}^{\mathrm{sm}}/A \longrightarrow \mathcal{S}$ be a formal moduli problem. Informally, a *quasi-coherent module* F on X is given by a collection of B-modules F_y for every $B \in \mathcal{C}^\infty\mathrm{Alg}^{\mathrm{sm}}/A$ and every $y \in X(B)$, together with a coherent family of equivalences

$$f^*F_y = B' \otimes_B F_y \xrightarrow{\sim} F_{f(y)}$$

for every $f\colon B \longrightarrow B'$ in $\mathcal{C}^\infty\mathrm{Alg}^{\mathrm{sm}}/A$. In particular, a quasi-coherent module F on X determines an A-module F_*, by restricting to the canonical point $* \in X(A)$. We will see that F_* carries a representation of $T_{A/X}$. In fact, we will prove the following:

Theorem 4.3.1. *Let A be an eventually coconnective $\mathcal{C}^\infty$-ring and consider a formal moduli problem $X\colon \mathcal{C}^\infty\mathrm{Alg}^{\mathrm{sm}}/A \longrightarrow \mathcal{S}$ with associated Lie algebroid $T_{A/X}$. Then there is a fully faithful left adjoint functor*

$$\Psi_X\colon \mathrm{Mod}(X) \longrightarrow \mathrm{Rep}_{T_{A/X}}.$$

from the ∞-category of quasi-coherent modules over X to the ∞-category of representations of $T_{A/X}$. The underlying A-module of $\Psi_X(F)$ is naturally equivalent to the restriction F_* to the canonical basepoint $* \in X(A)$. Furthermore, the functor Ψ_X induces an equivalence

$$\mathrm{Mod}(X)^{\geq 0} \simeq \mathrm{Rep}^{\geq 0}_{T_{A/X}}$$

between the connective quasi-coherent modules (i.e. those F for which each F_y is a connective chain complex) and the $T_{A/X}$-representations whose underlying chain complex is connective.

Remark 4.3.2. In general, the functor Ψ_X realizes $\mathrm{Mod}(X)$ as a proper subcategory of the ∞-category of $T_{A/X}$-representations . In algebro-geometric situations, one can often identify $\mathrm{Rep}_{T_{A/X}}$ geometrically with the ∞-category of Ind-coherent sheaves on X in the sense of [32]. We refer to [72] for further details (see [52] for a related discussion).

Our proof of the above theorem closely follows the discussion in [61, Section 2.4]: we first consider the behaviour of representations of good Lie algebroids (Section 4.3.1) and then extend our analysis to arbitrary Lie algebroids by a gluing construction in Section 4.3.3.

4.3.1 Koszul duality for modules. Recall that for any representation of an dg-Lie algebroid, the Chevalley-Eilenberg complex $C^*(\mathfrak{g}, E)$ (Definition 4.1.16) is a module over $C^*(\mathfrak{g})$.

Lemma 4.3.3. *Let $\mathfrak{g}$ be an A-cofibrant dg-Lie algebroid. Then there is a Quillen adjunction between the projective model structures*

$$K(\mathfrak{g}) \otimes_{C^*(\mathfrak{g})} (-) \colon \mathrm{Mod}^{\mathrm{dg}}_{C^*(\mathfrak{g})} \rightleftarrows \mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}} \colon C^*(\mathfrak{g}, -)$$

where $K(\mathfrak{g})$ is the Koszul complex of $\mathfrak{g}$ (Definition 4.1.14).

Proof. Recall that $C^*(\mathfrak{g}, E) \cong \mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(K(\mathfrak{g}), E)$. When $\mathfrak{g}$ is A-cofibrant, the Koszul complex $K(\mathfrak{g})$ is a cofibrant $\mathcal{U}(\mathfrak{g})$-module and the result follows. □

Let $\Phi_{\mathfrak{g}} \colon \mathrm{Mod}_{C^*(\mathfrak{g})} \longrightarrow \mathrm{Rep}_{\mathfrak{g}}$ be the left derived functor, sending a $C^*(\mathfrak{g})$-module V to the derived tensor product $A \otimes_{C^*(\mathfrak{g})} V$. We will analyze the behaviour of the adjunction $\Phi_{\mathfrak{g}} \colon \mathrm{Mod}_{C^*(\mathfrak{g})} \leftrightarrows \mathrm{Rep}_{\mathfrak{g}} \colon C^*(\mathfrak{g}, -)$ in the case where $\mathfrak{g}$ is a good Lie algebroid over A.

Lemma 4.3.4. *If $\mathfrak{g}$ is a very good dg-Lie algebroid over A, then*

$$\Phi_{\mathfrak{g}} = A \otimes_{C^*(\mathfrak{g})} (-) \colon \mathrm{Mod}_{C^*(\mathfrak{g})} \longrightarrow \mathrm{Rep}_{\mathfrak{g}} \tag{4.3.5}$$

is fully faithful.

Proof. Let $\mathcal{K}$ be the class of objects in $\mathrm{Mod}_{C^*(\mathfrak{g})}$ for which the derived unit map is an equivalence. Then $\mathcal{K}$ is closed under finite colimits and retracts and contains $C^*(\mathfrak{g})$ by construction. It follows that the unit map is an equivalence for all compact $C^*(\mathfrak{g})$-modules. To prove that $\Phi_{\mathfrak{g}}$ is fully faithful, observe that $\mathrm{Mod}_{C^*(\mathfrak{g})} \simeq \mathrm{Ind}(\mathrm{Mod}^{\omega}_{C^*(\mathfrak{g})})$ is the ind-completion of the category of compact $C^*(\mathfrak{g})$-modules. It therefore suffices to show that the right adjoint $C^*(\mathfrak{g}, -)$ preserves filtered colimits or equivalently, that the left adjoint $\Phi_{\mathfrak{g}}$ preserves compact objects. In fact, it suffices to show that $\Phi_{\mathfrak{g}}(C^*(\mathfrak{g})) \simeq A$ is a compact $\mathfrak{g}$-representation.

Recall from Remark 4.1.15 that A fits into a cofiber sequence of $\mathcal{U}(\mathfrak{g})$-modules

$$L_{\mathfrak{g}} \longrightarrow \mathcal{U}(\mathfrak{g}) \longrightarrow A.$$

It therefore suffices to prove that $L_{\mathfrak{g}}$ is a compact object. But $\mathfrak{g}$ is a compact object in the ∞-category of Lie algebroids (over A) and the cotangent complex functor preserves compact objects, since its right adjoint (taking square zero extensions) preserves filtered colimits. □

Lemma 4.3.6. *Let $\mathfrak{g}$ be a very good dg-Lie algebroid over A and let E be a left $\mathcal{U}(\mathfrak{g})$-module whose underlying chain complex is connective. Then there exists a map $\bigoplus_\alpha A \longrightarrow E$ in the ∞-category $\mathrm{Rep}_{\mathfrak{g}}$ which induces a surjection on π_0.*

Proof. Pick representatives $e_\alpha \in E$ for the generators of $\pi_0(E)$ and consider the associated map of $\mathfrak{g}$-representations $\bigoplus_\alpha \mathcal{U}(\mathfrak{g}) \longrightarrow E$. This map is clearly surjective on π_0, so it suffices to prove that it factors (up to homotopy) as

$$\bigoplus \mathcal{U}(\mathfrak{g}) \longrightarrow \bigoplus\nolimits_\alpha A \longrightarrow E.$$

Using the cofiber sequence $L_\mathfrak{g} \longrightarrow \mathcal{U}(\mathfrak{g}) \longrightarrow A$, we therefore have to provide a null-homotopy of each composite map

$$L_\mathfrak{g} \longrightarrow \mathcal{U}(\mathfrak{g}) \xrightarrow{e_\alpha} E.$$

Since $\mathfrak{g}$ is good, it admits a filtration $0 = \mathfrak{g}^{(0)} \longrightarrow \dots \longrightarrow \mathfrak{g}^{(n)} = \mathfrak{g}$ where each $\mathfrak{g}^{(i-1)} \longrightarrow \mathfrak{g}^{(i)}$ is a pushout of $\mathfrak{s}_{k_i} \longrightarrow 0$, for some $k_i \leq -2$. Consequently, its cotangent complex $L_\mathfrak{g}$ admits a filtration by $\mathcal{U}(\mathfrak{g})$-modules

$$0 = L_\mathfrak{g}^{(0)} \longrightarrow \dots \longrightarrow L_\mathfrak{g}^{(n)} = L_\mathfrak{g}.$$

where each $L_\mathfrak{g}^{(i-1)} \longrightarrow L_\mathfrak{g}^{(i)}$ has cofiber of the form $\mathcal{U}(\mathfrak{g}) \otimes_A A[k_i+1]$. An inductive argument now shows that any map $L_\mathfrak{g} \longrightarrow E$ to a connective $\mathfrak{g}$-representation is null-homotopic, which concludes the proof. □

Corollary 4.3.7. *Let $\mathfrak{g}$ be a very good dg-Lie algebroid over A. Then the fully faithful left adjoint $\Phi_\mathfrak{g}$ from* (4.3.5) *induces an equivalence of ∞-categories*

$$\Phi_\mathfrak{g} \colon \mathrm{Mod}^{\geq 0}_{C^*(\mathfrak{g})} \longrightarrow \mathrm{Rep}^{\geq 0}_\mathfrak{g}$$

between the full subcategories consisting of modules and representations whose underlying chain complex is connective.

Proof. $\mathrm{Mod}^{\geq 0}_{C^*(\mathfrak{g})}$ is the smallest subcategory of $\mathrm{Mod}_{C^*(\mathfrak{g})}$ which is closed under colimits and extensions and which contains $C^*(\mathfrak{g})$. As a consequence, the essential image of $\mathrm{Mod}^{\geq 0}_{C^*(\mathfrak{g})}$ under $\Phi_\mathfrak{g}$ is the smallest subcategory of $\mathrm{Rep}_\mathfrak{g}$ which is closed under colimits and extensions and which contains A. Let us denote this subcategory by $\mathcal{C}$. Clearly $\mathcal{C}$ is contained in $\mathrm{Rep}^{\geq 0}_\mathfrak{g}$, so it suffices to prove the reverse inclusion.

To this end, let E be a left $\mathcal{U}(\mathfrak{g})$-module whose underlying chain complex is connective. We will inductively construct a sequence of left $\mathcal{U}(\mathfrak{g})$-modules $0 = E^{(-1)} \longrightarrow E^{(0)} \longrightarrow \dots \longrightarrow E$ with the properties that each $E^{(n)} \in \mathcal{C}$ and that each map $E^{(n)} \longrightarrow E$ induces an isomorphism on homotopy groups in degrees $< n$ and a surjection on π_n. It follows that the map $\operatorname{colim} E^{(n)} \longrightarrow E$ is a weak equivalence, so that $E \in \mathcal{C}$.

To construct this sequence, suppose we have constructed $E^{(n-1)}$ and let F be the fiber of the map $E^{(n-1)} \longrightarrow E$. Then F is a left $\mathcal{U}(\mathfrak{g})$-module whose underlying chain complex is $(n-2)$-connective. In particular, it follows from (a shift of) Lemma 4.3.6 that there exists a map $\bigoplus_\alpha A[n-2] \longrightarrow F$ which induces a surjection on π_{n-2}. Now let $E^{(n)}$ be the cofiber of the map $\bigoplus_\alpha A[n-2] \longrightarrow F \longrightarrow E^{(n-1)}$. This cofiber is contained in $\mathcal{C}$ and it follows from the five lemma that the map $E^{(n)} \longrightarrow E$ induces an isomorphism on homotopy groups in degrees $< n$ and a surjection on π_n. □

4.3.2 Naturality. To deal with representations of general Lie algebroids, we will need to know that the functor $\Phi_\mathfrak{g}$ (4.3.5) depends on the Lie algebroid $\mathfrak{g}$ in a suitably functorial fashion. Unfortunately, this is not quite true at the point-set level: the adjunction

$$K(\mathfrak{g}) \otimes_{C^*(\mathfrak{g})} (-) \colon \mathrm{Mod}^{\mathrm{dg}}_{C^*(\mathfrak{g})} \rightleftarrows \mathrm{Rep}^{\mathrm{dg}}_\mathfrak{g} \colon C^*(\mathfrak{g}, -)$$

does not strictly intertwine restriction of $\mathfrak{g}$-representations with induction of $C^*(\mathfrak{g})$-modules. However, this does become true at the level of ∞-categories. To see this, we describe the dependence of the categories $\mathrm{Mod}^{\mathrm{dg}}_{C^*(\mathfrak{g})}$ and $\mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}}$ on the Lie algebroid $\mathfrak{g}$ in terms of fibrations.

Construction 4.3.8. Let $\mathrm{Mod}^{\mathrm{dg}}$ be the category with

- objects given by tuples (A, M) where A is a commutative dg-algebra and M is an A-module.
- morphisms $(A, M) \longrightarrow (B, N)$ given by a map of dg-algebras $A \longrightarrow B$ and an A-linear map $M \longrightarrow N$.

The obvious projection $\pi\colon \mathrm{Mod}^{\mathrm{dg}} \longrightarrow \mathrm{CAlg}^{\mathrm{dg}}$ is a cocartesian fibration.

Construction 4.3.9. Fix a dg-$\mathcal{C}^\infty$-ring A. Let $\mathrm{Rep}^{\mathrm{dg}}$ be the category with

- objects given by tuples $(\mathfrak{g}, E)$ where $\mathfrak{g}$ is an A-cofibrant dg-Lie-algebroid and E is a $\mathfrak{g}$-representation.
- a morphism $(\mathfrak{g}, E) \longrightarrow (\mathfrak{h}, F)$ is a map of dg-Lie algebroids $f\colon \mathfrak{h} \longrightarrow \mathfrak{g}$ and a map $f^!E \longrightarrow F$ of $\mathfrak{h}$-representations.

The obvious projection $\pi\colon \mathrm{Rep}^{\mathrm{dg}} \longrightarrow \mathrm{LieAlgd}^{\mathrm{dg}}_A$ is a cocartesian fibration.

These two cocartesian fibrations are intertwined by the Chevalley-Eilenberg functor: there is a commuting diagram

$$\begin{tikzcd}
\mathrm{Rep}^{\mathrm{dg}} \arrow[r, "C^*"] \arrow[d, "\pi"'] & \mathrm{Mod}^{\mathrm{dg}} \arrow[d] \\
\mathrm{LieAlgd}^{\mathrm{dg,op}}_A \arrow[r, "C^*"'] & \mathrm{CAlg}^{\mathrm{dg}}
\end{tikzcd}$$

where the top functor sends $(\mathfrak{g}, E)$ to $(C^*(\mathfrak{g}), C^*(\mathfrak{g}, E))$.

It follows from [42, Proposition 2.1.4] that inverting the quasi-isomorphisms yields a commuting square of cocartesian fibrations (see also Lemma 4.4.10)

$$\begin{tikzcd}
\mathrm{Rep} \arrow[r, "C^*"] \arrow[d] & \mathrm{Mod} \arrow[d] \\
\mathrm{LieAlgd}^{\mathrm{op}}_A \arrow[r, "C^*"'] & \mathrm{CAlg}.
\end{tikzcd}$$

Let $\mathrm{Mod}_{C^*} \longrightarrow \mathrm{LieAlgd}^{\mathrm{op}}_A$ be the base change of $\mathrm{Mod} \longrightarrow \mathrm{CAlg}$ along the functor C^*, so that we obtain a map of cocartesian fibrations over $\mathrm{LieAlgd}^{\mathrm{op}}_A$

$$\begin{tikzcd}
\mathrm{Rep} \arrow[rr, "C^*"] \arrow[dr] & & \mathrm{Mod}_{C^*} \arrow[dl] \\
& \mathrm{LieAlgd}^{\mathrm{op}}_A. &
\end{tikzcd}$$

Lemma 4.3.10. *The functor $C^*\colon \mathrm{Rep} \longrightarrow \mathrm{Mod}_{C^*}$ admits a left adjoint Φ. Furthermore, this left adjoint Φ preserves cocartesian edges.*

In other words, this lemma asserts that the functors $\Phi_{\mathfrak{g}}$ determine a natural transformation $\mathrm{Mod}_{C^*(-)} \longrightarrow \mathrm{Rep}_{(-)}$ between diagrams of ∞-categories.

Proof. For each Lie algebroid $\mathfrak{g}$, the functor $C^*\colon \mathrm{Rep}_{\mathfrak{g}} \longrightarrow \mathrm{Mod}_{C^*(\mathfrak{g})}$ admits a left adjoint $\Phi_{\mathfrak{g}}$ (4.3.5). By [62, Proposition 7.3.2.11], the existence of the global left adjoint $\Phi\colon \mathrm{Mod}_{C^*} \longrightarrow \mathrm{Rep}$ (as well as the fact that it preserves cocartesian edges) follows once we know that for any map of dg-Lie algebroids $\phi\colon \mathfrak{g} \longrightarrow \mathfrak{h}$ and any $C^*(\mathfrak{h})$-module M, the natural map

$$\Phi_{\mathfrak{g}}\big(C^*(\mathfrak{g}) \otimes_{C^*(\mathfrak{h})} M\big) \longrightarrow \Phi_{\mathfrak{h}}(M)$$

is an equivalence. Since both functors preserve colimits of modules, we can reduce to the case where M is equivalent to $C^*(\mathfrak{h})$. In that case, the map can be identified with the map

$$K(\mathfrak{g}) \longrightarrow K(\mathfrak{h})$$

between Koszul complexes. But both $K(\mathfrak{g})$ and $K(\mathfrak{h})$ were resolutions of the canonical representation A, so the result follows. □

Corollary 4.3.11. *Let A be a $\mathcal{C}^\infty$-ring and consider the restriction*

$$\begin{tikzcd} \mathrm{Rep}^{\mathrm{good},\geq 0} \arrow[rr, "C^*"] \arrow[dr] & & \mathrm{Mod}_{C^*}^{\mathrm{good},\geq 0} \arrow[dl] \\ & \mathrm{LieAlgd}_A^{\mathrm{good,op}} & \end{tikzcd}$$

of the functor C^* to the full subcategory of good Lie algebroids and connective modules over them (resp. over their Chevalley-Eilenberg complex). Then C^* is an equivalence and in particular preserves cocartesian edges.

Proof. Note that both C^* and its left adjoint Φ send the given subcategories to each other. The left adjoint Φ preserves cocartesian edges and gives a fiberwise equivalence of ∞-categories by Corollary 4.3.7, so that it is an equivalence [59, Proposition 3.1.3.5]. This implies that C^* is an equivalence as well. □

Remark 4.3.12. The above discussion can be extended to show that Φ determines a natural transformation between diagrams of *symmetric monoidal* ∞-categories. To this end, one extends the above fibrations to fibrations $\mathrm{Rep}^{\otimes} \longrightarrow \Gamma^{\mathrm{op}} \times \mathrm{LieAlgd}_A^{\mathrm{op}}$ encoding the symmetric monoidal structure. For more details, see [61].

4.3.3 Quasi-coherent modules. We can summarize the situation of Lemma 4.3.10 in terms of functors, rather than fibrations, as follows. Lemma 4.3.10 gives a natural transformation of between functors

$$\begin{tikzcd} \mathrm{LieAlgd}_A^{\mathrm{op}} \arrow[r, bend left, "\mathrm{Mod}_{C^*}"{name=U}] \arrow[r, bend right, "\mathrm{Rep}"'{name=D}] & \mathrm{Pr}^{\mathrm{L}} \arrow[from=U, to=D, Rightarrow, "\Phi"] \end{tikzcd}$$

taking values in presentable ∞-categories and left adjoint functors between them. The natural transformation Φ is given pointwise by the functor $\Phi_{\mathfrak{g}}$ of (4.3.5), which is fully faithful for good Lie algebroids (Lemma 4.3.4) and induces an equivalence between connective modules and representations (Corollary 4.3.7).

When A is eventually coconnective, we can precompose with the functor $\mathfrak{D}$ (4.2.21) and we obtain a natural transformation of functors

$$\begin{tikzcd} \mathcal{C}^\infty\mathrm{Alg}^{\mathrm{sm}}/A \arrow[rr, "\mathrm{forget}"] \arrow[d] & & \mathrm{CAlg}^{\geq 0} \arrow[d, "\mathrm{Mod}"] \arrow[dl, Rightarrow, "\Psi"] \\ \mathrm{Fun}(\mathcal{C}^\infty\mathrm{Alg}^{\mathrm{sm}}/A, \mathcal{S})^{\mathrm{op}} \arrow[r, "\mathfrak{D}_!"'] & \mathrm{LieAlgd}_A^{\mathrm{op}} \arrow[r, "\mathrm{Rep}"'] & \mathrm{Pr}^{\mathrm{L}} \end{tikzcd} \tag{4.3.13}$$

Here the left vertical functor is the inclusion of the corepresentable functors, so that the composite functor $\mathcal{C}^\infty\mathrm{Alg}^{\mathrm{sm}}/A \longrightarrow \mathrm{LieAlgd}_A^{\mathrm{op}}$ is just $\mathfrak{D}$. The functor Ψ is induced by the natural transformation Φ: for each A' in $\mathcal{C}^\infty\mathrm{Alg}^{\mathrm{sm}}/A$, it is given by the composite functor

$$\Psi_{A'}\colon \mathrm{Mod}_{A'} \longrightarrow \mathrm{Mod}_{C^*\mathfrak{D}(A')} \xrightarrow{\Phi_{\mathfrak{D}(A')}} \mathrm{Rep}_{\mathfrak{D}(A')} \tag{4.3.14}$$

where the first functor arises from the algebra map $A' \longrightarrow C^*\mathfrak{D}(A')$.

Since the ∞-category Pr^{L} admits small limits, we can form the right Kan extension of the functor Mod along the Yoneda embedding (see [59, Lemma 5.1.5.5])

$$\begin{tikzcd}
\mathcal{C}^\infty\mathrm{Alg}^{\mathrm{sm}}/A \arrow[r, "\mathrm{Mod}"] \arrow[d] & \mathrm{Pr}^{\mathrm{L}} \\
\mathrm{Fun}(\mathcal{C}^\infty\mathrm{Alg}^{\mathrm{sm}}/A, \mathcal{S})^{\mathrm{op}}. \arrow[ur, dotted] &
\end{tikzcd}$$

Definition 4.3.15. Let $X\colon \mathcal{C}^\infty\mathrm{Alg}^{\mathrm{sm}}/A \longrightarrow \mathcal{S}$ be a functor. The ∞-category $\mathrm{Mod}(X)$ of *quasi-coherent modules* on X is the value of the above right Kan extension on X.

By its universal property, we obtain a natural transformation

$$\begin{tikzcd}
\mathrm{Fun}\big(\mathcal{C}^\infty\mathrm{Alg}^{\mathrm{sm}}/A, \mathcal{S}\big)^{\mathrm{op}} \arrow[r, bend left, "\mathrm{Mod}"] \arrow[r, bend right, "\mathrm{Rep}_{\mathfrak{D}_!}"'] \arrow[r, phantom, "\Downarrow\Psi"] & \mathrm{Pr}^{\mathrm{L}}
\end{tikzcd}$$

which restricts to Ψ (4.3.13) on the corepresentable functors.

Remark 4.3.16. The ∞-category $\mathrm{Mod}(X)$ can be computed as a homotopy limit

$$\mathrm{Mod}(X) = \lim_{B\to A, y\in X(B)} \mathrm{Mod}_B.$$

In other words, an object of $\mathrm{Mod}(X)$ is given informally by a family of B-modules F_y for each $y \in X(B)$, together with a coherent family of equivalences $f^*(F_y) \longrightarrow F_{f(y)}$ for each map $f\colon B \longrightarrow B'$.

Remark 4.3.17. Each ∞-category Mod_B is locally presentable and any map $f\colon B \longrightarrow B'$ in $\mathcal{C}^\infty\mathrm{Alg}^{\mathrm{sm}}/A$ induces a left adjoint functor $f^*\colon \mathrm{Mod}_B \longrightarrow \mathrm{Mod}_{B'}$. It follows from [59, Proposition 5.5.3.13] that the ∞-category $\mathrm{Mod}(X)$ is locally presentable for any functor $X\colon \mathcal{C}^\infty\mathrm{Alg}^{\mathrm{sm}}/A \longrightarrow \mathcal{S}$.

By construction, the functor Ψ determines a (left adjoint) functor

$$\Psi_X\colon \mathrm{Mod}(X) \longrightarrow \mathrm{Rep}_{\mathfrak{D}_!(X)}.$$

When X is a formal moduli problem, the Lie algebroid $\mathfrak{D}_!(X)$ is naturally equivalent to the Lie algebroid $T_{A/X}$ (by Remark 4.2.20). To understand the functor Ψ_X, we use the following lemma:

Lemma 4.3.18. *Consider the functor* $\mathrm{Rep}\colon \mathrm{LieAlgd}_A^{\mathrm{op}} \longrightarrow \mathrm{Cat}_\infty$ *sending* $\mathfrak{g}$ *to the* ∞*-category of* $\mathfrak{g}$*-representations. This functor sends a sifted colimit diagram in* $\mathrm{LieAlgd}_A$ *to a limit diagram in* Cat_∞.

Proof. The functor Rep decomposes as the composite functor

$$\mathrm{LieAlgd}_A^{\mathrm{op}} \xrightarrow{\mathcal{U}} \mathrm{AssAlg}^{\mathrm{op}} \xrightarrow{\mathrm{LMod}} \widehat{\mathrm{Cat}}_\infty.$$

where $\mathcal{U}$ takes the enveloping algebra of a Lie algebroid and LMod takes ∞-categories of left modules. Recall that the enveloping algebra $\mathcal{U}(\mathfrak{g})$ was just the algebra of unary operations of the reduced enveloping operad $\overline{\mathrm{Env}}_{\mathfrak{g}}$ (Definition 3.2.4). It follows from Theorem 3.2.18 that $\mathcal{U}\colon \mathrm{LieAlgd}_A \longrightarrow \mathrm{AssAlg}$ preserves sifted colimits. The functor LMod sends a sifted colimit of algebras to a limit in $\widehat{\mathrm{Cat}}_\infty$ by [61, Lemma 2.4.32]. □

Proof (of Theorem 4.3.1). When X is representable by a small extension A', the functor $\mathrm{Mod}(X) \longrightarrow \mathrm{Rep}_{T_{A/X}}$ is given by the composite (4.3.14). The first functor is an equivalence by Proposition 4.1.2 and Theorem 4.3.1 reduces to Lemma 4.3.4 and Corollary 4.3.7.

The functor $\mathrm{Mod}\colon \mathrm{Fun}(\mathcal{C}^\infty\mathrm{Alg}^{\mathrm{sm}}/A, \mathcal{S})^{\mathrm{op}} \longrightarrow \widehat{\mathrm{Cat}}_\infty$ preserves limits by construction. Similarly, the functor $X \mapsto \mathrm{Rep}_{\mathfrak{D}_!(X)}$ preserves sifted limits, by Lemma 4.3.18. Since every formal moduli problem is a sifted colimit of representable functors (Lemma 4.2.29), it follows that $\Psi_X\colon \mathrm{Mod}(X) \longrightarrow \mathrm{Rep}_{T_{A/X}}$ is fully faithful (being a limit of fully faithful functors).

To identify the essential image of $\mathrm{Mod}(X)^{\geq 0}$, let $E \in \mathrm{Mod}(X)$. Then E is connective if and only if each $y^*E \in \mathrm{Mod}_B$ is connective. In terms of its image $\Psi_X(E)$, this means that for each $\mathfrak{D}(B) \longrightarrow T_{A/X}$, the restricted representation $\Psi_X(E) \in \mathrm{Rep}_{\mathfrak{D}(B)}$ is contained in the essential image of $\mathrm{Mod}_B^{\geq 0} \longrightarrow \mathrm{Rep}_{\mathfrak{D}(B)}$. By Corollary 4.3.7, this is equivalent to $\Psi_X(E)$ being a connective representation of $T_{A/X}$. □

Theorem 4.3.1 does not quite give an equivalence of categories: the free $\mathfrak{g}$-representation $\mathcal{U}(\mathfrak{g})$ is usually not contained in the essential image of Ψ_X. However, we do have the following:

Corollary 4.3.19. *Let $A \in \mathcal{C}^\infty\mathrm{Alg}^{\geq 0}$ be eventually coconnective and let X be a formal moduli problem such that $T_{A/X}$ is connective. Then the functor*

$$\Psi_X\colon \mathrm{Mod}(X) \longrightarrow \mathrm{Rep}_{T_{A/X}}$$

is an equivalence.

Proof. Since $T_{A/X}$ is connective, its enveloping algebra $\mathcal{U}(T_{A/X})$ is connective as well. It follows (see e.g. [62, Example 2.2.1.3]) that the stable ∞-category $\mathrm{Rep}_{T_{A/X}}$ carries a right complete t-structure, where $\mathrm{Rep}^{\geq 0}_{T_{A/X}}$ consists of those representations whose underlying chain complex is connective. Analogously, $\mathrm{Mod}(X)$ carries a right complete t-structure where $F \in \mathrm{Mod}(X)^{\geq 0}$ if and only if y^*F is a connective chain complex for all $y \in X(B)$.

The functor Ψ_X fits into a sequence of locally presentable ∞-categories and left adjoint functors between them

$$\begin{tikzcd}
\mathrm{Mod}(X)^{\geq 0} \arrow[r] \arrow[d, "\Psi_X^{\geq 0}"'] & \mathrm{Mod}(X)^{\geq -1} \arrow[r] \arrow[d, "\Psi_X^{\geq -1}"] & \dots \arrow[r] & \mathrm{Mod}(X) \arrow[d, "\Psi_X"] \\
\mathrm{Rep}^{\geq 0}_{T_{A/X}} \arrow[r] & \mathrm{Rep}^{\geq -1}_{T_{A/X}} \arrow[r] & \dots \arrow[r] & \mathrm{Rep}_{T_{A/X}}.
\end{tikzcd}$$

where the horizontal functors are the obvious inclusions. The horizontal sequences are (homotopy) colimit diagrams of locally presentable ∞-categories by right t-completeness; this means that the associated diagrams of right adjoint functors are limit diagrams of ∞-categories [59, Theorem 5.5.3.18]. Since the vertical functors $\Phi^{\geq -i}$ are equivalences by Theorem 4.3.1, the result follows. □

4.4 Quasi-coherent algebras

We have seen in Section 2.3.3 that for any connective $\mathcal{P}$-algebra R over a $\mathcal{C}^\infty$-ring A (in the sense of Definition 2.3.18), there is a formal moduli problem

$$\mathrm{Def}_R\colon \mathcal{C}^\infty\mathrm{Alg} \longrightarrow \mathcal{S};\ \ A' \longmapsto \mathcal{P}\mathrm{Alg}^{\geq 0}_{A'} \times_{\mathcal{P}\mathrm{Alg}^{\geq 0}_{A}} \{R\}$$

sending a small extension A' to the space of deformations of R to a $\mathcal{P}$-algebra over A'. The goal of this section is to show that the associated Lie algebroid is precisely the Atiyah Lie algebroid $\mathrm{At}_{\mathcal{P}}(R)$ of Example 3.1.3. More precisely, we will prove the following (folklore) result (cf. [41, 95] for a discussion of the case where A is a field):

Theorem 4.4.1. *Let A be a cofibrant dg-$\mathcal{C}^\infty$-ring and let R be a fibrant-cofibrant connective $\mathcal{P}$-algebra over A, as in Definition 2.3.18. Then there is an equivalence of Lie algebroids*

$$\theta\colon \mathrm{At}_{\mathcal{P}}(R) \xrightarrow{\sim} T_{A/\mathrm{Def}_R}.$$

Remark 4.4.2. For simplicity, we will assume all $\mathcal{P}$-algebras to be *connective* by definition throughout the rest of this section.

Remark 4.4.3. In many cases, one can also use Theorem 4.4.1 when R only has a cofibrant underlying A-module, using an operadic bar-cobar construction. For example, suppose that R is a non-unital associative algebra in $\mathrm{Mod}_A^{\geq 0,\mathrm{dg}}$ and let $\mathrm{bar}(R) = T_A(R[1])$ be its bar construction, which is a non-counital coalgebra. Let $\overline{\mathrm{HH}}_{\mathrm{At}}(R, R)[1]$ be the graded A-module of tuples

$$\big(v, \Delta_v\colon \mathrm{bar}(R) \longrightarrow \mathrm{bar}(R)\big)$$

where Δ_v is a both a connection and a coderivation on $\mathrm{bar}(R)$. The commutator bracket and the obvious projection make $\overline{\mathrm{HH}}_{\mathrm{At}}(R, R)[1]$ a (fibrant) dg-Lie algebroid over T_A.

Every such coderivation (v, Δ_v) determines a derivation (v, ∇_v) on the cobar construction $R^{\mathrm{cof}} = \Omega(\mathrm{bar}(R))$. When the underlying dg-A-module of R is cofibrant, the latter provides a cofibrant replacement of R and one obtains a map of transitive Lie algebroids

$$\overline{\mathrm{HH}}_{\mathrm{At}}(R, R)[1] \longrightarrow \mathrm{At}(R^{\mathrm{cof}}).$$

One can show that this map is a weak equivalence, since it induces a weak equivalence between the fibers over T_A. The latter are just the ordinary reduced Hochschild complex of R (no quotient by inner derivations) and the complex of A-linear derivations of R^{cof}. In particular, the 'Lie algebroid version' $\overline{\mathrm{HH}}_{\mathrm{At}}(R, R)[1]$ of the reduced Hochschild complex governs the (derived) deformation theory of the associative algebra R.

The proof of Theorem 4.4.1 is given in Section 4.4.3 and relies on an identification between *quasi-coherent $\mathcal{P}$-algebras* over a formal moduli problem X, and $\mathcal{P}$-algebras over A endowed with an action of the Lie algebroid $T_{A/X}$ (Theorem 4.4.16). This does not follow immediately from Theorem 4.3.1, or a monoidal refinement thereof: our notion of a $\mathcal{P}$-algebra also includes (diagrams of) $\mathcal{C}^\infty$-rings, which are not quite algebras over an operad and therefore require special care.

In Section 4.4.1, we will start by discussing the homotopy theory of $\mathcal{P}$-algebras in Mod_A endowed with an action of a dg-Lie algebroid. In Section 4.4.2, we then relate quasi-coherent $\mathcal{P}$-algebras over X to $\mathcal{P}$-algebras with an action by $T_{A/X}$.

4.4.1 Lie algebroid actions on algebras. Recall from Variant 3.3.10 that for any A-cofibrant dg-Lie algebroid $\mathfrak{g}$, the category $\mathrm{Rep}_{\mathfrak{g}}^{\geq 0,\mathrm{dg}}$ carries the *A-model structure*, in which a map is a weak equivalence (cofibration) if the underlying map of connective dg-A-modules is a weak equivalence (cofibration) in the projective model structure. This model structure makes $\mathrm{Rep}_{\mathfrak{g}}^{\geq 0,\mathrm{dg}}$ a monoidal model category for the tensor product of Lemma 3.3.12. Let

$$\mathcal{P}\mathrm{Rep}_{\mathfrak{g}}^{\geq 0,\mathrm{dg}} := \mathcal{P}\mathrm{Alg}(\mathrm{Rep}_{\mathfrak{g}}^{\geq 0,\mathrm{dg}})$$

denote the category of connective dg-$\mathcal{P}$-algebras with an action of $\mathfrak{g}$. More precisely, $\mathcal{P}\mathrm{Rep}_{\mathfrak{g}}^{\geq 0,\mathrm{dg}}$ consists of one of the following three types of objects, as in Definition 2.3.18:

(a) a connective algebra R over a dg-operad $\mathcal{P}$ with respect to the monoidal structure on $\mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}}$, as in Remark 3.3.14.

(b) a $\mathcal{J}$-indexed diagram of $\mathcal{C}^\infty$-rings $A \longrightarrow R_\bullet$, equipped with an action of $\mathfrak{g}$ on $R_\bullet$ by natural $\mathcal{C}^\infty$-derivations, as in Remark 3.3.15.

(c) a $\mathcal{J}$-indexed diagram of augmented $\mathcal{C}^\infty$-rings $A \longrightarrow R_\bullet \longrightarrow A$, equipped with an action of $\mathfrak{g}$ on $R_\bullet$ by natural $\mathcal{C}^\infty$-derivations, as in Remark 3.3.15.

For any map of dg-Lie algebroids $f\colon \mathfrak{g} \longrightarrow \mathfrak{h}$, the restriction functor $f^!$ preserves $\mathcal{P}$-algebras and fits into commuting squares

$$\begin{tikzcd} \mathcal{P}\mathrm{Rep}^{\geq 0,\mathrm{dg}}_{\mathfrak{h}} \arrow[r, "\text{forget}"] \arrow[d, "f^!"'] & \prod \mathrm{Rep}^{\geq 0,\mathrm{dg}}_{\mathfrak{h}} \arrow[d, "f^!"] \\ \mathcal{P}\mathrm{Rep}^{\geq 0,\mathrm{dg}}_{\mathfrak{g}} \arrow[r, "\text{forget}"'] & \prod \mathrm{Rep}^{\geq 0,\mathrm{dg}}_{\mathfrak{g}} \end{tikzcd} \qquad \begin{tikzcd} \prod \mathrm{Rep}^{\geq 0,\mathrm{dg}}_{\mathfrak{h}} \arrow[r, "\mathcal{P}"] \arrow[d, "f^!"'] & \mathcal{P}\mathrm{Rep}^{\geq 0,\mathrm{dg}}_{\mathfrak{h}} \arrow[d, "f^!"] \\ \prod \mathrm{Rep}^{\geq 0,\mathrm{dg}}_{\mathfrak{g}} \arrow[r, "\mathcal{P}"'] & \mathcal{P}\mathrm{Rep}^{\geq 0,\mathrm{dg}}_{\mathfrak{g}} \end{tikzcd} \tag{4.4.4}$$

Here $\mathcal{P}$ takes the free $\mathcal{P}$-algebra on a family of $\mathfrak{g}$-representations, indexed by the colours of the operad $\mathcal{P}$ or the objects of $\mathcal{J}$. To see that the right square commutes, let E be a collection of $\mathfrak{g}$-representations and let $\mathcal{P}(E)$ be the free $\mathcal{P}$-algebra on the underlying collection of dg-A-modules. There is a canonical action of $\mathfrak{g}$ on $\mathcal{P}(E)$ by derivations, determined uniquely by the condition that for any $X \in \mathfrak{g}$, the $\mathcal{P}$-algebra derivation

$$\nabla_X\colon \mathcal{P}(E) \longrightarrow \mathcal{P}(E)$$

is given on generators by the $\mathfrak{g}$-representation on E. It follows that free algebras can be computed at the level of the underlying dg-A-modules, which implies that the right square commutes.

A similar argument shows that $f^!$ preserves limits and colimits of $\mathcal{P}$-algebras in $\mathrm{Rep}^{\geq 0,\mathrm{dg}}_{\mathfrak{g}}$, since these can be computed at the level of $\mathcal{P}$-algebras in dg-A-modules. Indeed, let $\{R_i\}$ be a diagram of $\mathcal{P}$-algebras over A, equipped with compatible derivations $\nabla_{X,i}\colon R_i \longrightarrow R_i$ for each $X \in \mathfrak{g}$. These derivations determine unique derivations

$$\mathrm{colim}_i(\nabla_{X,i})\colon\ \mathrm{colim}_i\, R_i \longrightarrow \mathrm{colim}_i\, R_i$$

on the colimit of the diagram $\{R_i\}$ in the category of dg-$\mathcal{P}$-algebras over A.

Lemma 4.4.5. *The category $\mathcal{P}\mathrm{Rep}^{\geq 0,\mathrm{dg}}_{\mathfrak{g}}$ carries a model structure, in which a map is a weak equivalence (fibration) if the underlying map in $\mathcal{P}\mathrm{Rep}^{\geq 0,\mathrm{dg}}_{\mathfrak{g}}$ is a weak equivalence (fibration) in the A-model structure. For any map $f\colon \mathfrak{g} \longrightarrow \mathfrak{h}$ of A-cofibrant dg-Lie algebroids, restriction and coinduction along f give rise to a Quillen adjunction*

$$f^!\colon \mathcal{P}\mathrm{Rep}^{\geq 0,\mathrm{dg}}_{\mathfrak{h}} \rightleftarrows \mathcal{P}\mathrm{Rep}^{\geq 0,\mathrm{dg}}_{\mathfrak{g}} : f_!$$

which is a Quillen equivalence if f is a weak equivalence.

Proof. In order to transfer the A-model structure along the (right adjoint) forgetful functor $\mathcal{P}\mathrm{Rep}^{\geq 0,\mathrm{dg}}_{\mathfrak{g}} \longrightarrow \prod \mathrm{Rep}^{\geq 0,\mathrm{dg}}_{\mathfrak{g}}$, we have to verify the following: each transfinite composition of pushouts of maps

$$\mathcal{P}(i)\colon \mathcal{P}(E) \longrightarrow \mathcal{P}(F)$$

with $i\colon E \longrightarrow F$ a trivial cofibration in $\mathrm{Rep}^{\geq 0,\mathrm{dg}}_{\mathfrak{g}}$, is a quasi-isomorphism. Since free $\mathcal{P}$-algebras and colimits can be computed at the level of dg-A-modules, we can reduce to case where $\mathfrak{g} = 0$. In other words, it suffices to show that $\mathcal{P}\mathrm{Rep}_0^{\geq 0,\mathrm{dg}} = \mathcal{P}\mathrm{Alg}_A^{\geq 0,\mathrm{dg}}$ carries a transferred model structure, which is obvious.

The restriction functor $f^!\colon \mathrm{Rep}_{\mathfrak{h}}^{\geq 0,\mathrm{dg}} \longrightarrow \mathrm{Rep}_{\mathfrak{g}}^{\geq 0,\mathrm{dg}}$ is left Quillen functor and commutes with the free $\mathcal{P}$-algebra functor. It follows that it induces a left Quillen functor at the level of $\mathcal{P}$-algebras in $\mathfrak{g}$-representations, with right adjoint given by the coinduction functor $f_!$. The derived unit and counit of the resulting Quillen pair can be computed at the level of the underlying representations, which implies that $(f^!, f_!)$ is a Quillen equivalence whenever f is a weak equivalence. □

Let $\mathcal{P}\mathrm{Rep}_{\mathfrak{g}}^{\geq 0}$ denote the ∞-category associated to this model structure, consisting of connective $\mathcal{P}$-algebras with a $\mathfrak{g}$-action. It follows from Lemma 4.4.5 that there is a functor

$$\mathcal{P}\mathrm{Rep}^{\geq 0}\colon \mathrm{LieAlgd}_A^{\mathrm{op}} \longrightarrow \mathrm{Pr}^{\mathrm{L}};\ \ \mathfrak{g} \longmapsto \mathcal{P}\mathrm{Rep}_{\mathfrak{g}}^{\geq 0} \tag{4.4.6}$$

sending a map $f\colon \mathfrak{g} \longrightarrow \mathfrak{h}$ to the left adjoint functor $f^!$.

Corollary 4.4.7. *The functor* (4.4.6) *sends sifted colimits of Lie algebroids to sifted limits of ∞-categories.*

Proof. Consider the functor

$$\chi\colon \mathrm{LieAlgd}_A^{\mathrm{op}} \longrightarrow \mathrm{Fun}(\Delta[1], \mathrm{Pr}^{\mathrm{L}});\ \ \mathfrak{g} \longmapsto \Big(\mathcal{P}\colon \prod \mathrm{Rep}_{\mathfrak{g}}^{\geq 0} \longrightarrow \mathcal{P}\mathrm{Rep}_{\mathfrak{g}}^{\geq 0}\Big)$$

whose value on an arrow $f\colon \mathfrak{g} \longrightarrow \mathfrak{h}$ is given by the commuting square of left adjoints

$$\begin{tikzcd} \prod \mathrm{Rep}_{\mathfrak{h}}^{\geq 0} \arrow[r, "\mathcal{P}"] \arrow[d, "f^!"'] & \mathcal{P}\mathrm{Rep}_{\mathfrak{h}}^{\geq 0} \arrow[d, "f^!"] \\ \prod \mathrm{Rep}_{\mathfrak{g}}^{\geq 0} \arrow[r, "\mathcal{P}"'] & \mathcal{P}\mathrm{Rep}_{\mathfrak{g}}^{\geq 0}. \end{tikzcd}$$

The commuting diagrams (4.4.4) show that the forgetful functor commutes with $f^!$ as well, so that the above square is right adjointable in the sense of Definition 2.3.22. We therefore obtain a functor

$$\chi\colon \mathrm{LieAlgd}_A^{\mathrm{op}} \longrightarrow \mathrm{Fun}^{\mathrm{RAd}}(\Delta[1], \mathrm{Cat}_\infty) \subseteq \mathrm{Fun}(\Delta[1], \mathrm{Cat}_\infty)$$

with values in the subcategory from Definition 2.3.22.

Now let $\mathfrak{g}\colon \mathcal{I} \longrightarrow \mathrm{LieAlgd}_A$ be a sifted diagram of Lie algebroids over A with colimit $\mathfrak{g}_\infty = \mathrm{colim}_i\, \mathfrak{g}_i$ and let $f_i\colon \mathfrak{g}_i \longrightarrow \mathfrak{g}_\infty$ be the universal maps. By [62, Corollary 4.7.4.18], the canonical map $\chi(\mathfrak{g}_\infty) \longrightarrow \lim_i \chi(\mathfrak{g}_i)$ corresponds to the right adjointable square of ∞-categories

$$\begin{tikzcd} \prod \mathrm{Rep}_{\mathfrak{g}_\infty}^{\geq 0} \arrow[r, "\mathcal{P}"] \arrow[d] & \mathcal{P}\mathrm{Rep}_{\mathfrak{g}_\infty}^{\geq 0} \arrow[d] \\ \lim\limits_i \prod \mathrm{Rep}_{\mathfrak{g}_i}^{\geq 0} \arrow[r, "\mathcal{P}"'] & \lim\limits_i \mathcal{P}\mathrm{Rep}_{\mathfrak{g}_i}^{\geq 0}. \end{tikzcd}$$

The right adjoints of the horizontal functors are the forgetful functors, which detect equivalences. Remark 2.3.24 now implies that the right vertical functor is an equivalence whenever the left vertical functor is an equivalence. But the left vertical functor is a product of equivalences by Lemma 4.3.18. □

4.4.2 Quasi-coherent algebras. Let E be a representation of an (A-cofibrant) dg-Lie algebroid $\mathfrak{g}$ and let

$$\mathrm{c}^*(\mathfrak{g}, E) := \tau_{\geq 0} C^*(\mathfrak{g}, E)$$

be the connective cover of its Chevalley-Eilenberg complex. Remark 4.1.18 and Lemma 4.1.20 imply that $\mathrm{c}^*(\mathfrak{g}, E)$ is a $\mathcal{P}$-algebra whenever E is a $\mathcal{P}$-algebra. In other words, we obtain a functor

$$\mathrm{c}^*(\mathfrak{g}, -)\colon \mathcal{P}\mathrm{Rep}_{\mathfrak{g}}^{\geq 0,\mathrm{dg}} \longrightarrow \mathcal{P}\mathrm{Alg}_{\mathrm{c}^*(\mathfrak{g})}^{\geq 0,\mathrm{dg}}.$$

To study the naturality of this functor in $\mathfrak{g}$, let us consider the following constructions:

Construction 4.4.8. Let $\mathcal{P}\mathrm{Alg}^{\geq 0,\mathrm{dg}}$ be the category whose

- objects are given by tuples (B, R) where $B \in \mathcal{C}^\infty\mathrm{Alg}^{\mathrm{dg}}$ is a cofibrant dg-$\mathcal{C}^\infty$-ring and R is a dg-$\mathcal{P}$-algebra over B.
- morphisms $(B, R) \longrightarrow (C, S)$ are maps of cofibrant dg-$\mathcal{C}^\infty$-rings $f\colon B \longrightarrow C$, together with a map $R \longrightarrow f_*(S)$ of dg-$\mathcal{P}$-algebras over B.

Similarly, let $\mathcal{P}\mathrm{Rep}^{\geq 0,\mathrm{dg}}$ be the category whose

- objects are given by tuples $(\mathfrak{g}, R)$, where $\mathfrak{g}$ is an A-cofibrant dg-Lie algebroid and R is a $\mathcal{P}$-algebra with an action of $\mathfrak{g}$.
- morphisms $(\mathfrak{g}, R) \longrightarrow (\mathfrak{h}, S)$ are maps of dg-Lie algebroids $f\colon \mathfrak{h} \longrightarrow \mathfrak{g}$, together with a map $f^! R \longrightarrow S$ of $\mathcal{P}$-algebras in $\mathrm{Rep}_{\mathfrak{h}}^{\geq 0,\mathrm{dg}}$.

Let us fix a cofibrant replacement functor Q for dg-$\mathcal{C}^\infty$-rings (over A) and let $\tilde{\mathrm{c}}^*$ denote the composition $Q \circ \mathrm{c}^*$. If R is a dg-$\mathcal{P}$-algebra with an action of $\mathfrak{g}$, then the Chevalley-Eilenberg complex $\mathrm{c}^*(\mathfrak{g}, E)$ has the natural structure of a $\mathcal{P}$-algebra over $\tilde{\mathrm{c}}^*(\mathfrak{g})$ by restriction. We therefore obtain a commuting diagram

$$\begin{tikzcd} \mathcal{P}\mathrm{Rep}^{\geq 0,\mathrm{dg}} \arrow[r, "\mathrm{c}^*"] \arrow[d] & \mathcal{P}\mathrm{Alg}^{\geq 0,\mathrm{dg}} \arrow[d] \\ \left(\mathrm{LieAlgd}_A^{\mathrm{dg},A-\mathrm{cof}}\right)^{\mathrm{op}} \arrow[r, "\tilde{\mathrm{c}}^*"'] & \mathcal{C}^\infty\mathrm{Alg}^{\mathrm{dg,cof}} \end{tikzcd} \tag{4.4.9}$$

where the top functor sends $(\mathfrak{g}, R)$ to $(\tilde{\mathrm{c}}^*(\mathfrak{g}), \mathrm{c}^*(\mathfrak{g}, E))$ and the vertical functors are the obvious projections.

Lemma 4.4.10. *After inverting the quasi-isomorphisms, the square* (4.4.9) *induces a commuting square of ∞-categories*

$$\begin{tikzcd} \mathcal{P}\mathrm{Rep}^{\geq 0} \arrow[r, "\mathrm{c}^*"] \arrow[d, "p"'] & \mathcal{P}\mathrm{Alg}^{\geq 0} \arrow[d, "q"] \\ \mathrm{LieAlgd}_A^{\mathrm{op}} \arrow[r, "\mathrm{c}^*"'] & \mathcal{C}^\infty\mathrm{Alg} \end{tikzcd} \tag{4.4.11}$$

in which the vertical functors are cartesian and cocartesian fibrations.

Proof. Every functor in (4.4.9) preserves weak equivalences, so we indeed obtain a diagram of ∞-categories as asserted. To see that p is a cocartesian fibration, note that the functor

$$\mathcal{P}\mathrm{Rep}^{\geq 0,\mathrm{dg}} \longrightarrow \left(\mathrm{LieAlgd}_A^{\mathrm{dg},A-\mathrm{cof}}\right)^{\mathrm{op}}$$

is a cocartesian fibration classified by the functor

$$\left(\mathrm{LieAlgd}_A^{\mathrm{dg},A-\mathrm{cof}}\right)^{\mathrm{op}} \longrightarrow \mathrm{ModCat}^{\mathrm{L}};\ \mathfrak{g} \longmapsto \mathcal{P}\mathrm{Rep}_{\mathfrak{g}}^{\geq 0,\mathrm{dg}}.$$

The left Quillen functor $f^!$ associated to a map $f\colon \mathfrak{g} \longrightarrow \mathfrak{h}$ preserves all weak equivalences, and is a weak equivalence between relative categories whenever f is a weak equivalence, by Lemma 4.4.5. It follows from [42, Proposition 2.1.4] that $\mathcal{P}\mathrm{Rep}^{\geq 0} \longrightarrow \mathrm{LieAlgd}_A^{\mathrm{op}}$ is a cocartesian fibration. For every arrow $f\colon \mathfrak{g} \longrightarrow \mathfrak{h}$, the restriction functor

$$f^!\colon \mathcal{P}\mathrm{Rep}_{\mathfrak{h}}^{\geq 0} \longrightarrow \mathcal{P}\mathrm{Rep}_{\mathfrak{g}}^{\geq 0}$$

admits a right adjoint $f_!$, given by the right derived functor of the right Quillen functor $f_!$. It follows that $p\colon \mathcal{P}\mathrm{Rep}^{\geq 0} \longrightarrow \mathrm{LieAlgd}_A^{\mathrm{op}}$ is a cartesian fibration as well.

A similar argument applies to q: the functor $\mathcal{P}\mathrm{Alg}^{\geq 0,\mathrm{dg}} \longrightarrow \mathcal{C}^\infty\mathrm{Alg}^{\mathrm{dg,cof}}$ is a cartesian fibration, classified by the functor

$$\left(\mathcal{C}^\infty\mathrm{Alg}^{\mathrm{dg,cof}}\right)^{\mathrm{op}} \longrightarrow \mathrm{ModCat}^{\mathrm{R}};\ B \longmapsto \mathcal{P}\mathrm{Alg}_B^{\geq 0,\mathrm{dg}}.$$

The right Quillen functor f_* associated to a map $f\colon B \longrightarrow C$ preserves all weak equivalences, and is a weak equivalence between relative categories whenever f is a weak equivalence, by the discussion above Proposition 2.3.21. It follows from [42, Proposition 2.1.4] that the functor between localizations $\mathcal{P}\mathrm{Alg}^{\geq 0} \longrightarrow \mathcal{C}^\infty\mathrm{Alg}$ is a cartesian fibration. For every map f in $\mathcal{C}^\infty\mathrm{Alg}$, the functor between the fibers admits a left adjoint, given by the left derived functor of f^*. □

Proposition 4.4.12. *The restriction of* (4.4.11) *to good Lie algebroids over A*

$$\begin{tikzcd}
\mathcal{P}\mathrm{Rep}^{\geq 0}\times_{\mathrm{LieAlgd}_A^{\mathrm{op}}}\left(\mathrm{LieAlgd}_A^{\mathrm{good}}\right)^{\mathrm{op}} \arrow[rr, "\mathrm{c}^*"] \arrow[dr] & & \mathcal{P}\mathrm{Alg}^{\geq 0}\times_{\mathcal{C}^\infty\mathrm{Alg}}\left(\mathrm{LieAlgd}_A^{\mathrm{good}}\right)^{\mathrm{op}} \arrow[dl] \\
& \left(\mathrm{LieAlgd}_A^{\mathrm{good}}\right)^{\mathrm{op}} &
\end{tikzcd}$$

is an equivalence of cartesian and cocartesian fibrations.

Proof. Let $\mathcal{C} = \left(\mathrm{LieAlgd}_A^{\mathrm{good}}\right)^{\mathrm{op}}$ and note that there is a commuting square

$$\begin{tikzcd}
\mathcal{P}\mathrm{Rep}^{\geq 0}\times_{\mathrm{LieAlgd}_A^{\mathrm{op}}}\mathcal{C} \arrow[r, "\mathrm{c}^*"] \arrow[d, "\mathrm{forget}"'] & \mathcal{P}\mathrm{Alg}^{\geq 0}\times_{\mathcal{C}^\infty\mathrm{Alg}}\mathcal{C} \arrow[d, "\mathrm{forget}"] \\
\prod\mathrm{Rep}^{\geq 0}\times_{\mathrm{LieAlgd}_A^{\mathrm{op}}}\mathcal{C} \arrow[r, "\mathrm{c}^*"'] & \prod\mathrm{Mod}^{\geq 0}\times_{\mathcal{C}^\infty\mathrm{Alg}}\mathcal{C}
\end{tikzcd}$$

where the vertical functors forget the $\mathcal{P}$-algebra structure. The bottom functor is a product of equivalences, by Corollary 4.3.11.

For each of the above ∞-categories, the projection to $\left(\mathrm{LieAlgd}_A^{\mathrm{good}}\right)^{\mathrm{op}}$ is a cartesian fibration. Furthermore, the vertical functors preserve and detect cartesian edges. Indeed, let $f\colon \mathfrak{g} \longrightarrow \mathfrak{h}$ be a map of good Lie algebroids. A map $(\mathfrak{h}, F) \longrightarrow (\mathfrak{g}, E)$ in $\mathcal{P}\mathrm{Rep}^{\geq 0}$ covering f is cartesian if and only if $F \simeq f_! E$. Such maps are detected by the forgetful functor by the commuting square of right Quillen functors adjoint to the right square in (4.4.4).

On the other hand, consider a map $\phi\colon (\mathrm{c}^*(\mathfrak{h}), F) \longrightarrow (\mathrm{c}^*(\mathfrak{g}), E)$ in $\mathcal{P}\mathrm{Alg}^{\geq 0}$ covering the image of f under c^*. Then ϕ is a cartesian edge if and only if $F \simeq \mathrm{c}^*(f)_* E$ is obtained from E by restriction of scalars along $\mathrm{c}^*(f)$. Again, such maps are detected by the forgetful functor because of the commuting square (2.3.19). We conclude that the functor

$$\mathcal{P}\mathrm{Rep}^{\geq 0} \times_{\mathrm{LieAlgd}_A^{\mathrm{op}}} \mathcal{C} \xrightarrow{\mathrm{c}^*} \mathcal{P}\mathrm{Alg}^{\geq 0} \times_{\mathcal{C}^\infty\mathrm{Alg}} \mathcal{C} \tag{4.4.13}$$

preserves cartesian edges. It therefore suffices to verify that c^* induces equivalences between fibers [59, Proposition 2.4.4.4].

To see this, let $\mathfrak{g}$ be a good Lie algebroid and let $f\colon 0 \longrightarrow \mathfrak{g}$ be the canonical map. The vertical functors, forgetting the $\mathcal{P}$-algebra structure, also preserve and detect *cocartesian* edges that cover f. Indeed, an edge $(\mathfrak{g}, F) \longrightarrow (0, E)$ in $\mathcal{P}\mathrm{Rep}^{\geq 0}$ covering f is cocartesian if and only if $E \simeq f^! F$, which can be checked at the level of the underlying representations by the commuting square (4.4.4).

On the other hand, let $(\mathrm{c}^*(\mathfrak{g}), F) \longrightarrow (A, E)$ be an arrow in $\mathcal{P}\mathrm{Alg}^{\geq 0}$ covering the image of f under c^*. Such an arrow is cocartesian if and only if $E \simeq \mathrm{c}^*(f)^* F$. Since the map $\mathrm{c}^*(f)\colon \mathrm{c}^*(\mathfrak{g}) \longrightarrow A$ induces a surjection on π_0, Lemma 2.3.25 implies from such arrows are detected by the functor forgetting $\mathcal{P}$-algebra structures.

It follows that the functor c^* (4.4.13) preserves all cartesian and cocartesian edges that cover the map $f\colon 0 \longrightarrow \mathfrak{g}$, since it does so after forgetting $\mathcal{P}$-algebra structures. This means that for every good Lie algebroid $\mathfrak{g}$, the commuting square of right adjoints

$$\begin{tikzcd}
\mathcal{P}\mathrm{Rep}_0^{\geq 0} \arrow[r, "f_!"] \arrow[d, "\mathrm{c}^*"'] & \mathcal{P}\mathrm{Rep}_{\mathfrak{g}}^{\geq 0} \arrow[d, "\mathrm{c}^*"] \\
\mathcal{P}\mathrm{Alg}_A^{\geq 0} \arrow[r, "\mathrm{c}^*(f)_*"'] & \mathcal{P}\mathrm{Alg}_{\mathrm{c}^*(\mathfrak{g})}^{\geq 0}
\end{tikzcd}$$

is left adjointable (Definition 2.3.22). The left functor c^* is clearly an equivalence and the horizontal functors have left adjoints that detect equivalences, since they do at the level of representations and modules. It follows from Remark 2.3.24 (after passing to opposite categories) that the right functor c^* is an equivalence as well. □

Let Φ denote the inverse of the equivalence of Proposition 4.4.12. Under straightening, this determines a natural equivalence of diagrams of locally presentable ∞-categories

$$\begin{tikzcd}
\mathrm{LieAlgd}_A^{\mathrm{good,op}} \arrow[r, bend left=40, "\mathcal{P}\mathrm{Alg}^{\geq 0} \circ\, \mathrm{c}^*"{name=U}] \arrow[r, bend right=40, "\mathcal{P}\mathrm{Rep}^{\geq 0}"'{name=D}] & \mathrm{Pr}^{\mathrm{L}}. \arrow[from=U, to=D, Rightarrow, "\Phi", "\sim"']
\end{tikzcd}$$

We can now repeat the argument of Section 4.3.3: when A is eventually coconnective, we can precompose with the functor $\mathfrak{D}$ (4.2.21) and we obtain a natural equivalence of functors

$$\begin{tikzcd}
\mathcal{C}^\infty\mathrm{Alg}^{\mathrm{sm}}/A \arrow[rr, "\mathrm{forget}"] \arrow[d] & & \mathrm{CAlg}^{\geq 0} \arrow[d, "\mathcal{P}\mathrm{Alg}^{\geq 0}"] \\
\mathrm{Fun}(\mathcal{C}^\infty\mathrm{Alg}^{\mathrm{sm}}/A, \mathcal{S})^{\mathrm{op}} \arrow[r, "\mathfrak{D}_!"'] \arrow[urr, phantom, "\Leftarrow\Psi"] & \mathrm{LieAlgd}_A^{\mathrm{op}} \arrow[r, "\mathcal{P}\mathrm{Rep}^{\geq 0}"'] & \mathrm{Pr}^{\mathrm{L}}
\end{tikzcd}$$

where $\mathfrak{D}_!$ is the unique colimit-preserving functor whose restriction to the corepresentable functors is $\mathfrak{D}$. The functor Ψ is induced by the natural transformation Φ: for each A' in $\mathcal{C}^\infty\mathrm{Alg}^{\mathrm{sm}}/A$, it is given by the composite functor

$$\Psi_{A'}\colon \mathcal{P}\mathrm{Alg}_{A'}^{\geq 0} \longrightarrow \mathcal{P}\mathrm{Alg}_{C^*\mathfrak{D}(A')}^{\geq 0} \xrightarrow{\Phi_{\mathfrak{D}(A')}} \mathcal{P}\mathrm{Rep}_{\mathfrak{D}(A')}^{\geq 0} \tag{4.4.14}$$

where the first functor arises from the map $A' \longrightarrow C^*\mathfrak{D}(A')$.

Definition 4.4.15. Let $X\colon \mathcal{C}^\infty\mathrm{Alg}^{\mathrm{sm}}/A \longrightarrow \mathcal{S}$ be a functor. The locally presentable ∞-category $\mathcal{P}\mathrm{Alg}^{\geq 0}(X)$ of *connective quasi-coherent $\mathcal{P}$-algebras over X* is the value on X of the right Kan extension χ

$$\begin{tikzcd} \mathcal{C}^\infty\mathrm{Alg}^{\mathrm{sm}}/A \arrow[r, "\mathcal{P}\mathrm{Alg}^{\geq 0}"] \arrow[d] & \mathrm{Pr}^{\mathrm{L}} \\ \mathrm{Fun}(\mathcal{C}^\infty\mathrm{Alg}^{\mathrm{sm}}/A, \mathcal{S})^{\mathrm{op}} \arrow[ur, dotted, "\chi"'] & \end{tikzcd}$$

Informally, a quasi-coherent $\mathcal{P}$-algebra over X is a family of $\mathcal{P}$-algebras $R_y \in \mathcal{P}\mathrm{Alg}^{\geq 0}_B$ for every $y \in X(B)$, together with a coherent family of equivalences

$$f^*R_y \xrightarrow{\simeq} R_{f(y)}$$

for every $f\colon B \longrightarrow B'$.

By the universal property of $\mathcal{P}\mathrm{Alg}^{\geq 0}(X)$, the functor Ψ determines a natural left adjoint functor

$$\Psi_X\colon \mathcal{P}\mathrm{Alg}^{\geq 0}(X) \longrightarrow \mathcal{P}\mathrm{Rep}^{\geq 0}_{\mathfrak{D}_!(X)}.$$

When X is a formal moduli problem, the Lie algebroid $\mathfrak{D}_!(X)$ is naturally equivalent to the Lie algebroid $T_{A/X}$ (by Remark 4.2.20).

Theorem 4.4.16. *Let A be an eventually coconnective $\mathcal{C}^\infty$-ring and let X be a formal moduli problem under A. Then the natural functor*

$$\Psi_X\colon \mathcal{P}\mathrm{Alg}^{\geq 0}(X) \longrightarrow \mathcal{P}\mathrm{Rep}^{\geq 0}_{T_{A/X}}$$

is an equivalence of ∞-categories.

Proof. This is exactly as the proof of Theorem 4.3.1: when X is corepresentable by $A' \in \mathcal{C}^\infty\mathrm{Alg}^{\mathrm{sm}}/A$, the functor Ψ_X is the composite (4.4.14) of two equivalences, by Theorem 4.1.2 and Proposition 4.4.12.

The functor $\mathcal{P}\mathrm{Alg}^{\geq 0}(-)$ sends sifted colimits of functors to limits of ∞-categories by construction. Similarly, the composite $X \mapsto \mathcal{P}\mathrm{Rep}^{\geq 0}_{\mathfrak{D}_!(X)}$ sends sifted colimits to ∞-categories by Corollary 4.4.7. ☐

4.4.3 Deformations of algebras. We will now use the results of the previous section to study the formal deformations of a (connective) $\mathcal{P}$-algebra R over A and to prove Theorem 4.4.1. More precisely, consider the formal moduli problem from Example 2.3.35

$$\mathrm{Def}_R\colon \mathcal{C}^\infty\mathrm{Alg}^{\mathrm{sm}}/A \longrightarrow \mathcal{S};\ \ A' \longmapsto \mathcal{P}\mathrm{Alg}^{\geq 0}_{A'} \times_{\mathcal{P}\mathrm{Alg}^{\geq 0}_A} \{R\}. \tag{4.4.17}$$

This functor sends each small extension of A to the space of $\mathcal{P}$-algebras R' over A', endowed with an equivalence of $\mathcal{P}$-algebras $R' \otimes_{A'} A \simeq R$. On the other hand, let us consider the functor

$$\mathrm{Act}_R\colon \mathrm{LieAlgd}^{\mathrm{op}}_A \longrightarrow \widehat{\mathcal{S}};\ \ \mathfrak{g} \longmapsto \mathcal{P}\mathrm{Rep}^{\geq 0}_{\mathfrak{g}} \times_{\mathcal{P}\mathrm{Alg}^{\geq 0}_A} \{R\}$$

sending each Lie algebroid $\mathfrak{g}$ to the ∞-category of $\mathcal{P}$-algebras $\widetilde{R}$ with an action of $\mathfrak{g}$ by derivations, together with an equivalence $\widetilde{R} \simeq R$ of $\mathcal{P}$-algebras over A. Maps between objects are necessarily equivalences, so that Act_R takes values in (locally small) spaces. In other words, one can think of $\mathrm{Act}_R(\mathfrak{g})$ as the space of $\mathfrak{g}$-actions on R by derivations.

It follows from Corollary 4.4.7 that the functor Act_R sends colimits of Lie algebroids to limits of (locally small) spaces. For example, the value on each free Lie algebroid $\mathfrak{s}_n = F(A[n])$ (Definition 4.2.11) is given by

$$\mathrm{Act}_R(\mathfrak{s}_n) \simeq \Omega_0\mathrm{Act}_R(\mathfrak{s}_{n-1})$$

Because every Lie algebroid is a (small) colimit of $\mathfrak{s}_n$, it follows that Act_R does not only take values in locally small spaces, but in small spaces.

Lemma 4.4.18. *The limit-preserving functor* $\mathrm{Act}_R\colon \mathrm{LieAlgd}_A^{\mathrm{op}} \longrightarrow \mathcal{S}$ *is represented by the Lie algebroid* T_{A/Def_R} *associated to the formal moduli problem* (4.4.17).

Proof. By Proposition 4.4.12, the restriction of Act_R to the good Lie algebroids is naturally equivalent to the functor

$$\mathrm{LieAlgd}_A^{\mathrm{good,op}} \longrightarrow \mathcal{S};\ \mathfrak{g} \longmapsto \mathrm{Def}_R(\mathrm{c}^*(\mathfrak{g})).$$

By construction, this functor is represented by T_{A/Def_R} (see Section 4.2.2). □

Now suppose that A is a cofibrant dg-$\mathcal{C}^\infty$-ring and that R is presented by a cofibrant dg-$\mathcal{P}$-algebra over A. Recall from Example 3.1.3 that the Atiyah Lie algebroid $\mathrm{At}_{\mathcal{P}}(R)$ consists of tuples (v, ∇_v) consisting of an element v in T_A , together with an $\mathbb{R}$-linear $\mathcal{P}$-algebra derivation $\nabla_v\colon R \longrightarrow R$ such that

$$\nabla_v(a \cdot r) = v(a)\cdot r + a\cdot \nabla_v(r)$$

for all $a \in A$ and $r \in R$. In particular, there is a canonical action of $\mathrm{At}_{\mathcal{P}}(R)$ on R by $\mathcal{P}$-algebra derivations. This determines a point

$$R_{\mathrm{can}} \in \mathrm{Act}_R\big(\mathrm{At}_{\mathcal{P}}(R)\big)$$

which is classified by a map

$$\theta\colon \mathrm{At}_{\mathcal{P}}(R) \longrightarrow T_{A/\mathrm{Def}_R} \tag{4.4.19}$$

in the ∞-category of Lie algebroids over A. Theorem 4.4.1 asserts that the map (4.4.19) is an equivalence. To prove this, let us start with a few observations:

Lemma 4.4.20. *Let V be a chain complex of $\mathbb{R}$-vector spaces, let $\mathfrak{g} = F(V)$ be the free dg-Lie algebroid on $0\colon V \longrightarrow T_A$ and let $r\colon \mathfrak{g} \longrightarrow 0$ be the canonical map. The right Quillen functor of Lemma 4.4.5*

$$r_!\colon \mathcal{P}\mathrm{Rep}_{\mathfrak{g}}^{\geq 0,\mathrm{dg}} \longrightarrow \mathcal{P}\mathrm{Rep}_0^{\geq 0,\mathrm{dg}} = \mathcal{P}\mathrm{Alg}_A^{\geq 0,\mathrm{dg}}$$

has a right derived functor sending a $\mathcal{P}$-algebra R with a $\mathfrak{g}$-action to the square zero extension

$$R \oplus_{\nabla^\vee} \tau_{\geq 0}\mathrm{Hom}_{\mathbb{R}}(V[1], R) \tag{4.4.21}$$

with differential $\partial(r, \alpha) = (\partial r, \partial(\alpha) + \nabla_{(-)}r)$.

Proof. The functor $r_!$ sends a $\mathcal{P}$-algebra R with an action of $\mathfrak{g}$ to the subalgebra of $r \in R$ such that $\nabla_v r = 0$ for all $v \in V$. For any such dg-$\mathcal{P}$-algebra R with a $\mathfrak{g}$-action, there is a weak equivalence

$$R \xrightarrow[\sim]{(\mathrm{id},0,0)} R^{\mathrm{inj}} := R \oplus \tau_{\geq 0}\mathrm{Hom}(V[0,1], R) \tag{4.4.22}$$

to the (split) square zero extension of R by the connective cover of the contractible R-module $\mathrm{Hom}(V[0,1], R)$. This inclusion becomes $\mathfrak{g}$-equivariant if we endow R^{inj} with the action of $\mathfrak{g}$ determined on generators $v \in V \subseteq \mathfrak{g}$ by

$$\nabla_v\big(r, \alpha\big) = \big(\nabla_v(r) + \alpha(v), 0\big).$$

Unraveling the definitions, one sees that the V-fixed points $r_!(R^{\mathrm{inj}})$ are isomorphic to the (non-split) square zero extension (4.4.21). In particular, the functor $R \mapsto r_!(R^{\mathrm{inj}})$ preserves weak equivalences, so that it indeed describes the right derived functor of $r_!$. □

Lemma 4.4.23. *Let R be a cofibrant dg-$\mathcal{P}$-algebra over A and let $\mathfrak{s}_n = F(\mathbb{R}[n])$ be the free Lie algebroid. Then there is an isomorphism*

$$\pi_0\mathrm{Act}_R(\mathfrak{s}_n) \cong \pi_n\mathrm{Der}_A^{\mathcal{P}}(R, R). \tag{4.4.24}$$

Proof. Consider the pushout square in the ∞-category $\mathrm{LieAlgd}_A$

$$\begin{tikzcd} \mathfrak{s}_{n-1} \arrow[r, "r"] \arrow[d, "r"'] & 0 \arrow[d, "i"] \\ 0 \arrow[r, "i"'] & \mathfrak{s}_n \end{tikzcd} \tag{4.4.25}$$

We will use i to denote the map from the initial Lie algebroid 0. The functor Act_R sends colimits of Lie algebroids to limits of spaces, so that there is an equivalence

$$\sigma\colon \mathrm{Act}_R(\mathfrak{s}_n) \xrightarrow{\cong} \Omega\mathrm{Act}_R(\mathfrak{s}_{n-1}).$$

The loop space of $\mathrm{Act}_R(\mathfrak{s}_{n-1})$ is taken at $r^!R$, which is simply the zero action of $\mathfrak{s}_{n-1}$ on R. The loop space $\Omega\mathrm{Act}_R(\mathfrak{s}_{n-1})$ can be identified with the fiber of

$$\mathrm{Map}\big(r^!R, r^!R\big) \xrightarrow{i^!} \mathrm{Map}\big(i^!r^!R, i^!r^!R\big)$$

over the identity map of $i^!r^!R = R$. The restriction functors $r^!$ and $i^!$ have right adjoint coinduction functors $r_!, i_!$ satisfying $r_!i_! \simeq \mathrm{id}$. Using this, one can identify $\Omega\mathrm{Act}_R(\mathfrak{s}_{n-1})$ with the space of dotted sections

$$\begin{tikzcd} R \arrow[r, dotted] \arrow[dr, "\simeq"'] & r_!r^!R \arrow[d, "r_!(\eta)"] \\ & r_!i_!i^!r^!R \end{tikzcd}$$

in $\mathcal{P}\mathrm{Rep}_0^{\geq 0} \simeq \mathcal{P}\mathrm{Alg}_A^{\geq 0}$. By Lemma 4.4.20, we can identify $r_!(\eta)$ with the projection

$$R \oplus \tau_{\geq 0}(R[-n]) \longrightarrow R.$$

The space of dotted sections is therefore equivalent to the space of A-linear $\mathcal{P}$-algebra derivations of R with coefficients in $R[-n]$, so that we obtain the desired isomorphism (4.4.24). ☐

Lemma 4.4.26. *Under the isomorphism (4.4.24), an element $v \in \pi_n\mathrm{Der}_A^{\mathcal{P}}(R, R)$ corresponds to (the equivalence class of) the $\mathfrak{s}_n$-action on R where the generator of $\mathfrak{s}_n$ acts by v.*

Proof. The pushout (4.4.25) can be modeled by the homotopy pushout square of dg-Lie algebroids over A

$$\begin{tikzcd} \mathfrak{s}_{n-1} = F\big(\mathbb{R}[n-1]\big) \arrow[r] \arrow[d, "r"'] & \mathfrak{s}_{n-1,n} = F\big(\mathbb{R}[n-1, n]\big) \arrow[d] \\ 0 \arrow[r] & \mathfrak{s}_n = F\big(\mathbb{R}[n]\big). \end{tikzcd} \tag{4.4.27}$$

Let h be the generator of $\mathfrak{s}_n$ and let z be the generator of $\mathfrak{s}_{n-1}$, so that $\partial h = z$ in $\mathfrak{s}_{n-1,n}$. Let us use $R_{v,w}$ to denote the action of $\mathfrak{s}_{n-1,n}$ on R where the generator h acts by $v\colon R \longrightarrow R[-n]$ and z acts by $w = \partial(v)\colon R \longrightarrow R[1-n]$. Similarly, we will use R_v to denote the representation of $\mathfrak{s}_n$ where h acts by $v\colon R \longrightarrow R[-n]$.

Let us fix an element $v \in \pi_n \mathrm{Der}_A^{\mathcal{P}}(R,R)$ and let R_v be the associated action of $\mathfrak{s}_n$. We then have a commuting square in the category $\mathcal{P}\mathrm{Rep}^{\geq 0,\mathrm{dg}}$ (see Construction 4.4.8) of the form

$$\begin{tikzcd} (R_0)^{\mathrm{inj}} & (R_{0,0})^{\mathrm{inj}} \arrow[l, "\mathrm{id}"'] \\ R \arrow[u, "{(\mathrm{id},v,0)}"] & R_v \arrow[l, "\mathrm{id}"] \arrow[u, "{(\mathrm{id},v,0)}"'] \end{tikzcd} \tag{4.4.28}$$

whose image under the cocartesian fibration $p\colon \mathrm{Rep}^{\mathrm{dg}} \longrightarrow \mathrm{LieAlgd}_A^{\mathrm{dg,op}}$ is the pushout square (4.4.27). The left hand side of (4.4.28) is simply obtained from the right hand side by forgetting the action of the generator h. Let us therefore describe the right vertical map.

The $\mathfrak{s}_{n-1,n}$-representation $\big(R_{0,0}\big)^{\mathrm{inj}}$ is the resolution of the *trivial* representation on R provided by (4.4.22). Unraveling the definitions, this resolution is given by $R \oplus \tau_{\geq 0} R[-n, 1-n]$, where the generators h and z of $\mathfrak{s}_{n-1,n}$ act by the derivations

$$\nabla_h(r_0, r_1, r_2) = (r_2, 0, 0) \qquad\qquad \nabla_z(r_0, r_1, r_2) = (r_1, 0, 0).$$

The restriction of the representation R_v along $\mathfrak{s}_{n-1,n} \longrightarrow \mathfrak{s}_n$ is given by $R_{v,0}$. The right vertical map in (4.4.28) then arises from the map of $\mathfrak{s}_{n-1,n}$-representations

$$(\mathrm{id}, v, 0)\colon R_{v,0} \longrightarrow R \oplus \big(\tau_{\geq 0} R[-n]\big) \oplus \big(\tau_{\geq 0} R[1-n]\big) = \big(R_{0,0}\big)^{\mathrm{inj}}.$$

This map is an $\mathfrak{s}_{n-1,n}$-equivariant weak equivalence. It follows that $\big(R_{0,0}\big)^{\mathrm{inj}}$ is a model for the (derived) restriction of R_v along $\mathfrak{s}_{n-1,n} \longrightarrow \mathfrak{s}_n$. In other words, the right vertical map in (4.4.28) determines a p-cocartesian edge in the ∞-category $\mathcal{P}\mathrm{Rep}^{\geq 0}$. Similarly, the left vertical map determines a p-cocartesian edge, so that the entire square (4.4.28) consists of p-cocartesian arrows.

The left vertical map $R \longrightarrow (R_0)^{\mathrm{inj}}$ is adjoint to a map of $\mathcal{P}$-algebras over A, which is just

$$(\mathrm{id}, v)\colon R \longrightarrow r_!\Big(R_0^{\mathrm{inj}}\Big) = R \oplus \tau_{\geq 0} R[-n] \tag{4.4.29}$$

Because (4.4.28) consists of p-cocartesian edges, unwinding the definitions from Lemma 4.4.23 shows that the image of $R_v \in \pi_0 \mathrm{Act}_R(\mathfrak{s}_n)$ under the isomorphism (4.4.24) is the derivation classifying (4.4.29). This derivation is exactly v, which proves the assertion. □

Proof (of Theorem 4.4.1). To show that the map θ (4.4.19) is an equivalence, it suffices to show that for any $n \in \mathbb{Z}$, the map

$$[\mathfrak{s}_n, \mathrm{At}_{\mathcal{P}}(R)] \xrightarrow{\ \theta_*\ } [\mathfrak{s}_n, T_{A/\mathrm{Def}_R}] \xrightarrow{\ \cong\ } \pi_0 \mathrm{Act}_R(\mathfrak{s}_n) \tag{4.4.30}$$

is an isomorphism. Since R is a cofibrant dg-$\mathcal{P}$-algebra over A, $\mathrm{At}_{\mathcal{P}}(R)$ is a fibrant dg-Lie algebroid (Example 3.1.11). Evaluation on the generator then defines an isomorphism

$$\big[\mathfrak{s}_n, \mathrm{At}_{\mathcal{P}}(R)\big] \xrightarrow{\ \cong\ } \big[\mathbb{R}[n], \mathrm{Der}_A^{\mathcal{P}}(R,R)\big] \cong \pi_n \mathrm{Der}_A^{\mathcal{P}}(R,R).$$

The map θ_* (4.4.30) sends $f\colon \mathfrak{s}_n \longrightarrow \mathrm{At}_{\mathcal{P}}(R)$ to the restriction $f^!(R_{\mathrm{can}})$ of the canonical $\mathrm{At}_{\mathcal{P}}(R)$-action on R. For the map associated to an element $v \in \pi_n \mathrm{Der}_A^{\mathcal{P}}(R,R)$, this is precisely the action of $\mathfrak{s}_n$ on R where the generator acts by v. Lemma 4.4.26 now shows that θ_* is an isomorphism. □

Chapter 5

Derived differential topology

The aim of the next chapters is to study the relation between Lie algebroids and the global geometry of moduli spaces. The equivalence between Lie algebroids and formal moduli problems already takes a step in this direction: we can think of formal moduli problems as formal moduli spaces, built out of nilpotent extensions of $\mathcal{C}^\infty$-rings. To compare these objects to geometric objects like smooth manifolds, it will be convenient to work in a setting where manifolds and (derived) infinitesimal extensions thereof are treated on an equal footing. One of the purposes of derived differential topology is to provide such a setting.

There are various treatments of derived differential topology in the literature [92, 14, 51], which emphasize the importance of derived geometry to intersection theory. In this chapter, we further develop the theory of derived differential topology, so that it also incorporates some of the nilpotent aspects relevant for deformation theory (see Chapter 6). Our treatment is based on the discussion of $\mathcal{C}^\infty$-rings in Chapter 2 and closely follows the algebro-geometric work of Toën and Vezzosi [97] and Lurie [60].

Section 5.1 discusses the basic geometric objects in derived differential topology: derived manifolds. For us, a derived manifold will simply be a topological space equipped with a structure sheaf that is locally equivalent to the spectrum of a (derived) $\mathcal{C}^\infty$-ring, i.e. we impose no a priori finiteness conditions. A large part of Section 5.1 studies the relation between $\mathcal{C}^\infty$-rings and their associated spectra, which differs from its algebro-geometric analogue: a general $\mathcal{C}^\infty$-ring A cannot be retrieved from $\mathrm{Spec}(A)$ and a general A-module cannot be retrieved from the associated sheaf on $\mathrm{Spec}(A)$.

Many moduli spaces are not quite derived manifolds, but instead arise as singular *quotients* of derived manifolds. Such quotients still exhibit good geometric behaviour, which can be described concretely in terms of stacks. Section 5.2 recalls the language of higher (derived) Lie groupoids and their associated stacks, following [97].

From the perspective of homotopy theory, smooth manifolds are very well-behaved because their local topology is very simple: they are locally just given by the contractible space $\mathbb{R}^n$. In particular, it follows that smooth manifolds have a good theory of locally constant sheaves (and the associated sheaf cohomology), which is controlled by their homotopy type. In Section 5.3, we will study a relative version of this for smooth maps between derived stacks. As an application, we show that any (higher, derived) Lie groupoid admits a 'source n-connected cover'.

5.1 Derived manifolds

Somewhat informally, derived manifolds are structured spaces that are locally equivalent to the spectrum of a $\mathcal{C}^\infty$-ring. The main purpose of this section is to recall the construction of the (Archimedean) spectrum of a (derived) $\mathcal{C}^\infty$-ring, due to [22, 66]. Following [50], we show that taking spectra of $\mathcal{C}^\infty$-rings behaves like a *localization* functor on $\mathcal{C}^\infty$-rings. In particular, different $\mathcal{C}^\infty$-rings can have equivalent spectra and among all $\mathcal{C}^\infty$-rings with the same spectrum, there is a terminal one. Furthermore, we discuss the relation between modules over $\mathcal{C}^\infty$-rings and module sheaves over their spectrum.

5.1.1 Locally $\mathcal{C}^\infty$-ringed spaces. Let us start by recalling the theory of structured spaces in higher category theory, as developed in [60] and [63].

Definition 5.1.1. Let $\mathcal{D}$ be a locally presentable ∞-category and let $\mathcal{C}$ be a small ∞-category with finite limits, equipped with a basis for a Grothendieck topology in the following sense: each object $U \in \mathcal{C}$ comes equipped with a collection of covers $\{U_i \longrightarrow U\}$, such that for any map $f\colon V \longrightarrow U$, the pullbacks $f^*U_i \longrightarrow V$ form a cover of V.

A $\mathcal{D}$-valued *sheaf* on $\mathcal{C}$ is a functor $F\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{D}$ with the property that for any open cover $\{U_i \longrightarrow U\}$, the augmented cosimplicial diagram in $\mathcal{D}$

$$F(U) \longrightarrow \prod_i F(U_i) \rightrightarrows \prod_{i,j} F\big(U_i \times_U U_j\big) \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \prod_{i,j,k} F\big(U_i \times_U U_j \times_U U_k\big) \dots$$

is a limit diagram. Let $\mathrm{Sh}(\mathcal{C};\mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathcal{D})$ be the full subcategory of $\mathcal{D}$-valued sheaves on $\mathcal{C}$. The inclusion of $\mathcal{D}$-valued sheaves into $\mathcal{D}$-valued presheaves admits a left adjoint (by the adjoint functor theorem [59, Corollary 5.5.2.9]), which takes the *associated sheaf*.

Example 5.1.2. Let X be a topological space and let $\mathcal{C}$ be the poset $\mathrm{Op}(X)$ of open subspaces of X. In this case, Definition 5.1.1 reproduces the usual notion of a sheaf on X. More generally, let $\mathcal{B}(X) \subseteq \mathrm{Op}(X)$ be a base for X *which is closed under finite intersections*. Then the ∞-category of $\mathcal{D}$-valued sheaves on X is equivalent to the ∞-category of $\mathcal{D}$-valued sheaves on $\mathcal{B}(X)$, with the induced topology (see [59, Warning 7.1.1.4] and the discussion above it).

Remark 5.1.3. A continuous function $f\colon X \longrightarrow Y$ induces an adjunction between categories of sheaves

$$f^{-1}\colon \mathrm{Sh}(Y;\mathcal{D}) \rightleftarrows \mathrm{Sh}(X;\mathcal{D})\colon f_*. \tag{5.1.4}$$

The right adjoint f_* sends a sheaf to its restriction along $f^{-1}\colon \mathrm{Op}(Y) \longrightarrow \mathrm{Op}(X)$. Alternatively, f_* can be described by restriction along $f^{-1}\colon \mathcal{B}(Y) \longrightarrow \mathcal{B}(X)$ if X and Y come equipped with a compatible basis of open subsets (closed under finite intersections).

If $\mathcal{M}$ is a combinatorial model category presenting the ∞-category $\mathcal{D}$, then $\mathrm{Sh}(X;\mathcal{D})$ can be modeled by a certain left Bousfield localization of the projective model structure on $\mathrm{Fun}(\mathrm{Op}(X)^{\mathrm{op}},\mathcal{M})$. The adjunction (5.1.4) arises from the Quillen pair which restricts and takes left Kan extension along $f^{-1}\colon \mathrm{Op}(Y) \longrightarrow \mathrm{Op}(X)$. In particular, this shows that the ∞-category $\mathrm{Sh}(X;\mathcal{D})$ depends on X in a functorial way, since it does so at the model-categorical level.

Construction 5.1.5. Let Top be the category of topological spaces and let $\mathcal{D}$ be a locally presentable ∞-category. In light of Remark 5.1.3, there is a functor $\mathrm{Sh}(-;\mathcal{D})^{\mathrm{op}}\colon \mathrm{Top} \longrightarrow \mathrm{Cat}_\infty$, sending a continuous function f to the functor f_* between the *opposites* of the categories of sheaves. This functor classifies a cocartesian fibration

$$\pi\colon \mathrm{Top}_{\mathcal{D}} \longrightarrow \mathrm{Top}$$

whose domain is the ∞-category of '$\mathcal{D}$-structured spaces': an object $(X,\mathcal{O}_X)$ is a space, equipped with a $\mathcal{D}$-valued sheaf and a map $(X,\mathcal{O}_X) \longrightarrow (Y,\mathcal{O}_Y)$ consists of a continuous function $f\colon X \longrightarrow Y$, together with a map of $\mathcal{D}$-valued sheaves $\mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$.

Consider the inclusion $\mathrm{Sh}(*;\mathcal{D})^{\mathrm{op}} \longrightarrow \mathrm{Top}_{\mathcal{D}}$ of the fiber over the terminal object of Top. Since π is a cocartesian fibration, this functor admits a left adjoint [5, Lemma 2.20]

$$\mathcal{O}\colon \mathrm{Top}_{\mathcal{D}} \longrightarrow \mathrm{Sh}(*;\mathcal{D})^{\mathrm{op}} = \mathcal{D}^{\mathrm{op}}$$

sending a $\mathcal{D}$-structured space $(X,\mathcal{O}_X)$ to the global sections $\mathcal{O}_X(X)$.

Definition 5.1.6 (cf. [63, Definition 1.1.5.3])**.** Let $\mathrm{Top}^{\mathrm{loc}}_{\mathcal{C}^\infty} \subseteq \mathrm{Top}_{\mathcal{C}^\infty}$ be the subcategory of the ∞-category of $\mathcal{C}^\infty$-ringed spaces whose

- objects are *locally* $\mathcal{C}^\infty$-ringed spaces $(X, \mathcal{O}_X)$, i.e. for each point $x \in X$, the stalk $\pi_0(\mathcal{O}_X)_x$ is a local discrete $\mathcal{C}^\infty$-ring with residue field $\mathbb{R}$.
- morphisms are the morphisms $(X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ of locally $\mathcal{C}^\infty$-ringed spaces, i.e. maps for which the map of stalks $\pi_0(\mathcal{O}_{X,x}) \longrightarrow \pi_0(\mathcal{O}_{Y,f(x)})$ is a map of local rings for each point $x \in X$.

Let $\mathcal{O}\colon \mathrm{Top}^{\mathrm{loc}}_{\mathcal{C}^\infty} \longrightarrow (\mathcal{C}^\infty\mathrm{Alg})^{\mathrm{op}}$ be the restriction of the global sections functor.

Remark 5.1.7. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be two locally $\mathcal{C}^\infty$-ringed spaces. The zeroth homotopy sheaf $\tilde{\pi}_0\mathcal{O}_X$ is a sheaf of discrete local $\mathcal{C}^\infty$-rings on X. Furthermore, a map $f\colon (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ of $\mathcal{C}^\infty$-ringed spaces is a map of *locally* $\mathcal{C}^\infty$-ringed spaces if and only if the map $f\colon (X, \tilde{\pi}_0\mathcal{O}_X) \longrightarrow (Y, \tilde{\pi}_0\mathcal{O}_Y)$ is a map of locally $\mathcal{C}^\infty$-ringed spaces. In other words, there is a pullback square of mapping spaces

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{Top}^{\mathrm{loc}}_{\mathcal{C}^\infty}}\Big((X, \mathcal{O}_X), (Y, \mathcal{O}_Y)\Big) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{Top}_{\mathcal{C}^\infty}}\Big((X, \mathcal{O}_X), (Y, \mathcal{O}_Y)\Big) \arrow[d] \\
\mathrm{Map}_{\mathrm{Top}^{\mathrm{loc}}_{\mathcal{C}^\infty}}\Big((X, \tilde{\pi}_0\mathcal{O}_X), (Y, \tilde{\pi}_0\mathcal{O}_Y)\Big) \arrow[r] & \mathrm{Map}_{\mathrm{Top}_{\mathcal{C}^\infty}}\Big((X, \tilde{\pi}_0\mathcal{O}_X), (Y, \tilde{\pi}_0\mathcal{O}_Y)\Big).
\end{tikzcd}$$

Proposition 5.1.8 ([60, Theorem 2.1.1])**.** *The functor $\mathcal{O}$ fits into an adjunction*

$$\mathcal{O}\colon \mathrm{Top}^{\mathrm{loc}}_{\mathcal{C}^\infty} \rightleftarrows (\mathcal{C}^\infty\mathrm{Alg})^{\mathrm{op}} : \mathrm{Spec}.$$

We will give an explicit description of $\mathrm{Spec}(A)$ in Section 5.1.2.

Definition 5.1.9. The ∞-category Aff of *affine derived manifolds*, or just affines, is the essential image of the functor Spec

$$\mathrm{Aff} \subseteq \mathrm{Top}^{\mathrm{loc}}_{\mathcal{C}^\infty}.$$

A locally $\mathcal{C}^\infty$-ringed space $(X, \mathcal{O})$ is a *derived manifold* if X admits a cover by opens U_i such that each $(U_i, \mathcal{O}|U_i) \simeq \mathrm{Spec}(A_i)$ for some $\mathcal{C}^\infty$-ring A_i. Let

$$\mathrm{dMfd} \subseteq \mathrm{Top}^{\mathrm{loc}}_{\mathcal{C}^\infty}$$

be the full subcategory of derived manifolds.

Example 5.1.10. The usual category of smooth manifolds is a full subcategory of dMfd. In fact, every embedded submanifold $M \subseteq \mathbb{R}^n$ is affine: it is equivalent to $\mathrm{Spec}(\mathcal{C}^\infty(M))$.

The inclusion $\mathrm{Mfd} \longrightarrow \mathrm{dMfd}$ does not preserve pullbacks, but it does preserve *transverse* pullbacks. Indeed, let $f\colon M \longrightarrow N$ be a map of smooth manifolds and let $N' \longrightarrow N$ be a closed submanifold which is transverse to f. To see that their (derived) intersection in dMfd agrees with the usual intersection, it suffices to work locally, where our diagram of smooth manifolds takes the form

$$\mathbb{R}^m \xrightarrow{\ f\ } \mathbb{R}^{(n-l)+l} \longleftarrow \{0\} \times \mathbb{R}^l.$$

It follows that the derived intersection $M \times^h_N N'$ is given locally by the vanishing locus of the function $(f_1, \ldots, f_{n-l})\colon \mathbb{R}^m \longrightarrow \mathbb{R}^{n-l}$. By Lemma 2.2.7, the vanishing locus of this map between affines is simply given by the spectrum of the derived pushout of $\mathcal{C}^\infty$-rings

$$\mathcal{C}^\infty(\mathbb{R}^m) \amalg^h_{\mathcal{C}^\infty(\mathbb{R}^{n-l})} \mathcal{C}^\infty(\{0\}) \simeq \mathcal{C}^\infty(\mathbb{R}^m) \otimes^h_{\mathcal{C}^\infty(\mathbb{R}^{n-l})} \mathcal{C}^\infty(\{0\}).$$

The f_i form a regular sequence, because $(f_1, \dots, f_{n-l})$ is transverse to zero. It follows that this derived pushout of $\mathcal{C}^\infty$-rings is simply given by the discrete $\mathcal{C}^\infty$-ring $\mathcal{C}^\infty(\mathbb{R}^m)/(f_1, \dots, f_{n-l})$, which is the ring of smooth functions on the smooth manifold $f^{-1}(0)$ by [68, Proposition 2.5].

Remark 5.1.11. Let us say that a derived manifold M is *quasi-smooth* if it is locally equivalent to the derived zero locus of a map $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$. Equivalently, M is locally of finite presentation and has a cotangent complex with Tor-amplitude contained in $[0,1]$ (see Remark 6.2.5). The quasi-smooth derived manifolds are exactly the types of derived manifolds that are studied in [92] (and in [51]).

5.1.2 Spectra of $\mathcal{C}^\infty$-rings. Let us now provide an explicit description of the spectrum of a $\mathcal{C}^\infty$-ring.

Definition 5.1.12. Let A be a $\mathcal{C}^\infty$-ring and let $a \in \pi_0(A)$. The *localization* of A at a is the universal map of $\mathcal{C}^\infty$-rings $A \longrightarrow A\{a^{-1}\}$ with the property that for any $\mathcal{C}^\infty$-ring B, the map

$$\mathrm{Map}_{\mathcal{C}^\infty\mathrm{Alg}}(A\{a^{-1}\}, B) \longrightarrow \mathrm{Map}_{\mathcal{C}^\infty\mathrm{Alg}}(A, B)$$

is an inclusion of path components, with essential image consisting of the maps $f\colon A \longrightarrow B$ for which $f(a) \in \pi_0(B)$ is invertible.

Lemma 5.1.13. *For any A be a $\mathcal{C}^\infty$-ring and $a \in \pi_0(A)$, the localization $A \longrightarrow A\{a^{-1}\}$ exists. Furthermore, the map $A \longrightarrow A\{a^{-1}\}$ is a flat map of commutative $\mathbb{R}$-algebras and induces isomorphisms on homotopy groups*

$$\pi_0(A)\{a^{-1}\} \otimes_{\pi_0(A)} \pi_n(A) \longrightarrow \pi_n(A\{a^{-1}\})$$

where $\pi_0(A)\{a^{-1}\}$ is the localization of the discrete $\mathcal{C}^\infty$-ring $\pi_0(A)$.

Proof. The map $A \longrightarrow A\{a^{-1}\}$ can be obtained as a (derived) pushout of $\mathcal{C}^\infty$-rings

$$\begin{tikzcd}
\mathcal{C}^\infty(\mathbb{R}) \arrow[r] \arrow[d, "a"'] & \mathcal{C}^\infty(\mathbb{R}^2) \arrow[r] \arrow[d] & \mathcal{C}^\infty(\mathbb{R}^\times) \arrow[d] \\
A \arrow[r] & A\{y\} \arrow[r] & A\{a^{-1}\}.
\end{tikzcd}$$

The map a is classified by the element $a \in \pi_0(A)$ and $A\{y\}$ is obtained from A by freely adding a variable. The top sequence of $\mathcal{C}^\infty$-rings is the image of the sequence of commutative $\mathbb{R}$-algebras

$$\mathbb{R}[x] \longrightarrow \mathbb{R}[x,y] \longrightarrow \mathbb{R}[x,y]/(xy-1) \simeq \mathbb{R}[x,x^{-1}]$$

under the free functor $\mathrm{CAlg}_{\mathbb{R}}^{\geq 0} \longrightarrow \mathcal{C}^\infty\mathrm{Alg}$. This composite map of commutative algebras has the universal property that

$$\mathrm{Map}_{\mathrm{CAlg}_{\mathbb{R}}^{\geq 0}}(\mathbb{R}[x,x^{-1}], B) \longrightarrow \mathrm{Map}_{\mathrm{CAlg}_{\mathbb{R}}^{\geq 0}}(\mathbb{R}[x], B)$$

is an inclusion of path components on those maps $f\colon \mathbb{R}[x] \longrightarrow B$ for which $f(x) \in \pi_0(B)$ is invertible ([60, p. 4.1.18]).

The class of $A \in \mathcal{C}^\infty\mathrm{Alg}$ for which the second assertion holds is closed under filtered colimits and retracts, since these commute with homotopy groups and tensor products. We may therefore assume that A is finitely presented (Definition 2.2.26) and present it by a dg-$\mathcal{C}^\infty$-ring $A = \mathcal{C}^\infty(\mathbb{R}^n)[\xi_i]$ which is free on a finite set of generators, but with nontrivial

differential. Given an element $a \in A_0$, unwinding the definitions and using Example 5.1.10, one can identify
$$A \amalg_{\mathcal{C}^\infty(\mathbb{R})} \mathcal{C}^\infty(\mathbb{R}^\times) \simeq A\{y\}/(ay-1) \cong \mathcal{C}^\infty(U)[\xi_i]$$
with $\mathcal{C}^\infty(U)[\xi_i]$, where $U \subseteq \mathbb{R}^n$ is the open subset $a^{-1}(\mathbb{R}^\times)$.

The map $\mathcal{C}^\infty(\mathbb{R}^n) \longrightarrow \mathcal{C}^\infty(U)$ is the ring-theoretic localization of $\mathcal{C}^\infty(\mathbb{R}^n)$ at the set of all functions $\mathbb{R}^n \longrightarrow \mathbb{R}$ that are nonzero on U [69]. It is in particular flat, so that the map

$$\pi_0(A\{a^{-1}\}) \otimes_{\pi_0(A)} \pi_n(A) \longrightarrow \pi_n(A\{a^{-1}\})$$

is an isomorphism. The fact that $\pi_0(A\{a^{-1}\})$ coincides with the localization $\pi_0(A)\{a^{-1}\}$ follows from the universal property. $\square$

Corollary 5.1.14. *Let $A \in \mathcal{C}^\infty\mathrm{Alg}$ and let us denote by*

$$\mathcal{B}(A)^{\mathrm{op}} \subseteq A/\mathcal{C}^\infty\mathrm{Alg}$$

the full subcategory on the localization maps $A \longrightarrow A\{a^{-1}\}$. The functor

$$\pi_0 \colon \mathcal{B}(A) \longrightarrow \mathcal{B}(\pi_0(A)); \ \big(A \longrightarrow B\big) \longmapsto \big(\pi_0(A) \longrightarrow \pi_0(B)\big)$$

is an equivalence of ∞-categories and in fact (up to equivalence) of posets.

Proof. The functor π_0 is well defined because $\pi_0(A\{a^{-1}\}) \cong \pi_0(A)\{a^{-1}\}$ is a localization of $\pi_0(A)$. It is essentially surjective by Lemma 5.1.13. To see that it is fully faithful, let $f\colon A \longrightarrow A\{a^{-1}\}$ and $g\colon A \longrightarrow A\{b^{-1}\}$ be two localization maps. The mapping space $\mathrm{Map}_{A/}(f,g)$ in $\mathcal{B}(A)^{\mathrm{op}}$ fits into a pullback square

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{B}(A)^{\mathrm{op}}}(f,g) \arrow[r] \arrow[d] & \{g\} \arrow[d] \\
\mathrm{Map}_{\mathcal{C}^\infty\mathrm{Alg}}(A\{a^{-1}\}, A\{b^{-1}\}) \arrow[r, "f^*"'] & \mathrm{Map}_{\mathcal{C}^\infty\mathrm{Alg}}(A, A\{b^{-1}\}).
\end{tikzcd}$$

Since f is a localization map, the map f^* is an inclusion of connected components. It follows that $\mathrm{Map}_{\mathcal{B}(A)^{\mathrm{op}}}(f,g)$ is either empty or contractible, depending on whether the map $\pi_0(A) \longrightarrow \pi_0(A\{b^{-1}\})$ factors over $\pi_0(A\{a^{-1}\})$. This implies that $\pi_0\colon \mathcal{B}(A) \longrightarrow \mathcal{B}(\pi_0(A))$ is fully faithful. $\square$

Construction 5.1.15. Let $B \in \mathcal{B}(A)^{\mathrm{op}}$ and let $b\colon \mathcal{C}^\infty(\mathbb{R}) \longrightarrow B$ be an element in $\pi_0(B)$. For any open cover $\{U_i\}$ of $\mathbb{R}$ in the usual Euclidean topology, the maps

$$B \longrightarrow B \coprod_{\mathcal{C}^\infty(\mathbb{R})} \mathcal{C}^\infty(U_i)$$

define a cover of B in $\mathcal{B}(A)$. We endow the poset $\mathcal{B}(A)$ with the Grothendieck topology generated by these covers (see [66] or [60, Notation 2.2.6]). There is an obvious presheaf of $\mathcal{C}^\infty$-rings

$$\mathcal{O}_{A,\mathrm{pre}}\colon \mathcal{B}(A)^{\mathrm{op}} \longrightarrow A/\mathcal{C}^\infty\mathrm{Alg} \longrightarrow \mathcal{C}^\infty\mathrm{Alg}$$

sending an object $A \longrightarrow A\{a^{-1}\}$ to $A\{a^{-1}\}$. Let $\mathcal{O}_A$ be its associated sheaf.

Remark 5.1.16. As in [66], one can identify the poset $\mathcal{B}(A) \simeq \mathcal{B}(\pi_0(A))$, together with its Grothendieck topology, in rather concrete terms: there is a topological space $\mathrm{Spec}(A)$ for which it forms a basis of open neighbourhoods, closed under finite intersections. The set underlying $\mathrm{Spec}(A)$ is the set of maps of (discrete) $\mathcal{C}^\infty$-rings $\pi_0(A) \longrightarrow \mathbb{R}$. When $\pi_0(A) \cong \mathcal{C}^\infty(\mathbb{R}^E)/I$, the topological space $\mathrm{Spec}(A)$ is simply the vanishing locus of I inside $\mathbb{R}^E$, equipped with the product topology.

Definition 5.1.17. The (Archimedean) *spectrum* of $A \in \mathcal{C}^\infty\mathrm{Alg}$ is the $\mathcal{C}^\infty$-ringed space $\mathrm{Spec}(A) = \big(\mathrm{Spec}(A), \mathcal{O}_A\big)$.

Example 5.1.18. The spectrum of a discrete $\mathcal{C}^\infty$-ring A is just the spectrum of A as defined in [22, 66]. In particular, when M is a smooth manifold we have that $\mathrm{Spec}(\mathcal{C}^\infty(M))$ is just M.

Example 5.1.19. For any $\mathcal{C}^\infty$-ring A, Lemma 5.1.13 implies that the spectra of its truncations are given by

$$\mathrm{Spec}(\tau_{\leq n}(A)) \simeq \Big(\mathrm{Spec}(A), \tilde{\tau}_{\leq n}\mathcal{O}_A\Big).$$

Here $\tilde{\tau}_{\leq n}\mathcal{O}_{\mathrm{Spec}(A)}$ is the associated sheaf of the pointwise truncation of $\mathcal{O}_A$. In particular, it follows that $\mathrm{Spec}(A)$ is a locally $\mathcal{C}^\infty$-ringed space, since the usual spectrum of a discrete $\mathcal{C}^\infty$-ring is a locally $\mathcal{C}^\infty$-ringed space [66].

Example 5.1.20. Let $U \in \mathcal{B}(A)$ be a basic open subspace of $\mathrm{Spec}(A)$, corresponding to a localization map $A \longrightarrow A\{a^{-1}\}$. Then $\mathrm{Spec}(A\{a^{-1}\})$ is simply given by $(U, \mathcal{O}\big|U)$.

The presheaf $\mathcal{O}_{A,\mathrm{pre}}$ is a presheaf of $\mathcal{C}^\infty$-rings under A, so that the associated sheaf $\mathcal{O}$ takes values in $\mathcal{C}^\infty$-rings under A. In particular, there is a natural map $A \longrightarrow \mathcal{O}\big(\mathrm{Spec}(A)\big)$.

Proposition 5.1.21 ([60, Theorem 2.2.12], [63, Proposition 1.1.5.5])**.** *Let A be a $\mathcal{C}^\infty$-ring and let $(X, \mathcal{O}_X)$ be a locally $\mathcal{C}^\infty$-ringed space. Precomposition with the canonical map $A \longrightarrow \mathcal{O}\big(\mathrm{Spec}(A)\big)$ induces an equivalence of mapping spaces*

$$\mathrm{Map}_{\mathrm{Top}^{\mathrm{loc}}_{\mathcal{C}^\infty}}\big((X, \mathcal{O}_X), \mathrm{Spec}(A)\big) \xrightarrow{\ \simeq\ } \mathrm{Map}_{\mathcal{C}^\infty\mathrm{Alg}}(A, \mathcal{O}(X)). \tag{5.1.22}$$

In other words, the spectrum of Definition 5.1.17 indeed describes the right adjoint in the adjunction of Proposition 5.1.8. Before proving Proposition 5.1.21, let us make the following observation:

Remark 5.1.23. Let A be a $\mathcal{C}^\infty$-ring with spectrum $\mathrm{Spec}(A) = \big(\mathrm{Spec}(A), \mathcal{O}\big)$ and suppose that $\mathcal{R}$ is another sheaf of $\mathcal{C}^\infty$-rings on $\mathrm{Spec}(A)$. Then the map

$$\mathrm{Map}(\mathcal{O}, \mathcal{R}) \longrightarrow \mathrm{Map}\Big(A, \mathcal{R}\big(\mathrm{Spec}(A)\big)\Big)$$

is an inclusion of path components. The essential image consists of maps $A \longrightarrow \mathcal{R}(\mathrm{Spec}(A))$ such that each composite map

$$\pi_0 A \longrightarrow \pi_0\mathcal{R}\big(\mathrm{Spec}(A)\big) \longrightarrow \pi_0\mathcal{R}\big(\mathrm{Spec}(A\{a^{-1}\})\big)$$

inverts the element $a \in \pi_0(A)$. To see this, note that the space of maps of sheaves $\mathcal{O} \longrightarrow \mathcal{R}$ is equivalent to the space of maps of presheaves $\mathcal{O}_{A,\mathrm{pre}} \longrightarrow \mathcal{R}$ over the category $\mathcal{B}(A)$ of basic opens of $\mathrm{Spec}(A)$. Each map

$$A = \mathcal{O}_{A,\mathrm{pre}}\big(\mathrm{Spec}(A)\big) \longrightarrow \mathcal{O}_{A,\mathrm{pre}}\big(\mathrm{Spec}(A\{a^{-1}\})\big) = A\{a^{-1}\}$$

is just the localization of A at a, so that the result follows from the universal property of localizations.

Proof (of Proposition 5.1.21). Fix a map $\phi\colon A \longrightarrow \mathcal{O}(X)$ and let F be the fiber of (5.1.22) over ϕ. By [22, 66], the induced map $\pi_0(A) \longrightarrow \tilde{\pi}_0\mathcal{O}_X(X)$ of discrete $\mathcal{C}^\infty$-rings is classified by a unique map of locally $\mathcal{C}^\infty$-ringed spaces

$$f\colon (X, \tilde{\pi}_0\mathcal{O}_X) \longrightarrow \mathrm{Spec}(\pi_0(A))$$

By Remark 5.1.7, the fiber F is equivalent to the fiber of the map

$$\mathrm{Map}_{\mathrm{Top}_{\mathcal{C}^\infty}}\big((X, \mathcal{O}_X), \mathrm{Spec}(A)\big)_f \longrightarrow \mathrm{Map}\big(A, \mathcal{O}(X)\big) \tag{5.1.24}$$

The domain is the union of path components consisting of maps of $\mathcal{C}^\infty$-ringed spaces $(X, \mathcal{O}_X) \longrightarrow \mathrm{Spec}(A)$ that induce the map f at the level of π_0. This fixes the continuous function on the underlying spaces, so this domain is a union of path components

$$\mathrm{Map}_{\mathrm{Top}_{\mathcal{C}^\infty}}\big((X, \mathcal{O}_X), \mathrm{Spec}(A)\big)_f \subseteq \mathrm{Map}(\mathcal{O}_A, f_*\mathcal{O}_X)$$

consisting of maps of sheaves $\mathcal{O}_A \longrightarrow f_*\mathcal{O}_X$ inducing the map f on π_0. It then follows from Remark 5.1.23 that (5.1.24) is an inclusion of path components. To see that the fiber F is contractible, it therefore suffices to check that ϕ is contained in the essential image of (5.1.22). By Remark 5.1.23 this comes down to verifying that each map

$$A \longrightarrow \mathcal{O}_X(X) \longrightarrow \mathcal{O}_X\big(f^{-1}\mathrm{Spec}(A\{a^{-1}\})\big)$$

inverts the element $a \in \pi_0(A)$. But this is just a condition on π_0, where it holds by definition, since f was adjoint to the map of discrete $\mathcal{C}^\infty$-rings $\pi_0(A) \longrightarrow \tilde{\pi}_0\mathcal{O}_X(X)$. □

The functor $\mathrm{Spec}\colon \mathcal{C}^\infty\mathrm{Alg}^{\mathrm{op}} \longrightarrow \mathrm{Top}^{\mathrm{loc}}_{\mathcal{C}^\infty}$ is a bit more complicated than its algebro-geometric counterpart, because it is not fully faithful (or equivalently, $\mathcal{O}(\mathrm{Spec}(A))$ is not equivalent to A). Nonetheless, the category of affine derived manifolds can be studied purely in terms of $\mathcal{C}^\infty$-rings. To see this, let us first make some cohomological remarks about the structure sheaves on affines.

Lemma 5.1.25. *Let A be a $\mathcal{C}^\infty$-ring and let $\mathcal{O}_A\colon \mathcal{B}(A)^{\mathrm{op}} \longrightarrow \mathcal{C}^\infty\mathrm{Alg}$ be the structure sheaf of its spectrum. For any module sheaf $F\colon \mathcal{B}(A)^{\mathrm{op}} \longrightarrow \mathrm{Mod}$ over $\mathcal{O}_A$ (Definition 5.1.31), the homotopy presheaves $\pi_n F$ are (discrete) sheaves.*

Consequently, the pointwise *truncation $\tau_{\leq n}F$ is a sheaf and the module sheaf F is a hypersheaf, i.e. satisfies hyperdescent (see [59, Section 6.5]).*

Proof. Fix an element $f \in \pi_0(A)$ and consider the induced continuous function

$$f\colon \mathrm{Spec}(A) \longrightarrow \mathrm{Spec}(\mathcal{C}^\infty(\mathbb{R})) = \mathbb{R}.$$

Let us first consider the behaviour of the sheaf f_*F on $\mathbb{R}$. Since $\mathbb{R}$ has finite covering dimension, f_*F is a hypersheaf (see [59, Theorem 7.2.3.6]), so that sections over an open subspace $V \subseteq \mathbb{R}$ can be computed in terms of a hypercohomology spectral sequence (see e.g. [94, Proposition 1.36])

$$E_2^{p,q} = H^p(V, \tilde{\pi}_q(f_*F)) \Longrightarrow \pi_{q-p}\big(f_*F(V)\big).$$

Here $\tilde{\pi}_q(f_*F))$ are the sheaves over $\mathbb{R}$ associated to the presheaves $\pi_q(f_*F)$. We have a map of sheaves of $\mathcal{C}^\infty$-rings $\mathcal{O}_\mathbb{R} \longrightarrow f_*\mathcal{O}_A$ from the sheaf of smooth functions on $\mathbb{R}$, so that the sheaves $\tilde{\pi}_q(f_*F)$ are modules over $\mathcal{O}_\mathbb{R}$. The latter has partitions of unity, so that their higher sheaf cohomology groups vanish. The spectral sequence therefore degenerates and one finds that $\tilde{\pi}_q(f_*F)(V) \cong \pi_q(f_*F(V))$.

In other words, the homotopy presheaves $\pi_q(f_*F)$ are already sheaves. It follows that the homotopy presheaves $\pi_q F$ on $\mathcal{B}(A)$ satisfy descent with respect to all covers of $\mathrm{Spec}(A)$ of the form $\mathcal{U} = f^{-1}\mathcal{V}$, where $f\colon \mathrm{Spec}(A) \longrightarrow \mathbb{R}$ is induced by an element $f \in \pi_0(A)$. But these were exactly the basic covers of $\mathrm{Spec}(A)$ in $\mathcal{B}(A)$. Repeating this argument for $A\{a^{-1}\}$ instead of A shows that the presheaves $\pi_q\mathcal{O}_A$ satisfy descend with respect to all basic covers in $\mathcal{B}(A)$, so that they are sheaves on $\mathcal{B}(A)$.

A similar argument shows that each (pointwise) truncation $\tau_{\leq n}F$ is a sheaf. Since these truncated sheaves are automatically hypersheaves, it follows that F is a limit of hypersheaves and thus satisfies hyperdescent itself. □

Proposition 5.1.26 ([50, Proposition 4.34]). *Let A be a $\mathcal{C}^\infty$-ring. Then the unit map*

$$f = \mathrm{Spec}(\eta)\colon \mathrm{Spec}(A) \longrightarrow \mathrm{Spec}\Big(\mathcal{O}(\mathrm{Spec}(A))\Big)$$

is an equivalence.

Proof. Let us denote $S = \mathrm{Spec}(A)$ and $B = \mathcal{O}(\mathrm{Spec}(A))$. By Example 5.1.19 and Lemma 5.1.25, the functor $\mathcal{O}\circ\mathrm{Spec}$ commutes with the operation of taking truncations. Consequently, the map of topological spaces $S \longrightarrow \mathrm{Spec}(B)$ underlying f is simply the map induced by $\mathrm{Spec}(\pi_0(A)) \longrightarrow \mathrm{Spec}\big(\mathcal{O}(\mathrm{Spec}(\pi_0 A))\big)$. This map is a homeomorphism by [50, Proposition 4.34].

It remains to identify the structure sheaves. Since both structure sheaves are hypersheaves, it suffices to show that f induces isomorphisms on homotopy sheaves. The homotopy sheaves $\tilde{\pi}_q\mathcal{O}_A$ of $\mathrm{Spec}(A)$ are the associated sheaves of the presheaves $\pi_q\mathcal{O}_A$ sending a basic open $\mathrm{Spec}(A\{a^{-1}\})$ to

$$\pi_q(A\{a^{-1}\}) \cong \pi_0(A)\{a^{-1}\} \otimes_{\pi_0(A)} \pi_q(A).$$

On the other hand, $\pi_q(B) = \tilde{\pi}_q\mathcal{O}_A\big(S\big)$ agrees with the global sections of the q-th homotopy sheaf of $\mathcal{O}_A$, by Lemma 5.1.25. It follows that the homotopy sheaves $\tilde{\pi}_q\mathcal{O}_B$ are associated to the presheaves sending an open $\mathrm{Spec}(A\{a^{-1}\})$ to the tensor product of global sections

$$\Gamma(\tilde{\pi}_0\mathcal{O}_A)\{a^{-1}\} \otimes_{\Gamma(\tilde{\pi}_0\mathcal{O}_A)} \Gamma(\tilde{\pi}_q\mathcal{O}_A).$$

The isomorphism on π_0-sheaves now follows from the case of discrete $\mathcal{C}^\infty$-rings [50, Proposition 4.34] and the isomorphism on higher homotopy sheaves follows from [50, Proposition 5.20]. $\square$

Definition 5.1.27 ([50, Definition 4.35]). A $\mathcal{C}^\infty$-ring A is *complete* if the unit map $A \longrightarrow \mathcal{O}(\mathrm{Spec}(A))$ is an equivalence.

Corollary 5.1.28. *The adjunction $\mathcal{O}\colon \mathrm{Aff} \leftrightarrows \mathcal{C}^\infty\mathrm{Alg}^{\mathrm{op}}\colon \mathrm{Spec}$ realizes the ∞-category Aff as the opposite of the full subcategory of $\mathcal{C}^\infty\mathrm{Alg}$ on the complete $\mathcal{C}^\infty$-rings.*

Example 5.1.29. A finitely generated discrete $\mathcal{C}^\infty$-ring $A = \mathcal{C}^\infty(\mathbb{R}^n)/I$ is complete iff it is germ-determined (see [68]). If $\pi_0(A)$ is finitely generated and germ-determined and each $\pi_n(A)$ is a finitely presented module over $\pi_0(A)$, then A is complete: by Lemma 5.1.25 the map $\pi_n(A) \longrightarrow \pi_n\mathcal{O}(\mathrm{Spec}(A))$ can be identified with the map $\pi_n(A) \longrightarrow \Gamma(\tilde{\pi}_n\mathcal{O}_A)$, which is an isomorphism by [50, Proposition 5.27].

Example 5.1.30. Any finitely presented $\mathcal{C}^\infty$-ring is complete. To see this, let $A = \mathcal{C}^\infty(\mathbb{R}^n)[x_i]$ be a finitely presented dg-$\mathcal{C}^\infty$-ring and consider the presheaf F on $\mathbb{R}^n$ sending an open $U \subseteq \mathbb{R}^n$ to $\mathcal{C}^\infty(U)[x_i]$. Using that F is given in each homological degree by a finite direct sum of the structure sheaf $\mathcal{O}_{\mathbb{R}^n}$ of $\mathbb{R}^n$, a spectral sequence argument shows that F satisfies descent.

The restriction of F to the basic open subspaces of $\mathrm{Spec}(A)$ agrees with the presheaf $\mathcal{O}_{A,\mathrm{pre}}\colon \mathcal{B}(A)^{\mathrm{op}} \longrightarrow \mathcal{C}^\infty\mathrm{Alg}$ of Construction 5.1.15. It follows that $\mathcal{O}_A = \mathcal{O}_{A,\mathrm{pre}}$, so that $A \simeq \mathcal{O}(\mathrm{Spec}(A))$.

5.1.3 Quasicoherent sheaves.

The discussion from the previous section has an analogue for modules over $\mathcal{C}^\infty$-rings. To this end, consider the presentable ∞-category of $\mathcal{C}^\infty$-rings and modules over them

$$\mathrm{Mod}_{\mathcal{C}^\infty} := \mathrm{Mod} \times_{\mathrm{CAlg}^{\geq 0}_{\mathbb{R}}} \mathcal{C}^\infty\mathrm{Alg}$$

where $\mathcal{C}^\infty\mathrm{Alg} \longrightarrow \mathrm{CAlg}^{\geq 0}_{\mathbb{R}}$ is the forgetful functor and $\mathrm{Mod} \longrightarrow \mathrm{CAlg}^{\geq 0}_{\mathbb{R}}$ is the cartesian and cocartesian fibration arising from Construction 4.3.8. Unraveling the definitions, one sees that a $\mathrm{Mod}_{\mathcal{C}^\infty}$-valued sheaf on a topological space X (Definition 5.1.1) is given by

- a sheaf $\mathcal{O}\colon \mathrm{Op}(X)^{\mathrm{op}} \longrightarrow \mathcal{C}^\infty\mathrm{Alg}$ of $\mathcal{C}^\infty$-rings.
- an $\mathcal{O}$-module sheaf F, i.e. the data of an $\mathcal{O}(U)$-module $F(U)$ for each U, such that the restriction maps $F(U) \longrightarrow F(V)$ are $\mathcal{O}(U)$-linear.

Remembering only the underlying sheaf of $\mathcal{C}^\infty$-rings determines a functor

$$\pi\colon \mathrm{Sh}(X; \mathrm{Mod}_{\mathcal{C}^\infty}) \longrightarrow \mathrm{Sh}(X; \mathcal{C}^\infty\mathrm{Alg})$$

which is both a left and a right adjoint.

Definition 5.1.31. If $(X, \mathcal{O})$ is a $\mathcal{C}^\infty$-ringed space, we define the ∞-category $\mathrm{Sh}_{\mathcal{O}}(X)$ of $\mathcal{O}$-module sheaves to be the fiber of the functor π over $\mathcal{O}$.

Remark 5.1.32. The ∞-category $\mathrm{Sh}_{\mathcal{O}}(X)$ is locally presentable. For instance, $\mathrm{Sh}_{\mathcal{O}}(X)$ can be realized as a left Bousfield localization of the over-category $\mathrm{Sh}(X; \mathrm{Mod}_{\mathcal{C}^\infty})/(\mathcal{O}, 0)$, whose local objects are the pairs $(\mathcal{O}', F)$ consisting of an $\mathcal{O}'$-module sheaf F, together with an equivalence $\mathcal{O}' \longrightarrow \mathcal{O}$.

Using this, one can deduce that $\mathrm{Sh}_{\mathcal{O}}(X)$ can be presented as a Bousfield localization of the projective model structure on presheaves of dg-$\mathcal{O}^{\mathrm{dg}}$-modules, where $\mathcal{O}^{\mathrm{dg}}$ is a presheaf of dg-$\mathcal{C}^\infty$-rings on X presenting $\mathcal{O}$.

Construction 5.1.5 gives rise to a global sections functor

$$\Gamma\colon \mathrm{Top}_{\mathrm{Mod}_{\mathcal{C}^\infty}} \longrightarrow \mathrm{Mod}_{\mathcal{C}^\infty}^{\mathrm{op}}; \ (X, \mathcal{O}, F) \longmapsto (\mathcal{O}(X), \Gamma(X, F)) = (\mathcal{O}(X), F(X)).$$

For a $\mathcal{C}^\infty$-ringed space $(X, \mathcal{O})$, this functor restricts to a right adjoint functor $\Gamma\colon \mathrm{Sh}_{\mathcal{O}}(X) \longrightarrow \mathrm{Mod}_{\mathcal{O}(X)}$ between the fibers over $X \in \mathrm{Top}_{\mathcal{C}^\infty}$.

Lemma 5.1.33. *If A is a complete $\mathcal{C}^\infty$-ring, then $\Gamma\colon \mathrm{Sh}_{\mathcal{O}}(\mathrm{Spec}(A)) \longrightarrow \mathrm{Mod}_A$ is a fully faithful right adjoint functor.*

Proof. We can model $\mathcal{O}$ by a presheaf $\mathcal{O}\colon \mathcal{B}(A)^{\mathrm{op}} \longrightarrow \mathcal{C}^\infty\mathrm{Alg}^{\mathrm{dg}}$ of dg-$\mathcal{C}^\infty$-rings satisfying descent and present $\mathrm{Sh}_{\mathcal{O}}(\mathrm{Spec}(A))$ by the model category $\mathrm{Mod}_{\mathcal{O},\mathrm{dg}}$ of Remark 5.1.32. In that case A can be modeled by the $\mathcal{C}^\infty$-ring $\mathcal{O}(X)$ and the functor Γ is modeled by the right Quillen functor $\Gamma\colon \mathrm{Mod}_{\mathcal{O},\mathrm{dg}} \longrightarrow \mathrm{Mod}_{A,\mathrm{dg}}$ sending F to $F(X)$. The left adjoint of this functor sends a dg-A-module E to the presheaf $E(U) = E \otimes_A \mathcal{O}(U)$.

Suppose that F is a fibrant object of $\mathrm{dgMod}_{\mathcal{O}}$, so that F satisfies descent. Since all $\mathcal{O}$-module sheaves on $\mathrm{Spec}(A)$ satisfy hyperdescent by Lemma 5.1.25, it suffices to show that the map of presheaves $E(-) := F(X) \otimes_A \mathcal{O}(-) \longrightarrow F(-)$ induces stalkwise isomorphisms on homotopy groups.

The stalk of E at a point x is given by the filtered colimit

$$E_x = \operatorname*{colim}_{a_1,\dots,a_n} F(X) \otimes_A A\{a_1^{-1}, \dots, a_n^{-1}\}$$

indexed by finite tuples of elements $a_1, \dots, a_n \in \pi_0(A)$ such that $a_i(x) \neq 0$. Each map $A \longrightarrow A\{a_1^{-1}, \dots, a_n^{-1}\}$ is flat by Lemma 5.1.13, so that $\pi_q(E_x) \cong \pi_q(F(X)) \otimes_{\pi_0(A)} \pi_0(\mathcal{O}_x)$. Since $\pi_q F$ is an $\mathcal{O}$-module sheaf by Lemma 5.1.25, the map

$$\pi_q(F(X)) \otimes_{\pi_0(A)} \pi_0(\mathcal{O}_x) \longrightarrow \pi_q F_x$$

is an isomorphism by [50, Proposition 5.20]. □

Definition 5.1.34. Let M be a derived manifold. We define $\mathrm{QC}(M)$ to be the ∞-category of $\mathcal{O}$-module sheaves on M and refer to them as *quasi-coherent sheaves* on M.

When M is affine, the global sections functor $\Gamma\colon \mathrm{QC}(M) \longrightarrow \mathrm{Mod}_{\mathcal{O}(M)}$ is fully faithful. We will refer to its essential image as the *complete* $\mathcal{O}(M)$-modules and to its left adjoint $\mathrm{Mod}_{\mathcal{O}(M)} \longrightarrow \mathrm{QC}(M)$ as the *associated sheaf* functor.

We will freely switch between the interpretation of a quasi-coherent sheaf over an affine as an $\mathcal{O}$-module sheaf and its interpretation as a complete module over a complete $\mathcal{C}^\infty$-ring. For example, let $f\colon M \longrightarrow N$ be a map of affine derived manifolds, dual to a map $\phi = f^*\colon \mathcal{O}(N) \longrightarrow \mathcal{O}(M)$. Then there is a left adjoint functor $f^*\colon \mathrm{QC}(N) \longrightarrow \mathrm{QC}(M)$, which equivalently sends

(a) an $\mathcal{O}_N$-module sheaf E to the inverse image sheaf $f^*E = f^{-1}E \otimes_{f^{-1}\mathcal{O}_N} \mathcal{O}_M$.
(b) a complete $\mathcal{O}(N)$-module E to the associated sheaf of the $\mathcal{O}(M)$-module $E \otimes_{\mathcal{O}(N)} \mathcal{O}(M)$.

Remark 5.1.35. Let M be an affine derived manifold and let $A = \mathcal{O}(M)$ be the corresponding complete $\mathcal{C}^\infty$-ring. An A-module $E \in \mathrm{Mod}_A$ is complete if and only if all $\pi_n(E)$ are complete. It follows that the t-structure on Mod_A descends to a t-structure on the full subcategory $\mathrm{QC}(M) \subseteq \mathrm{Mod}_A$.

Similarly, suppose that $f\colon E \longrightarrow E'$ is a map of A-modules which induces an equivalence on associated sheaves. For any other A-module F, the map $E \otimes_A F \longrightarrow E' \otimes_A F$ also induces an equivalence on the associated sheaves, since the associated map on stalks can be identified with

$$(E \otimes_A \mathcal{O}_x) \otimes_{\mathcal{O}_x} (F \otimes_A \mathcal{O}_x) \xrightarrow{f\otimes\mathrm{id}} (E' \otimes_A \mathcal{O}_x) \otimes_{\mathcal{O}_x} (F \otimes_A \mathcal{O}_x)$$

The symmetric monoidal structure on Mod_A therefore descends to $\mathrm{QC}(M)$.

Example 5.1.36. The subcategory $\mathrm{QC}(M) \subseteq \mathrm{Mod}_{\mathcal{O}(M)}$ is closed under limits and retracts. Since it contains the unit $\mathcal{O}(M)$, it also contains all finitely presented $\mathcal{O}(M)$-modules, as well as retracts of those.

Remark 5.1.37. Let $M = (M, \mathcal{O}_M)$ be a locally $\mathcal{C}^\infty$-ringed space. Then the $\mathcal{C}^\infty$-ring $\mathcal{O}(M)$ is complete (Definition 5.1.27). Indeed, it is a retract of the complete $\mathcal{C}^\infty$-ring $\mathcal{O}\big(\mathrm{Spec}(\mathcal{O}(M))\big)$ by the triangle identities.

Similarly, if E is an $\mathcal{O}_M$-module sheaf, then $\Gamma(E)$ is a complete $\mathcal{O}(M)$-module: it can be identified with the complete module of global sections of $f_*(E)$, where $f\colon M \longrightarrow \mathrm{Spec}(\mathcal{O}(M))$ is the canonical map.

5.2 Derived stacks

A situation one frequently encounters in geometry is that a collection of geometric objects can itself be organized into a space. Such moduli spaces of geometric objects are not quite (derived) manifolds, but *quotients* of manifolds where equivalent objects are identified. These quotients tend to be badly behaved because of the existence of objects with nontrivial automorphisms. Stacks and higher stacks provide a method for dealing with such singular quotients of manifolds by equivalences.

Higher stacks have been defined in great generality by Simpson [89] (see also [97] for a detailed discussion), starting from any reasonable setting of spaces and smooth maps between them. In this section we recall the notion of a stack and the closely related notion of a higher (derived) Lie groupoid in the setting of derived differential topology.

5.2.1 Functor of points. Let $\mathrm{Sh}(\mathrm{Aff})$ be the ∞-category of sheaves

$$\mathrm{Aff}^{\mathrm{op}} \longrightarrow \mathcal{S}$$

on the category of affine derived manifolds (Definition 5.1.9), equipped with the topology generated by the basic open covers $\mathrm{Spec}(A\{a_i^{-1}\}) \longrightarrow \mathrm{Spec}(A)$ from Construction 5.1.15.

Remark 5.2.1. There is a set-theoretic issue with our definition of $\mathrm{Sh}(\mathrm{Aff})$, since Aff is not small. We will tacitly assume that Aff is the opposite of the (small) ∞-category of κ-small objects in the locally presentable ∞-category of complete $\mathcal{C}^\infty$-rings, for some large enough κ. Similarly, a derived manifold is locally the spectrum of a κ-small $\mathcal{C}^\infty$-ring.

Example 5.2.2. For every affine M, the representable presheaf $\mathrm{Map}_{\mathrm{Aff}}(-, M)$ is a sheaf. Indeed, if $S = \{U \longrightarrow V\}$ is a covering sieve of an affine V, then there is commuting square

$$\begin{tikzcd} \mathrm{Map}_{\mathrm{Aff}}(V, M) \arrow[r] \arrow[d, "\sim"'] & \lim_{U\in S} \mathrm{Map}_{\mathrm{Aff}}(U, M) \arrow[d, "\sim"] \\ \mathrm{Map}_{\mathcal{C}^\infty\mathrm{Alg}}(\mathcal{O}(M), \mathcal{O}(V)) \arrow[r, "\sim"'] & \lim_{U\in S} \mathrm{Map}_{\mathcal{C}^\infty\mathrm{Alg}}(\mathcal{O}(M), \mathcal{O}(U)) \end{tikzcd}$$

where the bottom map is an equivalence since $\mathcal{O}$ is a sheaf of $\mathcal{C}^\infty$-rings on V. Similarly, every (possibly non-connective) $B \in \mathrm{CAlg}^{\geq 0}$ determines a sheaf

$$\mathrm{Spec}(B)\colon \mathrm{Aff}^{\mathrm{op}} \longrightarrow \mathcal{S};\ U \longmapsto \mathrm{Map}_{\mathrm{CAlg}^{\geq 0}}(B, \mathcal{O}(M)).$$

Proposition 5.2.3 ([60, Theorem 2.4.1])**.** *Every derived manifold M determines a sheaf*

$$h_M = \mathrm{Map}_{\mathrm{dMfd}}(-, M)\colon \mathrm{Aff}^{\mathrm{op}} \longrightarrow \mathcal{S}.$$

This produces a fully faithful inclusion $h\colon \mathrm{dMfd} \longrightarrow \mathrm{Sh}(\mathrm{Aff})$.

To see that derived manifolds represent sheaves, let us first consider a slightly different descent problem:

Lemma 5.2.4. *For any locally presentable ∞-category $\mathcal{D}$, the functor*

$$\mathrm{Sh}(-, \mathcal{D})\colon \mathrm{Top}^{\mathrm{op}} \longrightarrow \mathrm{Pr}^{\mathrm{L}}$$

satisfies descent, where the category of topological spaces is endowed with the usual open cover topology.

Proof. Since the forgetful functor $\mathrm{Pr}^{\mathrm{L}} \longrightarrow \widehat{\mathrm{Cat}}_\infty$ to (large) ∞-categories preserves small limits [59, Proposition 5.5.3.13], it suffices to show that $\mathrm{Sh}(-, \mathcal{D})\colon \mathrm{Top}^{\mathrm{op}} \longrightarrow \widehat{\mathrm{Cat}}_\infty$ satisfies descent. To see this, it suffices to show that for a fixed topological space X, the functor $\mathrm{Sh}(-, \mathcal{D})\colon \mathrm{Op}(X)^{\mathrm{op}} \longrightarrow \widehat{\mathrm{Cat}}_\infty$ satisfies descent. Any open inclusion $j\colon U \longrightarrow V$ induces a left adjoint functor $j^*\colon \mathrm{Sh}(V, \mathcal{D}) \longrightarrow \mathrm{Sh}(U, \mathcal{D})$, which itself admits a left adjoint $j_!$, given by extension by zero. We can therefore think of $\mathrm{Sh}(-, \mathcal{D})$ as a diagram

$$\mathrm{Sh}(-; \mathcal{D})\colon \mathrm{Op}(X) \longrightarrow \left(\mathrm{Pr}^{\mathrm{R}}\right)^{\mathrm{op}} \simeq \mathrm{Pr}^{\mathrm{L}}$$

sending an inclusion j to $j_!$. Since limits in Pr^{R} are computed at the level of the underlying ∞-categories [59, Theorem 5.5.3.18], it suffices to show that this diagram sends the Čech nerve of an open cover in $\mathrm{Op}(X)$ to a colimit diagram in Pr^{L}.

Now recall from [60, Remark 1.1.5] that there is a natural equivalence

$$\mathrm{Sh}(-, \mathcal{D}) \simeq \mathrm{Sh}(-) \otimes \mathcal{D} = \mathrm{Fun}^{\mathrm{R}}(\mathrm{Sh}(-)^{\mathrm{op}}, \mathcal{D}),$$

so that we can reduce to the case where $\mathcal{D} = \mathcal{S}$ is the ∞-category of spaces. In that case, the diagram of locally presentable ∞-categories associated to the Čech nerve of an open cover $\{U_i \longrightarrow U\}$ can be identified with the augmented simplicial diagram of over-categories

$$\mathrm{Sh}(X)/U \longleftarrow \mathrm{Sh}(X)/V \Lleftarrow \mathrm{Sh}(X)/V \times_U V \Lleftarrow \ldots$$

where $V = \coprod U_i$ is the coproduct of sheaves on X represented by the U_i. This is a colimit diagram of locally presentable ∞-categories by [59, Theorem 6.1.3.9]. $\square$

Corollary 5.2.5. *Let V be a derived manifold and let $S = \{U \longrightarrow V\}$ be a covering sieve of V (in the usual sense of topology). Then the map*

$$\mathrm{Map}_{\mathrm{dMfd}}(V, M) \longrightarrow \lim_{U \in S} \mathrm{Map}_{\mathrm{dMfd}}(U, M) \tag{5.2.6}$$

is an equivalence.

Proof. Consider the commuting square

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{Top}_{\mathcal{C}^\infty}}(V, M) \arrow[r] \arrow[d] & \lim_{U \in S} \mathrm{Map}_{\mathrm{Top}_{\mathcal{C}^\infty}}(U, M) \arrow[d] \\
\mathrm{Map}_{\mathrm{Top}}(V, M) \arrow[r] & \lim_{U \in S} \mathrm{Map}_{\mathrm{Top}}(U, M).
\end{tikzcd}$$

A map $(V, \mathcal{O}_V) \longrightarrow (M, \mathcal{O}_M)$ of $\mathcal{C}^\infty$-ringed spaces is a map of *locally* $\mathcal{C}^\infty$-ringed spaces iff the restrictions to all $U \in S$ are maps of locally $\mathcal{C}^\infty$-ringed spaces. To see that the map (5.2.6) is an equivalence, it therefore suffices to show that the top horizontal map is an equivalence of spaces.

Since the bottom map is a bijection of sets, it suffices to compare the fibers over a continuous map $f : V \longrightarrow M$. The map between fibers can be identified with the map

$$\mathrm{Map}_{\mathrm{Sh}(V;\mathcal{C}^\infty\mathrm{Alg})}(f^{-1}\mathcal{O}_M, \mathcal{O}_V) \longrightarrow \lim_{U \in S} \mathrm{Map}_{\mathrm{Sh}(U;\mathcal{C}^\infty\mathrm{Alg})}(f^{-1}\mathcal{O}_M|U, \mathcal{O}_V|U)$$

between spaces of maps between sheaves of $\mathcal{C}^\infty$-rings. This map is an equivalence by Lemma 5.2.4. □

Proof (of Proposition 5.2.3). For every derived manifold M, the representable presheaf h_M on Aff is a sheaf by Corollary 5.2.5. To see that h is fully faithful, observe that the functor h preserves pullbacks and sends any open cover $\{U_i \longrightarrow M\}$ to a local surjection of sheaves $\coprod h_{U_i} \longrightarrow h_M$. Consequently, any open cover induces a commuting square

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{dMfd}}(M, N) \arrow[r, "\theta"] \arrow[d, "\sim"'] & \mathrm{Map}_{\mathrm{Sh}(\mathrm{Aff})}(h_M, h_N) \arrow[d, "\sim"] \\
\lim\limits_{i_0 \dots i_n} \mathrm{Map}_{\mathrm{dMfd}}(U_{i_0 \dots i_n}, N) \arrow[r, "\theta"'] & \lim\limits_{i_0 \dots i_n} \mathrm{Map}_{\mathrm{Sh}(\mathrm{Aff})}(h_{U_{i_0 \dots i_n}}, h_N).
\end{tikzcd}$$

Here the maps θ are induced by the functor h and each $U_{i_0 \dots i_n}$ denotes a finite intersection of U_i. The vertical maps are equivalences by descent.

The map θ is an equivalence when M is affine, by the Yoneda lemma. When $M \subseteq \mathrm{Spec}(A)$ is an open subspace of an affine, let $S = \{U_i \longrightarrow M\}$ be the cover by all opens $\mathrm{Spec}(A\{a^{-1}\}) \subseteq M$. Each finite intersection of such opens is again affine, so that the above square shows that θ is an equivalence for any open subspace $M \subseteq \mathrm{Spec}(A)$. Repeating the same argument and using an affine open cover $\{U_i \longrightarrow M\}$ shows that θ is an equivalence for any derived manifold M. □

Remark 5.2.7. The above proof gives the following description of the sheaves representable by derived manifolds. First, a sheaf is representable by an open subspace of an affine if it is the essential image of a map of sheaves

$$\coprod \mathrm{Spec}(A\{a_i^{-1}\}) \longrightarrow \mathrm{Spec}(A)$$

where each $\mathrm{Spec}(A\{a_i^{-1}\}) \longrightarrow \mathrm{Spec}(A)$ is an affine open subspace. A sheaf X is representable by a derived manifold if there exists a surjection $\coprod \mathrm{Spec}(A_i) \longrightarrow X$ such that each $\mathrm{Spec}(A_i) \times_X \mathrm{Spec}(A_j) \longrightarrow \mathrm{Spec}(A_i)$ is the inclusion of an open subspace.

5.2.2 Groupoids and stacks. Remark 5.2.7 describes derived manifolds as sheaves obtained by gluing affines along open inclusions. This description fits into a general pattern for defining spaces in terms of local models by means of gluing constructions, where instead of gluing along open inclusions, one glues along more general types of maps.

Definition 5.2.8. Let us say that a map $f\colon M \longrightarrow N$ of derived manifolds is

- a *closed* (*open*) *immersion* if the underlying map of topological spaces is a closed (open) embedding and the map of sheaves $f^{-1}\mathcal{O}_N \longrightarrow \mathcal{O}_M$ induces a surjection on π_0-sheaves (is an equivalence).
- *étale* if f is a local homeomorphism and the map $f^{-1}\mathcal{O}_N \longrightarrow \mathcal{O}_M$ is an equivalence of sheaves.
- *smooth* if for every point $x \in M$, there are affine open neighbourhoods $x \in U$ and $f(x) \in V$ such that the restricted map $f\colon U \longrightarrow V$ is equivalent to a projection map $\mathbb{R}^n \times V \longrightarrow V$.
- *locally finitely presented* if for every point $x \in M$, there are affine open neighbourhoods $x \in U$ and $f(x) \in V$ such that the restricted map $f\colon U \longrightarrow V$ is contained in the smallest subcategory of Aff/V which contains $V \times \mathbb{R} \longrightarrow V$ and is closed under finite limits.

Lemma 5.2.9. *Let $f\colon M \longrightarrow N$ be a map of derived manifolds. The following are equivalent:*

(1) f is locally finitely presented.
(2) f is locally given by a map $\mathrm{Spec}(\phi)\colon \mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$, where $\phi\colon A \longrightarrow B$ is a finitely presented map of $\mathcal{C}^\infty$-rings.

Proof. If $\phi\colon A \longrightarrow B$ is a finitely presented map of $\mathcal{C}^\infty$-rings, then the map $\mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$ is locally finitely presented, so that (2) implies (1). For the converse, suppose that $\phi\colon A \longrightarrow B$ is finitely presented and that $A \longrightarrow C$ is another map of $\mathcal{C}^\infty$-rings. There is a map of presheaves $\mathcal{B}(C)^{\mathrm{op}} \longrightarrow \mathcal{S}$

$$\mathrm{Map}_{A/}(B, \mathcal{O}_{C,\mathrm{pre}}) \longrightarrow \mathrm{Map}_{A/}(B, \mathcal{O}_C) \simeq \mathrm{Map}_{/\mathrm{Spec}(A)}(-, \mathrm{Spec}(B)).$$

Since the map $A \longrightarrow B$ is finitely presented, this map arises as a finite limit of the local weak equivalence $\mathcal{O}_{C,\mathrm{pre}} \longrightarrow \mathcal{O}_C$ from Construction 5.1.15. In particular, it follows that the above natural transformation induces a local surjection between π_0-presheaves. In other words, any map $g\colon \mathrm{Spec}(C) \longrightarrow \mathrm{Spec}(B)$ over $\mathrm{Spec}(A)$ is locally given by a map $\mathrm{Spec}(\psi)\colon \mathrm{Spec}(C\{c^{-1}\}) \longrightarrow \mathrm{Spec}(B)$, for some map ψ of $\mathcal{C}^\infty$-rings under A.

Now let $\mathcal{C} \subseteq \mathrm{Aff}/\mathrm{Spec}(A)$ be the subcategory of maps $V \longrightarrow \mathrm{Spec}(A)$ that can locally (on V) be described by finitely presented maps of $\mathcal{C}^\infty$-rings $A \longrightarrow B$. This category contains $\mathrm{Spec}(A) \times \mathbb{R}$ and is closed under pullbacks. Indeed, we can compute pullbacks of affines locally, where they can be described by pushout diagrams of finitely presented $\mathcal{C}^\infty$-rings under A, by the above argument.

Unwinding the definitions, this implies that any locally finitely presented map $f\colon M \longrightarrow N$ is locally given by a map in $\mathcal{C}$, which in turn means that it is locally given by the spectrum of a finitely presented map of $\mathcal{C}^\infty$-rings. □

Lemma 5.2.10. *Let P be one of the properties of a map from Definition 5.2.8.*

(a) The composition of two maps with property P has property P.
(b) If $f\colon N \longrightarrow M$ has property P and $g\colon M' \longrightarrow M$ is any map, then the base change $f'\colon M' \times_M N \longrightarrow N$ has property P.
(c) If $f\colon N \longrightarrow M$ is a map and $\{U_i \longrightarrow M\}$ is an open cover such that each base change $U_i \times_M N \longrightarrow U_i$ has property P, then f has property P.

(d) Let $f\colon L \longrightarrow M$ be a smooth (resp. étale) surjection and let $g\colon M \longrightarrow N$ be a map. If gf is locally finitely presented or smooth (resp. étale), then so is g.

Proof. Properties (a) - (c) are easily verified, using that pullbacks of derived manifolds can be computed locally. For assertion (d), note that the map f (which is locally finitely presented) admits local sections s_i which are themselves locally finitely presented. It follows that the map g is locally given by the finitely presented maps gfs_i and hence is locally finitely presented itself. Furthermore, note that there is a cofiber sequence of (connective) cotangent complexes (see Definition 6.1.20)

$$f^*L_{M/N} \longrightarrow L_{L/N} \longrightarrow L_{L/M}.$$

When f and gf are smooth, the cotangent complexes $L_{L/N}$ and $L_{L/M}$ are locally free of finite rank (Definition 5.2.28). Then $f^*L_{M/N}$ and $L_{M/N}$ are locally free of finite rank as well, since f was surjective. Corollary 6.2.4 now implies that g is smooth. The étale case is straightforward. $\square$

Definition 5.2.11 (Simpson [89])**.** Let $p\colon Y \longrightarrow X$ be a morphism in Sh(Aff) and let P be the class of smooth or locally finitely presented maps. We will say that:

(a_0) p is *0-representable* (resp. 0-P) if for any map $\mathrm{Spec}(A) \longrightarrow X$ from an affine space, the pullback $Y \times_X \mathrm{Spec}(A)$ is a derived manifold (and the map of derived manifolds $Y \times_Y \mathrm{Spec}(A) \longrightarrow \mathrm{Spec}(A)$ is of class P).

(a_n) X is *n-representable* (resp. n-P) if for any map $\mathrm{Spec}(A) \longrightarrow X$ from an affine space, there exists an $(n-1)$-smooth surjection $M \longrightarrow Y \times_X \mathrm{Spec}(A)$ whose domain is a derived manifold (and such that the composite map of derived manifolds $M \longrightarrow \mathrm{Spec}(A)$ is of class P).

A map $Y \longrightarrow X$ is smooth if it is n-smooth for some n, and similarly for being locally finitely presented. A sheaf X is a *derived* (resp. smooth) *n-stack* if the map $X \longrightarrow *$ is n-representable (resp. smooth).

Remark 5.2.12. The above definition has an immediate analogue where the role of the smooth maps is taken by the étale maps. We will call the resulting classes of morphisms étale representable and étale. A sheaf X is a derived étale stack if $X \longrightarrow *$ is étale representable (it is usually not an étale map).

Remark 5.2.13. In the definition of an n-smooth map, one may equivalently demand that for *any* choice of atlas $V \longrightarrow X \times_Y M$, the composite map $V \longrightarrow M$ is a smooth map between derived manifolds.

The classes of n-smooth maps are stable under base change and a map $p\colon Y \longrightarrow X$ is n-smooth if and only if its base change $Y \times_X Z \longrightarrow Z$ along an effective epimorphism $Z \longrightarrow X$ is n-smooth. For $n \leq k$, a k-smooth map between derived n stacks is automatically n-smooth (see e.g. [97, Section 1.3.3]).

Derived stacks are closely related to *groupoids*. Indeed, to verify that a sheaf X is a derived n-stack, one has to pick a derived manifold M and an $(n-1)$-smooth surjection $M \longrightarrow X$. The Čech nerve of this map is a groupoid object in Sh(Aff)

$$\dots \overset{\longrightarrow}{\underset{\longrightarrow}{\overset{\longrightarrow}{\longrightarrow}}} \mathcal{G} \times_M \mathcal{G} \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} \mathcal{G} \rightrightarrows M.$$

The source and target map are both given by the base change of $M \longrightarrow X$ and are therefore $(n-1)$-smooth. This implies that $\mathcal{G} \rightrightarrows M$ is a groupoid object which has a derived manifold of objects, a derived $(n-1)$-stack of morphisms and smooth source and target maps.

Conversely, consider a groupoid object $\mathcal{G} \rightrightarrows M$ with an $(n-1)$-stack of arrows, for which the source and target maps are smooth. Then the map $M \longrightarrow M/\mathcal{G}$ to the quotient, i.e. the colimit of the above simplicial diagram, provides an atlas for $M/\mathcal{G}$ [97, Proposition 1.3.4.2]. In particular, $M/\mathcal{G}$ is a derived n-stack.

This provides an equivalence between smooth surjections $M \longrightarrow X$ from a derived manifold to a derived n-stack and smooth groupoids $\mathcal{G} \rightrightarrows M$ over M, which have a derived $(n-1)$-stack of arrows. In particular, verifying that a sheaf X is a derived stack involves inductively finding atlases: given the map $M_0 \longrightarrow X$, one has to produce an atlas $M_1 \longrightarrow \mathcal{G}$ to show that $\mathcal{G}$ is a derived $(n-1)$-stack, etcetera. Alternatively, one can make all these choices of atlases all at the same time:

Definition 5.2.14. Consider a simplicial diagram $X_\bullet\colon \mathbf{\Delta}^{\mathrm{op}} \longrightarrow \mathrm{dMfd}$ of derived manifolds. We will say that $X_\bullet$ is a *derived Lie n-groupoid* if for any horn inclusion $\Lambda^i[k] \subseteq \Delta[k]$, the map $X(\Delta[k]) \longrightarrow X(\Lambda^i[k])$ from the derived manifold of k-simplices to the derived manifold of i-th horns is a smooth surjection and an equivalence for $k > n$.

A map $p\colon Y_\bullet \longrightarrow X_\bullet$ between derived Lie n-groupoids is called a *Kan fibration* if for each horn inclusion $\Lambda^i[k] \subseteq \Delta[k]$, the matching map

$$Y(\Delta[k]) \simeq Y_k \longrightarrow X(\Delta[k]) \times_{X(\Lambda^i[k])} Y(\Lambda^i[k]) \tag{5.2.15}$$

is a smooth surjection between derived manifolds. If in addition the map $Y_0 \longrightarrow X_0$ is smooth, we will say that the map p is a *smooth* Kan fibration.

Similarly, a map $p\colon Y_\bullet \longrightarrow X_\bullet$ of derived Lie n-groupoids is a *smooth hypercover* if all maps

$$Y(\Delta[k]) \simeq Y_k \longrightarrow X(\Delta[k]) \times_{X(\partial\Delta[k])} Y(\partial\Delta[k])$$

are smooth surjections. There are similar notions of derived étale Lie n-groupoids, Kan fibrations and hypercovers.

Remark 5.2.16. Let $\mathcal{X}$ be an ∞-topos and let n be a natural number. An *n-groupoid* in $\mathcal{X}$ is a simplicial diagram $X_\bullet\colon \mathbf{\Delta}^{\mathrm{op}} \longrightarrow \mathcal{X}$ with the property that each map $X(\Delta[k]) \longrightarrow X(\Lambda^j[k])$ is an effective epimorphism and an equivalence for $k > n$. Similarly, a map $Y_\bullet \longrightarrow X_\bullet$ is a Kan fibration if each map (5.2.15) is an effective epimorphism.

As one would expect, Kan fibrations are useful for computing homotopy limits. For example, consider a pullback diagram of n-groupoids

$$\begin{tikzcd} Y'_\bullet \arrow[r] \arrow[d] & Y_\bullet \arrow[d, "p"] \\ X'_\bullet \arrow[r] & X_\bullet \end{tikzcd}$$

where p is a Kan fibration. Then the map of colimits $|Y'_\bullet| \longrightarrow |Y_\bullet| \times_{|X_\bullet|} |X'_\bullet|$ is an equivalence.

The relation between n-stacks and derived Lie n-groupoids can be summarized as follows:

Proposition 5.2.17 ([77])**.** *There is a functor* $|-|\colon \mathrm{dLie}_n \longrightarrow \mathrm{Sh}(\mathrm{Aff})$ *sending each derived Lie n-groupoid* $X_\bullet\colon \mathbf{\Delta}^{\mathrm{op}} \longrightarrow \mathrm{dMfd}$ *to the colimit of the composite diagram* $\mathbf{\Delta}^{\mathrm{op}} \longrightarrow \mathrm{dMfd} \longrightarrow \mathrm{Sh}(\mathrm{Aff})$*. This functor has the following properties:*

(i) The essential image of the functor $|-|$ is the subcategory of derived n-stacks.

(ii) The functor $|-|$ sends smooth Kan fibrations to smooth maps.

(iii) Let $X'_\bullet \longrightarrow X_\bullet \longleftarrow Y_\bullet$ be a diagram of derived Lie n-groupoids and suppose that $Y_\bullet \longrightarrow X_\bullet$ is a Kan fibration. Then the levelwise pullback $X'_\bullet \times_{X_\bullet} Y_\bullet$ is a derived Lie n-groupoid and the natural map $|X'_\bullet \times_{X_\bullet} Y_\bullet| \longrightarrow |X'_\bullet| \times_{|X_\bullet|} |Y_\bullet|$ is an equivalence of sheaves.

(iv) The functor $|-|$ *sends smooth hypercovers to equivalences and realizes the essential image of* $|-|$ *as the universal* ∞*-category obtained from the* ∞*-category* dLie_n *of derived Lie* n*-groupoids by inverting the smooth hypercovers.*

The same result holds in the étale case.

In other words, an object $X \in \mathrm{Sh}(\mathrm{Aff})$ is a derived n-stack if and only if there exists a derived Lie n-groupoid $X_\bullet \colon \mathbf{\Delta}^{\mathrm{op}} \longrightarrow \mathrm{dMfd}$ whose colimit $|X_\bullet|$ is equivalent to X. We will refer to $|X_\bullet|$ as the *associated stack* of the derived Lie n-groupoid $X_\bullet$. The associated stack of a derived Lie n-groupoid $X_\bullet$ has a canonical atlas, given by the map $X_0 \longrightarrow |X_\bullet|$.

Remark 5.2.18. It follows from the description of the localization of dLie_n at its hypercovers (see [77] or [70]) that for two derived Lie n-groupoids $X_\bullet$ and $Y_\bullet$, any map between their associated stacks $f \colon |X_\bullet| \longrightarrow |Y_\bullet|$ fits into a homotopy-commuting triangle

$$\begin{tikzcd} & {|U_\bullet|} \arrow[dl, "|p|"'] \arrow[dr, "|g|"] & \\ {|X_\bullet|} \arrow[rr, "f"'] & & {|Y_\bullet|} \end{tikzcd}$$

where $p \colon U_\bullet \longrightarrow X_\bullet$ is a smooth hypercover and $g \colon U_\bullet \longrightarrow Y_\bullet$ is a Kan fibration. We will use this to freely replace maps of n-stacks by maps of derived Lie n-groupoids and vice versa.

5.2.3 Examples.

Example 5.2.19. A (smooth) Lie 1-groupoid is just a Lie groupoid. If A is a chain complex of abelian Lie groups concentrated in degrees $[0, n]$, then the associated Eilenberg-Maclane object $K(A)$ is a Lie n-groupoid, whose classifying space is the *Eilenberg-Maclane stack* of A.

Example 5.2.20. Let $\mathcal{G}$ be a proper Lie groupoid (e.g. a smooth manifold or a compact Lie group) and let $\eta \in H^{n+1}(B\mathcal{G}, \mathbb{Z})$ be a class in the cohomology of the classifying space of $\mathcal{G}$. Then η can be represented by a $U(1)$-valued Čech cocycle, i.e. by the datum of a hypercover $U_\bullet \longrightarrow \mathcal{G}$ of $\mathcal{G}$, together with a map of simplicial manifolds $\eta\colon U_\bullet \longrightarrow K(U(1), n)$ (see e.g. [16]). The pullback of η along the path fibration of the map $* \longrightarrow K(U(1), n)$ is a Lie n-groupoid, which provides an extension of M by $K(U(1), n-1)$.

This gives a general procedure for higher stacks: given a stack X and a cohomology class in $H^n(X, U(1))$, one can form a higher stack $X \times_{K(U(1),n)} *$ by annihilating this cohomology class. A well-known example of this construction is classifying stack of the string group associated to a compact, simple, simply connected Lie group [84].

Example 5.2.21. Let $\mathcal{G}$ be a Lie groupoid with associated stack X. The *free loop stack* $\mathcal{L}X$ is given by the derived self-intersection $X \times_{X\times X} X$ of the diagonal and is modeled by the derived *inertia groupoid* of $\mathcal{G}$, whose objects are the automorphisms of $\mathcal{G}$. Alternatively, one can think of $\mathcal{L}X \simeq \mathrm{Map}(S^1, X)$ as the stack of X-valued local systems on the circle. More generally, for any finite simplicial set K and any derived n-stack X, there is a derived n-stack of X-valued local systems on K.

Remark 5.2.22. Work of Joyce [51] shows that many analytical constructions of moduli spaces (such as spaces of J-holomorphic curves) produce derived manifolds.

Example 5.2.23. Other examples of derived Lie groupoids (or derived 1-stacks) are given by action groupoids of (Lie) groups acting on derived manifolds, which arise as soon as one tries to intersect two non-transverse G-invariant submanifolds of a manifold with G-action. In general, there is no equivariant way of deforming two submanifolds until they are transverse. For example, if $\mu\colon M \longrightarrow \mathfrak{g}^*$ is the moment map of a Hamiltonian G-space M, then the symplectic reduction $\mu^{-1}(0)/G$ always exists as a derived stack.

Remark 5.2.24. Consider the composite left adjoint functor from affine derived $\mathbb{R}$-schemes to affine derived manifolds

$$(-)_\mathbb{R}\colon \mathrm{AffSch}_\mathbb{R}^{\mathrm{op}} := \mathrm{CAlg}_\mathbb{R}^{\geq 0} \xrightarrow{F} \mathcal{C}^\infty\mathrm{Alg} \xrightarrow{\mathrm{Spec}} \mathrm{Aff}^{\mathrm{op}}$$

Here F is the free functor, left adjoint to the forgetful functor from $\mathcal{C}^\infty$-rings to connective commutative $\mathbb{R}$-algebras, and Spec is the left adjoint of Corollary 5.1.28. This composite sends an affine derived $\mathbb{R}$-scheme S to its $\mathbb{R}$-points $S_\mathbb{R}$, endowed with the analytic topology.

The free functor F preserves cotangent complexes: $L_{F(A)/F(B)}$ is equivalent to $F(A) \otimes_A L_{A/B}$. It follows that $(-)_\mathbb{R}$ preserves smooth and étale maps, as well as étale covers which are surjective on $\mathbb{R}$-points. Let τ denote the topology on derived $\mathbb{R}$-schemes consisting of such étale covers which are surjective on $\mathbb{R}$-points. Then the above functor extends to a left exact left adjoint

$$(-)_\mathbb{R}\colon \mathrm{Sh}(\mathrm{AffSch}_\mathbb{R}, \tau) \longrightarrow \mathrm{Sh}(\mathrm{Aff})$$

taking $\mathbb{R}$-points of sheaves. This functor preserves smooth and étale maps, so that $X_\mathbb{R}$ is a derived stack in the $\mathcal{C}^\infty$-sense as soon as X is a derived Artin stack with respect to the topology τ, instead of the étale topology.

Similarly, the $\mathbb{C}$-points of a derived Artin stack give rise to a derived stack in the $\mathcal{C}^\infty$-sense. See e.g. [97, 85] for many examples, notably stacks of maps from a flat and proper scheme to a locally finitely presented derived stack.

5.2.4 Perfect complexes. Recall that each affine space M comes with a reflective full subcategory $\mathrm{QC}(M) \subseteq \mathrm{Mod}_{\mathcal{O}(M)}$ of ($\mathcal{O}(M)$-modules that arise as the global sections of) quasi-coherent sheaves on M. Every map $f\colon M \longrightarrow N$ between affines induces an adjunction $f^*\colon \mathrm{Mod}_{\mathcal{O}(N)} \leftrightarrows \mathrm{Mod}_{\mathcal{O}(M)}\colon f_*$ where f_* preserves quasi-coherent sheaves. We therefore obtain a functor

$$\mathrm{QC}\colon \mathrm{Aff}^{\mathrm{op}} \longrightarrow \mathrm{Pr}^{\mathrm{L}};\ \ M \longmapsto \mathrm{QC}(M).$$

Lemma 5.2.25. *The functor* $\mathrm{QC}\colon \mathrm{Aff}^{\mathrm{op}} \longrightarrow \mathrm{Pr}^{\mathrm{L}}$ *satisfies descent and therefore determines a limit-preserving functor* $\mathrm{QC}\colon \mathrm{Sh}(\mathrm{Aff})^{\mathrm{op}} \longrightarrow \mathrm{Pr}^{\mathrm{L}}$*. We refer to* $\mathrm{QC}(X)$ *as the ∞-category of* quasi-coherent sheaves *on X; when $X = M$ is a derived manifold, this agrees with Definition 5.1.34.*

Proof. Let M be a derived manifold and consider the cartesian square consisting of presheaves of (large) ∞-categories on M

$$\begin{tikzcd} \mathrm{Sh}_{\mathcal{O}}(-) \arrow[r] \arrow[d] & \mathrm{Sh}(-, \mathrm{Mod}) \arrow[d] \\ * \arrow[r, "\mathcal{O}"'] & \mathrm{Sh}(-, \mathcal{C}^\infty\mathrm{Alg}). \end{tikzcd}$$

The pullback $\mathrm{Sh}_{\mathcal{O}}(-)$ sends each open subspace $U \subseteq M$ to the ∞-category of $\mathcal{O}$-module sheaves over U and each inclusion $V \subseteq U$ is simply given by the restriction functor $\mathrm{Sh}_{\mathcal{O}}(U) \longrightarrow \mathrm{Sh}_{\mathcal{O}}(V)$. By Lemma 5.2.4, the presheaf $\mathrm{Sh}_{\mathcal{O}}(-)$ is a sheaf.

By Lemma 5.1.33, the restriction of $\mathrm{Sh}_{\mathcal{O}}(-)$ to the full subcategory of affine opens is equivalent to the functor sending an affine open $U \subseteq M$ to $\mathrm{QC}(U)$. In particular, taking M to be an affine derived manifold shows that $\mathrm{QC}\colon \mathrm{Aff}^{\mathrm{op}} \longrightarrow \widehat{\mathrm{Cat}}_\infty$ satisfies descent. Furthermore, for a general derived manifold M, there is an equivalence

$$\mathrm{Sh}_{\mathcal{O}}(M) \xrightarrow{\sim} \lim_I \mathrm{Sh}_{\mathcal{O}}(U_I) \xrightarrow{\sim} \lim_I \mathrm{QC}(U_I) \xleftarrow{\sim} \mathrm{QC}(M)$$

where $\{U_I\}$ is some diagram of affine open subspaces of M whose colimit in $\mathrm{Sh}(\mathrm{Aff})$ is a model for M. Indeed, the first equivalence follows from the descent of $\mathrm{Sh}_{\mathcal{O}}(-)$ and the last equivalence uses that QC is extended to all sheaves on Aff by colimits. $\square$

Remark 5.2.26. Let X be a sheaf on Aff and let $\mathrm{QC}(X) = \lim_{U\in \mathrm{Aff}/X} \mathrm{QC}(U)$ be its category of quasi-coherent sheaves. For each map $x\colon U \longrightarrow X$ from an affine, the category $\mathrm{QC}(U)$ carries a t-structure where $\mathrm{QC}^{\geq 0}(U)$ consist of the quasi-coherent sheaves F whose homotopy sheaves vanish below degree 0 or equivalently, for which $F(U)$ is connective.

Given a map $f\colon V \longrightarrow U$, the functor $f^*\colon \mathrm{QC}(U) \longrightarrow \mathrm{QC}(V)$ preserves the subcategory of connective sheaves. Using this, one sees that $\mathrm{QC}(X)$ carries a t-structure where $\mathrm{QC}^{\geq 0}(X)$ consists of the quasi-coherent sheaves F such that for any point $x\colon U \longrightarrow X$, the sheaf x^*F is connective over $\mathrm{Spec}(A)$.

Remark 5.2.27. By Remark 5.1.35, the (closed) symmetric monoidal structure on $\mathrm{Mod}_{\mathcal{O}(M)}$ descends to a unique closed symmetric monoidal structure on $\mathrm{QC}(M)$ with the property that the localization functor $\mathrm{Mod}_{\mathcal{O}(M)} \longrightarrow \mathrm{QC}(M)$ is symmetric monoidal. Each map between affines $f\colon M \longrightarrow N$ induces a symmetric monoidal functor $f^*\colon \mathrm{QC}(N) \longrightarrow \mathrm{QC}(M)$, so that $\mathrm{QC}(X)$ is closed symmetric monoidal for any sheaf X.

Definition 5.2.28. Let M be a derived manifold. We will say that an object $E \in \mathrm{QC}(M)$ is a *perfect complex* if it is locally finitely presentable: every point of M admits an open neighbourhood U such that the restriction $E\big|U$ can be obtained from the structure sheaf $\mathcal{O}\big|U$ by finite limits and colimits. Let $\mathrm{Perf}(M) \subseteq \mathrm{QC}(M)$ be the full subcategory on the perfect complexes.

A sheaf is *locally free of finite rank* if E is locally equivalent to a finite direct sum $\mathcal{O}^{\oplus n}$.

Example 5.2.29. Let M be a smooth manifold. Every finite chain complex of finite rank vector bundles

$$\dots \longrightarrow 0 \longrightarrow E_a \longrightarrow E_{a-1} \longrightarrow \dots \longrightarrow E_{b+1} \longrightarrow E_b \longrightarrow 0 \longrightarrow \dots.$$

determines a perfect complex. Conversely, any perfect complex over M is locally quasi-isomorphic to such a complex of vector bundles. In fact, any perfect complex over a *compact* manifold admits a global presentation by a finite complex of finite rank vector bundles, by the existence of global resolutions [88, Exposé II, Proposition 2.3.2].

Example 5.2.30. The subcategory $\mathrm{Perf}(M) \subseteq \mathrm{QC}(M)$ is closed under finite limits and colimits, as well as the tensor product. Indeed, these can be computed locally, so that the result follows from the fact that finitely presented modules over a commutative algebra are closed under these operations.

Note that every perfect complex E is dualizable. Indeed, the mapping sheaf $E^\vee = \mathrm{Map}(E, \mathcal{O})$ in $\mathrm{QC}(M)$ can be computed locally. But on small enough opens U we can present E by a finitely presented $\mathcal{O}$-module, so that $E^\vee$ is finitely presented and serves as the dual of E.

Recall that a perfect complex $E \in \mathrm{Perf}(M)$ has *Tor-amplitude* contained in $[a, b]$ if the associated sheaf of $E \otimes_{\mathcal{O}} \pi_0(\mathcal{O})$ has homotopy sheaves vanishing in degrees strictly below a and strictly above b. Let $\mathrm{Perf}^{[a,b]}(M) \subseteq \mathrm{Perf}(M)$ be the subcategory of perfect complexes with Tor-amplitude contained in $[a, b]$.

Example 5.2.31. The following properties are immediate (see e.g. [96]):

(a) If E is a perfect complex such that $E \otimes_{\mathcal{O}} \pi_0(\mathcal{O}) \simeq 0$, then $E \simeq 0$.

(b) A perfect complex E is locally equivalent to the sheaf $\mathcal{O}^{\oplus n}[a]$ if and only if $E \in \mathrm{Perf}^{[a,a]}(M)$.

(c) If $E \in \mathrm{Perf}^{[a,b]}(M)$, then $E \in \mathrm{Perf}^{[a-1,b+1]}(M)$.

(d) If $f\colon E \longrightarrow F$ is a map in $\mathrm{Perf}^{[a,b]}(M)$, then the cofiber $\mathrm{cof}(f)$ is contained in $\mathrm{Perf}^{[a,b+1]}(M)$

(e) If $E \in \mathrm{Perf}^{[a,b]}(M)$ and $f\colon N \longrightarrow M$ is a map of affine spaces, then $f^*E \in \mathrm{Perf}^{[a,b]}(N)$.

(f) If $E \in \mathrm{Perf}^{[a,b]}(M)$ and $F \in \mathrm{Perf}^{[a',b']}(M)$, then the tensor product $E \otimes_A F$ is contained in $\mathrm{Perf}^{[a+a',b+b']}(M)$ and $E^\vee \in \mathrm{Perf}^{[-b,-a]}(M)$.

(g) If $E \in \mathrm{Perf}^{[a,b]}(M)$, then locally on M there exists a map $\mathcal{O}_U[a]^{\oplus n} \longrightarrow E\big|U$ whose cofiber is contained in $\mathrm{Perf}^{[a+1,b]}(U)$.

Lemma 5.2.32. *The functors* Perf *and* $\mathrm{Perf}^{[a,b]}\colon \mathrm{Aff}^{\mathrm{op}} \longrightarrow \mathrm{Cat}_\infty$ *satisfy descent.*

Proof. This follows from descent of quasi-coherent sheaves, since a quasi-coherent sheaf is perfect (with Tor-amplitude contained in $[a, b]$) if and only if it is locally perfect (with Tor-amplitude contained in $[a, b]$). □

Every sheaf $X\colon \mathrm{Aff}^{\mathrm{op}} \longrightarrow \mathcal{S}$ therefore comes equipped with full subcategories

$$\mathrm{Perf}^{[a,b]}(X) \subseteq \mathrm{Perf}(X) \subseteq \mathrm{QC}(X)$$

of perfect complexes over X (with Tor-amplitude contained in $[a, b]$), obtained as limits of the form $\mathrm{Perf}(X) = \lim_{U \in \mathrm{Aff}/X} \mathrm{Perf}(U)$.

Remark 5.2.33. Let $X_\bullet$ be a Lie n-groupoid and let X be its associated stack. The ∞-category $\mathrm{Perf}(X)$ of perfect complexes is closely related to the ∞-category of *representations up to homotopy* of $X_\bullet$, in the sense of [4]. Indeed, for every smooth manifold M, consider the dg-category of bounded complexes of (finite rank) vector bundles on M, whose dg-nerve (see e.g. [62, Section 1.3.1]) we will denote by $\mathrm{Vect}^{\mathrm{dg}}(M)$. This ∞-category admits a natural fully faithful inclusion $\mathrm{Vect}^{\mathrm{dg}}(M) \longrightarrow \mathrm{Perf}(M)$.

Now suppose that $X_\bullet$ is a simplicial manifold and consider the underlying semi-cosimplicial diagram of fully faithful functors $\mathrm{Vect}^{\mathrm{dg}}(X_\bullet) \longrightarrow \mathrm{Perf}(X_\bullet)$. This induces a fully faithful functor on homotopy limits

$$\operatorname{holim} \mathrm{Vect}^{\mathrm{dg}}(X_\bullet) \longrightarrow \mathrm{Perf}(X).$$

The homotopy limit of $\mathrm{Vect}^{\mathrm{dg}}(X_\bullet)$ can be computed explicitly: an object is given by a collection of simplices $\alpha_n\colon \Delta[n] \longrightarrow \mathrm{Vect}^{\mathrm{dg}}(X_n)$ taking values in the equivalences and such that $\alpha_n \circ \partial_i = d_i^* \alpha_{n-1}$. By the definition of the dg-nerve, each α_n is given by the datum of

- objects $E_0, \ldots, E_n \in \mathrm{Vect}^{\mathrm{dg}}(X_n)$.
- for each subset $I = \{i_- < i_m < \cdots < i_1 < i_+\} \subseteq [n]$, a degree m element f_I in the complex $\mathrm{Map}_{\mathrm{Vect}^{\mathrm{dg}}(X_n)}(X_{i_-}, X_{i_+})$, such that

$$\partial f_I = \sum_{j=1}^{m} (-1)^j \Big(f_{I\setminus\{i_j\}} - f_{\{i_j < \cdots < i_+\}} \circ f_{\{i_- < \cdots < i_j\}} \Big).$$

  and such that each $f_{\{i_-, i_+\}}$ is a quasi-isomorphism.

Using the relations between the α_n, one sees that a vertex of $\operatorname{holim} \mathrm{Vect}^{\mathrm{dg}}(X_\bullet)$ is precisely a representation up to homotopy in the sense of [4, Proposition 3.2] (with the extra condition that the action is by quasi-isomorphisms). A similar analysis can be applied to the higher simplices of the homotopy limit.

The following results are well-known (see e.g. [97, Section 1.3.7, 2.2.6] and [96]):

Lemma 5.2.34. *Let $E \in \mathrm{QC}(X)$ be a quasi-coherent sheaf over an sheaf X and consider the functor*

$$\mathrm{Spec}_X(E)\colon (\mathrm{Aff}/X)^{\mathrm{op}} \longrightarrow \mathcal{S}; \ \ \big(U \xrightarrow{f} X\big) \longmapsto \mathrm{Map}_{\mathcal{O}(U)}\big(f^*E, \mathcal{O}(U)\big).$$

This determines a map of sheaves $p\colon \mathrm{Spec}_X(E) \longrightarrow X$ over Aff, together with a zero section $0\colon X \longrightarrow \mathrm{Spec}_X(E)$.

(1) If E is connective, then $p\colon \mathrm{Spec}_X(E) \longrightarrow X$ is 0-representable.
(2) If E is perfect with Tor-amplitude contained in $[-a, b]$ with $a \geq 0$, then p is a-representable and locally finitely presented.
(3) If in addition $b \leq 0$, then the map p is smooth and if $b < 0$, then the zero section is a $(a-1)$-smooth surjection.

Proof. Clearly the functor $\mathrm{Spec}_X(E)\colon (\mathrm{Aff}/X)^{\mathrm{op}} \longrightarrow \mathcal{S}$ satisfies descent, so that it determines an object in $\mathrm{Sh}(\mathrm{Aff}/X) \simeq \mathrm{Sh}(\mathrm{Aff})/X$. The zero section $X \longrightarrow \mathrm{Spec}_X(E)$ sends each affine $f\colon U \longrightarrow X$ to the zero map $f^*E \longrightarrow \mathcal{O}(U)$. For any map $f\colon Y \longrightarrow X$ of sheaves, the functor $\mathrm{Spec}_Y(f^*E)$ is simply given by the restriction of $\mathrm{Spec}_X(E)$ along the canonical functor $\mathrm{Aff}/Y \longrightarrow \mathrm{Aff}/X$. It follows that there is a natural commuting square of functors

$$\begin{tikzcd}
\mathrm{QC}(X)^{\mathrm{op}} \arrow[r, "f^*"] \arrow[d, "\mathrm{Spec}_X"'] & \mathrm{QC}(Y)^{\mathrm{op}} \arrow[d, "\mathrm{Spec}_Y"] \\
\mathrm{Sh}(\mathrm{Aff})/X \arrow[r, "f^*"'] & \mathrm{Sh}(\mathrm{Aff})/Y.
\end{tikzcd}$$

Since all statements are local in X, we may therefore assume that $X = M$ is an affine. In that case, (1) is immediate since $\mathrm{Spec}_M(E)$ is corepresentable by the free $\mathcal{C}^\infty$-ring under $\mathcal{O}(M)$ generated by the connective $\mathcal{O}(M)$-module E.

For (2) and (3), we proceed by induction on a. The case $a = 0$ follows immediately from (1). Next, suppose that E has Tor-amplitude contained in $[-a, b]$ with $a > 0$. To see that $\mathrm{Spec}_M(E)$ is a derived a-stack, it suffices to work locally on M and assume that there exists a map $F[-a] \longrightarrow E$ where F is free and with a cofiber E' of Tor-amplitude $[-a+1, b]$. It follows that $\mathrm{Spec}_M(E)$ fits into a homotopy pullback of sheaves over M of the form

$$\begin{tikzcd} \mathrm{Spec}_M(E') \arrow[r] \arrow[d] & M \arrow[d, "0"] \\ \mathrm{Spec}_M(E) \arrow[r] & \mathrm{Spec}_M(F[-a]). \end{tikzcd}$$

The zero section $0\colon M \longrightarrow \mathrm{Spec}_M(F[-a])$ is an $(a-1)$-smooth surjection, since it is the base change of the $(a-1)$-smooth surjection $* \longrightarrow K(\mathbb{R}^k, a)$. It follows that the map $\mathrm{Spec}_M(E') \longrightarrow \mathrm{Spec}_M(E)$ is an $(a-1)$-smooth surjection. Since $\mathrm{Spec}_M(E')$ is an $(a-1)$-stack, locally finitely presented over M (smooth when $b \leq 0$), it follows that $\mathrm{Spec}_M(E)$ is an a-stack, locally finitely presented over M (smooth when $b \leq 0$).

Finally if $b < 0$, then the zero section of $\mathrm{Spec}_M(E)$ is the composition of the zero section of $\mathrm{Spec}_M(E')$ and the $(-a-1)$-smooth surjection $\mathrm{Spec}_M(E') \longrightarrow \mathrm{Spec}_M(E)$. Since E' has Tor-amplitude contained in $[-a+1, b]$, an inductive argument reduces the statement to the case where E has Tor-amplitude contained in $[b, b]$. In that case, E is locally free and the zero section $M \longrightarrow \mathrm{Spec}_M(E)$ can locally be identified with the base change of $* \longrightarrow K(\mathbb{R}^k, b)$, which is smooth. □

Example 5.2.35. Let X be a derived stack and let $L_X \in \mathrm{QC}(X)$ be its cotangent complex (see Definition 6.1.20). For $n \geq 0$, the shifted tangent bundles of X are the derived stacks

$$T[-n]X := \mathrm{Spec}_X(L_X[n]).$$

This definition extends for negative n when X is locally finitely presented, so that L_X is perfect.

Lemma 5.2.36 ([97, Proposition 2.3.3.1], [96])**.** *Consider the sheaves*

$$\mathcal{M}_0 = \mathrm{Ob}\big(\mathrm{Perf}^{[a,b]}\big)\colon \mathrm{Aff}^{\mathrm{op}} \longrightarrow \mathcal{S}$$
$$\mathcal{M}_1 = \mathrm{Ar}\big(\mathrm{Perf}^{[a,b]}\big)\colon \mathrm{Aff}^{\mathrm{op}} \longrightarrow \mathcal{S}$$

sending an affine U to the spaces of objects and arrows in $\mathrm{Perf}^{[a,b]}(U)$*. Both sheaves are derived $(b-a+1)$-stacks.*

Proof. The second assertion can be deduced from the first, since the map taking domain and codomain

$$(d_0, d_1)\colon \mathcal{M}_1 \longrightarrow \mathcal{M}_0 \times \mathcal{M}_0$$

is $(b-a)$-representable. Indeed, consider a pullback square of the form

$$\begin{tikzcd} P \arrow[r] \arrow[d] & \mathcal{M}_1 \arrow[d] \\ M \arrow[r, "(E,F)"'] & \mathcal{M}_0 \times \mathcal{M}_0 \end{tikzcd}$$

where M is affine and the bottom map classifies two perfect complexes E and F over M. Unraveling the definitions, one sees that the sheaf P sends an affine U to the space of maps $f\colon U \longrightarrow M$, together with a map of perfect complexes $f^*E \longrightarrow f^*F$. Since perfect complexes are dualizable, the latter is equivalent to the data of a map $f^*(E \otimes F^\vee) \longrightarrow \mathcal{O}_U$. It follows that P is simply the stack $\mathrm{Spec}_M(E \otimes F^\vee)$. Since $E \otimes F^\vee$ has Tor-amplitude contained in $[a-b, b-a]$, it follows from Lemma 5.2.34 that the projection to M is $(b-a)$-representable.

To show that $\mathcal{M}_0$ is a derived stack, we will realize it as the $\mathbb{R}$-points of a derived Artin stack, following Remark 5.2.24. Let $Z \in \mathrm{Sh}(\mathrm{AffSch}_{\mathbb{R}})$ be the stack of perfect modules with Tor-amplitude in $[a, b]$, which is a derived Artin $(b-a+1)$-stack by [96, Proposition 3.7]. In fact, all coverings used in the proof of loc. cit. induce surjections on $\mathbb{R}$-points, so that the $\mathbb{R}$-points $Z_{\mathbb{R}}$ form a derived $(b-a+1)$-stack as well, by Remark 5.2.24.

The sheaf $Z_{\mathbb{R}}$ is the associated sheaf of the presheaf sending an affine derived manifold M to $Z(\mathcal{O}(M))$ (where $\mathcal{O}(M)$ is considered as an $\mathbb{R}$-algebra). Every perfect $\mathcal{O}(M)$-module determines a perfect complex over M in the $\mathcal{C}^\infty$-sense of Definition 5.2.28, so that there is a canonical map $Z_{\mathbb{R}} \longrightarrow \mathcal{M}_0$. This map induces a surjection on π_0-sheaves, since by definition every perfect complex can be presented locally by a finitely presented $\mathcal{O}$-module. On the other hand, the map $Z_{\mathbb{R}} \longrightarrow \mathcal{M}_0$ is a fully faithful inclusion, since the quasicoherent sheaves on M form a full subcategory of the ∞-category of $\mathcal{O}(M)$-modules. We conclude that $\mathcal{M}_0 \simeq Z_{\mathbb{R}}$ is a derived $(b-a+1)$-stack. $\square$

Variant 5.2.37. Replacing each structure sheaf by its complexification, one obtains a derived $(b-a+1)$-stack $\mathrm{Perf}_{\mathbb{C}}^{[a,b]}$ of perfect complexes over $\mathbb{C}$. Similarly, there are derived stacks of perfect complexes equipped with (non-degenerate or skew-) symmetric forms, defined as certain pullbacks of the stacks $\mathcal{M}_0$ and $\mathcal{M}_1$ [102].

5.3 Homotopy theory of stacks

Because smooth manifolds have a very simple (local) topology, they come with a well-behaved theory of *locally constant sheaves*, which is controlled by a simple invariant: their underlying homotopy type (rather than something more complicated, like a pro-homotopy type). For example, the category of locally constant set-valued sheaves on M is equivalent to the category of functors $\tau_{\leq 1}(M) \longrightarrow \mathrm{Set}$ indexed by the fundamental groupoid of M. The purpose of this section is to describe an analogue of this result that applies not only a smooth manifold, but also in relative situations, where we have a smooth map $p\colon Y \longrightarrow X$ between two sheaves on Aff.

5.3.1 Sheaves on stacks. Every derived manifold M comes equipped with an ∞-topos $\mathrm{Sh}(M)$ of sheaves on its underlying topological space, as well as an ∞-topos $\mathrm{Sh}(\mathrm{Aff})/M$ of 'sheaves over M'. These two categories of sheaves are closely connected (this is treated in great generality in [48]): for any map $f\colon M \longrightarrow N$ between affine derived manifolds, there is a commuting square

$$\begin{tikzcd} \mathrm{Op}(N) \arrow[r, "\iota"] \arrow[d, "f^{-1}"'] & \mathrm{Sh}(\mathrm{Aff})/N \arrow[d, "f^*"] \\ \mathrm{Op}(M) \arrow[r, "\iota"'] & \mathrm{Sh}(\mathrm{Aff})/M \end{tikzcd} \tag{5.3.1}$$

where the horizontal functors are the obvious inclusions (using Proposition 5.2.3) and the vertical functors take inverse images (resp. the pullback) along f.

Lemma 5.3.2. *Let $\iota^*\colon \mathrm{PSh}(\mathrm{Aff})/M \longrightarrow \mathrm{Sh}(\mathrm{Op}(M))$ be the functor restricting along ι. Then ι^* has the following two properties:*

(1) it preserves sheaves.

(2) if $\{V_i \longrightarrow V\}$ is an open cover of a derived manifold V over M, then the map of presheaves $\coprod \iota^*(V_i) \longrightarrow \iota^*(V)$ induces a surjection on the associated sheaves.

Proof. The functor ι sends an open cover $\{V_i \longrightarrow V\}$ in $\mathrm{Op}(M)$ to a surjection $\coprod \iota(V_i) \longrightarrow \iota(V)$ in $\mathrm{Sh}(\mathrm{Aff})/M$, so ι^* preserves sheaves. For any derived manifold V over M, the sheaf ι^*V on M is simply the sheaf sending an open $U \subseteq M$ to $\mathrm{Map}_{\mathrm{dMfd}/M}(U, V)$. Clearly $\coprod \iota^*(V_i) \longrightarrow \iota^*(V)$ will induce a surjection on associated sheaves: given a map $U \longrightarrow V$ over M, one can always find a cover $\{U_\alpha \longrightarrow U\}$ such that each $U_\alpha \longrightarrow U \longrightarrow V$ factors over some open V_i. □

Lemma 5.3.3. *Restriction along ι determines a functor*

$$\mathrm{et}^*\colon \mathrm{Sh}(\mathrm{Aff})/M \longrightarrow \mathrm{Sh}(M)$$

which admits a left adjoint $\mathrm{et}_!$ and right adjoint et_*. The left adjoint $\mathrm{et}_!$ is fully faithful and preserves finite limits. Furthermore, it depends functorially on the affine manifold M, in the sense that any map $f\colon M \longrightarrow N$ induces a (natural) commuting square

$$\begin{tikzcd} \mathrm{Sh}(M) \arrow[r, "\mathrm{et}_!"] \arrow[d, "f^{-1}"'] & \mathrm{Sh}(\mathrm{Aff})/M \arrow[d, "f^*"] \\ \mathrm{Sh}(N) \arrow[r, "\mathrm{et}_!"'] & \mathrm{Sh}(\mathrm{Aff})/N. \end{tikzcd} \tag{5.3.4}$$

Proof. By (1) of Lemma 5.3.2, restriction along ι determines a functor between categories of sheaves, whose left adjoint $\mathrm{et}_!$ sends a sheaf to the associated sheaf of its left Kan extension along ι. On the other hand, (2) implies that right Kan extension along ι preserves sheaves and provides a right adjoint et_* to et^*.

Since et^* and $\mathrm{et}_!$ both preserve colimits, to verify that $\mathrm{et}_!$ is fully faithful it suffices to show that the map $U \longrightarrow \mathrm{et}^*\mathrm{et}_!(U)$ is an equivalence for any open subspace $U \subseteq M$, which is immediate. Since the functor ι preserves finite limits, the functor $\mathrm{et}_!$ preserves finite limits. Finally, the commuting square (5.3.4) arises from (5.3.1) by passing to the associated left Kan extension functors (and taking associated sheaves). □

Remark 5.3.5. One can think of the functor $\mathrm{et}_!$ as an analogue of the étale space (or espace étalé) construction from classical sheaf theory [36, Chapitre II.1.2]. To see this, suppose that F is a sheaf of sets on M and pick a surjection $\coprod U_i \longrightarrow F$ where all $U_i \subseteq M$ are open subspaces. The sheaf $\mathrm{et}_!(F)$ is obtained from the sheaves represented by the U_i, by gluing along their intersection $U_i \times_F U_j$. These intersections are representable by open subspaces of U_i and U_j, so that $\mathrm{et}_!(F)$ is representable by a derived manifold over M. This derived manifold is simply the usual étale space over M associated to the sheaf F.

In particular, $\mathrm{et}_!(F) \longrightarrow M$ is an étale map (since it locally given by the open inclusion $\mathrm{et}_!(U_i) \longrightarrow M$). Similarly, if $F \longrightarrow G$ is a (surjective) map of sheaves of sets over M, then the map $\mathrm{et}_!(F) \longrightarrow \mathrm{et}_!(G)$ is an étale (surjection) of derived manifolds over M. Using this, one sees that for any n-truncated sheaf F on M, its image $\mathrm{et}_!(F) \longrightarrow M$ is an n-étale map. Indeed, if F is modeled by an n-groupoid object $F_\bullet$ of discrete sheaves on M, then $\mathrm{et}_!(F)$ is the colimit of a derived étale n-groupoid $\mathrm{et}_!(F_\bullet)$ over M.

Using Lemma 5.2.4 and [59, Theorem 6.1.3.9], we can extend the natural geometric morphism (5.3.4) to a natural geometric morphism

$$\mathrm{et}_!\colon \mathrm{Sh}(-) \rightleftarrows \mathrm{Sh}(\mathrm{Aff})/(-)\colon \mathrm{et}^*$$

between colimit-preserving functors $\mathrm{Sh}(\mathrm{Aff}) \longrightarrow \mathrm{Topos}^{\mathrm{R}}$ to the ∞-category of ∞-toposes and (the right adjoints of) geometric morphisms between them. For any sheaf X on Aff, we will refer to $\mathrm{Sh}(X)$ as the ∞-category of *sheaves on X*.

Example 5.3.6. Let M be a derived manifold and consider the composite

$$\mathrm{Sh}(M) \xrightarrow{\mathrm{et}_!} \mathrm{Sh}(\mathrm{Aff})/M \xrightarrow{\mathrm{Sh}(-)} \mathrm{Topos}^{\mathrm{R}}.$$

This functor preserves colimits and its restriction to the open subspaces $\mathrm{Op}(M)$ is naturally equivalent to the functor sending $U \subseteq M$ to the ∞-topos $\mathrm{Sh}(U) \simeq \mathrm{Sh}(M)/U$. It follows from [59, Theorem 6.1.3.9] that for any sheaf F on M, there is an equivalence $\mathrm{Sh}(\mathrm{et}_!(F)) \simeq \mathrm{Sh}(M)/F$.

Similarly, taking colimits one sees that for any sheaf $X\colon \mathrm{Aff}^{\mathrm{op}} \longrightarrow \mathcal{S}$ and any $F \in \mathrm{Sh}(X)$, there is an equivalence of ∞-categories

$$\mathrm{Sh}(\mathrm{et}_!(F)) \simeq \mathrm{Sh}(X)/F.$$

Corollary 5.3.7. *For every sheaf $X\colon \mathrm{Aff} \longrightarrow \mathcal{S}$, the left adjoint $\mathrm{et}_!\colon \mathrm{Sh}(X) \longrightarrow \mathrm{Sh}(\mathrm{Aff})/X$ is fully faithful and sends n-truncated sheaves over X to n-étale maps over X.*

Proof. The colimit of a diagram of ∞-toposes is computed as the limit of the associated diagram of ∞-categories and left adjoints between them. It follows that $\mathrm{et}_!$ is the limit of the fully faithful functors $\mathrm{et}_!\colon \mathrm{Sh}(M) \longrightarrow \mathrm{Sh}(\mathrm{Aff})/M$ for every affine $M \longrightarrow X$, so that it is fully faithful itself.

To see that $\mathrm{et}_!(F) \longrightarrow X$ is n-étale for any n-truncated sheaf F on X, it suffices to verify that for any map $f\colon M \longrightarrow X$ from an affine, the map

$$f^*\mathrm{et}_!(F) \simeq \mathrm{et}_!(f^{-1}F) \longrightarrow M$$

is n-étale. This follows from Remark 5.3.5, since $f^{-1}F$ remains n-truncated. ☐

Remark 5.3.8. Quasi-coherent sheaves on $X \in \mathrm{Sh}(\mathrm{Aff})$ cannot generally be considered as sheaves on X (with algebraic structure) in the above sense. A notable exception is the case where X is a derived étale stack: in this case the structure sheaves on an affine étale cover of X determine a structure sheaf $\mathcal{O}$ of (local) $\mathcal{C}^\infty$-rings on the ∞-topos $\mathrm{Sh}(X)$, whose category of module sheaves can be identified with $\mathrm{QC}(X)$.

5.3.2 Locally contractible maps.

Classically, one can identify locally constant set-valued sheaves on a smooth manifold M with presheaves on its fundamental groupoid, using that M is locally simply connected. Similarly, to make sure that a map $p\colon Y \longrightarrow X$ in $\mathrm{Sh}(\mathrm{Aff})$ has a simple theory of fiberwise locally constant sheaves (of spaces), we need to require p to be locally contractible:

Definition 5.3.9. Let $p\colon Y \longrightarrow X$ be a map in $\mathrm{Sh}(\mathrm{Aff})$. We will say that p is *almost locally contractible* if for each pullback diagram in $\mathrm{Sh}(\mathrm{Aff})$ on the left

$$\begin{tikzcd} X & X' \arrow[l, "f"'] \\ Y \arrow[u, "p"] & Y' \arrow[l, "f'"] \arrow[u, "p'"'] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Sh}(X) \arrow[r, "f^{-1}"] \arrow[d, "p^{-1}"'] & \mathrm{Sh}(X') \arrow[d, "(p')^{-1}"] \\ \mathrm{Sh}(Y) \arrow[r, "(f')^{-1}"'] & \mathrm{Sh}(Y'). \end{tikzcd} \tag{5.3.10}$$

the right square of ∞-categories is right adjointable (Definition 2.3.22), i.e. the base change morphism $p^{-1}f_* \longrightarrow f'_*(p')^{-1}$ is a natural equivalence.

A map $p\colon Y \longrightarrow X$ is *locally contractible* if for every map $X' \longrightarrow X$ in $\mathrm{Sh}(\mathrm{Aff})$, the base change $p'\colon Y \times_X X' \longrightarrow X'$ is almost locally contractible.

Remark 5.3.11. Definition 5.3.9 arises more naturally as a condition at the level of ∞-toposes. The above definition suffices for our purposes and avoids working with the ∞-category of ∞-toposes.

The class of locally contractible maps is closed under composition and it is closed under base change by construction.

Example 5.3.12. Let $K \in \mathrm{Sh}(\mathrm{Aff})$ be the constant sheaf on a space K. For any sheaf X, the projection $p\colon K \times X \longrightarrow X$ is almost locally contractible, and thus locally contractible. Indeed, note that there is a natural equivalence of ∞-categories $\mathrm{Sh}(K \times X) \simeq \mathrm{Fun}(K, \mathrm{Sh}(X))$. For any $f\colon X' \longrightarrow X$, the square of toposes (5.3.10) can then be identified with

$$\begin{tikzcd} \mathrm{Sh}(X) \arrow[r, "f^{-1}"] \arrow[d, "\mathrm{cst}"'] & \mathrm{Sh}(X') \arrow[d, "\mathrm{cst}"] \\ \mathrm{Fun}(K, \mathrm{Sh}(X)) \arrow[r, "\mathrm{Fun}(K, f^{-1})"'] & \mathrm{Fun}(K, \mathrm{Sh}(X')) \end{tikzcd}$$

This is easily seen to be right adjointable, since the right adjoints of the horizontal functors simply apply f_* pointwise.

Example 5.3.13. Suppose that $p_i\colon Y_i \longrightarrow X$ is a set of (almost) locally contractible maps. Then the induced map $p\colon Y = \coprod Y_i \longrightarrow X$ is (almost) locally contractible as well. Indeed, the base change morphism associated to a pullback square (5.3.10) is a natural transformation

$$\begin{tikzcd} \mathrm{Sh}(X') \arrow[r, bend left, "p^{-1}f'_*"] \arrow[r, bend right, "f_*(p')^{-1}"'] \arrow[r, phantom, "\Downarrow\mu"] & \mathrm{Sh}(Y) = \prod_i \mathrm{Sh}(Y_i) \end{tikzcd}$$

whose i-th component is the base change morphism μ_i associated to p_i. Each of these components is a natural equivalence.

Suppose that $p\colon Y \longrightarrow X$ is an almost locally contractible map and let I be a set. Taking $X' = \coprod_I X \longrightarrow X$ to be the fold map, the base change morphism can be identified with the natural transformation

$$\begin{tikzcd} \mathrm{Sh}(X) \arrow[d, "p^{-1}"'] & \mathrm{Fun}(I, \mathrm{Sh}(X)) \arrow[l, "\lim"'] \arrow[d, "\mathrm{Fun}(I, p^{-1})"] \arrow[dl, phantom, "\Rightarrow"] \\ \mathrm{Sh}(Y) & \mathrm{Fun}(I, \mathrm{Sh}(Y)). \arrow[l, "\lim"] \end{tikzcd}$$

It follows that the functor p^{-1} preserves infinite products. Consequently, it preserves all limits [59, Proposition 4.4.2.7] and admits a left adjoint $p_!$.

Remark 5.3.14. If p is locally contractible, then for every every pullback square (5.3.10), the functors p^{-1} and $(p')^{-1}$ have left adjoints. It follows from [62, Remark 4.7.4.14] that the transposed square of (5.3.10) is left adjointable, i.e. the natural transformation of left adjoint functors $p'_!(f')^{-1} \longrightarrow f^{-1}p_!$ is an equivalence.

For every locally contractible map $p\colon Y \longrightarrow X$, there is a simple theory of fiberwise locally constant sheaves on Y:

Definition 5.3.15. If $p\colon Y \longrightarrow X$ is a locally contractible map, we define the following sheaves on X

$$\mathrm{Sing}(Y/X) := p_!(*) \qquad\qquad \mathrm{Sing}_{\leq n}(Y/X) := \tau_{\leq n}\big(p_!(*)\big).$$

Informally, one can think of $\mathrm{Sing}(Y/X)$ as the sheaf of homotopy types of the fibers of p.

Lemma 5.3.16. *Let $p\colon Y \longrightarrow X$ be a locally contractible map. The functor $p_!\colon \mathrm{Sh}(Y) \longrightarrow \mathrm{Sh}(X)\big/\mathrm{Sing}(Y/X)$ admits a right adjoint ψ^*, which is fully faithful and preserves colimits.*

Proof. The right adjoint ψ^* sends a map $F \longrightarrow p_!(*) = \mathrm{Sing}(Y/X)$ to the pullback $p^{-1}F \times_{p^{-1}p_!(*)} \{*\}$ along the unit map. This preserves colimits since colimits are universal in $\mathrm{Sh}(Y)$. To see that it is fully faithful, let $f\colon F \longrightarrow G$ be a map of sheaves in $\mathrm{Sh}(X)$. Consider the square (5.3.10) in the case where

- $X' \longrightarrow X$ is the map $\mathrm{et}_!(G) \longrightarrow \mathrm{et}_!(F)$
- $Y \longrightarrow X$ is the locally contractible map $\mathrm{et}_!(p^{-1}F) \longrightarrow \mathrm{et}_!(F)$.

Example 5.3.6 then identifies the right square in (5.3.10) with

$$\begin{tikzcd}
\mathrm{Sh}(X)/F \arrow[r, "f^*"] \arrow[d, "p^{-1}"'] & \mathrm{Sh}(X)/G \arrow[d, "p^{-1}"] \\
\mathrm{Sh}(Y)/p^{-1}F \arrow[r, "p^{-1}(f)^*"'] & \mathrm{Sh}(Y)/p^{-1}G.
\end{tikzcd}$$

For any $A \longrightarrow p^{-1}G$, the map $p_!(p^{-1}F \times_{p^{-1}G} A) \longrightarrow F \times_G p_!A$ is an equivalence by Remark 5.3.14. Taking $G = p_!(*)$ and $A = *$, this implies that $p_!\psi^*(F) \longrightarrow F$ is an equivalence. $\square$

We will refer to the essential image of $\psi^*\colon \mathrm{Sh}(X)\big/\mathrm{Sing}(Y/X) \longrightarrow \mathrm{Sh}(Y)$ as the *fiberwise locally constant sheaves* on Y.

Example 5.3.17. Suppose that X is locally contractible, i.e. the map $X \longrightarrow *$ is locally contractible. In this case, $F \in \mathrm{Sh}(X)$ is locally constant iff there exists an epimorphism $\coprod U_\alpha \longrightarrow *$ in $\mathrm{Sh}(X)$ such that each $U_\alpha \times F$ is a constant sheaf, i.e. lies in the image of the left adjoint to the global sections functor (see [62, Appendix A.1]). The ∞-category of locally constant sheaves on X can be identified with the ∞-category of diagrams $\mathrm{Sing}(X) \longrightarrow \mathcal{S}$.

Similarly, if $p\colon Y \longrightarrow X$ is locally contractible, then every fiberwise constant sheaf $p^{-1}(F)$ is fiberwise locally constant. The proof of [62, Theorem A.1.15] shows that a sheaf $F \in \mathrm{Sh}(Y)$ is fiberwise locally constant if there exists an epimorphism $\coprod U_\alpha \longrightarrow *$ such that each $U_\alpha \times F$ is fiberwise constant.

Lemma 5.3.18. *Let $p\colon Y \longrightarrow X$ be a locally contractible map. For any sheaf $F \in \mathrm{Sh}(X)$ and any sheaf $G \in \mathrm{Sh}(Y)$, there is an equivalence between hom sheaves*

$$p_*\mathrm{Hom}\big(G, p^{-1}F\big) \simeq \mathrm{Hom}\big(p_!(G), F\big).$$

In particular, there is an equivalence of sheaves $p_*p^{-1}F \simeq \mathrm{Hom}\big(\mathrm{Sing}(Y/X), F\big)$.

Proof. The proof of Lemma 5.3.16 provides a natural equivalence

$$p_!(p^{-1}(A) \times G) \xrightarrow{\sim} A \times p_!(G)$$

for any $A \in \mathrm{Sh}(X)$. The desired equivalence is obtained by adjunction. $\square$

The remainder of this section is devoted to a proof of the following:

Theorem 5.3.19. *Let $p\colon Y \longrightarrow X$ be a smooth map between sheaves over* Aff*. Then p is locally contractible.*

Example 5.3.20. Let X be a smooth n-stack X and present X by a (smooth) Lie n-groupoid $X_\bullet$. Then $\mathrm{Sing}(X)$ is the realization of the bisimplicial set $\mathrm{Sing}(X_\bullet)$, where $\mathrm{Sing}(X_k)$ is the usual singular complex of the manifold X_k.

Similarly, let $p\colon Y \longrightarrow X$ be a submersion between two finite-dimensional smooth manifolds. Then a sheaf on Y is fiberwise locally constant if and only if its restriction to each fiber of p is locally constant in the sense of Example 5.3.17. The stalk of the sheaf $\mathrm{Sing}(Y/X)$ at $x \in X$ is given by the singular complex of the fiber Y_x.

The proof of Theorem 5.3.19 uses a few properties of locally contractible maps:

Lemma 5.3.21. *Let $p\colon Y \longrightarrow X$ be a map of sheaves over* Aff *and consider a diagram $U\colon \mathcal{I} \longrightarrow \mathrm{Sh}(\mathrm{Aff})$ with colimit X. If each base change $p_i\colon V_i := Y \times_X U_i \longrightarrow U_i$ is locally contractible, then p is locally contractible.*

Proof. It suffices to show that p is almost locally contractible. Indeed, given a map $f\colon X' \longrightarrow X$, the base change $p'\colon Y \times_X X' \longrightarrow X'$ and the diagram $U'_\bullet = U_\bullet \times_X X'$ also satisfy the conditions of the lemma: each base change $Y' \times_{X'} U'_i \longrightarrow U'_i$ is the base change of $p_i\colon V_i \longrightarrow U_i$ and hence locally contractible. Replacing p by p', the result then follows.

To see that p is almost locally contractible, let us fix a map $f\colon X' \longrightarrow X$ and let us consider the diagram $\mathcal{J} \times \Delta[1]^{\times 2} \longrightarrow \mathrm{Sh}(\mathrm{Aff})$ whose value at $i \in \mathcal{I}$ is given by the pullback square

$$\begin{tikzcd} U_i & X' \times_X U_i \arrow[l, "f_i"'] \\ Y \times_X U_i \arrow[u, "p_i"] & Y' \times_X U_i. \arrow[l, "f'_i"] \arrow[u, "p'_i"'] \end{tikzcd}$$

The vertical maps are locally contractible by assumption and taking the colimit over $\mathcal{J}$ produces the cartesian square (5.3.10). Let

$$\chi\colon \mathcal{J}^{\mathrm{op}} \times \Delta[1]^{\times 2} \longrightarrow \widehat{\mathrm{Cat}}_\infty$$

be the postcomposition of the above diagram with the limit-preserving functor $\mathrm{Sh}(\mathrm{Aff})^{\mathrm{op}} \longrightarrow \widehat{\mathrm{Cat}}_\infty$ sending a map of sheaves $f\colon X' \longrightarrow X$ to the left adjoint $f^{-1}\colon \mathrm{Sh}(X) \longrightarrow \mathrm{Sh}(X')$. The limit of this diagram over $\mathcal{J}$ is the right square in (5.3.10). To see that this limiting square is right adjointable, we can equivalently consider the diagram χ as a diagram

$$\chi'\colon \mathcal{J}^{\mathrm{op}} \times \Delta[1] \longrightarrow \mathrm{Fun}(\Delta[1], \widehat{\mathrm{Cat}}_\infty)$$

sending $(i, 0)$ to the functor $(p_i)^{-1}$ and $(i, 1)$ to the functor $(p'_i)^{-1}$. Since the maps p_i and p'_i are all locally contractible by assumption, the diagram χ' takes values in the category $\mathrm{Fun}^{\mathrm{RAd}}(\Delta[1], \widehat{\mathrm{Cat}}_\infty)$ of Definition 2.3.22. By [62, Corollary 4.7.4.18], the inclusion

$$\mathrm{Fun}^{\mathrm{RAd}}(\Delta[1], \widehat{\mathrm{Cat}}_\infty) \subseteq \mathrm{Fun}(\Delta[1], \widehat{\mathrm{Cat}}_\infty)$$

is a right adjoint. Consequently, taking the limit of χ' over $\mathcal{J}^{\mathrm{op}}$ produces an arrow $\Delta[1] \longrightarrow \mathrm{Fun}^{\mathrm{RAd}}(\Delta[1], \widehat{\mathrm{Cat}}_\infty)$. This means exactly that (5.3.10) is right adjointable. □

Lemma 5.3.22. *Consider a sequence $Z \xrightarrow{q} Y \xrightarrow{p} X$ such that q and pq are locally contractible. If q is a surjection, then p is locally contractible as well.*

Proof. Since the conditions of the lemma are stable under base change along a map $X' \longrightarrow X$, it suffices to verify that p is almost locally contractible, as in the proof of Lemma 5.3.21. To see this, let $f\colon X' \longrightarrow X$ be a map and consider the commuting diagram

$$\begin{tikzcd}
\mathrm{Sh}(X) \arrow[r, "f^{-1}"] \arrow[d, "p^{-1}"'] & \mathrm{Sh}(X') \arrow[d, "(p')^{-1}"] \\
\mathrm{Sh}(Y) \arrow[r, "(f')^{-1}"] \arrow[d, "q^{-1}"'] & \mathrm{Sh}(Y') \arrow[d, "(q')^{-1}"] \\
\mathrm{Sh}(Z) \arrow[r, "(f'')^{-1}"'] & \mathrm{Sh}(Z')
\end{tikzcd}$$

The base change morphism μ associated to the composite square decomposes as

$$\mu\colon q^{-1}p^{-1}f_* \xrightarrow{q^{-1}(\mu_1)} q^{-1}f'_*(p')^{-1} \xrightarrow{\mu_2\circ(p')^{-1}} f''_*(p'q')^{-1}$$

where μ_1 and μ_2 are the base change morphisms of the top and bottom square. By assumption, μ and μ_2 are equivalences, so that $q^{-1}(\mu_1)$ is an equivalence. Since $q\colon Z \longrightarrow Y$ is an effective epimorphism, the functor q^{-1} detects equivalences. We conclude that μ_1 is an equivalence, so that p is almost locally contractible. □

Lemma 5.3.23. *Let $F \in \mathrm{Sh}(X)$ be a sheaf and consider the map $p\colon \mathrm{et}_!(F) \longrightarrow X$. This map is locally contractible.*

Proof. The base change of p along a map $f\colon X' \longrightarrow X$ can be identified with the map $p'\colon \mathrm{et}_!(f^{-1}F) \longrightarrow X'$. It therefore suffices to show that p is almost locally contractible. Given a map $f\colon X' \longrightarrow X$, it follows from Example 5.3.6 that the right square in (5.3.10) can be identified with

$$\begin{tikzcd}
\mathrm{Sh}(X) \arrow[r, "f^{-1}"] \arrow[d, "(-)\times F"'] & \mathrm{Sh}(X') \arrow[d, "(-)\times f^{-1}F"] \\
\mathrm{Sh}(X)/F \arrow[r, "f^{-1}"'] & \mathrm{Sh}(X')/f^{-1}F.
\end{tikzcd}$$

One easily verifies that this square is right adjointable, using that f_* preserves limits. □

Proof (of Theorem 5.3.19). We will prove by induction on n that an n-smooth map $p\colon Y \longrightarrow X$ is locally contractible. By Lemma 5.3.21, we may assume that X is affine, so that Y is a derived n-stack.

For the induction step, suppose that p is n-smooth and let $q\colon Z \longrightarrow Y$ be an $(n-1)$-smooth atlas for the derived n-stack Y. Then q is locally contractible and pq is a 0-smooth map, hence locally contractible by the base of the induction. Lemma 5.3.22 then implies that p is locally contractible as well.

It therefore remains to treat the base case, where $p\colon Y \longrightarrow X$ is a (0-)smooth map between derived manifolds. Using Lemmas 5.3.21, 5.3.22 and 5.3.23, as well as Example 5.3.13, we can work locally on Y and X and thus reduce to the case of a projection map $X \times \mathbb{R}^n \longrightarrow X$.

It therefore suffices to verify that for any sheaf X, the map $p\colon X \times \mathbb{R}^n \longrightarrow X$ is almost locally contractible. The ∞-category $\mathrm{Sh}(X \times \mathbb{R}^n)$ has a very simple description: it follows from Proposition 7.3.1.11 and Theorem 7.3.3.9 of [59] that for any topological space M, there is a natural equivalence

$$\mathrm{Sh}(M \times \mathbb{R}^n) \simeq \mathrm{Sh}(M, \mathrm{Sh}(\mathbb{R}^n))$$

between the ∞-category of sheaves on $M \times \mathbb{R}^n$ and the category of sheaves on M with values in $\mathrm{Sh}(\mathbb{R}^n)$. If $X \simeq \operatorname{colim} M_i$ is a colimit of affine derived manifolds, we obtain a natural equivalence of ∞-categories

$$\mathrm{Sh}(X \times \mathbb{R}^n) \simeq \lim_i \mathrm{Sh}(M_i \times \mathbb{R}^n) \simeq \lim_i \mathrm{Sh}(M_i, \mathrm{Sh}(\mathbb{R}^n)) \simeq \mathrm{Sh}(X, \mathrm{Sh}(\mathbb{R}^n))$$

where the limits are taken along the left adjoints of the geometric morphisms. For any map of sheaves $f\colon X' \longrightarrow X$, the square (5.3.10) can then be identified with the commuting square of ∞-categories

$$\begin{tikzcd} \mathrm{Sh}(X, \mathcal{S}) \arrow[r, "f^{-1}"] \arrow[d, "\mathrm{Sh}(X,\mathrm{cst})"'] & \mathrm{Sh}(X', \mathcal{S}) \arrow[d, "\mathrm{Sh}(X',\mathrm{cst})"] \\ \mathrm{Sh}(X, \mathrm{Sh}(\mathbb{R}^n)) \arrow[r, "f^{-1}"'] & \mathrm{Sh}(X', \mathrm{Sh}(\mathbb{R}^n)). \end{tikzcd} \tag{5.3.24}$$

The horizontal functors simply take inverse image sheaves (with coefficients) along f and the vertical functors send a sheaf of spaces to $\mathrm{Sh}(\mathbb{R}^n)$-valued sheaf whose values are constant sheaves.

To see that this square is right adjointable, note that the functor $\mathrm{cst}\colon \mathcal{S} \longrightarrow \mathrm{Sh}(\mathbb{R}^n)$ admits a left adjoint σ, because $\mathbb{R}^n$ is a contractible topological space (see e.g. [62, Remark A.1.4]). Consequently, the vertical functors admit left adjoints $\mathrm{Sh}(X, \sigma)$ and $\mathrm{Sh}(X', \sigma')$. These left adjoints clearly commute with restriction of sheaves along f, so that the transpose of (5.3.24) is left adjointable. It then follows from Remark 2.3.24 that (5.3.24) is right adjointable, which shows that $p\colon X \times \mathbb{R}^n \longrightarrow X$ is almost locally contractible. ☐

5.3.3 Application: higher connected covers. Any Lie groupoid $\mathcal{G} \rightrightarrows M$ admits a source-simply connected cover $\widetilde{\mathcal{G}} \rightrightarrows M$. The discussion from the previous paragraphs provides an alternative description of this Lie groupoid. Indeed, suppose that $p\colon Y \longrightarrow X$ is a smooth map and let

$$\mathrm{Sing}_{\leq n}(Y/X) \longrightarrow X$$

be the image of sheaf $\mathrm{Sing}_{\leq n}(Y/X)$ from Definition 5.3.15 under the functor $\mathrm{et}_!$. This map is n-étale by Corollary 5.3.7. Furthermore, the map of sheaves on Y

$$* \longrightarrow p^{-1}(p_!(*)) \longrightarrow p^{-1}\tau_{\leq n}(p_!(*))$$

decomposes the map $p\colon Y \longrightarrow X$ as $Y \xrightarrow{\tilde{p}} \mathrm{Sing}_{\leq n}(Y/X) \longrightarrow X$.

Lemma 5.3.25. *Suppose that $p\colon Y \longrightarrow X$ is m-smooth. Then the induced map $\tilde{p}\colon Y \longrightarrow \mathrm{Sing}_{\leq n}(Y/X)$ is an m-smooth surjection for all $-1 \leq n \leq m$.*

Proof. The fact that $\tilde{p}$ is m-smooth follows immediately from the fact that the map $\mathrm{Sing}_{\leq n}(Y/X) \longrightarrow X$ is n-étale. To see that $\tilde{p}$ is a surjection, we may replace X by $\mathrm{Sing}_{\leq n}(Y/X)$, so that $\tilde{p} = p$ has n-connected fibers. Working locally on X, we may assume that X is a derived manifold and take a smooth map $p_\bullet\colon Y_\bullet \longrightarrow X$ between derived Lie n-groupoids modeling p.

For any point $x \in X$, the stalk $p_!(*)_x$ is given by the homotopy type $\mathrm{Sing}(Y_x)$ of the fiber of Y over x. By assumption, this homotopy type is nonempty so that Y_x is nonempty. It follows that the map $p_0\colon Y_0 \longrightarrow X$ is a smooth surjective map of derived manifolds, so that $Y \longrightarrow X$ is a surjection of sheaves. ☐

Lemma 5.3.26. *Suppose that $p\colon Y \longrightarrow X$ is smooth and let $\tilde{x}\colon * \longrightarrow \mathrm{Sing}_{\leq n}(Y/X)$ be a point with image x in X. Then the natural map $Y_{\tilde{x}} \longrightarrow Y_x$ realizes the fiber $Y_{\tilde{x}}$ over $\tilde{x}$ as an n-connected cover of the fiber Y_x over x.*

Proof. There is a diagram of pullback squares of sheaves

$$\begin{tikzcd}
Y_{\tilde{x}} \arrow[r] \arrow[d] & Y_x \arrow[r] \arrow[d] & Y \arrow[d] \\
* \arrow[r, "\tilde{x}"'] & \mathrm{Sing}_{\leq n}(Y/X)_x \arrow[r] \arrow[d] & \mathrm{Sing}_{\leq n}(Y/X) \arrow[d] \\
& * \arrow[r, "x"'] & X.
\end{tikzcd}$$

The sheaf $\mathrm{Sing}_{\leq n}(Y/X)_x$ is the *constant* sheaf on the stalk of $\tau_{\leq n}p_!(*) \in \mathrm{Sh}(X)$, which can be identified with the space $\tau_{\leq n}\mathrm{Sing}(Y_x)$. The top left pullback square therefore realizes $Y_{\tilde{x}}$ as an n-connected cover of the fiber Y_x. $\square$

Example 5.3.27. In terms of groupoids, the above construction takes source n-connected covers. For example, let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and let $p\colon M \longrightarrow X$ be the induced smooth map to its classifying stack. Then $\tilde{p}\colon M \longrightarrow \mathrm{Sing}_{\leq n}(M/X)$ is an n-smooth surjection, whose Čech nerve $\widetilde{\mathcal{G}}$ fits into a commuting diagram

$$\begin{tikzcd}
\widetilde{\mathcal{G}} \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & M \arrow[d] \\
\mathcal{G} \arrow[r, shift left] \arrow[r, shift right] & M.
\end{tikzcd}$$

Here $\widetilde{\mathcal{G}}$ is no longer a manifold, but a smooth $(n-1)$-stack. For each point $x \in M$, the map on source fibers $\widetilde{\mathcal{G}}_{s^{-1}(x)} \longrightarrow \mathcal{G}_{s^{-1}(x)}$ realizes the former as the n-connected cover of the latter.

Chapter 6

Deformation theory

A derived stack $X\colon \mathrm{Aff}^{\mathrm{op}} \longrightarrow \mathcal{S}$ is a particular kind of sheaf that still exhibits a reasonable geometric behaviour. In particular, just like (derived) manifolds, derived stacks have a good infinitesimal theory, which is controlled by their cotangent complex.

The purpose of this section is to study the infinitesimal structure of derived stacks. In particular, we will establish the following version of Theorem 0 from the introduction:

Theorem 0. *Let $f\colon M \longrightarrow X$ be a map from a derived manifold (or a derived étale stack) to a derived stack. Then the fiberwise tangent bundle of M over X has the structure of a sheaf of Lie algebroids over M, with anchor given by the canonical map*

$$T_{M/X} = \mathrm{Hom}_{\mathcal{O}_M}(L_{M/X}, \mathcal{O}_M) \longrightarrow T_M.$$

Remark 6.0.1. The behaviour of the Lie algebroid $T_{M/X}$ depends on the map $f\colon M \longrightarrow X$. If f is a smooth map, then $T_{M/X}$ is a connective Lie algebroid over M, whose global sections can be thought of as the Lie algebra of the group of automorphisms of M over X. When $M \longrightarrow X$ is a closed embedding, the Lie algebroid $T_{M/X}$ behaves like the (shifted) normal bundle of M inside X.

Theorem 2 is a formal consequence of the results of Chapter 4, together with some elementary properties of derived stacks that we will discuss in this chapter. More precisely, this chapter can be outlined as follows:

(1) We will start in Section 6.1 by recalling various conditions on a sheaf $X\colon \mathrm{Aff}^{\mathrm{op}} \longrightarrow \mathcal{S}$ that guarantee that it has good infinitesimal behaviour. The most important of these conditions is being *infinitesimally cohesive* (Definition 6.1.1): this guarantees that the deformations of a map $\mathrm{Spec}(A) \longrightarrow X$ are controlled by a formal moduli problem under A.

(2) In Section 6.2, we verify that derived stacks have all the infinitesimal properties discussed in Section 6.1, using a well-known inductive argument. The main technical result is a version of the inverse function theorem for derived manifolds (Proposition 6.2.1). This was already used in Section 5.2 to guarantee a reasonable theory of derived stacks.

(3) In Section 6.3 we describe a variant of Theorem 4.2.1 which relates *sheaves* of formal moduli problems and sheaves of Lie algebroids (Corollary 6.3.15). Using this, we prove Theorem 2.

(4) Finally, Section 6.4 discusses the *deformation theory* of derived stacks: when $p\colon X \longrightarrow \mathrm{Spec}(A)$ is a derived stack over $\mathrm{Spec}(A)$, there is a Lie algebroid controlling the deformations of p to a map $X' \longrightarrow \mathrm{Spec}(A')$ over an infinitesimal extension of A. We prove this by verifying (some of) the infinitesimal properties of Section 6.1 for the moduli space of derived stacks (Theorem 6.4.3).

Remark 6.0.2. The material of Section 6.1 and Section 6.2 is extensively discussed in the algebro-geometric setting, see e.g. [97, Chapter 1.4], [63, Chapter 17] and [33, Chapter III.1]. We closely follow the treatments of loc. cit. in the setting of derived differential topology.

6.1 Infinitesimal properties of sheaves

As we have seen in Section 2.3, there is good obstruction theory for deformations of algebraic objects, such as (connective) modules, along square zero extensions of $\mathcal{C}^\infty$-rings: the spaces of such deformations can be identified with spaces of null-homotopies of an obstruction class inside a certain spectrum. In this section, we recall some conditions on a sheaf

$$X\colon \mathrm{Aff} \longrightarrow \mathcal{S}$$

that provide it with a similar *deformation theory*, i.e. with a (linear) obstruction theory for extending a point $x\colon \mathrm{Spec}(A) \longrightarrow X$ to a point $x'\colon \mathrm{Spec}(A_\eta) \longrightarrow X$ along a square zero extension of A.

6.1.1 Infinitesimally cohesive maps. The axioms of a formal moduli problem (Definition 2.3.34) have a simple global analogue:

Definition 6.1.1 ([63, Definition 17.3.1.4])**.** A (pre)sheaf $X\colon \mathrm{Aff}^{\mathrm{op}} \longrightarrow \mathcal{S}$ is *infinitesimally cohesive* if it preserves each pullback square of complete $\mathcal{C}^\infty$-rings

$$\begin{tikzcd} A'_\eta \arrow[r] \arrow[d] & A \arrow[d, "0"] \\ A' \arrow[r, "\eta"'] & A \oplus I[1] \end{tikzcd} \tag{6.1.2}$$

where I is a complete connective A-module and $\pi_0(A') \longrightarrow \pi_0(A)$ is a surjection of (discrete) $\mathcal{C}^\infty$-rings with nilpotent kernel.

More generally, a map $p\colon X \longrightarrow S$ of (pre)sheaves is *infinitesimally cohesive* if for each $S' = \mathrm{Spec}(B) \longrightarrow S$, the base change $X \times_S S'$ is infinitesimally cohesive.

Example 6.1.3. If X is representable by an affine derived manifold, then X is infinitesimally cohesive. Similarly, if X is representable by a (non-connective) commutative dg-algebra B (Example 5.2.2), then it is infinitesimally cohesive.

Example 6.1.4. The class of infinitesimally cohesive sheaves is closed under limits in $\mathrm{Sh}(\mathrm{Aff})$. Consequently, a sheaf X is infinitesimally cohesive if and only if the map $X \longrightarrow *$ is infinitesimally cohesive.

Definition 6.1.5. Given an infinitesimally cohesive sheaf X and a map $x\colon M = \mathrm{Spec}(A) \longrightarrow X$, the *deformation functor of x* is the formal moduli problem

$$\widehat{X}\colon \mathcal{C}^\infty\mathrm{Alg}^{\mathrm{sm}}/A \longrightarrow \mathcal{S};\ \ A' \longmapsto X(\mathrm{Spec}(A')) \times_{X(\mathrm{Spec}(A))} \{x\}.$$

We denote by $T_{M/X} := T_{A/\hat{X}}$ the corresponding Lie algebroid from Theorem 4.2.1.

Lemma 6.1.6. *Let X be a presheaf with associated sheaf $\tilde{X}$. If X is infinitesimally cohesive, so is $\tilde{X}$.*

Proof. Let A'_η be a square zero extension of A' by an A-module I. For any (pre)sheaf $X\colon \mathrm{Aff}^{\mathrm{op}} \longrightarrow \mathcal{S}$ and a map of $\mathcal{C}^\infty$-rings $A'_\eta \longrightarrow B$, consider the (pre)sheaf on the basis of opens $\mathcal{B}(A'_\eta)$ of $\mathrm{Spec}(A'_\eta)$

$$X^B\colon \mathcal{B}(A'_\eta)^{\mathrm{op}} \longrightarrow \mathcal{S};\ \ U \longmapsto X\Big(U \times_{\mathrm{Spec}(A'_\eta)} \mathrm{Spec}(B)\Big).$$

The associated sheaf of X^B is simply $(\tilde{X})^B$. A (pre)sheaf X is infinitesimally cohesive if and only if each pullback square (6.1.2) induces a pullback square of sheaves on $\mathcal{B}(A'_\eta)$

$$\begin{tikzcd} X^{A'_\eta} \arrow[r] \arrow[d] & X^A \arrow[d] \\ X^{A'} \arrow[r] & X^{A\oplus E[1]}. \end{tikzcd}$$

Since taking associated sheaves is left exact, the result follows. □

Example 6.1.7. Let X_i be a set of infinitesimally cohesive sheaves. Then the sheaf $\coprod_i X_i$ is infinitesimally cohesive as well, since the pointwise coproduct is infinitesimally cohesive.

Construction 6.1.8. For each affine $U = \mathrm{Spec}(A)$ and each $E \in \mathrm{QC}^{\geq 0}(U)$, let

$$U \longrightarrow U_E := \mathrm{Spec}(A \oplus E) \longrightarrow U$$

denote the retract diagram of affines dual to $A \longrightarrow A \oplus E \longrightarrow A$. This determines a right adjoint functor from quasi-coherent sheaves to retract diagrams of affines. For every map $f\colon U \longrightarrow V$ between affines, there is a natural commuting square of right adjoint functors

$$\begin{tikzcd} \mathrm{QC}^{\geq 0}(V)^{\mathrm{op}} \arrow[r, "V_{(-)}"] \arrow[d, "f^*"'] & V/\mathrm{Sh}(\mathrm{Aff})/V \arrow[d, "f^*"] \\ \mathrm{QC}^{\geq 0}(U)^{\mathrm{op}} \arrow[r, "U_{(-)}"'] & U/\mathrm{Sh}(\mathrm{Aff})/U \end{tikzcd} \tag{6.1.9}$$

where the left and right functors take pullbacks along f. If $X\colon \mathrm{Aff} \longrightarrow \mathcal{S}$ is a presheaf, the limit of these functors over all $U \in \mathrm{Aff}/X$ yields a right adjoint functor

$$\mathrm{QC}^{\geq 0}(X)^{\mathrm{op}} \longrightarrow X/\mathrm{Sh}(\mathrm{Aff})/X;\ \ E \longmapsto \big(X \longrightarrow X_E \longrightarrow X\big).$$

We will call the sheaf X_E the *square zero extension* of X by E.

Remark 6.1.10. For *any* map $f\colon U \longrightarrow V$ in $\mathrm{Sh}(\mathrm{Aff})$, there is a commuting square of right adjoints as in (6.1.9). For every $E \in \mathrm{QC}^{\geq 0}(U)$, there is a base change morphism $f^*U_E \longrightarrow V_{f^*E}$, which arises from a map of retract diagrams

$$\begin{tikzcd} U \arrow[r] \arrow[d] & U_E \arrow[r] \arrow[d] & U \arrow[d] \\ V \arrow[r] & V_{(f_*E)} \arrow[r] & V. \end{tikzcd}$$

When f arises from a map $A \longrightarrow B$ of $\mathcal{C}^\infty$-rings, this vertical map arises from the obvious map $A \oplus E \longrightarrow B \oplus E$.

Construction 6.1.11. Let $p\colon X \longrightarrow S$ be a map of sheaves and fix a map $x\colon Y \longrightarrow X$. For any $E \in \mathrm{QC}^{\geq 0}(Y)$, let $\mathrm{Der}_x(X/S; E)$ be the space of dotted sections of the square

$$\begin{tikzcd} Y \arrow[rr, "x"] \arrow[d] & & X \arrow[d, "p"] \\ Y_E \arrow[r] \arrow[urr, dotted] & Y \arrow[r, "p(x)"'] & S. \end{tikzcd}$$

Given a map of quasi-coherent sheaves $E \longrightarrow F$, we can restrict such sections along $Y_F \longrightarrow Y_E$. Consequently, we obtain a functor

$$\mathrm{Der}_x(X/S; -)\colon \mathrm{QC}^{\geq 0}(Y) \longrightarrow \mathcal{S}.$$

Remark 6.1.12. Consider a pullback square

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S. \end{tikzcd}$$

For any map $x'\colon Y \longrightarrow X'$, there is an equivalence of functors

$$\mathrm{Der}_{x'}(X'/S';-) \xrightarrow{\sim} \mathrm{Der}_{f(x')}(X/S;-).$$

Remark 6.1.13. Let $f\colon U \longrightarrow V$ be a map of sheaves and let $x\colon V \longrightarrow X$ be a point. Restriction along the (natural) map of square zero extensions from Remark 6.1.10 yields a natural transformation

$$\mathrm{Der}_x\big(X/S; f_*(-)\big) \longrightarrow \mathrm{Der}_{f^*(x)}\big(X/S;-\big). \tag{6.1.14}$$

Lemma 6.1.15. *Suppose that $Y = \operatorname{colim} Y_i$ is a colimit of sheaves and let $f_i\colon Y_i \longrightarrow Y$ be the canonical maps. For every $p\colon X \longrightarrow S$, $x\colon Y \longrightarrow X$ and $E \in \mathrm{QC}^{\geq 0}(Y)$, there is a natural equivalence of spaces*

$$\mathrm{Der}_x(X/S;E) \longrightarrow \lim \mathrm{Der}_{f_i^*(x)}(X/S; f_i^*E).$$

Proof. There is a natural equivalence of sheaves

$$\operatorname{colim}\Big((Y_i)_{f_i^*(E)}\Big) \xrightarrow{\sim} \operatorname{colim}\big(Y_i \times_Y Y_E\big) \xrightarrow{\sim} Y_E.$$

The result follows by restricting along this equivalence. □

Lemma 6.1.16. *Suppose that $p\colon X \longrightarrow S$ be infinitesimally cohesive. For every $x\colon Y \longrightarrow X$, the functor $\mathrm{Der}_x(X/S;-)$ naturally extends to an exact functor*

$$\mathrm{Der}_x(X/S;-)\colon \mathrm{QC}^-(Y) \longrightarrow \mathrm{Sp}$$

from the eventually connective quasi-coherent sheaves on Y.

Proof. Consider the functor

$$\mathrm{QC}^{\geq 0}(Y) \times \mathcal{S}_*^{\mathrm{fin}} \longrightarrow \mathcal{S};\ \ (E,K) \longmapsto \mathrm{Der}_x\big(X/S; C_*(K,E)\big)$$

that sends a finite pointed space and quasi-coherent sheaf E to the value of $\mathrm{Der}_x(X/S;-)$ on the reduced chains of K with values in E.

The functor $\mathrm{Der}_x(X/S;E)\colon \mathcal{S}_*^{\mathrm{fin}} \longrightarrow \mathcal{S}$ is reduced excisive for each E and cofiber sequences in $\mathrm{QC}^{\geq 0}(Y)$ are sent to fiber sequences of reduced excisive functors. Indeed, when $Y = \mathrm{Spec}(A)$ is affine, this follows immediately from the fact that $p\colon X \longrightarrow S$ is infinitesimally cohesive. Lemma 6.1.15 implies that it holds for general Y as well.

By [62, Lemma 1.3.3.11], the resulting right exact functor $F\colon \mathrm{QC}^{\geq 0}(Y) \longrightarrow \mathrm{Sp}$ admits a unique extension to an exact functor

$$\mathrm{Der}_x(X/S;-)\colon \mathrm{QC}^-(Y) \longrightarrow \mathrm{Sp}$$

which sends $E \in \mathrm{QC}^{\geq -n}(Y)$ to $\Omega^n F(E[n])$. □

Remark 6.1.17. Let $X' \xrightarrow{q} X \xrightarrow{p} S$ be a sequence of infinitesimally cohesive maps. For any point $x'\colon Y \longrightarrow X'$, there is a fiber sequence of functors $\mathrm{QC}^-(Y) \longrightarrow \mathrm{Sp}$

$$\begin{tikzcd} \mathrm{Der}_{x'}(X'/X;-) \arrow[r] \arrow[d] & \mathrm{Der}_{x'}(X'/S;-) \arrow[d] \\ * \arrow[r, "0"'] & \mathrm{Der}_{q(x')}(X/S;-). \end{tikzcd}$$

6.1.2 The cotangent complex. If $p\colon X \longrightarrow S$ is infinitesimally cohesive and $x\colon Y \longrightarrow X$ is a map, a *relative cotangent space* of p at the point x is an object

$$L_{X/S,x} \in \mathrm{QC}^{-}(Y)$$

that corepresents the functor $\mathrm{Der}_x(X/S;-)\colon \mathrm{QC}^{\geq 0}(Y) \longrightarrow \mathcal{S}$ (see e.g. [97, Definition 1.4.1.5]). Such a relative cotangent space automatically corepresents the spectrum-valued functor $\mathrm{Der}_x(X/S;-)\colon \mathrm{QC}^{\geq 0}(Y) \longrightarrow \mathrm{Sp}$ from Lemma 6.1.16, so that $L_{X/S,x}$ is unique if it exists.

Informally speaking, $p\colon X \longrightarrow S$ admits a relative cotangent complex if it admits a cotangent space at each affine point of X, and these cotangent spaces together determine a quasi-coherent sheaf on X. To make this somewhat more explicit, let us start with the following observation:

Lemma 6.1.18. *For any map $p\colon X \longrightarrow S$, the following are equivalent:*

(1) p is infinitesimally cohesive and for every map $f\colon V \longrightarrow U$ between affines, the map

$$\mathrm{Der}_x\big(X/S; f_*(-)\big) \longrightarrow \mathrm{Der}_{f^*(x)}\big(X/S;-\big). \tag{6.1.19}$$

from Remark 6.1.13 is an equivalence.

(2) For any map $S' \longrightarrow S$ from an affine, the pullback $X \times_S S'\colon \mathrm{Aff}^{\mathrm{op}} \longrightarrow \mathcal{S}$ preserves all *pullback squares of complete $\mathcal{C}^\infty$-rings of the form*

$$\begin{tikzcd} A'_\eta \arrow[r] \arrow[d] & A \arrow[d, "0"] \\ A' \arrow[r] & A \oplus I[1]. \end{tikzcd}$$

Proof. We may assume that $S' = S$ is affine. The above square can be decomposed into pullback squares

$$\begin{tikzcd} A'_\eta \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d] & A \arrow[d] \\ A' \arrow[r] & A' \oplus I[1] \arrow[r] & A \oplus I[1]. \end{tikzcd}$$

We claim that the map (6.1.19) is an equivalence if and only if X preserves the right pullback square. The result follows immediately from this.

Consider the commuting square of spectra

$$\begin{tikzcd} \mathrm{Der}_x(X/S; f_*I[1]) \arrow[r] \arrow[d] & \mathrm{Der}_{f^*(x)}(X/S; I[1]) \arrow[d] \\ \mathrm{Der}_x(X; f_*I[1]) \arrow[r] & \mathrm{Der}_{f^*(x)}(X; I[1]). \end{tikzcd}$$

Unwinding the definitions, one sees that X preserves the right pullback square if and only if the bottom map is an equivalence. To see that this is equivalent to the top map being an equivalence, it suffices to show that the induced map between the fibers of the vertical maps

$$\mathrm{Der}_{p(x)}(S; f_*I) \longrightarrow \mathrm{Der}_{f^*(p(x))}(S; I).$$

is an equivalence. This is obvious, since $S = \mathrm{Spec}(B)$ was assumed affine. ☐

Definition 6.1.20 ([97, Definition 1.4.1.15])**.** Let $p\colon X \longrightarrow S$ be infinitesimally cohesive. We will say that p admits a *relative cotangent complex* if it satisfies the following conditions:

(1) for each map $x\colon U \longrightarrow X$ from an affine, there exists a relative cotangent space $L_{X/S,x} \in \mathrm{QC}^{\geq -n}(U)$ of p at x, for some fixed n.

(2) the map p satisfies the equivalent conditions of Lemma 6.1.18.

Remark 6.1.21. Let $f\colon U \longrightarrow V$ be a map of affines and let $x\colon V \longrightarrow X$. If $p\colon X \longrightarrow S$ satisfies the equivalent conditions of Lemma 6.1.18 and p admits a cotangent space at x, then p admits a cotangent space at $f^*(x)$, given by $f^*L_{X/S,x}$.

Example 6.1.22. Let $X = \mathrm{Spec}(A)$ be an affine and let $x\colon \mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$ be a map of affines. Clearly X satisfies the second condition of Lemma 6.1.18. Using Remark 6.1.21, we see that X admits a cotangent space at x, given by the associated sheaf of the B-module $B \otimes_A L_A$, where L_A is the $\mathcal{C}^\infty$-algebraic cotangent complex of A.

Remark 6.1.23. A map of sheaves $p\colon X \longrightarrow S$ admits a relative cotangent complex if and only if its base change $X' \longrightarrow S'$ admits a relative cotangent complex for every map $S' \longrightarrow S$ from an affine.

Lemma 6.1.24. *Let $p\colon X \longrightarrow S$ be a map satisfying the equivalent conditions of Lemma 6.1.18. Then the following are equivalent:*

(1) For any map $x\colon Y \longrightarrow X$ from a sheaf Y, there exists a (unique) cotangent space $L_{X/S,x} \in \mathrm{QC}^{\geq -n}(Y)$ at x.
(2) p admits a cotangent complex (which is $(-n)$-connective).
(3) For any map $x\colon U \longrightarrow X$ from an affine U, there exists a cover $\{U_i \longrightarrow U\}$ such that p admits a relative cotangent space $L_{X/S,x_i} \in \mathrm{QC}^{\geq -n}(U_i)$ at each $x_i\colon U_i \longrightarrow U \longrightarrow X$.

Proof. Obviously (1) ⇒ (2) ⇒ (3). The converse follows from the following assertion. Fix a point $x\colon Y \longrightarrow X$ and suppose that $Y \simeq \mathrm{colim}\, Y_i$ is a colimit of affines, with canonical maps $f_i\colon Y_i \longrightarrow Y$. If p admits a cotangent space at each point $x_i := f_i^*(x)$, then p admits a cotangent space at x.

To see this, note that for any map $\alpha\colon Y_i \longrightarrow Y_j$ over Y, the map (6.1.14) induces a natural map $L_{x_i} \longrightarrow \alpha^* L_{x_j}$, which is an equivalence by the conditions of Lemma 6.1.18. It follows that the cotangent spaces L_{x_i} form a matching family of quasi-coherent sheaves over the diagram of Y_i. Since

$$\mathrm{QC}^{\geq -n}(Y) \simeq \lim \mathrm{QC}^{\geq -n}(Y_i)$$

there exists a unique object $L_{X/S,x}$ in $\mathrm{QC}^{\geq -n}(Y)$ such that $f_i^* L_{X/S,x} \simeq L_{X/S,x_i}$. To see that this object corepresents $\mathrm{Der}_x(X/S; -)$, note that there are natural equivalences

$$\begin{aligned} \mathrm{Map}\big(L_{X/S,x}, E\big) &\simeq \lim \mathrm{Map}\big(f_i^* L_{X/S,x}, f_i^* E\big) \\ &\simeq \lim \mathrm{Map}\big(L_{X/S,x_i}, f_i^* E\big) \\ &\simeq \lim \mathrm{Der}_{x_i}(X/S; f_i^* E) \simeq \mathrm{Der}_x(X/S; E) \end{aligned}$$

where the last equivalence is Lemma 6.1.15. □

Definition 6.1.25. If $p\colon X \longrightarrow S$ admits a relative cotangent complex, we refer to $L_{X/S} := L_{X/S,\mathrm{id}} \in \mathrm{QC}(X)$ as its relative cotangent complex.

Remark 6.1.26. Let $X' \xrightarrow{q} X \xrightarrow{p} S$ be a sequence of infinitesimally cohesive maps. By Remark 6.1.17, if two out of the three maps p, q and pq admits a relative cotangent space at a point $x'\colon Y \longrightarrow X'$ (resp. $q(x')$), then so does the third. In fact, these cotangent spaces are related to each other by a cofiber sequence in $\mathrm{QC}(Y)$

$$L_{X/S,q(x')} \longrightarrow L_{X'/S,x'} \longrightarrow L_{X'/X,x'}.$$

Example 6.1.27. Let $Y' = X' \times_X Y$ be a pullback of infinitesimally cohesive sheaves, each of which has a cotangent complex. Then Y' has a cotangent complex as well:one easily verifies that Y' satisfies the second condition of Lemma 6.1.18 and for any point $y' \colon U \longrightarrow Y'$, the functor $\mathrm{Der}_{y'}(Y'; -)$ is corepresented by the pushout $L_{Y',y'} \simeq L_{X',x'} \oplus_{L_{X,x}} L_{Y,y}$ in $\mathrm{QC}^{-}(U)$.

Let us now describe how the obstruction theory for deformations along square zero extensions arises in the above terms. Consider a lifting problem

$$\begin{tikzcd}
\mathrm{Spec}(A) \arrow[r] \arrow[dr, "="'] & \mathrm{Spec}(A \oplus I[1]) \arrow[r, "\eta"] \arrow[d, "q"] & \mathrm{Spec}(A') \arrow[r, "x"] \arrow[d] & X \arrow[d, "p"] \\
& \mathrm{Spec}(A) \arrow[r] & \mathrm{Spec}(A'_\eta) \arrow[r] \arrow[ur, dotted, "\tilde{x}"'] & S
\end{tikzcd} \tag{6.1.28}$$

where the left square arises from the pullback square (6.1.2). When p is infinitesimally cohesive, the space of diagonal lifts $\tilde{x}$ is equivalent to the space of diagonal lifts $\mathrm{Spec}(A) \longrightarrow X$ of the composite rectangle. To identify the space of such lifts, note that there is a unique such lift making the outer part of the diagram commute, given by the total horizontal composite $x_0 \colon \mathrm{Spec}(A) \longrightarrow X$. The space of lifts can then be identified with the space of homotopies between the two maps

$$\mathrm{Spec}\big(A \oplus I[1]\big) \xrightarrow{q} \mathrm{Spec}(A) \xrightarrow{x_0} X \qquad \mathrm{Spec}\big(A \oplus I[1]\big) \xrightarrow{\eta} \mathrm{Spec}(A') \xrightarrow{x} X$$

over S and *relative to* $\mathrm{Spec}(A)$. These two maps correspond to the zero element and a certain obstruction element 'ob' in $\mathrm{Der}_{x_0}\big(X/S, I[1]\big)$. We conclude that the space of diagonal lifts $\tilde{x}$ in (6.1.28) is equivalent to the space of null-homotopies of

$$\mathrm{ob} \in \mathrm{Der}_{x_0}(X/S, I[1]).$$

When $p \colon X \longrightarrow S$ admits a cotangent complex, this means that the space of lifts $\tilde{x}$ is equivalent to the space of null-homotopies of a certain map

$$\mathrm{ob} \colon x_0^* L_{X/S} \longrightarrow I[1].$$

A particular situation in which there exists such a null-homotopy, is the case where $\mathrm{Der}_{x_0}(X/S, I[1])$ is connected:

Definition 6.1.29. A map $p \colon X \longrightarrow S$ is *formally smooth* if for each point $x \colon \mathrm{Spec}(A) \longrightarrow X$ and $E \in \mathrm{QC}^{\geq 1}(A)$, the space $\mathrm{Der}_x(X/S; E)$ is connected. When p is infinitesimally cohesive, it is formally smooth iff the spectrum $\mathrm{Der}_x(X/S; E)$ is connective for every $E \in \mathrm{QC}^{\geq 0}(A)$.

6.1.3 Convergence. There above obstruction theory can be applied to a particular system of square zero extensions: if A is a complete $\mathcal{C}^\infty$-ring, then its Postnikov tower

$$A \longrightarrow \dots \longrightarrow \tau_{\leq n+1} A \longrightarrow \tau_{\leq n} A \longrightarrow \dots \longrightarrow \tau_{\leq 0} A = \pi_0(A)$$

consists of square zero extensions by the complete $\pi_0(A)$-modules $\pi_n(A)[n]$. Ideally, one would like to use the above theory to study the ways of extending maps from $\pi_0(A)$ to A, or more generally, lifting problems of the form

$$\begin{tikzcd}
\mathrm{Spec}(\pi_0(A)) \arrow[r] \arrow[d] & X \arrow[d] \\
\mathrm{Spec}(A) \arrow[r] & S.
\end{tikzcd}$$

This lifting problem can be decomposed into lifting problems for the various stages of the Postnikov tower of A. However, if one inductively finds a compatible sequence of lifts $\operatorname{Spec}(\tau_{\leq n}A) \longrightarrow X$, it may not be true that these lifts together determine a lift $\operatorname{Spec}(A) \longrightarrow X$.

Definition 6.1.30 ([63, Definition 17.3.2.1])**.** A (pre)sheaf $X\colon \mathrm{Aff}^{\mathrm{op}} \longrightarrow \mathcal{S}$ is *convergent* if for every complex $\mathcal{C}^\infty$-ring A, the natural map

$$X(A) \xrightarrow{\sim} \lim_n X(\tau_{\leq n}A)$$

is an equivalence. A map $p\colon X \longrightarrow S$ is convergent if $S' \times_S X$ is convergent for any affine S'.

Example 6.1.31. If $X = \operatorname{Spec}(B)$ is a representable sheaf then X is convergent, and similarly when X is represented by a (possibly non-connective) commutative dg-algebra. Indeed, this follows from the fact that

$$\operatorname{Map}(B, A) \simeq \operatorname{Map}(B, \lim \tau_{\leq n}A) \simeq \lim \operatorname{Map}(B, \tau_{\leq n}A).$$

Remark 6.1.32. The full subcategory of convergent sheaves is closed under limits in $\operatorname{Sh}(\mathrm{Aff})$. In particular, this implies that X is a convergent sheaf if and only if the map $X \longrightarrow *$ is convergent.

Let $\operatorname{Sh}_{\mathrm{conv}}(\mathrm{Aff}) \subseteq \operatorname{Sh}(\mathrm{Aff})$ be the full subcategory of the convergent sheaves. This fully faithful inclusion admits a left adjoint, sending a sheaf X to the sheaf $A \mapsto \lim X(\tau_{\leq n}A)$. In fact, one can describe the ∞-category of convergent sheaves more concisely as follows (see e.g. [33, Chapter I.2]): let $i\colon \mathrm{Aff}^{<\infty} \subseteq \mathrm{Aff}$ be the full subcategory on the eventually coconnective complete $\mathcal{C}^\infty$-rings, i.e. those $\operatorname{Spec}(A)$ for which A has homotopy groups vanishing above a certain degree. Then the adjunction

$$i^*\colon \operatorname{Sh}(\mathrm{Aff}) \rightleftarrows \operatorname{Sh}(\mathrm{Aff}^{<\infty})\colon i_*$$

yields an equivalence between $\operatorname{Sh}(\mathrm{Aff}^{<\infty})$ and the full subcategory $\operatorname{Sh}_{\mathrm{conv}}(\mathrm{Aff})$. Indeed, unwinding the definitions one sees that i_* sends a sheaf X defined on eventually coconnective complete $\mathcal{C}^\infty$-rings to the sheaf $i_*X(A) = \lim X(\tau_{\leq n}A)$.

Let us conclude with a local criterion for being infinitesimally cohesive or convergent:

Proposition 6.1.33. *Let $p\colon U \longrightarrow X$ be a surjective map of sheaves which is formally smooth. If U and p are infinitesimally cohesive, then X is infinitesimally cohesive. If U and p are also convergent, then so is X.*

Proof. Let $U_\bullet$ be the Čech nerve of p. By assumption, each U_n is infinitesimally cohesive (and convergent). If A'_η is a square zero extension of A' by an A-module I, we therefore obtain a pullback square of simplicial spaces

$$\begin{tikzcd} U_\bullet(A'_\eta) \arrow[r] \arrow[d] & U_\bullet(A) \arrow[d] \\ U_\bullet(A') \arrow[r] & U_\bullet\big(A \oplus I[1]\big) \end{tikzcd}$$

Since $p\colon U \longrightarrow X$, the right vertical map is a Kan fibration of simplicial spaces (5.2.16). It follows that the induced square of realizations is cartesian as well. The realization $|U_\bullet|$ computed in presheaves is therefore infinitesimally cohesive. Lemma 6.1.6 implies that X is infinitesimally cohesive as well.

For the second part, let $X_{\mathrm{conv}} = i_*i^*X$ be the convergent sheaf associated to X. There is a commuting diagram

$$\begin{tikzcd}
\operatorname{colim} U_\bullet \arrow[r, "\sim"] \arrow[d, "\sim"'] & X \arrow[d] \\
\operatorname{colim}(U_\bullet)_{\mathrm{conv}} \arrow[r] & X_{\mathrm{conv}}
\end{tikzcd}$$

where the top and left maps are equivalences by assumption. The bottom map is the canonical map from the realization of the Čech nerve of $U_{\mathrm{conv}} \longrightarrow X_{\mathrm{conv}}$ to X_{conv} itself, because taking convergent sheaves preserves limits. It therefore suffices to check that $U_{\mathrm{conv}} \longrightarrow X_{\mathrm{conv}}$ induces a surjection on π_0-sheaves.

To see this, let $x\colon \mathrm{Spec}(A) \longrightarrow X_{\mathrm{conv}}$ be a point, corresponding to a natural family of points $x_n\colon \mathrm{Spec}(\tau_{\leq n}A) \longrightarrow X$. Since $U \longrightarrow X$ is surjective, we may shrink $\mathrm{Spec}(A)$ and assume that the point x_0 lifts to a point $u_0\colon \mathrm{Spec}(\tau_{\leq 0}A) \longrightarrow U$.

Now suppose that we have found a matching family of points $u_0, \dots, u_{n-1}$ lifting the points $x_0, \dots, x_{n-1}$. We claim that there is a point $u_n\colon \mathrm{Spec}(\tau_{\leq n}A) \longrightarrow U$ which lifts x_n and restricts to u_{n-1}. In other words, there exists a diagonal lift in the diagram

$$\begin{tikzcd}
\mathrm{Spec}(\tau_{\leq n-1}A) \arrow[r, "u_{n-1}"] \arrow[d] & U \arrow[d, "p"] \\
\mathrm{Spec}(\tau_{\leq n}A) \arrow[r, "x_n"'] \arrow[ur, dotted, "u_n"] & X.
\end{tikzcd}$$

This follows from the fact that p is formally smooth, since $\tau_{\leq n}A \longrightarrow \tau_{\leq n-1}A$ is a square zero extension. Proceeding inductively, we obtain a point u in $U_{\mathrm{conv}}(A) = \lim U(\tau_{\leq n}A)$ which lifts the point x (up to homotopy). $\square$

6.2 Derived stacks

We will now turn to verifying the above properties for derived stacks. These results and arguments are well-known (see e.g. [97]) and proceed by induction, starting from the case of affine derived manifolds.

6.2.1 The cotangent complex of an affine. For every $\mathcal{C}^\infty$-ring A, the affine $X = \mathrm{Spec}(A)$ is infinitesimally cohesive, convergent and admits a cotangent complex, which is simply the associated sheaf of the $\mathcal{C}^\infty$-algebraic cotangent complex L_A. More generally, let $A \longrightarrow B$ be a map of $\mathcal{C}^\infty$-rings. Then the relative cotangent complex of $f\colon \mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$ is the associated sheaf of

$$\mathcal{O}(\mathrm{Spec}(B)) \otimes_B L_{B/A} \in \mathrm{Mod}_{\mathcal{O}(\mathrm{Spec}(B))}.$$

In particular, if f is locally finitely presented (resp. smooth, étale), it follows from Lemma 5.2.9 that $L_{X/S} \in \mathrm{QC}^{\geq 0}(X)$ is a perfect complex (resp. locally free of finite rank, zero). Conversely, we have the following:

Proposition 6.2.1 (Inverse function theorem)**.** *Let $f\colon X \longrightarrow S$ be a locally finitely presented map between affines and let $x \in X$. If $x^*L_{X/S} \simeq 0$, then f is étale (Definition 5.2.8) on an neighbourhood of x.*

Lemma 6.2.2. *Let $X = \mathrm{Spec}(A)$ be the spectrum of a $\mathcal{C}^\infty$-ring A and let $x \in X$. Suppose that E is a finitely presented A-module such that $x^*E \simeq 0$. Then there exists a localization $A\{a^{-1}\}$ with $x \in \mathrm{Spec}(A\{a^{-1}\})$ such that $E \otimes_A A\{a^{-1}\} \simeq 0$.*

In particular, if a perfect complex is trivial at a point, then it is trivial on an open neighbourhood of that point.

Proof. It suffices to prove the result when A is discrete: indeed, if we know the result for $E \otimes_A \pi_0(A)$, then there exists a $a \in \pi_0(A)$ such that

$$\begin{aligned} 0 &\simeq \big(E \otimes_A \pi_0(A)\big) \otimes_{\pi_0(A)} \pi_0(A)\{b^{-1}\} \\ &\simeq \big(E \otimes_A A\{b^{-1}\}\big) \otimes_{A\{b^{-1}\}} \pi_0(A\{b^{-1}\}) \end{aligned}$$

The lowest nontrivial homotopy group of $E \otimes_A A\{b^{-1}\}$ (which exists since E is eventually connective) is isomorphic to the corresponding homotopy group of the above $\pi_0(A\{b^{-1}\})$-module. This implies that $E \otimes_A A\{b^{-1}\} \simeq 0$.

When A is discrete and $x^*E \simeq 0$, it follows from Nakayama's lemma that the stalk $E_x = E \otimes_A A_x$ is trivial as well. By assumption, E is given by a chain complex

$$0 \longrightarrow A^{\oplus n_b} \longrightarrow \dots \longrightarrow A^{\oplus n_{a+1}} \longrightarrow A^{\oplus n_a} \longrightarrow 0$$

in degrees $[a, b]$. Since the stalk E_x is null-homotopic, the map $A^{\oplus n_{a+1}} \longrightarrow A^{\oplus n_a}$ induces a surjection on stalks. We can therefore find a localization $A\{a^{-1}\}$ with $x \in \operatorname{Spec}(A\{a^{-1}\})$ so that $A\{a^{-1}\}^{\oplus n_{a+1}} \longrightarrow A\{a^{-1}\}^{\oplus n_a}$ is surjective. The kernel of this map is projective, so we can find a further localization $A\{b^{-1}\}$ on which it becomes free. We can therefore replace the above complex by an equivalent complex of free modules in degrees $[a+1, b]$ after tensoring with $A\{b^{-1}\}$. Proceeding inductively, one finds a neighbourhood of x on which $E \simeq 0$. $\square$

Proof (of Proposition 6.2.1). Since the statement is local, we can assume that f arises from a finitely presented map of $\mathcal{C}^\infty$-rings $\phi\colon A \longrightarrow B$, by Lemma 5.2.9. Then the cotangent complex is the sheaf associated to the finitely presented B-module $L_{B/A}$. It follows from Lemma 6.2.2 that $L_{B/A}$ vanishes after tensoring with some localization $B\{b^{-1}\}$, so replacing B by $B\{b^{-1}\}$ we may that $L_{B/A} \simeq 0$.

Let us first prove the proposition in the special case where ϕ induces a surjection $\pi_0(A) \longrightarrow \pi_0(B)$. In that case, we can present the map $A \longrightarrow B$ by a map of dg-$\mathcal{C}^\infty$-rings $A \longrightarrow A[\eta_i]$, with finitely many generators in degrees > 0. Consequently, the $\mathcal{C}^\infty$-algebraic cotangent complex $L_{B/A}$ agrees with the cotangent complex of $A \longrightarrow B$, considered as a map of $\mathbb{R}$-algebras.

It follows that $A \longrightarrow B$ is an étale map of $\mathbb{R}$-algebras, which means that $\pi_0(A) \longrightarrow \pi_0(B)$ is étale and that the natural maps $\pi_0(B) \otimes_{\pi_0(A)} \pi_n(A) \longrightarrow \pi_n(B)$ are isomorphisms (see e.g. [97, Theorem 2.2.2.6]). Since $\pi_0(A) \longrightarrow \pi_0(B)$ is surjective, it follows that there exists an idempotent element $e \in \pi_0(A)$ such that $\pi_0(B) \cong \pi_0(A)[e^{-1}] \cong \pi_0(A)/(1-e)$. Since this already has the natural structure of a $\mathcal{C}^\infty$-ring, it follows that the natural map $A\{e^{-1}\} \longrightarrow B$ is an equivalence of connective $\mathcal{C}^\infty$-rings. But this means that the map $\operatorname{Spec}(A) \longrightarrow \operatorname{Spec}(B)$ is an affine open inclusion, which is in particular étale.

For the general case, suppose that $A = \mathbb{R}\{x_i, \xi_i\}$ is a free dg-$\mathcal{C}^\infty$-ring on generators x_i of degree zero and ξ_i of degree > 0 (but with nontrivial differential) and that $B = \mathbb{R}\{x_i, y_i, \xi_i, \eta_i\}$ is obtained from A by adding finitely many extra generators $y_1, \dots, y_n$ (of degree 0) and $\eta_1, \dots, \eta_{n'}$ (of degree > 0) to it. Our goal will be to prove that the map $A \longrightarrow B$ induces a surjection on π_0 after localizing A and B, so that we can apply the previous argument.

The module $L_{B/A}$ is freely generated as a graded B-module by the elements dy_i and $d\eta_j$. Since $L_{B/A}$ is acyclic, we have for each $i = 1, \dots, n$ that

$$dy_i = \partial \left(\sum_{j=1}^{n} f_{ij}(x, y, \eta, \xi) \cdot dy_j + \sum_{k=1}^{n'} g_{ik}(x, y) \cdot d\eta_k \right)$$

for some finite set of elements $f_{ij}, g_{ik} \in B$. In particular, we can fix a *finite* set of generators $x_1, \dots, x_m$ such that the elements f_{ij} and g_{ik}, as well as the boundaries $\partial(\eta_k)$, depend only on these. Together with the n generators $y_1, \dots, y_n$ of B, these generators determine a map of $\mathcal{C}^\infty$-rings $\mathcal{C}^\infty(\mathbb{R}^{m+n}) \longrightarrow B$. This map induces a map of affine locally $\mathcal{C}^\infty$-ringed spaces

$$\Phi := (x_1, \dots, y_n)\colon \mathrm{Spec}(B) \longrightarrow \mathbb{R}^{m+n} = \mathrm{Spec}(\mathcal{C}^\infty(\mathbb{R}^{m+n})). \tag{6.2.3}$$

Writing out the above formula for dy_i, we find that

$$dy_i = \sum_{j=1}^{n} \left(h_{ij}(x,y) + \sum_k g_{ik} \frac{\partial(\partial\eta_k)}{\partial y_j} \right) \cdot dy_j + \sum_{l=1}^{m} \sum_k g_{ik} \frac{\partial(\partial\eta_k)}{\partial x_l} \cdot dx_l.$$

where the g_{ik} and $h_{ij} = \partial(f_{ij})$ are smooth functions on $\mathbb{R}^{m+n}$.

Since the elements h_{ij} are boundaries of elements in B, it follows that their value at each point in $|\mathrm{Spec}(B)|$ is zero. Consequently, the smooth matrix-valued function

$$(\mathrm{id} - h_{ij})\colon \mathbb{R}^{m+n} \longrightarrow \mathrm{Mat}(n, \mathbb{R})$$

sends the image $\Phi(|\mathrm{Spec}(B)|)$ of the map (6.2.3) to the identity matrix. It follows that $(\mathrm{id} - h_{ij})$ is invertible on an open neighbourhood of the image of $\Phi(|\mathrm{Spec}(B)|)$ inside $\mathbb{R}^{m+n}$.

But this smooth matrix-valued function agrees with the matrix valued function

$$\Big(\sum_k g_{ik} \frac{\partial(\partial\eta_k)}{\partial y_j} \Big)_{ij},$$

which implies that the Jacobi matrix $\big(\frac{\partial(\partial\eta_k)}{\partial y_j}\big)_{kj}$ has maximal rank on an open neighbourhood of $\Phi(|\mathrm{Spec}(B)|)$. The usual inverse function now shows that for each point $b \in |\mathrm{Spec}(B)|$, there exists an open subset $U \times V \subseteq \mathbb{R}^m \times \mathbb{R}^n$ containing $\Phi(b)$, on which there is a new set of coordinates $(x_1, ..., x_m, \tilde{y}_1, \dots, \tilde{y}_n)$ with the property that $\tilde{y}_1 = \partial\eta_{i_1}, \dots, \tilde{y}_n = \partial\eta_{i_n}$.

It follows that all extra generators y_i in degree 0 are boundaries when restricted to the open $U \times V$. Now consider the localizations

$$A' = A \amalg_{\mathcal{C}^\infty(\mathbb{R}^n)} \mathcal{C}^\infty(U) \qquad\qquad \tilde{B} = B \amalg_{\mathcal{C}^\infty(\mathbb{R}^{m+n})} \mathcal{C}^\infty(U \times V).$$

By construction, the map $A' \longrightarrow \tilde{B}$ induces a surjection on π_0 and its relative cotangent complex is just the restriction of $L_{B/A}$, so that we can apply the first part of the argument. ☐

Corollary 6.2.4. *Let $f\colon X \longrightarrow S$ be a locally finitely presented map of affines. If $x \in X$ is a point such that the fiber $x^*(L_{X/S})$ is a finite-dimensional vector space (in degree 0), then f is smooth at x, i.e. there are open neighbourhoods around x and $f(x)$ on which f takes the form $U \times \mathbb{R}^n \longrightarrow U$.*

Proof. There exist functions $g_1, \dots, g_n\colon V \longrightarrow \mathbb{R}$ defined on an open neighbourhood of x inside X whose differentials generate $x^*(L_{X/S})$. It follows that the map $(f, g_1, \dots, g_n)\colon V \longrightarrow S \times \mathbb{R}^n$ has a cotangent complex that vanishes at x. By the inverse function theorem, we can identify an open neighbourhood of x with an open neighbourhood of $S \times \mathbb{R}^n$, which implies the result. ☐

Remark 6.2.5. By a similar argument, if $x^*L_{X/S}$ is concentrated in degrees $[0,1]$, then locally around x the map f can be identified with a composition

$$h^{-1}(0) \longrightarrow U \times \mathbb{R}^n \longrightarrow U$$

where the first map is the inclusion of the zero locus of some $h\colon U \times \mathbb{R}^n \longrightarrow \mathbb{R}^m$. In particular, locally around the point x the fibers of $f\colon X \longrightarrow S$ are quasi-smooth (Remark 5.1.11).

6.2.2 Derived stacks. The deformation-theoretic properties of derived stacks can immediately be deduced from the properties of affines:

Proposition 6.2.6. *Any derived stack X satisfies the equivalent conditions of Lemma 6.1.18. In particular, it is infinitesimally cohesive.*

Proof. Let $X_\bullet \longrightarrow X$ be a derived Lie n-groupoid whose colimit is X. Each map $X(\Delta[n]) \longrightarrow X(\Lambda^i[n])$ is smooth, so in particular formally smooth. The same argument as in the proof of the Proposition 6.1.33 shows that X satisfies the second condition of Lemma 6.1.18. □

It follows that every map $x\colon \mathrm{Spec}(A) \longrightarrow X$ to a derived stack gives rise to a Lie algebroid $T_{\mathrm{Spec}(A)/X}$ over A, as in Definition 6.1.5. The anchor map of $T_{\mathrm{Spec}(A)/X}$ is the linear dual of the map of cotangent complexes $L_{\mathrm{Spec}(A)} \longrightarrow L_{\mathrm{Spec}(A)/X}$, which can often be identified more explicitly as follows:

Lemma 6.2.7. *Let $X_\bullet$ be a derived Lie n-groupoid and let $x\colon U \longrightarrow X_\bullet$ be a map from an affine into $X_\bullet$, inducing a map $x\colon U \longrightarrow X$ into the associated stack. Then the cotangent space $L_{X,x}$ at the point x exists and is given by the limit of the cosimplicial diagram $L_{X_\bullet,x}$ of quasi-coherent sheaves over U. More generally, given maps $x\colon U \longrightarrow X_\bullet$ and $p\colon X_\bullet \longrightarrow S_\bullet$ of derived Lie n-groupoids, $L_{X/S,y}$ is the limit of the cosimplicial diagram of quasi-coherent sheaves $L_{X_\bullet/S_\bullet,x}$.*

Proof. The relative cotangent space $L_{X/S,x}$ is the cofiber of the map $L_{S,p(x)} \longrightarrow L_{X,x}$. Since taking limits over $\mathbf{\Delta}$ commutes with taking cofibers, the description of the relative cotangent space follows from the absolute case.

We will prove by induction on n that $L_{X,x}$ exists and is given by

$$L_{X,x} \simeq \lim_{\mathbf{\Delta}} \big(x^* L_{X_\bullet}\big).$$

When $n = 0$, X is just a derived manifold. Working locally on U, we may replace X by one of its affine open subspaces, where the result is obvious.

For $n > 0$, consider the pullback diagram of derived Lie n-groupoids, each of which is equipped with a natural map from U

$$\begin{tikzcd} \mathcal{G}_\bullet \arrow[r] \arrow[d] & \mathrm{Dec}_0(X_\bullet) \arrow[d] \\ X_0 \arrow[r] & X_\bullet \end{tikzcd} \tag{6.2.8}$$

Here $\mathrm{Dec}_0(X_\bullet)$ is the décalage of $X_\bullet$, whose value on $[n] \in \mathbf{\Delta}$ is the value of $X_\bullet$ at the join $[1+n] = [0] \star [n]$ (see e.g. [46, Chapitre VI]). Since the right vertical map is a smooth, surjective Kan fibration, the induced square of colimits is simply the pullback of stacks $\mathcal{G} \simeq X_0 \times_X X_0$.

Let us first prove that the cotangent space $L_{X,x}$ exists. To see this, note that there is a fiber sequence of functors

$$\mathrm{Der}_x(X_0/X; -) \longrightarrow \mathrm{Der}_x(X_0; -) \longrightarrow \mathrm{Der}_x(X; -).$$

In light of Remark 6.1.26, it suffices to show that $\mathrm{Der}_x(X_0/X; -)$ is corepresentable. By Remark 6.1.12, there is an equivalence

$$\mathrm{Der}_x(\mathcal{G}/X_0; -) \xrightarrow{\sim} \mathrm{Der}_x(X_0/X; -)$$

Both $\mathcal{G}$ and X_0 are derived $(n-1)$-stacks, so $\mathrm{Der}_x(\mathcal{G}/X_0; -)$ is corepresentable by inductive hypothesis. This implies that $\mathrm{Der}_x(X_0/X; -)$ is corepresentable.

Let us now show that $L_{X,x} \simeq \lim L_{X_\bullet,x}$. In each simplicial degree, the pullback square of (6.2.8) gives rise to a pushout square on cotangent spaces (Example 6.1.27). Because taking homotopy limits over $\mathbf{\Delta}$ preserve pushout squares of quasi-coherent sheaves, the natural map

$$L_{X,x} \longrightarrow \lim_{\mathbf{\Delta}} L_{X_\bullet,x}$$

is a weak equivalence as soon as it is an equivalence for the other three simplicial sheaves $X_0, \mathrm{Dec}_0(X_\bullet)$ and $\mathcal{G}_\bullet$. For the derived Lie $(n-1)$-groupoids X_0 and $\mathcal{G}_\bullet$ this holds by inductive hypothesis.

The simplicial diagram $\mathrm{Dec}_0(X_\bullet)$ can be extended to a split augmented simplicial object, augmented by X_0. Such a split augmented simplicial diagram is a colimit diagram that is preserved by any functor, from which it follows that $L_{X_0,x} \simeq \lim L_{\mathrm{Dec}_0(X_\bullet),x}$. □

Corollary 6.2.9. *Let $p\colon X \longrightarrow S$ be an n-representable map of sheaves. Then p has a relative cotangent complex, which is $-n$-connective. If the map p is locally of finite presentation (smooth, étale), then $L_{X/S}$ is a perfect complex over X (with Tor-amplitude contained in $[-n,0]$, zero).*

Proof. By Lemma 6.1.24, it suffices to verify the existence of a cotangent space at a point $x\colon \mathrm{Spec}(A) \longrightarrow X$, where we can work locally on $\mathrm{Spec}(A)$. We may therefore assume that S is affine, that $X_\bullet \longrightarrow S$ is a Kan fibration whose domain is a derived Lie n-groupoid and that $\mathrm{Spec}(A) \longrightarrow X$ factors over $X_\bullet$. By Lemma 6.2.7, the relative cotangent complex at x exists and is given by the limit of the cosimplicial diagram $L_{X_\bullet/S,x}$.

To verify the properties of the cotangent complex, observe that the limit of the cosimplicial diagram $L_{X_\bullet/S,x}$ can be computed as the limit of a tower of quasi-coherent sheaves

$$\ldots \longrightarrow L(2) \longrightarrow L(1) \longrightarrow L(0)$$

where each $L(k)$ is the limit over $\mathbf{\Delta}_{\leq n}$. By the Dold-Kan correspondence of [62, Section 1.2.4] that for $k \geq 1$, the fiber $F(k) = \mathrm{fib}(L(k) \longrightarrow L(k-1))$ of this tower fits into a fiber sequence of modules

$$L_{X(\Lambda^0[k])/S,x} \longrightarrow L_{X(\Delta[k])/S,x} \longrightarrow F(k)[k].$$

In other words, $F(k)$ is the k-fold desuspension of the relative cotangent complex of $X(\Delta[k]) \longrightarrow X(\Lambda^0[k])$ at the point x. Since $X_\bullet$ is a derived Lie n-groupoid, the above tower stabilizes after degree n. Furthermore, each fiber $F(k)$ is $(-k)$-connective, so that the limit $L(n)$ is $(-n)$-connective.

Because the maps $X(\Delta[k]) \longrightarrow X(\Lambda^0[k])$ are smooth, the fibers $F(k)$ for $k \geq 1$ are locally free modules. If p is locally of finite presentation (smooth), then we may assume that the map $X_0 \longrightarrow S$ is locally of finite presentation (smooth), so that $F(0) \simeq L_{X_0/S,x}$ is perfect (locally free). Using this, one sees that $L_{X/S}$ is perfect (with Tor-amplitude contained in $[-n,0]$). Finally, if p is étale, then all $X(\Delta[k]) \longrightarrow X(\Lambda^0[k])$ are étale and $X_0 \longrightarrow S$ is étale, so that the relative cotangent complex vanishes. □

Lemma 6.2.10. *Any derived stack X is convergent and satisfies hyperdescent.*

Proof. Let X be a derived n-stack. If X is convergent, then it satisfies hyperdescent: indeed, since X is a derived n-stack, the value of X on a complete, k-truncated $\mathcal{C}^\infty$-ring A is an $(n+k)$-truncated space. This implies that each of the sheaves $A \mapsto X(\tau_{\leq k}A)$ are $(n+k)$-truncated sheaves, and in particular satisfy hyperdescent. Since X is convergent, it can be obtained as the limit of these hypersheaves, so that it is a hypersheaf itself.

We will prove by induction on n that X is convergent. We will assume the case $n = 0$ (i.e. where X is a derived manifold); one can use the exact same method as we will employ here to deduce this case from the case of a coproduct of affines. For the inductive step, let X be a derived n-stack and let $p\colon U \longrightarrow X$ be an affine atlas for X. Since U and $U \times_X U$ are convergent by inductive hypothesis and p is formally smooth by Corollary 6.2.9, it follows from Proposition 6.1.33 that X is convergent. $\square$

Example 6.2.11. Let $X_\bullet$ be a (smooth) Lie n-groupoid and let $p\colon M = X_0 \longrightarrow X$ be the canonical atlas of its associated stack. The anchor map of the Lie algebroid $T_{M/X}$ can then be identified as follows. Consider the simplicial vector bundle $TX_\bullet\big|M$, obtained by restricting the tangent bundle of each X_n along the degeneracy $M = X_0 \longrightarrow X_n$. Its normalized chains form a chain complex of vector bundles

$$\dots \longrightarrow T^{d_1,d_2,d_3}X_3\big|M \xrightarrow{d_0} T^{d_1,d_2}X_2\big|M \xrightarrow{d_0} T^{d_1}X_1\big|M \xrightarrow{d_0} T_M.$$

Here $T^{d_1,d_2}X_2 \subseteq TX_2$ denotes the sub-bundle of tangent vectors whose images under $d_1, d_2\colon TX_2 \longrightarrow TX_1$ vanish. There is an obvious inclusion of T_M into the above chain complex. Lemma 6.2.7 identifies the anchor map $T_{M/X} \longrightarrow T_M$ with the mapping fiber of this inclusion, which is given up to quasi-isomorphism by

$$\Big[\dots \longrightarrow T^{d_1,d_2}X_2\big|M \longrightarrow T^{d_1}X_1\big|M\Big] \xrightarrow{d_0} \big[\dots \longrightarrow 0 \longrightarrow 0 \longrightarrow T_M\big].$$

6.3 Sheaves of Lie algebroids

For every point $x\colon M = \mathrm{Spec}(A) \longrightarrow X$ of an infinitesimally cohesive sheaf gives rise, the deformation functor of x from Definition 6.1.5

$$\widehat{X}\colon \mathcal{C}^\infty\mathrm{Alg}^{\mathrm{sm}}/A \longrightarrow \mathcal{S};\ \ A' \longmapsto X(\mathrm{Spec}(A')) \times_{X(\mathrm{Spec}(A))} \{x\}.$$

is a formal moduli problem and gives rise to a Lie algebroid $T_{M/X}$ over A. Since X is a sheaf, one would expect this formal moduli problem and this Lie algebroid to arise as the global sections of *sheaves* of formal moduli problems and Lie algebroids over M.

The purpose of this section is to describe such sheaves of formal moduli problems and the associated sheaves of Lie algebroids. This allows us to study Lie algebroids and formal moduli problems locally on $\mathrm{Spec}(A)$. As an immediate consequence, we find that maps $f\colon M \longrightarrow X$ from non-affine derived manifolds (or étale stacks) give rise to sheaves of Lie algebroids over M as well.

6.3.1 Sheaves of formal moduli problems. Throughout this section, let us fix an injectively fibrant-cofibrant diagram of complete dg-$\mathcal{C}^\infty$-rings

$$\mathcal{O}^{\mathrm{dg}}\colon \mathcal{I}^{\mathrm{op}} \longrightarrow \mathcal{C}^\infty\mathrm{Alg}^{\mathrm{dg}}.$$

Let $\mathcal{O}$ be the corresponding diagram in with values in $\mathcal{C}^\infty\mathrm{Alg}$. We assume that for each $i \longrightarrow j$, the associated sheaf of $L_{\mathcal{O}(j)/\mathcal{O}(i)}$ is trivial.

Example 6.3.1. Let M be a derived manifold and let $\mathrm{Op}_{\mathrm{aff}}(M)$ be the category of affine open subspaces of M. The structure sheaf $\mathcal{O}\colon \mathrm{Op}_{\mathrm{aff}}(M)^{\mathrm{op}} \longrightarrow \mathcal{C}^\infty\mathrm{Alg}$ has the property that each restriction map is étale.

Example 6.3.2. Similarly, suppose that $X_\bullet$ is a derived étale Lie n-groupoid and let $\mathrm{Op}_{\mathrm{aff}}(X_\bullet)$ be the category of tuples $([n], U)$ where $U \subseteq X_n$ is an affine open subspace. A map $([n], U) \longrightarrow ([m], V)$ is a map $\alpha\colon [m] \longrightarrow [n]$ such that $\alpha^*(U) \subseteq V$. The structure sheaf

$$\mathcal{O}\colon \mathrm{Op}_{\mathrm{aff}}(X_\bullet)^{\mathrm{op}} \longrightarrow \mathcal{C}^\infty\mathrm{Alg};\ \ ([n], U) \longmapsto \mathcal{O}(U)$$

sends each restriction map to an étale map (together, these determine a locally $\mathcal{C}^\infty$-ringed ∞-topos which models the étale stack X).

Construction 6.3.3. Let us consider the category $\mathcal{C}^{\mathrm{dg}}_{/\mathcal{O}}$ with

- objects given by tuples (i, A), where $i \in \mathcal{I}$ and $A \longrightarrow \mathcal{O}(i)$ is a dg-$\mathcal{C}^\infty$-rings over $\mathcal{O}(i)$.
- maps $(i, A) \longrightarrow (j, B)$ given by a map $i \longrightarrow j$ in $\mathcal{I}$ and a commuting square

$$\begin{tikzcd} B \arrow[r] \arrow[d] & A \arrow[d] \\ \mathcal{O}^{\mathrm{dg}}(j) \arrow[r] & \mathcal{O}^{\mathrm{dg}}(i). \end{tikzcd}$$

The obvious projection $\mathcal{C}^{\mathrm{dg}}_{/\mathcal{O}} \longrightarrow \mathcal{I}$ is a cartesian and cocartesian fibration. After inverting the quasi-isomorphisms, this yields a cartesian and cocartesian fibration

$$\mathcal{C}_{/\mathcal{O}} \longrightarrow \mathcal{I}.$$

Let $\mathrm{Inf}_{\mathcal{O}} \subseteq \mathcal{C}_{/\mathcal{O}}$ be the full subcategory consisting of tuples (i, A) for which $A \longrightarrow \mathcal{O}(i)$ is a small extension. In particular, each $A \longrightarrow \mathcal{O}(i)$ in $\mathrm{Inf}_{\mathcal{O}}$ is a map between *complete* $\mathcal{C}^\infty$-rings.

Definition 6.3.4. Let M be a derived manifold (or a derived étale stack). We define the *infinitesimal site* of M to be the ∞-category $\mathrm{Inf}_M = \mathrm{Inf}_{\mathcal{O}}$ associated to the structure sheaf $\mathcal{O}$ of Example 6.3.1.

Lemma 6.3.5. *The projection* $\mathrm{Inf}_{\mathcal{O}} \longrightarrow \mathcal{I}$ *is a cartesian fibration.*

Proof. Let $i \longrightarrow j \longrightarrow k$ be a sequence of maps in $\mathcal{I}$ and let $A' \longrightarrow \mathcal{O}(k)$. We have to find a small extension $\widetilde{B} \longrightarrow \mathcal{O}(j)$ with a map from A', with the following property: for every solid diagram

$$\begin{tikzcd} A' \arrow[r] \arrow[d] \arrow[rr, bend left] & \widetilde{B} \arrow[r, dotted] \arrow[d] & \widetilde{C} \arrow[d] \\ \mathcal{O}(k) \arrow[r] & \mathcal{O}(j) \arrow[r] & \mathcal{O}(i) \end{tikzcd} \tag{6.3.6}$$

where $\widetilde{C} \longrightarrow \mathcal{O}(i)$ is a small extension, there exists a unique dotted lift, as indicated.

Suppose that there is a map $\widetilde{B} \longrightarrow \mathcal{O}(j)$ making the left square in (6.3.6) cocartesian. By Proposition 6.4.7, this map is a small extension so that $\widetilde{B} \in \mathrm{Inf}_{\mathcal{O}}$. An inductive application of Proposition 6.4.13 shows that $L_{\tilde{B}/A'}$ has vanishing associated sheaf.

We claim that this $\widetilde{B}$ has the desired universal property. Indeed, the map $\widetilde{C} \longrightarrow \mathcal{O}(i)$ is a composition of square zero extensions by complete modules. Since $L_{\tilde{B}/A'}$ has vanishing associated sheaf, there is a unique lift of $A' \longrightarrow \widetilde{B}$ against each of these square zero extensions. Proceeding inductively, one obtains a unique dotted lift in (6.3.6).

It therefore remains to find such a $\widetilde{B}$, i.e. a deformation of $\mathcal{O}(k) \longrightarrow \mathcal{O}(j)$ along A'. To this end, consider the formal moduli problem

$$\mathrm{Def}_{\mathcal{O}(j)} \colon \mathcal{C}^\infty\mathrm{Alg}^{\mathrm{sm}}/\mathcal{O}(k) \longrightarrow \mathcal{S}.$$

By Theorem 4.4.1, this formal moduli problem is classified by the Atiyah Lie algebroid $\mathrm{At}_{\mathcal{O}(j)} \longrightarrow T_{\mathcal{O}(k)}$. Since $L_{\mathcal{O}(j)/\mathcal{O}(k)}$ has vanishing associated sheaf and $\mathcal{O}(k)$ is complete, the above formal moduli problem is trivial and the desired deformation exists. $\square$

Definition 6.3.7. A *presheaf of formal moduli problems* over $\mathcal{O}$ is a functor

$$F \colon \mathrm{Inf}^{\mathrm{op}}_{\mathcal{O}} \longrightarrow \mathcal{S}$$

which restricts to a formal moduli problem $F_i\colon \mathcal{C}^\infty\mathrm{Alg}^{\mathrm{sm}}/\mathcal{O}(i) \longrightarrow \mathcal{S}$ on each fiber.

When $\mathcal{I}$ comes equipped with a Grothendieck topology and $\mathcal{O}$ is a sheaf of $\mathcal{C}^\infty$-rings, there is an induced topology on $\mathrm{Inf}_{\mathcal{O}}$, where a collection of maps is a cover if they are cartesian lifts of a cover in $\mathcal{I}$. In that case, we will say that F is a *sheaf of formal moduli problems* if it satisfies descent.

Remark 6.3.8. Let $F\colon \mathrm{Inf}_{\mathcal{O}}^{\mathrm{op}} \longrightarrow \mathcal{S}$ be a presheaf of formal moduli problems. Each map $\alpha\colon i \longrightarrow j$ in $\mathcal{I}$ induces a functor $\mathcal{C}^\infty\mathrm{Alg}^{\mathrm{sm}}/\mathcal{O}(j) \longrightarrow \mathcal{C}^\infty\mathrm{Alg}^{\mathrm{sm}}/\mathcal{O}(i)$ between the fibers. The proof of Proposition 6.3.5 shows that this functor preserves pullbacks along the maps $\mathcal{O}(j) \longrightarrow \mathcal{O}(j) \oplus \mathcal{O}(j)[n]$. Consequently, restriction along this functor induces a right adjoint

$$\alpha_*\colon \mathrm{FormMod}_{\mathcal{O}(i)} \longrightarrow \mathrm{FormMod}_{\mathcal{O}(j)}.$$

The functoriality of F gives rise to a natural map $F_j \longrightarrow \alpha_* F_i$ from the restriction of F to the fiber over j to its restriction over i.

If $\mathcal{I}$ carries a Grothendieck topology and $\mathcal{O}$ is a sheaf of $\mathcal{C}^\infty$-rings, then F is a sheaf of formal moduli problems if and only if it satisfies the following condition: for every covering sieve $S = \{\alpha\colon i \longrightarrow j\}$ on $\mathcal{I}$, the natural map

$$F_j \longrightarrow \lim_{\alpha\in S} \alpha_* F_i$$

is an equivalence of formal moduli problems over $\mathcal{O}(j)$.

6.3.2 Sheaves of Lie algebroids.

The presheaf of dg-$\mathcal{C}^\infty$-rings $\mathcal{O}^{\mathrm{dg}}$ determines a presheaf of $\mathcal{O}$-modules $\Omega_{\mathcal{O}}$, sending each $i \in \mathcal{I}$ to the module $\Omega_{\mathcal{O}(i)}$ of $\mathcal{C}^\infty$-algebraic Kähler differentials on $\mathcal{O}(i)$. Let $T_{\mathcal{O}} = \mathrm{Hom}_{\mathcal{O}}(\Omega_{\mathcal{O}}, \mathcal{O})$ be the hom-presheaf, which sends each $i \in \mathcal{I}$ to the chain complex of natural derivations

$$v_j\colon \mathcal{O}(j) \longrightarrow \mathcal{O}(j) \qquad j \in \mathcal{I}/i.$$

For each $i \in \mathcal{I}$, the value $T_{\mathcal{O}}(i)$ has the natural structure of a dg-Lie algebroid over $\mathcal{O}(i)$, with bracket given by the commutator bracket and anchor map given by the map $T_{\mathcal{O}}(i) \longrightarrow T_{\mathcal{O}(i)}$ evaluating a natural derivation over $\mathcal{I}/i$ at i.

For each map $\alpha\colon i \longrightarrow j$, there is a natural map of dg-Lie algebroids $T_{\mathcal{O}}(j) \longrightarrow T_{\mathcal{O}}(i)$ over the map $\mathcal{O}(j) \longrightarrow \mathcal{O}(i)$, which restricts natural derivations on $\mathcal{I}/j$ to $\mathcal{I}/i$. By Lemma 3.1.40, there is a Quillen adjunction

$$\alpha^*\colon \mathrm{LieAlgd}^{\mathrm{dg}}_{\mathcal{O}(j)}/T_{\mathcal{O}}(j) \rightleftarrows \mathrm{LieAlgd}^{\mathrm{dg}}_{\mathcal{O}(i)}/T_{\mathcal{O}}(i)\colon \alpha_* \tag{6.3.9}$$

where $\alpha^*(\mathfrak{g}) = \mathfrak{g} \otimes_{\mathcal{O}(j)} \mathcal{O}(i)$. This produces a diagram of model categories and left Quillen functors

$$\mathrm{LieAlgd}^{\mathrm{dg}}_{\mathcal{O}(-)}/T_{\mathcal{O}}(-)\colon \mathcal{I}^{\mathrm{op}} \longrightarrow \mathrm{ModCat}^{\mathrm{L}}.$$

Let $\mathrm{LieAlgd}^{\mathrm{dg}}_{/\mathcal{O}} \longrightarrow \mathcal{I}$ be the associated cartesian fibration and let $\mathrm{LieAlgd}_{/\mathcal{O}} \longrightarrow \mathcal{I}$ be the cartesian fibration of ∞-categories obtained from this by inverting the quasi-isomorphisms. Note that the fiber of this cartesian fibration over $i \in \mathcal{I}$ is equivalent to the ∞-category $\mathrm{LieAlgd}_{\mathcal{O}(i)}$, by the following observation:

Lemma 6.3.10. *For each i, the anchor map $T_{\mathcal{O}}(i) \longrightarrow T_{\mathcal{O}(i)}$ is a weak equivalence.*

Proof. There is a natural map of presheaves of cofibrant $\mathcal{O}$-modules over $\mathcal{I}/i$

$$\mathcal{O} \otimes_{\mathcal{O}(i)} \Omega_{\mathcal{O}(i)} \longrightarrow \Omega_{\mathcal{O}}.$$

Since each $L_{\mathcal{O}(j)/\mathcal{O}(i)}$ has a vanishing associated sheaf, the above map is a natural local weak equivalence. Since $\mathcal{O}$ is an injectively fibrant presheaf of complete (i.e. locally fibrant) $\mathcal{O}$-modules, the induced map between mapping complexes into $\mathcal{O}$ is a weak equivalence as well. But this map is precisely the map $T_{\mathcal{O}}(i) \longrightarrow T_{\mathcal{O}(i)}$ restricting a natural derivation to its component at i. □

Lemma 6.3.11. *Let* $\mathrm{LieAlgd}^{\mathrm{good}}_{/\mathcal{O}} \subseteq \mathrm{LieAlgd}_{/\mathcal{O}}$ *be the full subcategory on the tuples* $(i, \mathfrak{g})$ *where* $\mathfrak{g} \in \mathrm{LieAlgd}_{\mathcal{O}(i)}$ *is good. Then the following assertions hold:*

(1) The inclusion $\mathrm{LieAlgd}^{\mathrm{good}}_{/\mathcal{O}} \longrightarrow \mathrm{LieAlgd}_{/\mathcal{O}}$ *is a map of cartesian fibrations over* $\mathcal{I}$ *which preserves cartesian edges.*

(2) There is an equivalence between the ∞*-category* $\mathrm{LieAlgd}_{\mathcal{O}}$ *of sections of the functor* $\mathrm{LieAlgd}_{/\mathcal{O}} \longrightarrow \mathcal{I}$ *and the* ∞*-category of presheaves*

$$F\colon \mathrm{LieAlgd}^{\mathrm{good,op}}_{/\mathcal{O}} \longrightarrow \mathcal{S}$$

whose restriction to each fiber $\mathrm{LieAlgd}^{\mathrm{good}}_{\mathcal{O}(i)}$ *satisfies the conditions of Proposition 4.2.7.*

Proof. For assertion (1), let $i \longrightarrow j$ be a map in $\mathcal{I}$. The left Quillen functor (6.3.9) sends the free Lie algebroid on the map $0\colon \mathcal{O}(j)[n] \longrightarrow T_{\mathcal{O}}(j) \longrightarrow T_{\mathcal{O}(j)}$ to the free Lie algebroid on $\mathcal{O}(i)[n]$. Since it preserves all homotopy colimits of dg-Lie algebroids, this implies that the associated functor of ∞-categories preserves good Lie algebroids.

For assertion (2), consider the natural transformation of functors $\mathcal{I}^{\mathrm{op}} \longrightarrow \mathrm{Cat}_\infty$

$$j\colon \mathrm{LieAlgd}^{\mathrm{good}}_{\mathcal{O}(-)} \longrightarrow \mathrm{LieAlgd}_{\mathcal{O}(-)}.$$

obtained by straightening the map $\mathrm{LieAlgd}^{\mathrm{good}}_{/\mathcal{O}} \longrightarrow \mathrm{LieAlgd}_{/\mathcal{O}}$ of cartesian fibrations over $\mathcal{I}$. The codomain of this natural transformation is a diagram of locally presentable ∞-categories with left adjoint functors between them (induced by the left Quillen functors (6.3.9)). It follows from [59, Theorem 5.1.5.6] that there is a unique natural transformation of functors $\mathcal{I}^{\mathrm{op}} \longrightarrow \mathrm{Pr}^{\mathrm{L}}$

$$j_!\colon \mathrm{PSh}\Big(\mathrm{LieAlgd}^{\mathrm{good}}_{\mathcal{O}(-)}\Big) \longrightarrow \mathrm{LieAlgd}_{\mathcal{O}(-)}. \tag{6.3.12}$$

For each $i \in \mathcal{I}$, the functor $j_!$ has a fully faithful right adjoint, whose essential image consists of the presheaves satisfying the conditions of Proposition 4.2.7.

The diagram $\mathrm{PSh}\big(\mathrm{LieAlgd}^{\mathrm{good}}_{\mathcal{O}(-)}\big)$ is classified by a cartesian fibration over $\mathcal{I}$, whose ∞-category of sections is equivalent to the ∞-category of presheaves on $\mathrm{LieAlgd}^{\mathrm{good}}_{/\mathcal{O}}$, by [34]. We conclude that the above left adjoint functor induces an adjunction on ∞-categories of sections

$$j_!\colon \mathrm{PSh}\Big(\mathrm{LieAlgd}^{\mathrm{good}}_{/\mathcal{O}}\Big) \rightleftarrows \mathrm{LieAlgd}_{\mathcal{O}}\colon j^*$$

whose right adjoint is fully faithful. The essential image of j^* consists of those presheaves $\mathrm{LieAlgd}^{\mathrm{good,op}}_{/\mathcal{O}} \longrightarrow \mathcal{S}$ whose restriction to each fiber $\mathrm{LieAlgd}^{\mathrm{good}}_{\mathcal{O}(i)}$ is contained in the essential image of the right adjoint to (6.3.12). □

Remark 6.3.13. A section of $\mathrm{LieAlgd}_{/\mathcal{O}} \longrightarrow \mathcal{I}$ given by a collection of Lie algebroids $\mathfrak{g}_i$ over each $\mathcal{O}(i)$, together a coherent family of maps of Lie algebroids $\mathfrak{g}_j \longrightarrow \mathfrak{g}_i$ over $\mathcal{O}(j) \longrightarrow \mathcal{O}(i)$, for each $i \longrightarrow j$ in $\mathcal{I}$. We will therefore refer to an object of $\mathrm{LieAlgd}_{\mathcal{O}}$ as a *presheaf of Lie algebroids* over $\mathcal{O}$.

Suppose that $\mathcal{I}$ carries a Grothendieck topology and that $\mathcal{O}$ is a sheaf of $\mathcal{C}^\infty$-rings. If $\mathfrak{g}$ is a presheaf of Lie algebroids and $S = \{\alpha\colon i \longrightarrow j\}$ is a covering sieve of j, there is a natural map of Lie algebroids over $\mathcal{O}(j)$

$$\mathfrak{g}_j \longrightarrow \mathrm{holim}_{\alpha\in S}\, \alpha_*\mathfrak{g}_i.$$

Let us say that $\mathfrak{g}$ is a *sheaf of Lie algebroids* if each of these maps is an equivalence of Lie algebroids over $\mathcal{O}(j)$. This is equivalent to the assertion that the presheaf of $\mathcal{O}$-modules underlying $\mathfrak{g}$ is a sheaf.

6.3.3 Duality. A map $(i, \mathfrak{g}) \longrightarrow (j, \mathfrak{h})$ in the category $\mathrm{LieAlgd}^{\mathrm{dg}}_{/\mathcal{O}}$ is given by a map $\alpha\colon i \longrightarrow j$ in $\mathcal{I}$, together with a map dg-Lie algebroids

$$f\colon \mathfrak{g} \longrightarrow \alpha^*\mathfrak{h} = \mathfrak{h} \otimes_{\mathcal{O}(j)} \mathcal{O}(i)$$

over $T_{\mathcal{O}}(i)$. Such a map induces a map between (connective covers of) Chevalley-Eilenberg complexes

$$\begin{tikzcd} \mathrm{c}^*\big(\mathfrak{h}\big) \arrow[r, "\mathrm{c}^*(f)"] \arrow[d] & \mathrm{c}^*\big(\mathfrak{g}\big) \arrow[d] \\ \mathcal{O}(j) \arrow[r] & \mathcal{O}(i) \end{tikzcd}$$

where the top map is a composition

$$\mathrm{c}^*\big(\mathfrak{h}, \mathcal{O}(j)\big) \xrightarrow{\mathrm{c}^*(\mathfrak{h}, \alpha^*)} \mathrm{c}^*\big(\mathfrak{h}, \mathcal{O}(i)\big) \cong \mathrm{c}^*\big(\alpha^*\mathfrak{h}, \mathcal{O}(i)\big) \xrightarrow{f^*} \mathrm{c}^*\big(\mathfrak{g}, \mathcal{O}(i)\big).$$

The first map arises from the fact that $\mathfrak{h}$ is a dg-Lie algebroid over $T_{\mathcal{O}}(j)$, so that it acts on the map $\alpha^*\colon \mathcal{O}(j) \longrightarrow \mathcal{O}(i)$ by natural derivations. The last map restricts along f and the middle isomorphism sends an $\mathcal{O}(j)$-linear map $\mathrm{Sym}_{\mathcal{O}(j)}(\mathfrak{h}[1]) \longrightarrow \mathcal{O}(i)$ to its $\mathcal{O}(i)$-linear extension $\mathrm{Sym}_{\mathcal{O}(i)}(\alpha^*\mathfrak{h}[1]) \longrightarrow \mathcal{O}(i)$.

We therefore obtain a functor

$$\begin{tikzcd} \mathrm{LieAlgd}^{\mathrm{dg}}_{/\mathcal{O}} \arrow[rr, "\mathrm{c}^*"] \arrow[dr] & & \mathcal{C}^{\mathrm{dg}}_{/\mathcal{O}} \arrow[dl] \\ & \mathcal{I} & \end{tikzcd}$$

to the category of Construction 6.3.3. Since c^* preserves quasi-isomorphisms between cofibrant objects in each fiber, this induces a functor $\mathrm{c}^*\colon \mathrm{LieAlgd}_{/\mathcal{O}} \longrightarrow \mathcal{C}_{/\mathcal{O}}$ over $\mathcal{I}$. The functor between the fibers over some $i \in \mathcal{I}$ is just the functor taking the Chevalley-Eilenberg complex of a Lie algebroid over $\mathcal{O}(i)$, and therefore preserves colimits.

Lemma 6.3.14. *The functor* $\mathrm{c}^*\colon \mathrm{LieAlgd}_{/\mathcal{O}} \longrightarrow \mathcal{C}_{/\mathcal{O}}$ *restricts to a functor preserving cartesian edges* $\mathrm{c}^*\colon \mathrm{LieAlgd}^{\mathrm{good}}_{\mathcal{O}} \longrightarrow \mathrm{Inf}_{\mathcal{O}}$.

Proof. On each fiber, the functor c^* is the usual Chevalley-Eilenberg complex functor, which sends good Lie algebroids over $\mathcal{O}(i)$ to small extensions of $\mathcal{O}(i)$. This implies that it restricts to a functor $\mathrm{c}^*\colon \mathrm{LieAlgd}^{\mathrm{good}}_{\mathcal{O}} \longrightarrow \mathrm{Inf}_{\mathcal{O}}$.

To see that this functor preserves cartesian edges, fix a map $\alpha\colon i \longrightarrow j$ in $\mathcal{I}$ and let $\mathfrak{h}$ be a good Lie algebroid over $\mathcal{O}(j)$. By (the proof of) Proposition 6.3.5, it suffices to verify that the square

$$\begin{tikzcd} C^*(\mathfrak{h}) \arrow[r] \arrow[d] & C^*\big(\alpha^*\mathfrak{h}\big) \arrow[d] \\ \mathcal{O}(j) \arrow[r] & \mathcal{O}(i) \end{tikzcd}$$

realizes $C^*\big(\alpha^*\mathfrak{h}\big)$ as a deformation of the map $\mathcal{O}(j) \longrightarrow \mathcal{O}(i)$ along the small extension $C^*(\mathfrak{h}) \longrightarrow \mathcal{O}(j)$. When $\mathfrak{h}$ is free on $0\colon \mathcal{O}(j)[n] \longrightarrow T_{\mathcal{O}(j)}$, the above map can be identified with the map

$$\mathcal{O}(j) \oplus \mathcal{O}(j)[-n-1] \longrightarrow \mathcal{O}(i) \oplus \mathcal{O}(i)[-n-1]$$

which is clearly a deformation of the map $\mathcal{O}(j) \longrightarrow \mathcal{O}(i)$. A general good Lie algebroid $\mathfrak{h}$ can be obtained as an iterated pushout of such free Lie algebroids. The functor c^* sends these iterated pushouts to iterated pullbacks of deformations of $\mathcal{O}(i) \longrightarrow \mathcal{O}(j)$. An inductive application of Proposition 2.3.21 then shows that the map $C^*(\mathfrak{h}) \longrightarrow C^*(\alpha^*\mathfrak{h})$ is a deformation of $\mathcal{O}(i) \longrightarrow \mathcal{O}(j)$ along $C^*(\mathfrak{h})$. □

Corollary 6.3.15. *Let $\mathcal{O}\colon \mathcal{I}^{\mathrm{op}} \longrightarrow \mathcal{C}^\infty\mathrm{Alg}$ be a presheaf of complete $\mathcal{C}^\infty$-rings, such that each $\mathcal{O}(j) \longrightarrow \mathcal{O}(i)$ has vanishing relative cotangent complex. Then there is an adjunction of ∞-categories*

$$\mathrm{MC}\colon \mathrm{LieAlgd}_{\mathcal{O}} \rightleftarrows \mathrm{FormMod}_{\mathcal{O}}\colon T_{\mathcal{O}/}$$

between the ∞-category of presheaves of Lie algebroids over $\mathcal{O}$ (Remark 6.3.13) and presheaves of formal moduli problems over $\mathcal{O}$ (Definition 6.3.7). Furthermore, this adjunction is an equivalence if each $\mathcal{O}(i)$ is eventually coconnective.

Proof. Combine Lemma 6.3.14 and Lemma 6.3.11. □

Remark 6.3.16. If $\mathcal{I}$ carries a Grothendieck topology and $\mathcal{O}$ is a sheaf of $\mathcal{C}^\infty$-rings, Remark 6.3.8 and Remark 6.3.13 imply that for any sheaf of formal moduli problems $F\colon \mathrm{Inf}_{\mathcal{O}}^{\mathrm{op}} \longrightarrow \mathcal{S}$, the associated presheaf of Lie algebroids $T_{\mathcal{O}/F}$ is a sheaf of Lie algebroids over $\mathcal{O}$. When all $\mathcal{O}(i)$ are eventually coconnective, this furnishes an equivalence between the ∞-categories of sheaves of Lie algebroids and sheaves of formal moduli problems.

Example 6.3.17. Let M be a derived manifold and let Inf_M be its infinitesimal site (Definition 6.3.4). If $f\colon M \longrightarrow X$ is a map to an infinitesimally cohesive sheaf, then the assignment

$$\Big(M \supseteq \mathrm{Spec}(A) \longrightarrow \mathrm{Spec}(A')\Big) \longmapsto X\big(A'\big) \times_{X(A)} \{f\big|\mathrm{Spec}(A)\}$$

defines a sheaf of formal moduli problems $\widehat{X}\colon \mathrm{Inf}_M^{\mathrm{op}} \longrightarrow \mathcal{S}$. By Corollary 6.3.15, this determines a sheaf of Lie algebroids over M, which we will denote by $T_{M/X}$.

When the sheaf X has a cotangent complex, the underlying anchor map $T_{M/X} \longrightarrow T_M$ can be identified with the canonical map

$$\mathrm{Hom}(L_{M/X}, \mathcal{O}_M) \longrightarrow \mathrm{Hom}(L_M, \mathcal{O}_M).$$

Example 6.3.18. Similarly, suppose that $f\colon Y \longrightarrow X$ is a map from a derived étale stack to X. If $Y_\bullet \longrightarrow Y$ is a derived étale Lie n-groupoid modeling Y, then the map f gives rise to a sheaf of formal moduli problems over $\mathrm{Inf}_{Y_\bullet}$ (Example 6.3.2). By Corollary 6.3.15, it determines a sheaf of Lie algebroids $T_{Y/X}$ over $Y_\bullet$.

6.4 Deformations of stacks

In the previous section we have seen that derived stacks have good infinitesimal behaviour: they are infinitesimally cohesive, convergent and admit a cotangent complex which governs the obstruction theory for extending maps along square zero extensions. In this section, we will study a 'delooping' of this result: we show that the moduli space of all derived n-stacks is infinitesimally cohesive as well.

Definition 6.4.1. Let $\mathcal{E} \subseteq \mathrm{Fun}(\Delta[1], \mathrm{Sh}(\mathrm{Aff}))$ be the full subcategory of maps $X \longrightarrow M$ from a derived stack to an affine derived manifold M. The codomain projection $\mathcal{E} \longrightarrow \mathrm{Aff}$ is a cartesian fibration. Let

$$\mathrm{Stack}\colon \mathrm{Aff}^{\mathrm{op}} \longrightarrow \widehat{\mathrm{Cat}}_\infty$$

be the presheaf of (large) ∞-categories classifying $\mathcal{E} \longrightarrow \mathrm{Aff}$. In other words, $\mathrm{Stack}(U)$ is the ∞-category of derived stacks over U and each $f\colon U \longrightarrow V$ determines a functor

$$f^*\colon \mathrm{Stack}(V) \longrightarrow \mathrm{Stack}(U);\; \big(X \longrightarrow V\big) \longmapsto \big(f^*X = X \times_V U \longrightarrow U\big).$$

Remark 6.4.2. It follows from [59, Theorem 6.1.3.9] that for any covering sieve $S = \{U_i \longrightarrow U\}$, there is an equivalence of ∞-categories

$$\begin{aligned} \mathrm{Sh}(\mathrm{Aff})/U &\longrightarrow \lim_{U_i \in S} \mathrm{Sh}(\mathrm{Aff})/U_i; \\ \big(X \longrightarrow U\big) &\longmapsto \{X \times_U U_i \longrightarrow U_i\}_{i\in S} \end{aligned}$$

A sheaf $X \longrightarrow U$ over U is a derived n-stack if and only if each $U_i \times_U X$ is a derived n-stack. It follows that the above equivalence restricts to an equivalence between categories of derived stacks, so that Stack is a (large) sheaf of ∞-categories.

The aim of this section is to prove the following:

Theorem 6.4.3. *The sheaf* $\mathrm{Stack}\colon \mathrm{Aff}^{\mathrm{op}} \longrightarrow \widehat{\mathrm{Cat}}_\infty$ *is infinitesimally cohesive and convergent, i.e. each sheaf of spaces* $\mathrm{Map}(\Delta[n], \mathrm{Stack})\colon \mathrm{Aff}^{\mathrm{op}} \longrightarrow \widehat{\mathcal{S}}$ *is infinitesimally cohesive and convergent in the sense of Definition 6.1.1.*

In the algebro-geometric setting, this is proven in [76, Section 3.2.2] (based on results from [77]). We will give an alternative proof, which proceeds by induction. More precisely, note that Stack arises as a filtered colimit

$$\mathrm{Stack}_0 \longrightarrow \mathrm{Stack}_1 \longrightarrow \dots \longrightarrow \mathrm{Stack}$$

where Stack_n sends an affine U to the full subcategory of $\mathrm{Stack}_n(U)$ on the derived n-stacks over U. We will prove by induction on n that the sheaves Stack_n are infinitesimally cohesive. The case of derived manifolds ($n = 0$) is closely related to the deformation theory of $\mathcal{C}^\infty$-rings discussed in Section 2.3.

6.4.1 Deformations of derived manifolds.

A map of $\mathcal{C}^\infty$-ringed spaces

$$i\colon X = (\mathcal{X}, \mathcal{O}_X) \longrightarrow Y = (\mathcal{Y}, \mathcal{O}_Y)$$

is a *closed immersion* if the map of topological spaces $i\colon \mathcal{X} \longrightarrow \mathcal{Y}$ is a closed embedding and the map $i^{-1}\mathcal{O}_Y \longrightarrow \mathcal{O}_X$ induces a surjection on π_0-sheaves. Any closed immersion between locally $\mathcal{C}^\infty$-ringed spaces is a morphism of locally $\mathcal{C}^\infty$-ringed spaces.

Lemma 6.4.4 (cf. [63, Theorem 3.1.2.1])**.** *Let A be a complete $\mathcal{C}^\infty$-ring and let $i\colon X \longrightarrow$* $\mathrm{Spec}(A)$ *be a closed immersion of derived manifolds. Then X is affine.*

Proof. Since $i\colon \mathcal{X} \longrightarrow |\mathrm{Spec}(A)|$ is a closed embedding, the map $\mathcal{O}_A \longrightarrow i_*\mathcal{O}_X$ of sheaves of $\mathcal{O}_A$-modules induces a surjection on π_0. It follows that the induced map on global sections $A \longrightarrow B := \Gamma(X, \mathcal{O}_X)$ induces a surjection on π_0 as well.

The map i therefore factors as the composition of a map $f\colon X \longrightarrow \mathrm{Spec}(B)$ and a closed immersion $\mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$. In particular, $f\colon \mathcal{X} \longrightarrow |\mathrm{Spec}(B)|$ is a closed embedding and the map $\mathcal{O}_B \longrightarrow f_*\mathcal{O}_X$ is given on global sections by the equivalence $B \simeq \Gamma(\mathrm{Spec}(B), \mathcal{O}_B) \longrightarrow \Gamma(X, \mathcal{O}_X)$. But the global sections functor on $\mathcal{O}_B$-module sheaves is fully faithful, so it follows that the map $\mathcal{O}_B \longrightarrow f_*\mathcal{O}_X$ is an equivalence of sheaves. Since the support of $\mathcal{O}_B$ is the entire $\mathrm{Spec}(B)$, it follows that f is a homeomorphism so that $X \simeq \mathrm{Spec}(B)$. $\square$

Lemma 6.4.5. *Consider a solid square of derived manifolds in which the vertical arrows are closed immersions*

$$\begin{tikzcd} Y'_x \arrow[r, dotted] \arrow[d, dotted, tail] & Z' \arrow[r] \arrow[d, tail] & Z \arrow[d, tail] \\ \mathrm{Spec}(\mathcal{O}_{X,x}) \arrow[r, dotted] & X' \arrow[r] & X. \end{tikzcd}$$

This square is cartesian if for any point $x \in \mathcal{X}'$, the pullback $Y'_x = Z' \times_{X'} \mathrm{Spec}(\mathcal{O}_{X,x})$ is equivalent to the pullback $Y_x = Z \times_X \mathrm{Spec}(\mathcal{O}_{X,x})$.

Proof. Because Y'_x is nonzero iff x is contained in the image of Z', the map $Z' \longrightarrow X' \times_X Z$ is a homeomorphism as soon as each $Y'_x \longrightarrow Y_x$ is an equivalence. Since $\mathcal{O}_{Z'}$ and $\mathcal{O}_{X' \times_X Z}$ are both hypersheaves, it suffices to verify that their stalks are equivalent. But for a point $x \in \mathcal{Z}' \subseteq \mathcal{X}'$, the induced map on stalks can be identified with the map $\mathcal{O}(Y_x) \longrightarrow \mathcal{O}(Y_x)$, which is an equivalence by assumption. $\square$

Lemma 6.4.6 (cf. [63, Theorem 16.1.0.1])**.** *Consider a pushout square of $\mathcal{C}^\infty$-ringed spaces*

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, "h"] & X_0 \arrow[d, "f"] \\ X_1 \arrow[r, "g"'] & X_{01} \end{tikzcd}$$

where $X \longrightarrow X_0$ and $X \longrightarrow X_1$ are closed immersions of derived manifolds. Then X_{01} is a derived manifold and models the pushout of derived manifolds.

Proof. The $\mathcal{C}^\infty$-ringed space X_{01} is given by the pushout of topological spaces $\mathcal{X}_0 \coprod_{\mathcal{X}} \mathcal{X}_1 \cong \mathcal{X}_1$, together with the sheaf of $\mathcal{C}^\infty$-rings $f_*\mathcal{O}_{X_0} \times_{h_*\mathcal{O}_X} g_*\mathcal{O}_{X_1}$. For any open subspace $U \subseteq \mathcal{X}_{01}$, the $\mathcal{C}^\infty$-ringed space $(U, \mathcal{O}_{X_{01}}|U)$ is therefore given by the pushout of the diagram

$$(f^{-1}U, \mathcal{O}_{X_0}|f^{-1}U) \longleftarrow (h^{-1}U, \mathcal{O}_X|h^{-1}U) \longrightarrow (g^{-1}U, \mathcal{O}_{X_1}|g^{-1}U).$$

For any point $x \in \mathcal{X}_{01}$, one may choose an open U so that $f^{-1}U$ and $g^{-1}U$ are affine opens of X_0 and X_1. Lemma 6.4.4 implies that $h^{-1}U$ is affine as well.

We may therefore assume that X, X_0 and X_1 are affine and consider the complete $\mathcal{C}^\infty$-ring $A_{01} = \mathcal{O}(X_0) \times_{\mathcal{O}(X)} \mathcal{O}(X_1)$. There is a map of $\mathcal{C}^\infty$-ringed spaces $f\colon X_{01} \longrightarrow \mathrm{Spec}(A_{01})$, which is a closed embedding of topological spaces because each $X_i \longrightarrow \mathrm{Spec}(A_{01})$ is a closed immersion. Furthermore, the map $\mathcal{O}_{A_{01}} \longrightarrow f_*\mathcal{O}_{X_{01}}$ induces an equivalence on global sections. Arguing as in Lemma 6.4.4, one sees that f is an equivalence. $\square$

Proposition 6.4.7. *Let $A' \longrightarrow A$ be a map of complete $\mathcal{C}^\infty$-rings inducing a surjection on π_0 and let $\eta\colon A' \longrightarrow A \oplus E[1]$ classify a square zero extension A'_η of A' by a complete A-module E. Consider a cube of derived manifolds*

$$\begin{tikzcd} & X \arrow[rr] \arrow[dl] \arrow[dr, "h"] \arrow[dd, dotted] & & X_0 \arrow[dl, "f"'] \arrow[dd] \\ X_1 \arrow[rr, "g"] \arrow[dd] & & X_{01} \arrow[dd] & \\ & \mathrm{Spec}(A \oplus E[1]) \arrow[rr, dotted] \arrow[dl, dotted] & & \mathrm{Spec}(A) \arrow[dl] \\ \mathrm{Spec}(A') \arrow[rr] & & \mathrm{Spec}(A'_\eta) & \end{tikzcd} \tag{6.4.8}$$

such that the left and back square are pullback diagrams of derived manifolds. Then the following are equivalent:

(1) *The front and right square are cartesian.*
(2) *The top square is a pushout square of derived manifolds.*

Proof. At the level of the underlying topological spaces, conditions (1) and (2) are both equivalent to the condition that all maps from left to right are homeomorphisms. Assume (1), let $x \in X_{01}$ be a point and consider the map $\mathrm{Spec}(\mathcal{O}_{X_{01},x}) \longrightarrow X_{01}$. By pulling the top square of (6.4.8) back along this map, we obtain a cube of affine derived manifolds

$$\begin{tikzcd}
& \mathrm{Spec}(\mathcal{O}_{X,h^{-1}(x)}) \arrow[rr] \arrow[dl] \arrow[dr, "h_x"] \arrow[dd, dotted] & & \mathrm{Spec}(\mathcal{O}_{X_0,f^{-1}(x)}) \arrow[dl, "f_x"'] \arrow[dd] \\
\mathrm{Spec}(\mathcal{O}_{X_1,g^{-1}(x)}) \arrow[rr, "g_x"] \arrow[dd] & & \mathrm{Spec}(\mathcal{O}_{X_{01},x}) \arrow[dd] & \\
& \mathrm{Spec}(A \oplus E[1]) \arrow[rr, dotted] \arrow[dl, dotted] & & \mathrm{Spec}(A) \arrow[dl] \\
\mathrm{Spec}(A') \arrow[rr] & & \mathrm{Spec}(A'_\eta) &
\end{tikzcd} \tag{6.4.9}$$

where each $\mathcal{C}^\infty$-ring is either zero (if $f^{-1}(x) = \emptyset$) or local with residue field $\mathbb{R}$. Each of the vertical squares is a pullback of affine spaces, which dually means that it is a pushout square of $\mathcal{C}^\infty$-rings (the result is local, so in particular complete). Proposition 2.3.21 implies that the top square of (local) affine derived manifolds is a pushout, so that the map of hypersheaves $\mathcal{O}_{X_{01}} \longrightarrow f_*\mathcal{O}_{X_0} \times_{h_*\mathcal{O}_X} g_*\mathcal{O}_{X_1}$ induces equivalences on stalks. This implies (2).

Conversely, assuming condition (2) and pulling back the top row of (6.4.8) along $\mathrm{Spec}(\mathcal{O}_{X_{01},x}) \longrightarrow X_{01}$, we obtain the cube (6.4.9) in which the top square is a pushout. By Lemma 6.4.5, condition (1) follows if we show that the map

$$\mathrm{Spec}(\mathcal{O}_{X_{01},x}) \otimes_{A'_\eta} A' \longrightarrow \mathrm{Spec}(\mathcal{O}_{X_1,g^{-1}(x)})$$

is an equivalence of local (or zero) $\mathcal{C}^\infty$-rings, and similarly for X_0. This follows from Proposition 2.3.21. □

Remark 6.4.10. Suppose that $A_\eta \longrightarrow A$ is a square zero extension by an A-module E. Proposition 6.4.7 shows that a derived manifold $X \longrightarrow \mathrm{Spec}(A_\eta)$ is affine if and only if its base change $X \times_{\mathrm{Spec}(A_\eta)} \mathrm{Spec}(A)$ is affine.

Corollary 6.4.11. *The sheaf* $\mathrm{Map}(*, \mathrm{Stack}_0)\colon \mathrm{Aff}^{\mathrm{op}} \longrightarrow \widehat{\mathcal{S}}$ *is infinitesimally cohesive and convergent.*

Proof. Proposition 6.4.7 implies that $\mathrm{Map}(*, \mathrm{Stack}_0)$ is infinitesimally cohesive, using the same argument as in Example 2.3.35. To see that it is convergent, let $X_n \longrightarrow \mathrm{Spec}(\tau_{\leq n}A)$ be a compatible family of derived manifolds. The closed immersions $X_{n-1} \longrightarrow X_n$ are the pullbacks of $\mathrm{Spec}(\tau_{\leq n-1}A) \longrightarrow \mathrm{Spec}(\tau_{\leq n}A)$, which induce homeomorphisms on the underlying topological spaces. In other words, the X_n form a sequence of locally $\mathcal{C}^\infty$-ringed spaces $(\mathcal{X}, \mathcal{O}_{X_n})$ with the property that each map

$$\mathcal{O}_{X_n} \otimes_{\tau_{\leq n}A} \tau_{\leq n-1}A \longrightarrow \mathcal{O}_{X_{n-1}}$$

is an equivalence. There is a unique locally $\mathcal{C}^\infty$-ringed space $X = (\mathcal{X}, \lim_n \mathcal{O}_{X_n})$ such that $X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(\tau_{\leq n}A) \simeq X_n$.

To see that this locally $\mathcal{C}^\infty$-ringed space is a derived manifold, let $(U, \mathcal{O}_{X_0})$ be an affine open subspace of X_0. Each $(U, \mathcal{O}_{X_n}) \simeq \mathrm{Spec}(\mathcal{O}_{X_n}(U))$ is affine by Remark 6.4.10. It follows that $(U, \mathcal{O}_X) \simeq \mathrm{Spec}(\mathcal{O}_X(U))$ is affine as well, so that $\mathrm{Map}(*, \mathrm{Stack}_0)$ is convergent. □

6.4.2 Deformations of derived stacks. Let us now turn to the sheaves Stack_n of higher stacks. The mapping spaces of Stack_n are easily seen to be infinitesimally cohesive and convergent:

Lemma 6.4.12. *For each n, the map of sheaves (of spaces)*

$$(d_1, d_0)\colon \mathrm{Map}(\Delta[1], \mathrm{Stack}_n) \longrightarrow \mathrm{Map}(\{0\}, \mathrm{Stack}_n) \times \mathrm{Map}(\{1\}, \mathrm{Stack}_n)$$

is infinitesimally cohesive and convergent.

Proof. Let $A' \longrightarrow A$ be a map of complete $\mathcal{C}^\infty$-rings inducing a surjection on π_0 and let $\eta\colon A' \longrightarrow A \oplus E[1]$ classify a square zero extension A'_η of A' by a complete A-module E. It suffices to show that any commuting square of sheaves

$$\begin{tikzcd} \mathrm{Spec}(A') \coprod_{\mathrm{Spec}(A\oplus E[1])} \mathrm{Spec}(A) \arrow[r] \arrow[d] & \mathrm{Map}(\Delta[1], \mathrm{Stack}_n) \arrow[d, "{(d_1,d_0)}"] \\ \mathrm{Spec}(A'_\eta) \arrow[r, "{(X,Y)}"'] \arrow[ur, dotted] & \mathrm{Map}(\{0\}, \mathrm{Stack}_n) \times \mathrm{Map}(\{1\}, \mathrm{Stack}_n) \end{tikzcd}$$

has a contractible space of diagonal lifts. The bottom map classifies two derived stacks X, Y over $\mathrm{Spec}(A'_\eta)$. Unwinding the definitions, the space of diagonal lifts is equivalent to the space of dotted lifts in the square

$$\begin{tikzcd} X \times_{\mathrm{Spec}(A'_\eta)} \Big(\mathrm{Spec}(A') \coprod_{\mathrm{Spec}(A\oplus E[1])} \mathrm{Spec}(A)\Big) \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] \arrow[ur, dotted] & \mathrm{Spec}(A'_\eta). \end{tikzcd}$$

To see that this space is contractible, we may assume that $X = \mathrm{Spec}(B)$ is affine, since a general stack X arises as a colimit of affines. The fact that $A' \longrightarrow A$ induces a surjection on π_0 then implies that

$$X \times_{\mathrm{Spec}(A'_\eta)} \mathrm{Spec}(A) \simeq \mathrm{Spec}(B \otimes_{A'_\eta} A)$$

and similarly for the other pullbacks. It follows that B is a square zero extension of $B \otimes_{A'_\eta} A'$ by $B \otimes_{A'_\eta} E$. The space of diagonal lifts is then contractible because Y is infinitesimally cohesive.

Similarly, to see that (d_1, d_0) is convergent it suffices to show that any square

$$\begin{tikzcd} \mathrm{colim}_n \Big(\mathrm{Spec}(\tau_{\leq n} A)\Big) \arrow[r] \arrow[d] & \mathrm{Map}(\Delta[1], \mathrm{Stack}_n) \arrow[d, "{(d_1,d_0)}"] \\ \mathrm{Spec}(A) \arrow[r, "{(X,Y)}"'] \arrow[ur, dotted] & \mathrm{Map}(\{0\}, \mathrm{Stack}_n) \times \mathrm{Map}(\{1\}, \mathrm{Stack}_n) \end{tikzcd}$$

has a contractible space of diagonal lifts. Unwinding the definitions, the space of diagonal lifts is equivalent to the space of space of lifts

$$\begin{tikzcd} \mathrm{colim}_n \Big(X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(\tau_{\leq n} A)\Big) \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] \arrow[ur, dotted] & \mathrm{Spec}(A). \end{tikzcd}$$

Writing X as a colimit of affines, one sees that this space is contractible because Y is convergent. $\square$

Proposition 6.4.13. *Let $A' \longrightarrow A$ be a square zero extension of a complete $\mathcal{C}^\infty$-ring A by a complete A-module E and consider a composition of pullback squares of derived stacks*

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d] & X \arrow[r] \arrow[d] & \mathrm{Spec}(A) \arrow[d] \\ Y' \arrow[r, "f'"'] & X' \arrow[r] & \mathrm{Spec}(A'). \end{tikzcd}$$

If f is of one of the following types, then so is f': n-representable, locally finitely presented, smooth, smoothly surjective, étale, an equivalence.

Proof. Suppose that f' is m-representable. We will prove the proposition by induction on m. We may assume that $X' = \mathrm{Spec}(A')$ and that $X = \mathrm{Spec}(A)$, and we can work locally on $\mathrm{Spec}(A')$.

The case $\mathbf{m = 0}$. To see that f' is locally finitely presented (smooth, étale), we can work locally on Y'. In particular, we may assume that $Y' = \mathrm{Spec}(B')$ is affine, so that $Y = \mathrm{Spec}(B)$ is affine by Lemma 6.4.4. Furthermore, by Lemma 5.2.9, we may assume that f arises from a finitely presented map of $\mathcal{C}^\infty$-rings $A \longrightarrow B$.

By Lemma 2.2.28 and Proposition 6.2.1, it suffices to verify that $\pi_0(A') \longrightarrow \pi_0(B')$ is a locally finitely presented map of discrete $\mathcal{C}^\infty$-rings and that $L_{B'/A'}$ is perfect (resp. locally free of finite rank, zero). For the first condition, we can locally choose a map $q\colon \pi_0(A')\{x_1, \dots, x_n\} \longrightarrow \pi_0(B')$ which induces a surjection

$$\pi_0(A)\{x_1, \dots, x_n\} \longrightarrow \pi_0(B)$$

after tensoring along the square zero extension $\pi_0(A') \longrightarrow \pi_0(A)$. By Nakayama's lemma, the map q is surjective as well and $\pi_0(A') \longrightarrow \pi_0(B')$ is finitely presented.

Since $B' \longrightarrow B$ is a square zero extension, Nakayama's lemma has the following consequence: if E is an n-connective B'-module and $p\colon B'[n]^{\oplus k} \longrightarrow E$ induces a local surjection (isomorphism) $\pi_n(B)^{\oplus k} \longrightarrow \pi_n(B \otimes_{B'} E)$, then p induces a surjection (isomorphism) on π_n. Using this, we can inductively lift generators from $L_{B/A} \simeq L_{B'/A'} \otimes_{B'} B$ to $L_{B'/A'}$ to find that $L_{B'/A'}$ is perfect (resp. locally free of finite rank, zero).

Finally, to see that f' is a smooth surjection (equivalence) of derived manifolds when f is, note that f and f' induce the same maps on the underlying topological spaces. The result follows immediately from this.

The case $\mathbf{m > 0}$. Let us assume that the proposition is proven for $(m-1)$-representable maps f' and suppose that Y' is a derived m-stack. Let $U' \longrightarrow Y'$ be an atlas by a coproduct of affines. The pullback of this atlas yields an atlas $U \longrightarrow Y$ for Y. If Y is n-representable for $n < m$, then the map $U \longrightarrow Y$ is $(n-1)$-smooth. By inductive hypothesis, it follows that $U' \longrightarrow Y'$ is $(n-1)$-smooth, so that Y' is n-representable.

The assertions about locally finitely presented, smooth and smoothly surjective maps are immediate, since they can be verified for the composition $U' \longrightarrow X'$. When f is an equivalence, the map f' is an n-smooth surjection. To verify that f' is an equivalence, it therefore suffices to verify that the diagonal

$$Y' \longrightarrow Y' \times_{\mathrm{Spec}(A')} Y'$$

is an equivalence. This map is $(n-1)$-representable, so the assertion follows by inductive hypothesis.

Finally, suppose that f is étale. Let $U \longrightarrow Y$ be the smooth atlas for Y obtained by pulling back the smooth atlas $U' \longrightarrow Y'$. Since $Y \longrightarrow X = \mathrm{Spec}(A)$ is étale, we can refine this atlas by an étale atlas $\coprod V_i \longrightarrow U \longrightarrow Y$ given by a coproduct of affines. We therefore obtain a commuting diagram

$$\begin{tikzcd} \coprod V_i \arrow[r] \arrow[d, dotted] & U \arrow[r, two heads] \arrow[d] & Y \arrow[r] \arrow[d] & X = \mathrm{Spec}(A) \arrow[d] \\ \coprod V_i' \arrow[r, dotted] & U' \arrow[r, two heads] & Y' \arrow[r] & X' = \mathrm{Spec}(A') \end{tikzcd} \tag{6.4.14}$$

where the right two squares are cartesian. Each composite map of affines $V_i \longrightarrow \mathrm{Spec}(A)$ is étale, so by refining the atlas we can assume that it is presented by a map of $\mathcal{C}^\infty$-rings $A \longrightarrow B_i$ for which $L_{B_i/A}$ is zero.

By Example 2.3.29, the space of deformations of $A \longrightarrow B_i$ along the square zero extension $A' \longrightarrow A$ is given by the space of null-homotopies of a map

$$\mathrm{ob}\colon L_{B_i/A} \longrightarrow B_i \otimes_A E[2].$$

This space is contractible, so that there is a (unique) map $\coprod V_i' \longrightarrow X'$ making the outer rectangle in the above diagram cartesian. By Proposition 2.3.21, each map $V_i \longrightarrow V_i'$ is a square zero extension. Since the composite map $U' \longrightarrow X'$ is a smooth surjection, it follows that there exists a lift $\coprod V_i' \longrightarrow U'$ as indicated.

We can therefore extend diagram (6.4.14) as indicated, such that all squares are cartesian. But now the map $\coprod V_i' \longrightarrow Y'$ is $(n-1)$-representable and its base change is an $(n-1)$-étale surjection. It follows by inductive hypothesis that the map $\coprod V_i' \longrightarrow Y'$ provides an étale atlas for Y', such that $\coprod V_i' \longrightarrow \mathrm{Spec}(A')$ is étale. It follows that f' is étale. □

To inductively prove that the sheaves Stack_n are infinitesimally cohesive and convergent, we use the following construction.

Definition 6.4.15. Let $\mathrm{Atlas}_n\colon \mathrm{Aff}^{\mathrm{op}} \longrightarrow \widehat{\mathcal{S}}$ be the sheaf associating to each affine M the space of diagrams

$$U \xrightarrow{p} X \longrightarrow M$$

where X is a derived n-stack, U is a coproduct of affine derived manifolds and p is an $(n-1)$-smooth surjection.

Lemma 6.4.16. *The sheaf Atlas_n is equivalent to the sheaf Gpd_{n-1} sending each affine M to the space of groupoid objects over M*

$$\dots \mathcal{G} \times_U \mathcal{G} \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \mathcal{G} \substack{\longrightarrow\\ \longrightarrow} U \longrightarrow M$$

where U is a coproduct of affine derived manifolds, $\mathcal{G}$ is a derived $(n-1)$-stack and the source and target maps $\mathcal{G} \longrightarrow U$ are smooth.

Proof. Consider an augmented simplicial diagram over M

$$U_2 \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} U_1 \substack{\longrightarrow\\ \longrightarrow} U_0 \longrightarrow X \longrightarrow M. \tag{6.4.17}$$

By [59, Proposition 6.1.3.9], the following two assertions are equivalent:

(1) The entire diagram is a right Kan extension of its restriction $U_0 \longrightarrow X \longrightarrow M$, which is contained in $\mathrm{Atlas}_n(M)$.
(2) The entire diagram is a left Kan extension of its restriction $U_\bullet \longrightarrow M$ (i.e. $|U_\bullet| \simeq X$), which is contained in $\mathrm{Gpd}_{n-1}(M)$.

Let $\mathcal{E}$ be the sheaf associating to each M the space of diagrams (6.4.17) satisfying these equivalent conditions. Restriction determines natural equivalences $\mathrm{Atlas}_n \longleftarrow \mathcal{E} \longrightarrow \mathrm{Gpd}_{n-1}$. □

Proof (of Theorem 6.4.3). We will prove by induction that the sheaf of (large) ∞-categories $\mathrm{Stack}_n\colon \mathrm{Aff}^{\mathrm{op}} \longrightarrow \widehat{\mathrm{Cat}}_\infty$ is infinitesimally cohesive and convergent. For $n = 0$ this follows from Corollary 6.4.11 and Lemma 6.4.12. Assuming that Stack_{n-1} is infinitesimally cohesive, it suffices to verify that $\mathrm{Map}(*, \mathrm{Stack}_n)$ has deformation theory, by Lemma 6.4.12.

Consider the canonical map of (large) sheaves on Aff

$$G := \mathrm{Atlas}_n \longrightarrow \mathrm{Map}(*, \mathrm{Stack}_n) =: F.$$

This map is a surjection of sheaves, since every map $X \longrightarrow M$ from a derived n-stack to M can be equipped with an atlas $U \longrightarrow X$. We will show that the map $G \longrightarrow F$ satisfies the conditions Proposition 6.1.33, from which the result follows.

By Lemma 6.4.16, the sheaf G can equivalently be described as the sheaf of groupoid objects $\mathcal{G} \rightrightarrows U$ where U is a coproduct of affines and the source and target map are $(n-1)$-smooth. This sheaf is infinitesimally cohesive and convergent because Stack_{n-1} is and because deformations of smooth maps along a square zero extensions remain smooth, by Proposition 6.4.13.

To see that the map $G \longrightarrow F$ is infinitesimally cohesive (convergent), note that the map $G \longrightarrow F$ decomposes as

$$G \longrightarrow \mathrm{Stack}_{0 \longrightarrow n} \longrightarrow \mathrm{Map}(*, \mathrm{Stack}_0) \times \mathrm{Map}(*, \mathrm{Stack}_n) \longrightarrow F.$$

The second map is the base change of $\mathrm{Map}(\Delta[1], \mathrm{Stack}) \longrightarrow \mathrm{Map}(*, \mathrm{Stack}) \times \mathrm{Map}(*, \mathrm{Stack})$. In other words, $\mathrm{Stack}_{0\longrightarrow n}$ is the sheaf of maps from a derived manifold to a derived n-stack and the second map takes the domain and codomain. The last two maps are infinitesimally cohesive and convergent by Lemma 6.4.12 and Corollary 6.4.11.

To see that the first map is infinitesimally cohesive, we have to show that any diagram

$$\begin{tikzcd}
\mathrm{Spec}(A') \coprod_{\mathrm{Spec}(A \oplus E[1])} \mathrm{Spec}(A) \arrow[r] \arrow[d] & \mathrm{Atlas}_n \arrow[d] \\
\mathrm{Spec}(A'_\eta) \arrow[r, "U \longrightarrow X"'] \arrow[ur, dotted] & \mathrm{Stack}_{0 \longrightarrow n}
\end{tikzcd}$$

has a contractible space of diagonal lifts. Unwinding the definitions, the above square classifies a map $U \longrightarrow X \longrightarrow \mathrm{Spec}(A'_\eta)$ from a derived manifold U to a derived n-stack over $\mathrm{Spec}(A'_\eta)$, whose base change along $\mathrm{Spec}(A') \longrightarrow \mathrm{Spec}(A'_\eta)$ is a smooth surjection whose domain is a coproduct of affines. Proposition 6.4.7 and Proposition 6.4.13 then imply that $U \longrightarrow X$ is also a smooth surjection whose domain is a coproduct of affines, so that there is a unique diagonal lift.

A similar argument shows that $G \longrightarrow \mathrm{Stack}_{0\longrightarrow n}$ is convergent: let $U \longrightarrow X \longrightarrow \mathrm{Spec}(A)$ be a map between derived stacks whose base change $U_k \longrightarrow X_k$ along each $\mathrm{Spec}(\tau_{\leq k} A) \longrightarrow \mathrm{Spec}(A)$ is a smooth surjection. Then $U \longrightarrow X$ is a smooth surjection as well. Indeed, since being a smooth surjection is local for the smooth topology, we can reduce to the case where $U \longrightarrow X$ is a map between affines. In that case, the result follows in a straightforward way from Corollary 6.2.4.

Finally, we have to check that $G \longrightarrow F$ is formally smooth: for any point $x\colon M = \mathrm{Spec}(A) \longrightarrow G$ and any $E \in \mathrm{QC}^{\geq 2}(\mathrm{Spec}(A))$, we have to show that

$$\Omega_0 \mathrm{Der}_x(G/F, E)$$

is a connected space. The point x determines an atlas $U \xrightarrow{q} X \xrightarrow{p} M$ of a derived n-stack X and the basepoint $0 \in \mathrm{Der}_x(G/F, E)$ corresponds to the trivial deformation

$$U_{(pq)^*E} \longrightarrow X_{p^*E} \longrightarrow M_E \tag{6.4.18}$$

obtained by base change along the zero map $M_E \longrightarrow M$. The stacks appearing in (6.4.18) are simply the square zero extension from Construction 6.1.8, applied to the restrictions of E to U and X. The space $\Omega_0\mathrm{Der}_x(G/F, E)$ can then be identified with the space of natural equivalences of (6.4.18) over M_E which

(i) restrict to the identity when pulled back along $M \longrightarrow M_E$.

(ii) restrict to the identity $X_{p^*E} \longrightarrow X_{p^*E}$.

By Proposition 6.4.13, a map of stacks over M_E is an equivalence if and only if its pullback along $M \longrightarrow M_E$ is an equivalence. We can therefore identify $\Omega_0\mathrm{Der}_x(G/F, E)$ with the space of diagonal lifts

$$\begin{tikzcd} U \arrow[r] \arrow[d] & U_{(pq)^*E} \arrow[r] \arrow[d] & U \arrow[d] \\ U_{(pq)^*E} \arrow[r] \arrow[ur, dotted] & X_{p^*E} \arrow[r] & X \end{tikzcd}$$

Since $U_{(pq)^*E} \longrightarrow X_{p^*E}$ is the base change of $U \longrightarrow X$ along $M_E \longrightarrow M$, the space of such diagonal lifts is equivalent to the mapping space

$$\mathrm{Map}(L_{U/X}, (pq)^*E)$$

Since $U \longrightarrow X$ is smooth, $L_{U/X}$ is perfect with Tor-amplitude contained in $(-\infty, 0]$. Because U is a coproduct of affines, the space $\mathrm{Map}(L_{U/X}, (pq)^*E)$ is connected whenever E (hence $(pq)^*E$) is 1-connected. We conclude that $G \longrightarrow F$ is formally smooth, so that the sheaf $F = \mathrm{Map}(*, \mathrm{Stack}_n)$ is infinitesimally cohesive and convergent. □

6.4.3 Tangent spaces. Let $\chi\colon M \longrightarrow \mathrm{Map}(\mathcal{I}, \mathrm{Stack})$ be a map classifying an $\mathcal{I}$-diagram of stacks $p_\bullet\colon X_\bullet \longrightarrow M$ over an affine $M = \mathrm{Spec}(A)$. Consider the functor

$$\mathrm{Def}_{X_\bullet/M}\colon \mathcal{C}^\infty\mathrm{Alg}^{\mathrm{sm}}/A \longrightarrow \mathcal{S}$$

sending a small extension A' to the space of *deformations* of the diagram $X_\bullet$

$$\begin{tikzcd} X_\bullet \arrow[r] \arrow[d, "p"'] & \widetilde{X}_\bullet \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] & \mathrm{Spec}(A'). \end{tikzcd}$$

to a diagram of stacks over $\mathrm{Spec}(A')$. Theorem 6.4.3 implies that $\mathrm{Def}_{X_\bullet/M}$ is a formal moduli problem, which has an associated Lie algebroid over A by Theorem 4.2.1.

Proposition 6.4.19. *Let $\chi\colon M = \mathrm{Spec}(A) \longrightarrow \mathrm{Stack}$ classify a map $p\colon X \longrightarrow M$ and let $\mathfrak{g}$ be the Lie algebroid associated to the formal moduli problem $\mathrm{Def}_{X/M}$. Then the anchor map $\mathfrak{g} \longrightarrow T_M$ is equivalent to the projection map*

$$\mathrm{Hom}(L_X, \mathcal{O}_X) \times_{\mathrm{Hom}(p^*L_M, \mathcal{O}_X)} \mathrm{Hom}(L_M, \mathcal{O}_M) \longrightarrow \mathrm{Hom}(L_M, \mathcal{O}_M) \simeq T_M. \qquad (6.4.20)$$

In other words, one can think of the Lie algebroid $\mathfrak{g}$ as consisting of 'p-related vector fields' on X and M.

Proof. To provide the map from (6.4.20) to the anchor map $\mathfrak{g} \longrightarrow T_M$, it suffices to provide the following: for every (perfect) connective quasi-coherent sheaf E over M and every element $\eta \in \mathrm{Map}(L_M, E[1])$, we have to provide a natural map

$$\mathrm{Map}(L_X, p^*E[1]) \times_{\mathrm{Map}(p^*L_M, p^*E[1])} \{\eta\} \longrightarrow \mathrm{Def}_{X/M}(A_\eta) \qquad (6.4.21)$$

to the space of deformations over the square zero extension A_η of A classified by η. This determines a map of reduced excisive functors $L_A/\mathrm{Mod}_A^{\mathrm{f.p.},\geq 1} \longrightarrow \mathcal{S}$, whose domain is classified by (6.4.20) and whose codomain is classified by the anchor map $\mathfrak{g} \longrightarrow T_M$, by Example 4.2.24.

The map (6.4.21) can be described as follows. By the universal property of the cotangent complex of X, the domain of this map is given by the space of dotted lifts

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[rr, bend left, "="] & X_{p^*E[1]} \arrow[r, dotted, "\tilde{\eta}"'] \arrow[d] & X \arrow[d, "p"] \\ M \arrow[r] & M_{E[1]} \arrow[r, "\eta"'] & M \end{tikzcd} \tag{6.4.22}$$

from the square zero extension of X by $p^*E[1]$ (Construction 6.1.8), where the left vertical map $X_{p^*E[1]} \longrightarrow M_{E[1]}$ is the trivial deformation of $p\colon X \longrightarrow M$ over $M_{E[1]}$. Because $\tilde{\eta}$ restricts to the identity on X, Proposition 6.4.13 implies that the right square becomes cartesian. Together with the canonical map $0\colon X_{p^*E[1]} \longrightarrow X$ covering the zero map $M_{E[1]} \longrightarrow M$, the map $\tilde{\eta}$ therefore determines an element

$$\Big(X \xleftarrow{0} X_{p^*E[1]} \xrightarrow{\tilde{\eta}} X\Big) \in \mathrm{Def}_{X/M}(A) \times_{\mathrm{Def}_{X/M}(A\oplus E[1])} \mathrm{Def}_{X/M}(A).$$

Since $\mathrm{Def}_{X/M}$ is a formal moduli problem, this space is naturally equivalent to the space $\mathrm{Def}_{X/M}(\mathrm{Spec}(A_\eta))$. The above point therefore determines a unique deformation $X_{\tilde{\eta}} \in \mathrm{Def}_{X/M}(\mathrm{Spec}(A_\eta))$, depending functorially on the map $\tilde{\eta}$. This provides the natural map (6.4.21) and hence the map from (6.4.20) to $\mathfrak{g} \longrightarrow T_M$ over T_M.

To see that this map is an equivalence, it suffices to show that the map between the fibers over T_M is an equivalence. In other words, it suffices to show that the map (6.4.21) is an equivalence whenever $\eta = 0$ is the zero map. In that case, the map (6.4.21) reduces to a map

$$\mathrm{Map}\big(L_{X/M}, p^*E[1]\big) \longrightarrow \Omega_0\mathrm{Def}_{X/M}\big(A \oplus E[1]\big) \xrightarrow{\sim} \mathrm{Def}_{X/M}\big(A \oplus E\big).$$

The second map is the natural equivalence arising from the fact that $\mathrm{Def}_{X/M}$ is a formal moduli problem.

The first map sends $\tilde{\eta}\colon L_{X/M} \longrightarrow p^*E[1]$ to the lift (6.4.22) classified by it, which can equivalently be identified with a map $X_{p^*E[1]} \longrightarrow \eta^*X$ over $M_{E[1]}$ and under X. Since we are working over the fiber where $\eta = 0$, such a map is just an endomorphism of $X_{p^*E[1]}$ over $M_{E[1]}$, which restricts to the identity on X and is therefore an equivalence (Proposition 6.4.13). In other words, the universal property of the cotangent complex guarantees that the first map is an equivalence as well. □

Remark 6.4.23. Suppose that $\chi\colon * \longrightarrow \mathrm{Stack}$ classifies a stack X. It follows from Proposition 6.4.19 that the chain complex $\mathrm{Hom}_{\mathcal{O}_X}(L_X, \mathcal{O}_X)$ carries a natural Lie bracket. Informally, one can think of this Lie algebra as the Lie algebra of vector fields on X.

Example 6.4.24. Suppose that the map $X_\bullet \longrightarrow M = \mathrm{Spec}(A)$ is the opposite of a diagram of complete dg-$\mathcal{C}^\infty$-rings $A \longrightarrow B_\bullet$. Then the formal moduli problem $\mathrm{Def}_{X_\bullet/M}$ is equivalent to the formal moduli problem $\mathrm{Def}_{B_\bullet}$ from Example 2.3.35. In particular, the Lie algebroid from Proposition 6.4.19 can be identified with the Atiyah Lie algebroid of $A \longrightarrow B_\bullet$ (Example 3.1.3).

Example 6.4.25. Let $G = \mathrm{Map}(\Delta[1], \mathrm{Stack})$ be the sheaf of maps between stacks and suppose that $\chi\colon M = \mathrm{Spec}(A) \longrightarrow G$ classifies a diagram

$$X_0 \xrightarrow{f} X_1 \xrightarrow{p} M.$$

A similar argument as in Proposition 6.4.19 shows that for any $E \in \mathrm{QC}^{\geq 0}(M)$, the space $\Omega_0\mathrm{Der}_X(G; E)$ is equivalent to the space of natural diagonal lifts

$$\begin{tikzcd} X_\bullet \arrow[r, "="] \arrow[d] & X_\bullet \arrow[d] \\ \widetilde{X}_\bullet \arrow[r] \arrow[ur, dotted] & M. \end{tikzcd}$$

where $\widetilde{X}_\bullet$ is given by $(X_0)_{(pf)^*E} \longrightarrow (X_1)_{p^*E} \longrightarrow M_E$. The tangent complex of the formal moduli problem $\mathrm{Def}_{X_0 \longrightarrow X_1/M}$ is therefore given by

$$\mathrm{Hom}\big(L_{X_1/M}, \mathcal{O}_{X_1}[1]\big) \times_{\mathrm{Hom}(f^*L_{X_1/M}, \mathcal{O}_{X_0}[1])} \mathrm{Hom}\big(L_{X_0/M}, \mathcal{O}_{X_0}[1]\big). \tag{6.4.26}$$

There are canonical maps of formal moduli problems

$$\mathrm{Def}_{X_0/M} \xleftarrow{d_1} \mathrm{Def}_{X_\bullet/M} \xrightarrow{d_0} \mathrm{Def}_{X_1/M}$$

whose induced maps on tangent complexes are the obvious projection maps out of (6.4.26).

Corollary 6.4.27. *Let $X_0 \xrightarrow{f} X_1 \longrightarrow M = \mathrm{Spec}(A)$ be a map of derived stacks over an affine M. If f is smooth, then the map of formal moduli problems $\mathrm{Def}_{X_\bullet/M} \longrightarrow \mathrm{Def}_{X_1/M}$ induces a map of Lie algebroids with connective fibers.*

Proof. By Example 6.4.25, the fiber can be identified with $\mathrm{Map}(L_{X_0/X_1}, \mathcal{O}_{X_0})$, which is connective by the assumption that f is smooth. □

6.4.4 Lie algebroids from Lie groupoids. Let $\mathcal{G} \rightrightarrows M$ be an ordinary (smooth) Lie groupoid and let $p\colon M \longrightarrow X$ be the induced atlas of the associated stack. Let $T_{M/X}$ be the Lie algebroid associated to p by Definition 6.1.5. We will use Theorem 6.4.3 to show that $T_{M/X}$ is equivalent to the usual Lie algebroid of the Lie groupoid $\mathcal{G}$. In fact, we also provide a description of the Lie algebroid $T_{M/X}$ when M is an affine derived manifold.

Construction 6.4.28. Let $\mathcal{G}_\bullet\colon \mathbf{\Delta}^{\mathrm{op}} \longrightarrow \mathrm{Aff}$ be an affine derived Lie (1-)groupoid. The simplicial diagram $\mathcal{G}_\bullet$ is classified by a map $\{\mathcal{G}\} \longrightarrow \mathrm{Map}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Stack}_0)$ to the sheaf of simplicial diagrams of 0-stacks. Consider the sheaf

$$F_{\mathcal{G}} = \mathrm{Map}(\mathbf{\Delta}^{\mathrm{op}} \times [1], \mathrm{Stack}_0) \times_{\mathrm{Map}(\mathbf{\Delta}^{\mathrm{op}} \times \{1\}, \mathrm{Stack}_0)} \{\mathcal{G}\}$$

whose value on an affine U is given by the space of natural transformations

$$P_\bullet \longrightarrow U \times \mathcal{G}_\bullet$$

of derived manifolds over U. Let $\mathrm{Bun}_{\mathcal{G}} \subseteq F_{\mathcal{G}}$ be the sub-sheaf sending an affine U to the connected components on those diagrams that satisfy the following two conditions:

(a) The augmented simplicial diagram of derived manifolds $P_\bullet \longrightarrow U$ is the right Kan extension of its restriction $P_0 \longrightarrow U$. In other words, $P_\bullet$ is the Čech nerve of the map $P_0 \longrightarrow U$.

(b) The map $P_\bullet \longrightarrow U \times \mathcal{G}_\bullet$ of simplicial derived manifolds over U is *equifibered*, i.e. every map $\alpha\colon [n] \longrightarrow [m]$ induces a cartesian square

$$\begin{tikzcd} P_m \arrow[r] \arrow[d] & P_n \arrow[d] \\ U \times \mathcal{G}_m \arrow[r] & U \times \mathcal{G}_n. \end{tikzcd}$$

Unwinding the definitions, one sees that an object in $\mathrm{Bun}_{\mathcal{G}}(U)$ is given by a derived manifold P_0 equipped with an action of the groupoid $\mathcal{G}$, whose quotient is equivalent to U.

Lemma 6.4.29. *Let $\mathcal{G}$ be an affine derived Lie groupoid and let X be its associated stack. There is a canonical equivalence of sheaves*

$$X \longrightarrow \mathrm{Bun}_{\mathcal{G}}.$$

Proof. The groupoid $\mathcal{G}$ determines a natural diagram $\mathcal{G}_\bullet \longrightarrow X \times \mathcal{G}_\bullet$ of sheaves over X. Furthermore, each map $\mathcal{G}_n \longrightarrow X$ is 0-representable, the augmented simplicial diagram $\mathcal{G}_\bullet \longrightarrow X$ realizes $\mathcal{G}_\bullet$ as the Čech nerve of $\mathcal{G}_0 \longrightarrow X$ by [59, Proposition 6.1.3.9] and the map $\mathcal{G}_\bullet \longrightarrow X \times \mathcal{G}_\bullet$ is clearly equifibered. It follows that this diagram of 0-representable sheaves over X is classified by a map $X \longrightarrow \mathrm{Bun}_{\mathcal{G}}$.

To see that the map $f\colon X \longrightarrow \mathrm{Bun}_{\mathcal{G}}$ is an equivalence, fix an element $x \in \mathrm{Bun}_{\mathcal{G}}(U)$, corresponding to a diagram $P_\bullet \longrightarrow U \times \mathcal{G}_\bullet$ over U. Unwinding the definitions, the fiber $f^{-1}(x)$ can be identified with the space of dotted extensions

$$\begin{tikzcd} P_\bullet \arrow[r] \arrow[d] & \mathcal{G}_\bullet \arrow[d] \\ U \arrow[r, dotted] & X \end{tikzcd}$$

for which each induced map $P_n \longrightarrow U \times_X \mathcal{G}_n$ is an equivalence. Since the map $P_\bullet \longrightarrow \mathcal{G}_\bullet$ is equifibered, the space of such extensions is contractible by [59, Theorem 6.1.3.9] and it follows that f is an equivalence. □

The canonical atlas $M = \mathcal{G}_0 \longrightarrow X$ fits into a diagram of augmented simplicial objects

$$\begin{tikzcd} \mathrm{Dec}_0(\mathcal{G}) \arrow[r, "d_0"] \arrow[d, "d_{\{0\}}"'] & \mathcal{G}_\bullet \arrow[d] \\ M \arrow[r] & X. \end{tikzcd}$$

The groupoid object $\mathrm{Dec}_0(\mathcal{G})$ is the Čech nerve of the map $\mathrm{Dec}_0(\mathcal{G}) = \mathcal{G}_1 \longrightarrow M$ and each map $\mathrm{Dec}_0(\mathcal{G})_n \longrightarrow M \times_X \mathcal{G}_n$ is an equivalence. In other words, the composite map

$$M \longrightarrow X \xrightarrow{\sim} \mathrm{Bun}_{\mathcal{G}}$$

classifies the canonical $\mathcal{G}$-torsor over $M = \mathcal{G}_0$, given by the translation action of $\mathcal{G}_1$ on itself.

Proposition 6.4.30. *Let $\mathcal{G}$ be a derived Lie groupoid such that $M = \mathcal{G}_0$ and $\mathcal{G}_1$ are affine derived manifolds and let $M \longrightarrow X$ be the atlas of its associated stack. Consider the diagram of augmented simplicial derived manifolds*

$$\begin{tikzcd} \mathrm{Dec}_0(\mathcal{G}_\bullet) \arrow[r, "d_0"] \arrow[d, "d_{\{0\}}"'] & \mathcal{G}_\bullet \arrow[d] \\ M \arrow[r] & *. \end{tikzcd}$$

The the Lie algebroid $T_{M/X}$ can be identified with the $\mathcal{O}(M)$-module of natural vector fields on $\mathrm{Dec}_0(\mathcal{G}_\bullet) \longrightarrow M$ over $\mathcal{G}_\bullet \longrightarrow *$, equipped with the commutator bracket and with anchor map given by the map sending such a natural vector fields to its component at M.

Proof. By Lemma 6.4.29, the formal moduli problem associated to the map $M \longrightarrow X$ is equivalent to the formal moduli problem associated to the map $M \longrightarrow \mathrm{Bun}_{\mathcal{G}}$ classifying the canonical $\mathcal{G}$-torsor $\mathcal{G}_1 \longrightarrow M$. By Proposition 6.4.13, this formal moduli problem is equivalent to the formal moduli problem associated to the composite map $M \longrightarrow F_{\mathcal{G}}$ which classifies natural transformation of simplicial manifolds over M

$$\mathrm{Dec}_0(\mathcal{G}_\bullet) \longrightarrow M \times \mathcal{G}_\bullet. \tag{6.4.31}$$

Note that $F_{\mathcal{G}}$ is the fiber of a map of sheaves

$$\mathrm{Map}(\mathbf{\Delta}^{\mathrm{op}} \times [1], \mathrm{Stack}_0) \longrightarrow \mathrm{Map}(\mathbf{\Delta}^{\mathrm{op}} \times \{1\}, \mathrm{Stack}_0)$$

By Example 6.4.24 and Theorem 4.4.1, the Lie algebroids associated to these two sheaves are just given by Atiyah Lie algebroids, consisting of natural derivations on diagram 6.4.31 over M, resp. its restriction $M \times \mathcal{G}_\bullet$, equipped with the commutator bracket.

It follows that the Lie algebroid $T_{M/X}$ can be identified with the complex of natural vector fields over the diagram (6.4.31) (and over M) whose restriction to $M \times \mathcal{G}_\bullet$ is just induced by the the vector field on M. This is precisely the complex of natural vector fields on $\mathrm{Dec}_0(\mathcal{G}_\bullet) \longrightarrow M$ *over* $\mathcal{G}_\bullet$, together with the commutator bracket and with anchor map given by restriction to M. □

Remark 6.4.32. The description from Proposition 6.4.30 can be made more explicit at the level of algebra, as in Example 3.1.3: if the simplicial affine space $\mathcal{G}_\bullet$ is modeled by a projectively cofibrant cosimplicial diagram $A^\bullet \colon \mathbf{\Delta} \longrightarrow \mathcal{C}^\infty\mathrm{Alg}^{\mathrm{dg}}$, then the Lie algebroid $T_{M/X}$ can be modeled by the dg-Lie algebroid over A^0 consisting of natural $\mathbf{\Delta}_+$-diagrams of derivations $v_\bullet \colon A^{1+\bullet} \longrightarrow A^{1+\bullet}$ For all $n \geq 0$, the derivation $v_n \colon A^{1+n} \longrightarrow A^{1+n}$ is A^n-linear, where A^{1+n} is considered an A^n-algebra via the coface map δ_0. The anchor map takes the component v_{-1}.

Corollary 6.4.33. *Let $\mathcal{G} \rightrightarrows M$ be an ordinary Lie groupoid and let $M \longrightarrow X$ be the map to its associated stack. Then the Lie algebroid $T_{M/X}$ is equivalent to the usual Lie algebroid of $\mathcal{G}$.*

Proof. We already identified the anchor map of $T_{M/X}$ in Example 6.2.11. In particular, $T_{M/X}$ is just a non-derived, discrete Lie algebroid. The description of Proposition 6.4.30 can therefore be applied strictly (i.e. without taking the *derived* space of natural vector fields). Unwinding the definitions, this identifies $T_{M/X}$ with the vector space of tuples

$$v \in \Gamma(\mathcal{G}, T^t\mathcal{G})^{\mathcal{G}} \qquad\qquad w \in \Gamma(M, TM)$$

of a $\mathcal{G}$-invariant vector field v on $\mathcal{G}$, tangent to the target fiber, and a vector field w on M. Furthermore, these two vector fields have to be related under the source map $s \colon \mathcal{G} \longrightarrow M$. The vector field v then determines w uniquely: it is simply given by the restriction of $ds(v)$ to the unit section $M \longrightarrow \mathcal{G}$. We can therefore identify the Lie algebroid $T_{M/X}$ with $\Gamma(\mathcal{G}, T^t\mathcal{G})^{\mathcal{G}}$, endowed with the usual commutator bracket and anchor map. □

Chapter 7

Lie algebroids from stacks

In Chapter 6, we have seen that any map $f\colon M \longrightarrow X$ from a derived manifold to a derived stack X gives rise to a sheaf of Lie algebroids $T_{M/X}$ over M (see Definition 6.1.5 and Example 6.3.17). This Lie algebroid is associated to the sheaf of formal moduli problems $\widehat{X}$ sending a small extension of an affine open subspace of M to the space of extensions

$$\begin{tikzcd} \mathrm{Spec}(A) \arrow[r, "\text{open}"] \arrow[d] & M \arrow[r, "f"] & X \\ \mathrm{Spec}(\widetilde{A}). \arrow[urr, dotted] & & \end{tikzcd}$$

Somewhat informally, the formal moduli problem $\widehat{X}$ assembles a compatible family of infinitesimal neighbourhoods of M inside X, and hence describes a certain formal neighbourhood of X around M. However, $\widehat{X}$ does not contain the data of *all* infinitesimal neighbourhoods of M inside X. For example, $\widehat{X}$ does not describe deformations of f along a square zero extension $\mathrm{Spec}(A \oplus E)$ where E is infinitely generated. This also becomes apparent in terms of linear algebra: while the deformations of f along $\mathrm{Spec}(A \oplus I)$ are controlled by the cotangent complex $L_{M/X}$, the formal moduli problem $\widehat{X}$ is only controlled by the *dual* of $L_{M/X}$.

The entire family of infinitesimal neighbourhoods of M inside X can be organized into a sheaf $X^{\wedge}_M$, the *formal completion* of X at M. The purpose of this section is to study the relation between the stack X itself, the formal completion $X^{\wedge}_M$ and the formal moduli problem $\widehat{X}$, or equivalently, the Lie algebroid $T_{M/X}$.

In Section 7.1, we show that the formal completion of X at M can be retrieved from the Lie algebroid $T_{M/X}$ when the map $f\colon M \longrightarrow X$ is *locally finitely presented*. For example, when $M \longrightarrow X$ is smooth, one can think of the formal completion $X^{\wedge}_M$ as the infinitesimal quotient of M by the action of the Lie algebroid $T_{M/X}$.

In Section 7.2, we study the relation between this formal quotient and the global quotient X. Our main result is the following variant of the Van Est theorem:

Theorem 7.2.1. *Let $p\colon M \longrightarrow X$ be a smooth map from a smooth manifold to a smooth stack whose fibers are n-connected. If $f\colon M \longrightarrow Y$ is map to a derived m-stack, then the map*

$$\mathrm{Map}_{M/}(X, Y) \longrightarrow \mathrm{Map}_{\mathrm{LieAlgd}_M}(T_{M/X}, T_{M/Y})$$

is $(m-n-2)$-truncated (in particular, an equivalence if $m = n$).

Example 7.0.1. Suppose that X and Y arise from Lie groupoids $\mathcal{G} \rightrightarrows M$ and $\mathcal{H} \rightrightarrows M$. In light of Proposition 6.4.30, the above result reproduces 'Lie's second theorem' for Lie groupoids [64, 67]. When $Y = K(m, \mathbb{R})$, this reproduces the van Est theorem (with trivial coefficients) [28, 20].

Section 7.3 outlines some further examples and applications of Theorem 7.2.1. In particular, we describe how the cohomology of the Lie algebroid $T_{M/X}$ can often be identified without providing an explicit point-set model for $T_{M/X}$, using descent.

7.1 Formal completion

Let $f\colon Y \longrightarrow X$ be a map in Sh(Aff). Informally, the formal completion of X around Y consists of all A-points $\mathrm{Spec}(A) \longrightarrow X$ which are infinitesimal thickenings of points in Y. More precisely, let A be a complete $\mathcal{C}^\infty$-ring and define the *reduction* A_{red} to be the discrete $\mathcal{C}^\infty$-ring

$$A_{\mathrm{red}} = \pi_0(A)/\big(f : \text{locally } f^n = 0 \text{ for some } n\big).$$

By construction, A_{red} is a complete $\mathcal{C}^\infty$-ring, being the global sections of the quotient sheaf by the sheaf of locally nilpotent elements of $\pi_0\mathcal{O}$ (taking global sections of $\mathcal{O}$-module sheaves over $\mathrm{Spec}(\pi_0 A)$ is exact).

Remark 7.1.1. Our definition of A_{red} differs from the $\mathcal{C}^\infty$-algebraic reduction of a $\mathcal{C}^\infty$-ring, given by the quotient of $\pi_0(A)$ by the ideal containing all $f \in \pi_0(A)$ such that $f(x) = 0$ for all $x \in |\mathrm{Spec}(A)|$.

Definition 7.1.2 ([90])**.** Let $f\colon Y \longrightarrow X$ be a map in Sh(Aff). The *formal completion* $X^\wedge_Y$ of X at Y is the sheaf given by

$$X^\wedge_Y(A) = \left\{ \begin{tikzcd} \mathrm{Spec}(A_{\mathrm{red}}) \arrow[r] \arrow[d] & Y \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] & X \end{tikzcd} \right\}.$$

There are natural maps of sheaves $Y \longrightarrow X^\wedge_Y \longrightarrow X$.

Example 7.1.3. If $f\colon Y \longrightarrow X$ is a closed immersion of derived manifolds, $X^\wedge_Y \subseteq X$ is the subsheaf consisting of those maps $x\colon \mathrm{Spec}(A) \longrightarrow X$ for which the map $\mathrm{Spec}(A_{\mathrm{red}}) \longrightarrow X$ factors (uniquely) over Y. In particular, the map of topological spaces underlying x takes values in the closed subspace $Y \subseteq X$.

Remark 7.1.4. The formal completion of the terminal map $X \longrightarrow *$ is the *de Rham space* X_{dR} of X, defined by $X_{\mathrm{dR}}(A) = X(A_{\mathrm{red}})$ [90]. The functor $(-)_{\mathrm{dR}}\colon \mathrm{Sh}(\mathrm{Aff}) \longrightarrow \mathrm{Sh}(\mathrm{Aff})$ preserves limits and colimits and takes values in convergent, infinitesimally cohesive sheaves. For any map $Y \longrightarrow X$, the formal completion fits into a pullback square of sheaves

$$\begin{tikzcd} X^\wedge_Y \arrow[r] \arrow[d] & X \arrow[d] \\ Y_{\mathrm{dR}} \arrow[r] & X_{\mathrm{dR}}. \end{tikzcd}$$

Example 7.1.5. The canonical map $\mathbb{R}^n \longrightarrow \mathbb{R}^n_{\mathrm{dR}}$ is an epimorphism of sheaves (since any set of elements in $\pi_0(A)/I$ can be lifted to $\pi_0(A)$) and the associated Čech nerve is given by the groupoid object

$$\big(\mathbb{R}^n \times \mathbb{R}^n\big)^\wedge_{\mathbb{R}^n} \rightrightarrows \mathbb{R}^n \longrightarrow \mathbb{R}^n_{\mathrm{dR}}$$

whose space of arrows is the formal completion of $\mathbb{R}^n \times \mathbb{R}^n$ at its diagonal. In other words, $\mathbb{R}^n_{\mathrm{dR}}$ can be considered as the quotient of $\mathbb{R}^n$ by the formal completion of its *pair groupoid*, i.e. as the quotient where one identifies 'infinitesimally close points'.

More generally, if $p\colon Y \longrightarrow X$ is smooth then the map $Y \longrightarrow X^\wedge_Y$ is a surjection of sheaves and realizes $X^\wedge_Y$ as the quotient of Y by the formal completion of the Čech nerve of p. Indeed, it suffices to verify this locally on X and Y, where the map p is equivalent to a projection $U \times \mathbb{R}^k \longrightarrow U$. The map $Y \longrightarrow X^\wedge_Y$ can therefore be identified locally with the surjection $U \times \mathbb{R}^k \longrightarrow U \times \mathbb{R}^k_{\mathrm{dR}}$.

Our goal will be to prove the following:

Proposition 7.1.6. *Let $M \longrightarrow X$ be a locally finitely presented map from a derived manifold to a derived stack. Then the map $M \longrightarrow X_M^\wedge \longrightarrow X$ has the following universal property: for every map $M \longrightarrow Y$ to a convergent, infinitesimally cohesive sheaf, there is a natural commuting triangle*

$$\begin{tikzcd} & \mathrm{Map}_{M/}(X, Y) \arrow[dl] \arrow[dr, "\widehat{(-)}"] & \\ \mathrm{Map}_{M/}(X_M^\wedge, Y) \arrow[rr, "\simeq"'] & & \mathrm{Map}_{\mathrm{FormMod}_M}(\widehat{X}, \widehat{Y}) \end{tikzcd}$$

where $\mathrm{Map}_{\mathrm{FormMod}_M}(\widehat{X}, \widehat{Y})$ is the space of maps between the sheaves of formal moduli problems associated to X and Y.

When the structure sheaf $\mathcal{O}_M$ is locally eventually coconnective, Corollary 6.3.15 identifies the space of maps $\widehat{X} \longrightarrow \widehat{Y}$ with the space of Lie algebroid maps $T_{M/X} \longrightarrow T_{M/Y}$. Under this identification, the derivative $T_{M/X} \longrightarrow T_{M/Y}$ of a map of stacks $X \longrightarrow Y$ corresponds simply to its restriction to $X_M^\wedge$.

We will first study the formal completion of a map between affines and then use a gluing construction to deduce Proposition 7.1.6.

7.1.1 Formal completions of affines. When $M \longrightarrow X$ is a finitely presented map of affines, the formal completion $X_M^\wedge$ can be described relatively explicitly in terms of algebra.

Construction 7.1.7. Consider a cofibration $A \longrightarrow B = A[\xi_i]$ in $\mathcal{C}^\infty\mathrm{Alg}^{\mathrm{dg}}$, where A is cofibrant and there are finitely many generators ξ_i, each of which has degree ≥ 1. This cofibration determines a dg-Lie algebroid $\mathfrak{g} \subseteq T_B$ over B, spanned by the derivations $\partial/\partial\xi_i$. Let $\widetilde{A} = C^*_{\mathrm{sm}}(\mathfrak{g})$ be the global Chevalley-Eilenberg complex of this Lie algebroid, as in Construction 4.1.30.

Unwinding the definitions, one can explicitly describe $\widetilde{A}$ as follows. For each generator ξ_i, let $f_i = f_i(\xi)$ be the differential of ξ_i in B and let x_i be an additional generator of degree $|\xi_i| - 1$. Then $\widetilde{A}$ is the freely generated dg-$\mathcal{C}^\infty$-ring

$$\widetilde{A} = A\{x_i, \xi_i\} \qquad \partial\xi_i = f_i(\xi) - x_i \qquad \partial x_{i_k} = \sum \frac{\partial f_i}{\partial \xi_{i_{k-1}}} x_{i_{k-1}}.$$

When x_{i_k} is of degree $k-1$, the last sum runs over all generators $\xi_{i_{k-1}}$ of degree $k-1$. In other words, $\widetilde{A}$ is the dg-$\mathcal{C}^\infty$-ring obtained by attaching along each boundary f_i a cylinder instead of a cell. The canonical map $A \longrightarrow \widetilde{A}$ is a weak equivalence between cofibrant dg-$\mathcal{C}^\infty$-rings.

Let N be the total number of generators ξ_i and fix a number $n \geq 1$. For each generator x_i of degree k, let J_i be the set of all monomials of degree $(N+1)^k \cdot n$ satisfying the following two conditions:

- they are monomials in x_i, as well as all generators x_j of degree $< k$.
- they are divisible by x_i^n.

Let $J = \bigcup_i J_i$ and let $A^{(n)} := \widetilde{A}/J$ be the (strict) quotient of $\widetilde{A}$ by J. There are canonical maps

$$A \longrightarrow \cdots \longrightarrow A^{(n)} \longrightarrow A^{(n-1)} \longrightarrow \cdots A^{(1)} \longrightarrow B = A[\xi_i]$$

taking succesive quotients by lower degree polynomials in the generators x_i (and ultimately, by the generators x_i themselves).

Remark 7.1.8. The precise definition of the set J and the quotient $A^{(n)}$ is essentially irrelevant; we only need that J has the following properties:

(a) The set J contains the monomials x_0^n, for every generator x_0 of degree 0. All other elements in J have degrees $\geq n$.
(b) For any generator x_i of degree k, the set J contains its $n \cdot (N+1)^k$-th power. In particular, each map $A^{(n)} \longrightarrow B$ has a nilpotent kernel.
(c) The ideal generated by J is closed under the differential. Indeed, consider a monomial of the form $x_i^a \cdot p(x)$, where x_i is of degree k, $p(x)$ is some monomial in generators of lower degree and $n \leq a \leq (N+1)^k \cdot n$. The differential of $x_i^a \cdot p(x)$ is a linear combination of monomials of the form

$$x_i^a \cdot p'(x) \qquad \text{or} \qquad x_i^{a-1} \cdot q(x).$$

The first monomial is again contained in J and the second monomial is contained in J when $a > n$. When $a = n$, the monomial $q(x)$ must contain an $n \cdot (N+1)^{k-1}$-th power of a generator of lower degree, so that it is in the ideal generated by J as well.

Lemma 7.1.9. *Let $n \geq 2$ and suppose that C is an $(n-1)$-truncated $\mathcal{C}^\infty$-ring. Restriction along the map $A \longrightarrow A^{(n)}$ induces an inclusion of path components*

$$\mathrm{Map}(A^{(n)}, C) \longrightarrow \mathrm{Map}(A, C).$$

The essential image consists of those maps $\phi\colon A \longrightarrow C$ for which $\phi(f_{i_0})^n = 0$ in $\pi_0(C)$, for each $f_{i_0} = d\xi_{i_0}$ in $\pi_0(A)$.

Proof. The map $A \longrightarrow A^{(n)}$ decomposes as

$$A \xrightarrow{\sim} \widetilde{A} \longrightarrow \widetilde{A}/(x_{i_0}^n) \longrightarrow A^{(n)}$$

where $\widetilde{A} \longrightarrow \widetilde{A}/(x_{i_0}^n)$ takes the quotient by all n-th powers of generators $x_{i_0} \in \widetilde{A}$ of degree 0. The map $\widetilde{A}/(x_{i_0}^n) \longrightarrow A^{(n)}$ takes the quotient by the remaining monomials in the set J, each of which is of degree $\geq n$ (Remark 7.1.8(a)). Since C is $(n-1)$-truncated, the map

$$\mathrm{Map}(A^{(n)}, C) \longrightarrow \mathrm{Map}(\widetilde{A}/(x_{i_0}^n), C)$$

is an equivalence.

The sequence of $x_{i_0}^n$ is a regular sequence in $\widetilde{A}$, so that the quotient $\widetilde{A}/(x_{i_0}^n)$ is a model for the *homotopy* quotient of $\widetilde{A}$ by the elements $x_{i_0}^n$. For any $\mathcal{C}^\infty$-ring C, we therefore obtain a homotopy pullback square of spaces

$$\begin{tikzcd}
\mathrm{Map}(\widetilde{A}/(x_{i_0}^n), C) \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & * \arrow[d, "0"] \\
\mathrm{Map}(\widetilde{A}, C) \arrow[r, "\psi"'] & \prod_{i_0} C \arrow[r, "(-)^n"'] & \prod_{i_0} C
\end{tikzcd}$$

where ψ sends a map $\phi\colon \widetilde{A} \longrightarrow C$ to the various degree 0 cycles $\phi(x_{i_0})$ of C. The map $(-)^n$ takes the n-th power of elements in C. More precisely, given a map of pointed spaces $\alpha\colon K \longrightarrow C = (C, 0)$, the map $\alpha^n\colon K \longrightarrow C \longrightarrow C$ is given by the composite

$$K \xrightarrow{\Delta} K^{\wedge n} \xrightarrow{\alpha^{\wedge n}} C^{\wedge n} \xrightarrow{\times} C$$

where Δ is the diagonal map. When K is a k-sphere for $k > 0$, the diagonal map is null-homotopic, which implies that the map $(-)^n$ induces the zero map on homotopy groups in degree ≥ 1. It follows that the map $F \longrightarrow \prod C$ is an inclusion of path components, indexed by the elements $c \in \pi_0(C)$ for which $c^n = 0$ in $\pi_0(C)$.

The space $\mathrm{Map}(\widetilde{A}/(x_{i_0}^n), C)$ is therefore a union of the path components in $\mathrm{Map}(\widetilde{A}, C)$, on those maps $\phi\colon \widetilde{A} \longrightarrow C$ such that $\phi(x_{i_0}^n) = 0$ in $\pi_0(C)$. By construction, x_{i_0} is homotopic to f_{i_0} in $\widetilde{A}$ and the result follows. $\square$

Proposition 7.1.10. *Let $A \longrightarrow B$ be a cofibration of dg-$\mathcal{C}^\infty$-rings as in Construction 7.1.7 and let $i\colon M = \mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A) = X$ be the associated closed immersion. Then there is a map of presheaves* $\mathrm{Aff}^{\mathrm{op}} \longrightarrow \mathcal{S}$

$$\mathrm{colim}_{n\longrightarrow\infty}\big(\mathrm{Spec}(A^{(n)})\big) \longrightarrow X^\wedge_M$$

realizing $X^\wedge_M$ as the convergent sheaf *associated to the union of all* $\mathrm{Spec}(A^{(n)})$.

Proof. Recall that there is a finite set of elements $f_i \in \pi_0(A)$ such that

$$\pi_0(B) \cong \pi_0(A)/(f_i) \qquad\qquad \pi_0(A^{(n)}) \cong \pi_0(A)/(f_i^n).$$

The maps $A^{(n)} \longrightarrow B$ therefore induce isomorphisms $A^{(n)}_{\mathrm{red}} \longrightarrow B_{\mathrm{red}}$ and the maps $A_{\mathrm{red}} \longrightarrow A^{(n)}_{\mathrm{red}}$ factor naturally over B_{red}, so that there is a diagram of presheaves

$$\cdots \longrightarrow \mathrm{Spec}(A^{(n)}) \longrightarrow \mathrm{Spec}(A^{(n+1)}) \longrightarrow \cdots \longrightarrow X^\wedge_M \longrightarrow \mathrm{Spec}(A).$$

For any n-truncated complete $\mathcal{C}^\infty$-ring C, the maps

$$\mathrm{Spec}(A^{(n+1)})(C) \longrightarrow X^\wedge_M(C) \longrightarrow \mathrm{Spec}(A)(C)$$

are inclusions of path components: $X^\wedge_M(C)$ consists of maps $\phi\colon A \longrightarrow C$ such that each $\phi(f_i)$ is a locally nilpotent element of $\pi_0(C)$. On the other hand, $\mathrm{Spec}(A^{(n+1)})(C)$ consists of maps ϕ such that $\phi(f_i)^{n+1} = 0$ in $\pi_0(C)$, by Lemma 7.1.9. Because there are only finitely many elements f_i, the presheaf

$$\operatorname*{colim}_{n\longrightarrow\infty} \mathrm{Spec}(A^{(n)})$$

sends C to the path components of $\mathrm{Spec}(A)(C)$ on the maps $\phi\colon A \longrightarrow C$ for which each $\phi(f_i)$ is nilpotent. Consequently, its associated sheaf agrees with $X^\wedge_M$ on all truncated complete $\mathcal{C}^\infty$-rings. This means that the corresponding convergent sheaves are naturally equivalent as well (Remark 6.1.32). ☐

Example 7.1.11. If $0\colon * \longrightarrow \mathbb{R}^k$ is the inclusion of the origin, then the formal completion $(\mathbb{R}^k)^\wedge_0$ is given by the colimit

$$\operatorname*{colim}_{n\longrightarrow\infty} \mathrm{Spec}\Big(\mathbb{R}\{x_1, \dots, x_k\}/(x_1^n, \dots, x_k^n)\Big).$$

7.1.2 Formal completions of derived stacks. Let M be a derived manifold and let Inf_M be its infinitesimal site (Definition 6.3.4). There is a canonical functor

$$i\colon \mathrm{Inf}_M \longrightarrow M/\mathrm{Sh}_{\mathrm{inf.coh.,conv}}(\mathrm{Aff}) \tag{7.1.12}$$

to the category of convergent, infinitesimally cohesive sheaves with a map from M. For each affine open $\mathrm{Spec}(A) \subseteq M$, this functor sends a small extension $\mathrm{Spec}(A) \longrightarrow \mathrm{Spec}(\widetilde{A})$ to the pushout $M \coprod_{\mathrm{Spec}(A)} \mathrm{Spec}(\widetilde{A})$.

Since $M/\mathrm{Sh}_{\mathrm{inf.coh.,conv}}(\mathrm{Aff})$ is locally presentable, there is an adjunction

$$i_!\colon \mathrm{PSh}(\mathrm{Inf}_M) \rightleftarrows M/\mathrm{Sh}_{\mathrm{inf.coh.,conv}}(\mathrm{Aff})\colon i^*$$

where $i_!$ takes the left Kan extension of a sheaf of formal moduli problems $\mathrm{Inf}_M^{\mathrm{op}} \longrightarrow \mathcal{S}$ along i. The right adjoint i^* sends a map $M \longrightarrow X$ to the sheaf of formal moduli problems $\widehat{X}$.

If $f\colon M \longrightarrow X$ is a map to a convergent, infinitesimally cohesive sheaf, then the map $M \longrightarrow X^\wedge_M \longrightarrow X$ induces an equivalence of formal moduli problems

$$i^*(X^\wedge_M) \xrightarrow{\sim} i^*X = \widehat{X}.$$

The inverse of this equivalence is adjoint to a map $i_!\widehat{X} \longrightarrow X^\wedge_M$.

Proposition 7.1.13. *Let $A \longrightarrow B$ be a finitely presented map of $\mathcal{C}^\infty$-rings and let $f\colon M = \mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A) = X$ be the associated closed immersion. Then the canonical map*

$$i_!\widehat{X} \xrightarrow{\sim} i_!i^*(X^\wedge_M) \longrightarrow X^\wedge_M$$

is an equivalence of convergent, infinitesimally cohesive sheaves.

Proof. We can identify $X^\wedge_M$ with the sheaf $\operatorname{colim}\mathrm{Spec}(A^{(n)})$ from Proposition 7.1.10, at least when restricted to $\mathcal{C}^\infty$-rings that are eventually coconnective. We claim that each map $A^{(n)} \longrightarrow B$ is a small extension. Assuming this, the result follows: indeed, let

$$F = \operatorname{colim}\mathrm{Spf}(A^{(n)})\colon \mathrm{Inf}_M^{\mathrm{op}} \longrightarrow \mathcal{S}$$

be the ind-representable sheaf of formal moduli problems given by the colimit of the corepresentable functors on the $A^{(n)}$. Then $X^\wedge_M \simeq i_!F$ and the map $i_!i^*i_!F \longrightarrow i_!F$ admits a section, induced by the unit map $F \longrightarrow i^*i_!F$. This unit map is a filtered colimit of the unit maps

$$\mathrm{Spf}(A^{(n)}) \longrightarrow i^*\mathrm{Spec}(A^{(n)})$$

which are all equivalences. It follows that the map $i_!i^*(X^\wedge_M) \longrightarrow X^\wedge_M$ is an equivalence.

It therefore suffices to show that each

$$A^{(n)} = A\{x_i, \xi_i\}/(x_{i_0}^n, \dots) \longrightarrow B = A\{\xi_i\}$$

is a small extension of B. To see this, note that the quotient $A^{(n)}$ contains finitely many nonzero monomials in the generators x_{i_k}, since every generator x_{i_k} is nilpotent in $A^{(n)}$ (Remark 7.1.8(b)). Equip these remaining monomials with a linear order such that $p(x) \leq q(x)$ if and only if both

(1) the polynomial degree of p is less or equal than the polynomial degree of q.

(2) if the polynomial degrees of p and q agree, then the total homological degree of p is *greater* or equal than the homological degree of q.

The point of this ordering is that $x_i \cdot p(x) > p(x)$ and that the boundary of a monomial $p(x)$ is a linear combination of larger monomials. Now take successive quotients by these monomials, starting with the largest one and ending with the highest degree generator x_m. It follows from (2) that in each step, one takes the quotient by a cycle and it follows from (1) that each step is a *strict* square zero extension by a shifted copy of B (and therefore a square zero extension, by Lemma 2.2.21). We conclude that each $A^{(n)} \longrightarrow B$ is indeed a small extension. □

Lemma 7.1.14. *Let $f\colon M \longrightarrow X$ be a map from a derived manifold to an infinitesimally cohesive sheaf. Suppose that there exists a covering sieve $S \subseteq \mathrm{Aff}/M$ with the property that for each $U_\alpha \in S$, the map*

$$i_{\alpha!}i_\alpha^*(X^\wedge_{U_\alpha}) \longrightarrow X^\wedge_{U_\alpha}$$

is an equivalence, where i_α is the functor (7.1.12) for U_α. Then the map

$$i_!i^*(X^\wedge_M) \longrightarrow X^\wedge_M$$

is an equivalence as well.

Proof. There is a commuting square of convergent sheaves with deformation theory

$$\begin{tikzcd} \operatorname*{colim}_{\alpha\in S} M\coprod_{U_\alpha}\Big(i_{\alpha!}i_\alpha^*(X^\wedge_{U_\alpha})\Big) \arrow[r, "\sim"] \arrow[d] & \operatorname*{colim}_{\alpha\in S} M\coprod_{U_\alpha} X^\wedge_{U_\alpha} \arrow[d, "\sim"] \\ i_!i^*X^\wedge_M \arrow[r] & X^\wedge_M. \end{tikzcd}$$

The top map is an equivalence since each $i_{\alpha!}i_\alpha^*(X^\wedge_{U_\alpha}) \longrightarrow X^\wedge_{U_\alpha}$ is an equivalence. Because $M \simeq \operatorname{colim}_{\alpha\in S} U_\alpha$, there is an equivalence of sheaves

$$\operatorname*{colim}_{\alpha\in S} X^\wedge_{U_\alpha} \xrightarrow{\ \sim\ } \operatorname*{colim}_{\alpha\in S}\Big(X\times_{X_{\mathrm{dR}}}(U_\alpha)_{\mathrm{dR}}\Big) \xrightarrow{\ \sim\ } X\times_{X_{\mathrm{dR}}} M_{\mathrm{dR}} = X^\wedge_M.$$

This implies that the right vertical map is an equivalence as well. To see that the left vertical map is equivalence, note that each affine open subspace $U_\alpha \subseteq M$ defines a sheaf on Inf_M

$$\chi_\alpha\colon \mathrm{Inf}_M^{\mathrm{op}} \longrightarrow \mathcal{S};\ \ \big(V\longrightarrow \tilde V\big) \longmapsto \begin{cases} * & V\subseteq U_\alpha \\ \emptyset & \text{otherwise} \end{cases}$$

Unwinding the definitions, there is a natural equivalence $M\coprod_{U_\alpha} i_{\alpha!}i_\alpha^*(X) \simeq i_!(\chi_\alpha \times i^*X)$ for any $M \longrightarrow X$. There is an equivalence of sheaves $\operatorname{colim}_{\alpha\in S}\chi_\alpha \longrightarrow *$ over Inf_M, which implies that the left vertical map is an equivalence. It follows that the bottom map is an equivalence, as asserted. □

Proof (of Proposition 7.1.6). If $f\colon M \longrightarrow X$ is a map to an infinitesimally cohesive sheaf, then the sheaf $i_!(\widehat{X})$ is characterized by the property that

$$\mathrm{Map}_{\mathrm{FormMod}_M}(\widehat{X},\widehat{Y}) \simeq \mathrm{Map}_{M/}(i_!\widehat{X}, Y)$$

for any convergent sheaf with deformation theory. To prove the proposition, it therefore suffices to verify that for any locally finitely presented map $f\colon M\longrightarrow X$ to a derived stack, the canonical map

$$i_!i^*(X^\wedge_M) \longrightarrow X^\wedge_M$$

is an equivalence of convergent, infinitesimally cohesive sheaves.

To prove this, it suffices to work locally on M, by Lemma 7.1.14. We may therefore assume that the map $f\colon M\longrightarrow X$ arises from a map $M\longrightarrow X_\bullet$ to a derived Lie n-groupoid, which is locally finitely presented in each simplicial degree. In fact, replacing $X_\bullet$ by $X_\bullet\times\mathrm{Pair}(\mathbb{R}^k)$, we may assume that this map is given in each degree by a locally finitely presented embedding.

Since the maps $X(\Delta[k]) \longrightarrow X(\Lambda^j[k])$ are smooth, Example 7.1.5 shows that the map of simplicial sheaves $X_\bullet \longrightarrow (X_\bullet)_{\mathrm{dR}}$ is a Kan fibration. It follows that

$$X^\wedge_M \simeq \operatorname{colim}\Big((X_\bullet)^\wedge_M\Big).$$

Similarly, it follows that the sheaf of formal moduli problems $\widehat{X}$ is the colimit of the simplicial diagram of formal moduli problems $\widehat{X}_\bullet$. We therefore obtain a commuting square of sheaves

$$\begin{tikzcd} \operatorname*{colim}_{\mathbf{\Delta}^{\mathrm{op}}} i_!i^*(X_\bullet) \arrow[r] \arrow[d, "\sim"'] & \operatorname*{colim}_{\mathbf{\Delta}^{\mathrm{op}}}\Big((X_\bullet)^\wedge_M\Big) \arrow[d, "\sim"] \\ i_!i^*X_\bullet \arrow[r] & X^\wedge_M \end{tikzcd}$$

and it suffices to treat the case where f is a locally finitely presented embedding of derived manifolds. Working locally, we may assume that f arises from a finitely presented map of $\mathcal{C}^\infty$-rings $A\longrightarrow B$. In that case, the result follows from Proposition 7.1.13. □

Example 7.1.15. Consider the sheaf $\mathcal{O}\colon \mathrm{Aff}^{\mathrm{op}} \longrightarrow \mathrm{CAlg}$ of (unbounded) commutative algebras sending each affine space M to its ring of smooth functions. This extends to a limit-preserving functor $\mathcal{O}\colon \mathrm{Sh}(\mathrm{Aff})^{\mathrm{op}} \longrightarrow \mathrm{CAlg}$ which one can think of as sending a sheaf to its function algebra.

Suppose that M is a locally finitely presented derived manifold whose structure sheaf is locally eventually coconnective (e.g. M is a smooth manifold, or a derived intersection of two smooth submanifolds of a smooth manifold). If $F\colon \mathrm{Inf}_M^{\mathrm{op}} \longrightarrow \mathcal{S}$ is the terminal formal moduli problem, there is a canonical map of commutative algebras over $\mathcal{O}(M)$

$$\mathcal{O}(M_{\mathrm{dR}}) \longrightarrow \mathcal{O}(F) \longrightarrow C^*(T_M)$$

The second map is an equivalence by Remark 4.2.22. Taking the map $M \longrightarrow Y$ to be the zero map $0\colon M \longrightarrow K(n, \mathbb{R})$ in Proposition 7.1.6, one finds that the first map induces an equivalence between the fibers over $0 \in \mathcal{O}(M)$. The above map therefore provides an equivalence between the function algebra of the de Rham space $\mathcal{O}(M_{\mathrm{dR}})$ and $C^*(T_M)$.

Remark 7.1.16. Let $X_\bullet$ be a (smooth) Lie n-groupoid and let $M = X_0 \longrightarrow X$ be the canonical atlas. This map also arises as the colimit of the map of simplicial manifolds $X_0 \longrightarrow X_\bullet$. Proposition 7.1.6 shows that $C^*(T_{M/X})$ can also be computed as the homotopy limit

$$C^*(T_{M/X}) \simeq \lim_{\boldsymbol{\Delta}} C^*(T_{M/X_\bullet}) \simeq \lim_{\boldsymbol{\Delta}} \mathcal{O}((X_\bullet)^\wedge_{X_0}).$$

The cosimplicial diagram $\mathcal{O}((X_\bullet)^\wedge_{X_0})$ consists of discrete ($\mathcal{C}^\infty$-)rings and can be identified with the usual completion $(A_\bullet)^\wedge_{\mathfrak{m}_\bullet}$ of the cosimplicial ring $A_\bullet = \mathcal{C}^\infty(X_\bullet)$ at the kernel $\mathfrak{m}_\bullet \subseteq A_\bullet \longrightarrow A_0$ of the map restricting along $X_0 \longrightarrow X_\bullet$. A dg-algebra model for the above homotopy limit can then be obtained by applying the left derived functor of the Quillen equivalence

$$N^*\colon \mathrm{CAlg}_{\mathbb{R}}^{\boldsymbol{\Delta}} \rightleftarrows \mathrm{CAlg}_{\mathbb{R}}^{\mathrm{dg},\leq 0}\colon K$$

between cosimplicial commutative $\mathbb{R}$-algebras (in sets) and non-positively graded cdgas, induced by the Dold-Kan correspondence (see e.g. [30, Chapter 6]).

The resulting commutative dg-algebra is computed (implicitly) in the work of Ševera [87] (see also [57]) and is discussed more explicitly in [78, Section 3]. These works show that before deriving, there is a non-canonical isomorphism of graded-commutative algebras

$$N^*((A_\bullet)^\wedge_{\mathfrak{m}_\bullet}) \cong \mathrm{Sym}_{A_0}(N(\mathfrak{m}_\bullet/\mathfrak{m}_\bullet^2)).$$

The complex $N(\mathfrak{m}_\bullet/\mathfrak{m}_\bullet^2)$ is the normalization of the conormal bundle $\nu^\vee_{X_0/X_\bullet}$. By induction on the adic filtration, one can show that each unit map

$$\mathcal{O}(A_\bullet/\mathfrak{m}_\bullet^n) \xrightarrow{\sim} KN^*(\mathcal{O}(A_\bullet/\mathfrak{m}_\bullet^n))$$

is a weak equivalence. In the limit, this implies that the homotopy limit of $\mathcal{O}((X_\bullet)^\wedge_{X_0})$ can be modeled by the quasi-free cdga $\mathrm{Sym}_{A_0}(N(\mathfrak{m}_\bullet/\mathfrak{m}_\bullet^2))$.

The conormal bundle $N(\mathfrak{m}_\bullet/\mathfrak{m}_\bullet^2)$ is the shifted dual of $T_{M/X}$, as described in Example 6.2.11. At the point-set level, the differential on $\mathrm{Sym}_{A_0}(N(\mathfrak{m}_\bullet/\mathfrak{m}_\bullet^2))$ therefore determines an L_∞-algebroid structure on $T_{M/X}$. The underlying complex and the Chevalley-Eilenberg complex of this L_∞-algebroid are weakly equivalent to the ones of the dg-Lie algebroid $T_{M/X}$ that classifies the formal moduli problem $\widehat{X}$ under Theorem 4.2.1.

At least when M is a point, the L_∞-algebra obtained by the above computation is equivalent to the one classifying the formal moduli problem $\widehat{X}$. This follows from a refinement of the above construction (in the setting of pro-Artin dg-algebras) due to Pridham [75]; this refinement is part of the equivalence between L_∞-algebras and formal moduli problems constructed in loc. cit.

7.1.3 Lie algebroids from smooth maps. Suppose that $M \longrightarrow X$ is a map from a derived manifold to a derived stack and that $M \longrightarrow Y$ is a map to a convergent, infinitesimally cohesive sheaf. Then there is a sequence of maps

$$\mathrm{Map}_{M/}(X^{\wedge}_{M}, Y) \longrightarrow \mathrm{Map}_{\mathrm{FormMod}_M}(\widehat{X}, \widehat{Y}) \longrightarrow \mathrm{Map}_{\mathrm{LieAlgd}_M}(T_{M/X}, T_{M/Y}).$$

Proposition 7.1.6 asserts that the first map is an equivalence when the map $M \longrightarrow X$ is locally finitely presented and Corollary 6.3.15 asserts that the second map is an equivalence when the structure sheaf $\mathcal{O}_M$ is locally eventually coconnective. In fact, when $M \longrightarrow X$ is *smooth*, the second map is *always* an equivalence:

Proposition 7.1.17. *Let $M \longrightarrow X$ be a smooth map from a derived manifold to a derived stack and let $\widehat{X}$ be the associated sheaf of formal moduli problems over M. Then the counit map from Corollary 6.3.15*

$$\mathrm{MC}_{T_{M/X}} \longrightarrow \widehat{X}$$

is an equivalence (even when $\mathcal{O}_M$ is unbounded).

Proof. We can work locally on M. In particular, we can assume that $M \longrightarrow X$ is modeled by a map of derived Lie n-groupoids $M \longrightarrow X_\bullet$. Since $M \longrightarrow X$ is smooth, the map $f\colon M \longrightarrow X_0$ decomposes as

$$M \xrightarrow{(\mathrm{id}, s_0(f))} M \times_{X(\{0\})} X_1 \xrightarrow{d_0} X(\{1\})$$

where the second map is smooth. Pulling back the derived Lie n-groupoid $X_\bullet$ along the second map d_0, we obtain a hypercover $X'_\bullet \longrightarrow X_\bullet$ with the property that the map $M \longrightarrow X'_\bullet$ is the section of a smooth map $X'_0 \longrightarrow M$ in degree 0. Replacing $X_\bullet$ by $X'_\bullet$, we may therefore assume that each map $M \longrightarrow X_n$ is the section of a smooth map $X_n \longrightarrow M$.

Since the $X_\bullet \longrightarrow X$ is a smooth hypercover, the Lie algebroid $T_{M/X}$ is the colimit of the Lie algebroids $T_{M/X_\bullet}$. We therefore obtain a commuting square of formal moduli problems

$$\begin{tikzcd}
\operatorname{colim} \mathrm{MC}_{T_{M/X_\bullet}} \arrow[r] \arrow[d, "\sim"'] & \operatorname{colim} \widehat{X}_\bullet \arrow[d, "\sim"] \\
\mathrm{MC}_{T_{M/X}} \arrow[r] & \widehat{X}.
\end{tikzcd}$$

It hence suffices to verify that the map $\mathrm{MC}_{T_{M/X}} \longrightarrow \widehat{X}$ is an equivalence when $M \longrightarrow X$ is the section of a smooth map between derived manifolds. Working locally, we can assume that the map $M \longrightarrow X$ is given by the map $M \times \{0\} \longrightarrow M \times \mathbb{R}^n$, where $M = \mathrm{Spec}(A)$ is affine.

Let us therefore consider the deformation functor $\widehat{X}$ associated to the map $M = \mathrm{Spec}(A) \longrightarrow \mathrm{Spec}(A\{t_1, \dots, t_n\}) = X$. By Example 4.2.23, the Lie algebroid $\mathfrak{t} := T_{A/\widehat{X}}$ is given by the $0\colon A[-1]^{\oplus n} \longrightarrow T_A$, together with the trivial Lie bracket. To see that the counit map $\epsilon\colon \widehat{X} \longrightarrow \mathrm{MC}_{\mathfrak{t}}$ is an equivalence, note that the functor

$$T_{A/}\colon \mathrm{FormMod}_A \longrightarrow \mathrm{LieAlgd}_A$$

always detects equivalences and preserves filtered colimits. The image of ϵ under $T_{A/}$ has a section, given by the unit map $\eta\colon \mathfrak{t} \longrightarrow T_{A/\mathrm{MC}_{\mathfrak{t}}}$. It therefore suffices to verify that this unit map is an equivalence.

Consider the class $\mathcal{K}$ of Lie algebroids $\mathfrak{g}$ over A for which $\mathfrak{g} \longrightarrow T_{A/\mathrm{MC}_{\mathfrak{g}}}$ is an equivalence. Because $T_{A/}$ preserves filtered colimits, $\mathcal{K}$ is closed under filtered colimits. When $\mathfrak{g}$ is a good Lie algebroid and $\mathrm{MC}_{\mathfrak{g}}$ is corepresentable by $C^*(\mathfrak{g})$, Example 4.2.23 identifies the unit map with the unit map $\mathfrak{g} \longrightarrow \mathfrak{D}C^*(\mathfrak{g})$. Proposition 4.1.26 shows that this map is an equivalence for every good Lie algebroid $\mathfrak{g}$ whose underlying A-module is perfect.

It therefore suffices to show that we can write $\mathfrak{t}$ as a filtered colimit of good A-linear Lie algebras whose underlying A-module is perfect. In fact, it suffices to prove this for $A = \mathbb{R}$: once we have found such a sequence of Lie algebras for $\mathbb{R}$, we can simply take the tensor product of the entire sequence with A. Over $\mathbb{R}$, consider the sequence of small $\mathcal{C}^\infty$-rings $B_k = \mathbb{R}[t_1, \dots, t_n]/(t_1^k, t_2^k, \dots, t_n^k)$ and the induced sequence of Lie algebras

$$\mathfrak{D}\big(B_2\big) \longrightarrow \mathfrak{D}(B_3) \longrightarrow \dots \longrightarrow \mathfrak{D}\big(\mathbb{R}\{t_1, \dots, t_n\}\big) = \mathfrak{t}. \tag{7.1.18}$$

Since $\mathbb{R}$ is eventually coconnective, the functor $\mathfrak{D}$ yields an equivalence between small $\mathcal{C}^\infty$-rings and good Lie algebras over $\mathbb{R}$ (cf. Remark 4.2.20). In particular, each $\mathfrak{D}(B_k)$ is a good Lie algebra over $\mathbb{R}$, whose underlying module is given by

$$L^\vee_{\mathbb{R}/B_k}[-1] \simeq \mathbb{R}[-1]^{\oplus n} \oplus \mathbb{R}[-2]^{\oplus n}.$$

One can easily compute that the maps $\mathfrak{D}(B_k) \longrightarrow \mathfrak{D}(B_{k+1})$ are the identity on the summand $\mathbb{R}[-1]^{\oplus n}$ and zero on the summand $\mathbb{R}[-2]^{\oplus n}$. It follows that the sequence (7.1.18) realizes $\mathfrak{t}$ as a filtered colimit of good Lie algebras whose underlying module is perfect, which concludes the proof. □

7.2 An integrability result

Let $p\colon M \longrightarrow X$ be a smooth map from a derived manifold to a derived stack with associated (connective) Lie algebroid $T_{M/X}$ over M. When $f\colon M \longrightarrow Y$ is another map to a derived stack (or any other sheaf with deformation theory) with associated Lie algebroid $T_{M/Y}$, there is a natural map

$$\mathrm{Map}_{M/}(X, Y) \longrightarrow \mathrm{Map}(T_{M/X}, T_{M/Y}).$$

The aim of this section is to prove the following:

Theorem 7.2.1. *Let $p\colon M \longrightarrow X$ be a smooth map from a smooth manifold to a smooth stack whose fibers are n-connected. If $f\colon M \longrightarrow Y$ is map to a derived m-stack, then the map*

$$\mathrm{Map}_{M/}(X, Y) \longrightarrow \mathrm{Map}_{\mathrm{LieAlgd}}(T_{M/X}, T_{M/Y})$$

is $(m - n - 2)$-truncated (in particular, an equivalence if $m = n$).

In fact, we will prove something slightly more general (Theorem 7.2.14).

The proof of the above result is essentially the same as the usual proofs of Van Est-type theorems and boils down to a simple descent argument, which is described in Section 7.2.2. The main content of Theorem 7.2.1 is contained in its *local* version, where the map $M \longrightarrow X$ is simply a projection map $U \times \mathbb{R}^n \longrightarrow U$ to an affine U. In that case, Theorem 7.2.1 reduces to the analytical assertion that any smooth bundle $P \longrightarrow U \times \mathbb{R}^n$ (where P is a higher stack) with a flat connection along $\mathbb{R}^n$ admits parallel transport. This is proven in Section 7.2.1, using the deformation theory of stacks discussed in Section 6.4.

7.2.1 Poincaré lemma. Recall that in the end, smooth maps between derived stacks are constructed out of projection maps $U \times \mathbb{R}^n \longrightarrow U$ between affine derived manifolds. We will therefore start by studying Theorem 7.2.1 in the special case where $M \longrightarrow X$ is such a projection map. In this case, the result can be thought of as a version of the Poincaré lemma.

Definition 7.2.2. Let $\mathcal{C}$ be a locally presentable ∞-category and let $F\colon \mathrm{Aff}^{\mathrm{op}} \longrightarrow \mathcal{C}$ be $\mathcal{C}$-valued sheaf. We will say that F satisfies the *Poincaré lemma* if the natural map

$$F(U) \longrightarrow F(U \times \mathbb{R}_{\mathrm{dR}})$$

is an equivalence for any affine U.

Remark 7.2.3. The class of $\mathcal{C}$-valued sheaves satisfying the Poincaré lemma is closed under all limits. Furthermore, if F satisfies the Poincaré lemma, then $F(X) \longrightarrow F(X \times \mathbb{R}_{\mathrm{dR}})$ is an equivalence for every object $X \in \mathrm{Sh}(\mathrm{Aff})$. In particular, the maps $F(U) \longrightarrow F(U \times \mathbb{R}^n_{\mathrm{dR}})$ are equivalences for all n.

Remark 7.2.4. Let $\pi\colon U \times \mathbb{R} \longrightarrow U$ be the projection map and consider the ($\mathcal{C}$-valued) sheaf

$$F^\pi\colon \mathrm{Op}(U \times \mathbb{R})^{\mathrm{op}} \longrightarrow \mathcal{C};\;\; V \longmapsto F(U^\wedge_V).$$

The map $F(U) \longrightarrow F(U \times \mathbb{R}_{\mathrm{dR}})$ arises as the global sections of the map of sheaves $\pi^{-1}F \longrightarrow F^\pi$ over $U \times \mathbb{R}$, where $F \in \mathrm{Sh}(U)$ is the restriction of F to the open subspaces of U. It therefore suffices to verify the Poincaré lemma locally, on a cover of $U \times \mathbb{R}$ by opens of the form $V \times (a, b)$.

Remark 7.2.5. Let $f\colon U \times \mathbb{R} \longrightarrow X$ be a map from an affine to a convergent, infinitesimally cohesive sheaf. Since the map $U \times \mathbb{R} \longrightarrow U$ is smooth, Proposition 7.1.6 and Proposition 7.1.17 imply that there is an equivalence

$$\mathrm{Map}_{U\times\mathbb{R}/}(U \times \mathbb{R}_{\mathrm{dR}}, X) \longrightarrow \mathrm{Map}_{\mathrm{LieAlgd}}(U \times T\mathbb{R}, T_{U\times\mathbb{R}/X}).$$

In other words, one can think of a map $U \times \mathbb{R}_{\mathrm{dR}} \longrightarrow X$ as a map $f\colon U \times \mathbb{R} \longrightarrow X$, together with a (flat) connection along the direction of $\mathbb{R}$. The Poincaré lemma asserts that any such map is locally constant along the direction of $\mathbb{R}$.

Example 7.2.6. Let $F = \mathcal{O} \in \mathrm{Sh}(\mathrm{Aff}, \mathrm{CAlg})$ be the sheaf sending an affine to its function algebra, considered as an (unbounded) commutative algebra. Then $\mathcal{O}$ satisfies the Poincaré lemma. Indeed, for every affine U, there is a sequence of maps

$$\mathcal{O}(U \times \mathbb{R}_{\mathrm{dR}}) \longrightarrow \mathcal{O}(\widehat{U}) \longrightarrow C^*(T_{U\times\mathbb{R}/U}) \longrightarrow \mathcal{O}(U).$$

The first map is a weak equivalence by Remark 7.1.15, since $U \times \mathbb{R} \longrightarrow U$ is finitely presented. The second map is a weak equivalence by Proposition 7.1.17. The commutative dg-algebra $C^*(T_{U\times\mathbb{R}/U})$ can be identified with the usual de fiberwise de Rham complex of $U \times \mathbb{R}$, so that the last map is an equivalence by the usual Poincaré lemma.

We will prove the following:

Proposition 7.2.7. *Any derived stack satisfies the Poincaré lemma.*

Example 7.2.8. The sheaf $\mathrm{Perf}\colon \mathrm{Aff}^{\mathrm{op}} \longrightarrow \mathrm{Cat}_\infty$ satisfies the Poincaré lemma. To see this, recall that $\mathrm{Perf} \simeq \mathrm{colim}_{n\longrightarrow\infty} \mathrm{Perf}^{[-n,n]}$ is the union of the sheaves of perfect complexes with Tor-amplitude contained in $[-n, n]$. A perfect complex over $V \times \mathbb{R}_{\mathrm{dR}}$ is a perfect complex over $V \times \mathbb{R}$, together with an action of the groupoid $V \times (\mathbb{R} \times \mathbb{R})^\wedge_{\mathbb{R}} \rightrightarrows V \times \mathbb{R}$.

In particular, locally on $V \times \mathbb{R}$ such a perfect complex has Tor-amplitude contained in some $[-n, n]$. It follows that there is an equivalence of sheaves over $V \times \mathbb{R}$ (as in Remark 7.2.4)

$$\big(\mathrm{Perf}^{[-n,n]}\big)^\pi \longrightarrow \mathrm{Perf}^\pi.$$

It therefore suffices to verify that each sheaf of ∞-categories $\mathrm{Perf}^{[-n,n]}$ satisfies the Poincaré lemma. This follows from the fact that each $\mathrm{Map}(\Delta[n], \mathrm{Perf}^{[-n,n]})$ is a derived stack (Lemma 5.2.36) and satisfies the Poincaré lemma by Proposition 7.2.7.

The proof of Proposition 7.2.7 requires some preliminary observations:

Lemma 7.2.9. *Consider a pullback diagram of sheaves*

$$\begin{tikzcd} P \arrow[r] \arrow[d, "q"'] & \tilde{P} \arrow[d, "\tilde{q}"] \\ U\times\mathbb{R} \arrow[r] & U\times\mathbb{R}_{\mathrm{dR}} \end{tikzcd}$$

where $q\colon P\longrightarrow U\times\mathbb{R}$ is a smooth surjection between affines. Every point in $U\times\mathbb{R}$ admits an open neighbourhood $V\times(t-\epsilon,t+\epsilon)$ over which $\tilde{q}$ admits a section

$$V\times(t-\epsilon,t+\epsilon)_{\mathrm{dR}}\longrightarrow\tilde{P}.$$

Proof. Since the assertion is local, we may shrink $U\times\mathbb{R}$ whenever necessary. Because the map $U\times\mathbb{R}\longrightarrow U\times\mathbb{R}_{\mathrm{dR}}$ is a surjective map of sheaves, the map $\tilde{q}$ is a 0-smooth surjection and the above pullback diagram of 0-representable maps is classified by a sequence

$$U\times\mathbb{R}\longrightarrow U\times\mathbb{R}_{\mathrm{dR}}\longrightarrow\mathrm{Stack}_0.$$

The Lie algebroid associated to the composite map is given by the Atiyah Lie algebroid associated to the map of complete $\mathcal{C}^\infty$-rings $\mathcal{O}(U\times\mathbb{R})\longrightarrow\mathcal{O}(P)$ (see Example 6.4.24). By Remark 7.2.5, the map $U\times\mathbb{R}_{\mathrm{dR}}\longrightarrow\mathrm{Stack}_0$ can equivalently be described as a map of Lie algebroids

$$U\times T\mathbb{R}\longrightarrow\mathrm{At}(\mathcal{O}(P)).$$

In other words, the map $\tilde{q}$ is classified by a fiberwise vector field v on P, which lifts the vector field $\partial/\partial t$ on $U\times\mathbb{R}$. When $Q\subseteq P$ is an open subspace of P, the restriction of this vector field v determines a deformation of $Q\longrightarrow U\times\mathbb{R}$ over $U\times\mathbb{R}_{\mathrm{dR}}$. To find a local section, we may therefore work locally and assume that the map q is a projection map $q\colon U\times\mathbb{R}\times\mathbb{R}^n\longrightarrow U\times\mathbb{R}$.

Let $\mathbb{R}\{x_\alpha\}$ be a cofibrant dg-$\mathcal{C}^\infty$-ring which models $\mathcal{O}(U)$, let t be the coordinate in the direction of $\mathbb{R}$ and let y_i be the standard coordinates on $\mathbb{R}^n$. Then the vector field v takes the form

$$v=\frac{\partial}{\partial t}+\sum_{i=1}^n a_i(x,t,y)\frac{\partial}{\partial y_i}$$

where the functions a_i depend only finitely many coordinates (of degree 0) $x_1,\dots,x_k$ on U. We can (locally) find a solution $s=(s_1,\dots,s_n)\colon U\times\mathbb{R}\longrightarrow\mathbb{R}^n$ of the differential equation

$$\frac{\partial s_i}{\partial t}(x,t)=a_i(x,t,s(x,t)).$$

We then obtain a section of the map q

$$\sigma=(\mathrm{id},s)\colon U\times\mathbb{R}\longrightarrow U\times\mathbb{R}\times\mathbb{R}^n$$

with the property that the vector field v is σ-related to the vector field $\partial/\partial t$ on $U\times\mathbb{R}$. At the level of $\mathcal{C}^\infty$-rings, this section provides a retraction of the map $\mathcal{O}(U\times\mathbb{R})\longrightarrow\mathcal{O}(P)$ which intertwines the derivation v and the derivation $\partial/\partial t$. In other words, it determines commuting triangle of dg-Lie algebroids

$$\begin{tikzcd} & \mathrm{At}\big(\mathcal{O}(P)\xrightarrow{\sigma^*}\mathcal{O}(U\times\mathbb{R})\big) \arrow[d] \\ U\times T\mathbb{R} \arrow[ur, dotted, "\tilde{v}"] \arrow[r, "v"'] & \mathrm{At}(\mathcal{O}(P)). \end{tikzcd}$$

The top dg-Lie algebroid is the Atiyah Lie algebroid of the retract diagram $\mathcal{O}(U \times \mathbb{R}) \longrightarrow \mathcal{O}(P) \longrightarrow \mathcal{O}(U \times \mathbb{R})$ (Definition 3.1.3).

By Theorem 4.4.1 and Proposition 7.1.17, the lift $\tilde{v}$ gives rise to a deformation of the retract diagram of stacks $U \times \mathbb{R} \longrightarrow P \longrightarrow U \times \mathbb{R}$ to a retract diagram of sheaves

$$U \times \mathbb{R}_{\mathrm{dR}} \longrightarrow \tilde{P} \longrightarrow U \times \mathbb{R}_{\mathrm{dR}}.$$

and the result follows. □

Corollary 7.2.10. *Let $p\colon Y \longrightarrow X$ be a smooth surjection between derived stacks, let U be an affine derived manifold and let $Y^\pi \longrightarrow X^\pi$ be the induced map of sheaves over $U \times \mathbb{R}$, as in Remark 7.2.4. Then the map $Y^\pi \longrightarrow X^\pi$ is a surjection of sheaves.*

Proof. $U \times \mathbb{R}$ has a basis of opens of the form $W \times (a,b)$, for which $X^\pi(W \times (a,b)) \simeq X(W \times (a,b)_{\mathrm{dR}})$. Given a map $U \times \mathbb{R}_{\mathrm{dR}} \longrightarrow X$, it therefore suffices to find local lifts

$$\begin{tikzcd}
W \times (a,b)_{\mathrm{dR}} \arrow[r, dotted] \arrow[d] & Y \arrow[d, "p"] \\
U \times \mathbb{R}_{\mathrm{dR}} \arrow[r] & X
\end{tikzcd}$$

around any point of $U \times \mathbb{R}$. To prove that such a lift exists, we use the deformation theory of stacks discussed in Section 6.4. Consider the composite pullback diagram

$$\begin{tikzcd}
P \arrow[r] \arrow[d, "q"'] & \tilde{P} \arrow[r] \arrow[d] & Y \arrow[d, "p"] \\
U \times \mathbb{R} \arrow[r] & U \times \mathbb{R}_{\mathrm{dR}} \arrow[r] & X.
\end{tikzcd}$$

It suffices to find a local section of the map $\tilde{P} \longrightarrow U \times \mathbb{R}_{\mathrm{dR}}$. The left pullback square is classified by a sequence of maps

$$U \times \mathbb{R} \longrightarrow U \times \mathbb{R}_{\mathrm{dR}} \longrightarrow \mathrm{Stack}.$$

The composite map gives rise to a Lie algebroid $\mathfrak{g}$ over $U \times \mathbb{R}$, whose anchor map is described in Proposition 6.4.19. By Remark 7.2.5, the map $U \times \mathbb{R}_{\mathrm{dR}} \longrightarrow \mathrm{Stack}$ is classified by a map of Lie algebroids

$$U \times T\mathbb{R} \longrightarrow \mathfrak{g}.$$

Let $P_0 \longrightarrow P$ be an affine atlas for the derived stack P, so that the composite $P_0 \longrightarrow P \longrightarrow V \times \mathbb{R}$ is a smooth surjection between affines. There is a map of formal moduli problems (see Section 6.4.3)

$$\mathrm{Def}_{P_0 \longrightarrow P/U \times \mathbb{R}} \longrightarrow \mathrm{Def}_{P/U \times \mathbb{R}}.$$

The induced map of Lie algebroids $\tilde{\mathfrak{g}} \longrightarrow \mathfrak{g}$ over $U \times \mathbb{R}$ has connective fibers, by Corollary 6.4.27. Since the Lie algebroid $U \times T\mathbb{R}$ is the *free* Lie algebroid on a single generator $\partial/\partial t$, there exists a lift of the map $U \times T\mathbb{R} \longrightarrow \mathfrak{g}$ to a map $U \times T\mathbb{R} \longrightarrow \tilde{g}$. By Proposition 7.1.17, this map of Lie algebroids classifies a diagram of pullback squares of the form

$$\begin{tikzcd}
P_0 \arrow[r] \arrow[d] & \tilde{P}_0 \arrow[d] & \\
P \arrow[r] \arrow[d] & \tilde{P} \arrow[r] \arrow[d] & Y \arrow[d] \\
U \times \mathbb{R} \arrow[r] & U \times \mathbb{R}_{\mathrm{dR}} \arrow[r] & X.
\end{tikzcd}$$

It suffices to find a local section of the map $\tilde{P}_0 \longrightarrow U \times \mathbb{R}_{\mathrm{dR}}$. Such a local section is provided by Lemma 7.2.9. □

Proof (of Proposition 7.2.7). We have to verify that for any derived n-stack X and any derived manifold U, the map $Y(U) \longrightarrow Y(U \times \mathbb{R}_{\mathrm{dR}})$ is an equivalence. We proceed by induction on n.

For the case where X is affine, note that $\mathcal{C}^\infty\mathrm{Alg}$ is generated under colimits by $\mathcal{C}^\infty(\mathbb{R})$ and that the functor $\mathrm{Spec}\colon \mathcal{C}^\infty\mathrm{Alg}^{\mathrm{op}} \longrightarrow \mathrm{Sh}(\mathrm{Aff})$ preserves all limits. It therefore suffices to treat the case where X is simply $\mathbb{R}$. For any sheaf Z, the space $\mathrm{Map}(Z, \mathbb{R})$ is the underlying space of the connective cover of $\mathcal{O}(Z)$. The result then follows from Example 7.2.6.

For the inductive step, suppose we have proven the result for derived $(n-1)$-stacks and let X be a derived n-stack. In light of Remark 7.2.4, it suffices to show that for any derived manifold $U \times \mathbb{R}$, the map of sheaves $\pi^{-1}X \longrightarrow X^\pi$ over $U \times \mathbb{R}$ is an equivalence. To see this, let $X_0 \longrightarrow X$ be an atlas for X and let $X_\bullet \longrightarrow X$ be the associated Čech nerve. We obtain a natural transformation of augmented simplicial sheaves over $U \times \mathbb{R}$

$$\begin{tikzcd} \pi^{-1}X_\bullet \arrow[r] \arrow[d] & X_\bullet^\pi \arrow[d] \\ \pi^{-1}X \arrow[r] & X^\pi. \end{tikzcd}$$

The map $\pi^{-1}X_\bullet \longrightarrow \pi^{-1}X$ is a hypercover since π^{-1} is a left exact left adjoint and the map $X_\bullet^\pi \longrightarrow X^\pi$ is a hypercover by Corollary 7.2.10. Since $X_\bullet$ is a diagram of derived $(n-1)$-stacks, each map $\pi^{-1}X_n \longrightarrow X_n^\pi$ is an equivalence of sheaves. It follows that the map $\pi^{-1}X \longrightarrow X^\pi$ is an equivalence of sheaves as well. □

7.2.2 Descent. Recall that any smooth map $p\colon Y \longrightarrow X$ between derived stacks decomposes as

$$Y \xrightarrow{\tilde{p}} \mathrm{Sing}_{\leq n}(Y/X) \xrightarrow{p'} X$$

where $\mathrm{Sing}_{\leq n}(Y/X)$ is the 'fiberwise n-connective cover' of the map p (for some $n \leq \infty$). The étale map p' induces an equivalence upon completion at Y, so that the canonical map $X^\wedge_Y \longrightarrow X$ decomposes as

$$X^\wedge_Y \longrightarrow \mathrm{Sing}_{\leq n}(Y/X) \longrightarrow X.$$

Using a simple descent argument, one obtains the following global analogue of Proposition 7.2.7:

Proposition 7.2.11. *Let $F\colon \mathrm{Aff}^{\mathrm{op}} \longrightarrow \mathcal{C}$ be a sheaf satisfying the Poincaré lemma and let $p\colon Y \longrightarrow X$ be a smooth map between sheaves on* Aff*. Then the map*

$$F(\mathrm{Sing}(Y/X)) \longrightarrow F(X^\wedge_Y)$$

is an equivalence.

Proof. Let $f\colon \widetilde{X} \longrightarrow X$ be a map and $\tilde{p}\colon \widetilde{Y} = \widetilde{X} \times_X Y \longrightarrow \widetilde{X}$ be the base change of p. Then $\tilde{p}$ is smooth and the diagram of sheaves

$$\begin{tikzcd} \widetilde{Y} \arrow[r] \arrow[d] & \widetilde{X}^\wedge_{\widetilde{Y}} \arrow[r] \arrow[d] & \mathrm{Sing}(\widetilde{Y}/\widetilde{X}) \arrow[r] \arrow[d] & \widetilde{X} \arrow[d, "f"] \\ Y \arrow[r] & X^\wedge_Y \arrow[r] & \mathrm{Sing}(Y/X) \arrow[r] & X \end{tikzcd}$$

consists of pullback squares. Indeed, $\widetilde{X}^{\wedge}_{\widetilde{Y}}$ is the pullback of $X^{\wedge}_{Y}$ because taking de Rham spaces preserves limits. To see that $\mathrm{Sing}(\widetilde{Y}/\widetilde{X})$ is the pullback of $\mathrm{Sing}(Y/X)$, recall from Section 5.3 that $\mathrm{Sing}(Y/X)$ was defined as the étale space over X associated to the sheaf $p_!(*)$ over X. We then have that

$$\mathrm{Sing}(\widetilde{Y}/\widetilde{X}) = \mathrm{et}_!\tilde{p}_!(*) \simeq \mathrm{et}_!\big(f^{-1}p_!(*)\big) \simeq f^*\big(\mathrm{et}_!p_!(*)\big) = f^*(\mathrm{Sing}(Y/X))$$

where the first equivalence follows from smooth base change (Theorem 5.3.19) and the second equivalence follows from the fact that taking étale spaces commutes with pulling back sheaves (see Lemma 5.3.3).

Using this, one sees that the assertion is local on X, so that we may assume that X is a derived manifold. In this case, we can model the map p by a map of derived Lie n-groupoids $p_\bullet\colon Y_\bullet \longrightarrow X$ which is smooth in each simplicial degree. For each n, there is a ($\mathcal{C}$-valued) sheaf over Y_n

$$F^{p_n}\colon \mathrm{Op}(Y_n)^{\mathrm{op}} \longrightarrow \mathcal{C};\;\; U \longmapsto F(X^{\wedge}_{U}).$$

For each open $V \subseteq X$, there is a natural map $F(V) \longrightarrow F^{p_n}(p_n^{-1}(V))$. By adjunction, this determines a map of sheaves over Y_n

$$\psi_n\colon p_n^{-1}\big(\mathrm{et}^*F\big) \longrightarrow F^{p_n}$$

where et^*F is the restriction of F to a sheaf on X (see Lemma 5.3.3). By Proposition 7.2.7 (see Remark 7.2.4), ψ_n is an equivalence when restricted to an open of the form $V \times \mathbb{R}^n$, where V is an open subspace of X. It follows that each ψ_n is an equivalence of sheaves, so that the natural map

$$\mathrm{holim}_{\mathbf{\Delta}}\, \Gamma(Y_n, p_n^{-1}\mathrm{et}^*F) \longrightarrow \mathrm{holim}_{\mathbf{\Delta}}\, F(X^{\wedge}_{Y_n}) \simeq F(X^{\wedge}_{Y})$$

is an equivalence. Note that the matching family of sheaves $p_n^{-1}\mathrm{et}^*F$ can be identified with the inverse image sheaf

$$p^{-1}\mathrm{et}^*F \in \mathrm{Sh}(Y) \simeq \lim_{\mathbf{\Delta}} \mathrm{Sh}(Y_\bullet).$$

We therefore find that the natural map $\Gamma(Y, p^{-1}\mathrm{et}^*F) \longrightarrow F(X^{\wedge}_{Y})$ is an equivalence. Since $p\colon Y \longrightarrow X$ is smooth, Lemma 5.3.18 provides a sequence of equivalences

$$\Gamma\big(Y, p^{-1}\mathrm{et}^*F\big) \simeq \mathrm{Map}_{\mathrm{Sh}(X)}\big(p_!(*), \mathrm{et}^*F\big) \simeq F\big(\mathrm{et}_!p_!(*)\big).$$

But the sheaf $\mathrm{et}_!p_!(*)$ was exactly $\mathrm{Sing}(Y/X)$, and one therefore finds that the natural map $F(\mathrm{Sing}(Y/X)) \longrightarrow F(X^{\wedge}_{Y})$ is an equivalence. □

Corollary 7.2.12. *Let $p\colon M \longrightarrow X$ be a smooth map to a derived stack and let $f\colon M \longrightarrow Y$ be a map to a derived stack. Then there is an equivalence*

$$\mathrm{Map}_{M/}(\mathrm{Sing}(M/X), Y) \xrightarrow{\sim} \mathrm{Map}_{\mathrm{LieAlgd}_M}(T_{M/X}, T_{M/Y}) \tag{7.2.13}$$

sending each map $\mathrm{Sing}(M/X) \longrightarrow Y$ to the induced map of Lie algebroids over M. In particular, if the fibers of p are ∞-connected, then any map of Lie algebroids $T_{M/X} \longrightarrow T_{M/Y}$ integrates to a map of stacks $X \longrightarrow Y$ under M.

Proof. The map (7.2.13) is obtained as the composition

$$\mathrm{Map}_{M/}(\mathrm{Sing}(M/X), Y) \longrightarrow \mathrm{Map}_{M/}(X^{\wedge}_{M}, Y) \longrightarrow \mathrm{Map}(T_{M/X}, T_{M/Y}).$$

The first map restricts along $X^{\wedge}_{M} \longrightarrow \mathrm{Sing}(M/X)$ and is an equivalence by Proposition 7.2.11, since Y satisfies the Poincaré lemma by Proposition 7.2.7. The last map is an equivalence by Proposition 7.1.6 and Proposition 7.1.17. □

Theorem 7.2.14. *Let M be a derived manifold and let s be a number so that the following holds: there exists an open cover of M by affines U such that for all $q \geq 0$, $\mathcal{O}(U \times \mathbb{R}^q)$ is s-coconnective, i.e. $\pi_i(\mathcal{O}(U \times \mathbb{R}^q)) = 0$ for all $i > s$.*

Let $p\colon M \longrightarrow X$ be a smooth map to a derived stack, whose fibers are m-connected and let $f\colon M \longrightarrow Y$ be a map to a derived n-stack. Then the map

$$\mathrm{Map}_{M/}(X, Y) \longrightarrow \mathrm{Map}_{\mathrm{LieAlgd}_M}(T_{M/X}, T_{M/Y})$$

is an $(s + n - m - 2)$-truncated map of spaces.

Example 7.2.15. When M is a smooth manifold, one can take $s = 0$. If M is the derived zero locus of a function $f\colon \mathbb{R}^p \longrightarrow \mathbb{R}^q$, then one can take $s = q$.

Proof. By Corollary 7.2.12, it suffices to show that the map

$$\mathrm{Map}(X, Y) \longrightarrow \mathrm{Map}(\mathrm{Sing}(M/X), Y) \tag{7.2.16}$$

is $(s + n - m - 2)$-truncated.

To see this, let $X_\bullet \longrightarrow X$ be a derived Lie n-groupoid modeling X and let

$$p_i\colon F_i = X_i \times_X M \longrightarrow X_i$$

be the base change of p, whose fibers are m-connected. The base change of the map $\mathrm{Sing}(M/X) \longrightarrow X$ can be identified with the map $\mathrm{Sing}(F_i/X_i) \longrightarrow X_i$.

The derived n-stack Y restricts to a sheaf et^*Y on each derived manifold X_i. The condition on M guarantees that for any smooth map $V \longrightarrow M$, the sheaf $\mathcal{O}_V$ is locally s-coconnective. In particular, one can choose $X_\bullet$ such that each sheaf $\mathcal{O}_{X_i}$ is locally s-coconnective. In that case, the sheaf et^*Y is $(s + n)$-truncated, because Y is a derived n-stack.

The map $\mathrm{Map}(X_i, Y) \longrightarrow \mathrm{Map}(\mathrm{Sing}(F_i/X_i), Y)$ arises as the global sections of a map of sheaves over X_i

$$\phi\colon \mathrm{et}^*Y \simeq \mathrm{Hom}(*, \mathrm{et}^*Y) \longrightarrow \mathrm{Hom}((p_i)_!(*), \mathrm{et}^*Y).$$

The above map of sheaves induces a map of stalks at $x \in X_i$ (see also Lemma 5.3.18)

$$\phi_x\colon (\mathrm{et}^*Y)_x \longrightarrow \mathrm{Map}\Big(\mathrm{Sing}(p_i^{-1}(x)), (\mathrm{et}^*Y)_x\Big).$$

Since each fiber $p_i^{-1}(x)$ is m-connected and the stalk $(\mathrm{et}^*Y)_x$ is $(s + n)$-truncated, the map of spaces ϕ_x is $(s + n - m - 2)$-truncated. It follows that the map of sheaves ϕ is $(s+n-m-2)$-truncated as well, so that the induced map on global sections $\mathrm{Map}(X_i, Y) \longrightarrow \mathrm{Map}(\mathrm{Sing}(F_i/X_i), Y)$ is $(s + n - m - 2)$-truncated. Taking the limit over $\mathbf{\Delta}$, one finds that the map (7.2.16) is $(s + n - m - 2)$-truncated as well. □

7.3 Examples and applications

We have seen that Lie algebroids over a (derived) manifold M arise naturally from maps to (derived) stacks. However, the construction of these Lie algebroids is rather abstract and passes through the equivalence between Lie algebroids and formal moduli problems from Theorem 4.2.1. In particular, we do not know of a point-set description of the Lie algebroid associated to a stack, even if this stack is explicitly presented by a (derived) higher Lie groupoid.

The purpose of this section is to sketch how one may nonetheless study geometric structures (such as representations and differential forms) on such Lie algebroids in terms of descent data. This is based on the results of Section 7.1: for many purposes, studying the Lie algebroid of a map $M \longrightarrow X$ is equivalent to studying the formal completion $X^{\wedge}_M$. The latter can be decomposed into local pieces, which themselves can be described in terms of more elementary Lie algebroids.

7.3.1 Representations. Let A be a complete $\mathcal{C}^\infty$-ring and let $\mathfrak{g}$ be a Lie algebroid over A. Let $\mathrm{Perf}(\mathfrak{g}) \subseteq \mathrm{Rep}_{\mathfrak{g}}$ be the full subcategory of $\mathfrak{g}$-representations whose underlying module is contained in $\mathrm{Perf}(\mathrm{Spec}(A))$. A map from an affine to a derived stack $f\colon M = \mathrm{Spec}(A) \longrightarrow X$ gives rise to a sequence of functors

$$\mathrm{Perf}(X) \longrightarrow \mathrm{Perf}(X^{\wedge}_M) \longrightarrow \mathrm{Perf}(\widehat{X}) \longrightarrow \mathrm{Perf}(T_{M/X}) \tag{7.3.1}$$

where the first functor simply restricts along the inclusion $X^{\wedge}_M \longrightarrow X$. The second functor sends a perfect complex over $X^{\wedge}_M$ to the associated quasi-coherent module over the formal moduli problem $\widehat{X}$. The last functor is the functor Ψ_X from Theorem 4.3.1. The resulting $T_{M/X}$-representation associated to $E \in \mathrm{Perf}(X)$ is given as an $\mathcal{O}(M)$-module by f^*E.

Theorem 7.2.1 and Example 7.2.8 imply the following:

Corollary 7.3.2. *Let $p\colon M \longrightarrow X$ be a smooth map from a smooth manifold to a stack. For any $-\infty \leq a \leq b \leq \infty$, the functor of $(b-a+1)$-categories*

$$p^*\colon \mathrm{Perf}^{[a,b]}(X) \longrightarrow \mathrm{Perf}^{[a,b]}(T_{M/X}). \tag{7.3.3}$$

is an equivalence when p has $(b-a+1)$-connected fibers. If p has n-connected fibers, then p^* induces $(b-a-n-2)$-truncated maps of mapping spaces.

Remark 7.3.4. If $p\colon M \longrightarrow *$ is a smooth manifold, then the fiberwise ∞-connective cover $\mathrm{Sing}(M/*)$ can be identified with the constant sheaf on the singular complex of M. Using Example 7.2.8 and Proposition 7.2.11, one obtains an equivalence of ∞-categories

$$\mathrm{Perf}\big(\mathrm{Sing}(M)\big) \xrightarrow{\simeq} \mathrm{Perf}(T_M)$$

between the ∞-category of local systems of perfect complexes over M and the ∞-category of perfect T_M-representations. For an alternative treatment of this equivalence, which explicitly associates a local system over M to a representation of T_M, using iterated integrals, see [11].

Example 7.3.5. The functor p^* sends $\mathcal{O}_X \in \mathrm{Perf}(X)$ to the canonical representation of $T_{M/X}$ on $\mathcal{O}_M$, via the anchor map. At the level of mapping objects, the functor (7.3.3) therefore induces a map

$$\Gamma(X,E) = \mathrm{Map}_{\mathcal{O}_X}(\mathcal{O}_X, E) \longrightarrow \mathrm{Map}_{\mathcal{U}(T_{M/X})}(\mathcal{O}_M, p^*E) \simeq C^*(T_{M/X}, p^*E)$$

from the global sections of E over X to the Lie algebroid cohomology of $T_{M/X}$ with coefficients in p^*E.

The composite functor (7.3.1) is not particularly explicit, especially if we do not have a concrete description of the Lie algebroid $T_{M/X}$. One can use descent data to give an alternative description of (7.3.1), which avoids the Lie algebroid $T_{M/X}$. For simplicity, we will only treat the case where X is the colimit of a (smooth) Lie n-groupoid $X_\bullet$ and where $f\colon X_0 \longrightarrow X$ is the canonical map.

Construction 7.3.6. Let $X_\bullet$ be a (smooth) Lie n-groupoid with associated quotient stack X. The canonical atlas $X_0 \longrightarrow X$ arises as the colimit of the Kan fibration of simplicial manifolds

$$d_0 \colon X_{1+\bullet} = \mathrm{Dec}_0(X_\bullet) \longrightarrow X_\bullet$$

which is a degreewise surjective submersion. Let $\mathcal{U}$ denote the category whose

- objects are pairs $([n], U_n)$, which we will often abbreviate to U_n, where $[n] \in \mathbf{\Delta}_{\mathrm{inj}}$ and $U_n \subseteq X_{1+n}$ is an open subspace.
- morphisms $([n], U_n) \longrightarrow ([m], V_m)$ are given by a map $\alpha\colon [m] \longrightarrow [n]$ in $\mathbf{\Delta}_{\mathrm{inj}}$ such that $\alpha^*(U_n) \subseteq V_m$.

The category $\mathcal{U}$ carries a topology whose open covers are the maps $([n], U_{n,\alpha}) \longrightarrow ([n], U_n)$ where $U_{n,\alpha} \longrightarrow U_n$ is an open cover in X_{1+n}.

For each U_n in $\mathcal{U}$, let T_{U_n/X_n} be the Lie algebroid over U_n that describes the foliation on U_n by the fibers of the submersion

$$d_0 \colon U_n \subseteq X_{1+n} \longrightarrow X_n.$$

Let $f\colon U_n \longrightarrow V_m$ be a map in $\mathcal{U}$, determined by a map $\alpha\colon [m] \longrightarrow [n]$ in $\mathbf{\Delta}_{\mathrm{inj}}$. This fits into a commuting square

$$\begin{tikzcd} U_n \arrow[r, "f"] \arrow[d, "d_0"'] & V_m \arrow[d, "d_0"] \\ d_0(U_n) \arrow[r, "\alpha^*"'] & d_0(V_m) \end{tikzcd}$$

where $d_0(U_n)$ is the image of $U_n \subseteq X_{1+n}$ in X_n. Restriction these maps determines a commuting diagram of functors

$$\begin{tikzcd} \mathrm{Rep}^{\mathrm{dg}}_{\mathcal{O}(d_0(V_m))} \arrow[r, "d_0^*"] \arrow[d, "(\alpha^*)^*"'] & \mathrm{Rep}^{\mathrm{dg}}_{T_{V_m/X_m}} \arrow[r, "\text{forget}"] \arrow[d, "f^*"'] & \mathrm{Rep}^{\mathrm{dg}}_{T_{U_n/X_n}} \arrow[d, "f^*"] \\ \mathrm{Rep}^{\mathrm{dg}}_{\mathcal{O}(d_0(U_n))} \arrow[r, "d_0^*"'] & \mathrm{Mod}^{\mathrm{dg}}_{\mathcal{O}(V_m)} \arrow[r, "\text{forget}"'] & \mathrm{Mod}^{\mathrm{dg}}_{\mathcal{O}(U_n)} \end{tikzcd} \tag{7.3.7}$$

where f^* takes the tensor product over $\mathcal{O}(V_m) \longrightarrow \mathcal{O}(U_n)$, and similarly for d_0^* and $(\alpha^*)^*$. The functors with values in Lie algebroid representations are defined as follows:

- If E is a dg-module over $\mathcal{O}(d_0(U_n))$, then,

  $$d_0^* E = \mathcal{O}(U_n) \otimes_{\mathcal{O}(d_0(U_n))} E$$

  carries a T_{U_n/X_n}-representation given by $\nabla_v(a \otimes e) = v(a) \otimes e$.

- Note that there is a natural $\mathcal{O}(U_n)$-linear map

  $$T_{U_n/X_n} \longrightarrow f^* T_{V_m/X_m} = \mathcal{O}(U_n) \otimes_{\mathcal{O}(V_m)} T_{V_m/X_m}. \tag{7.3.8}$$

  Geometrically, this is simply the map that applies the derivative of $f\colon U_n \longrightarrow V_m$. Algebraically, this map restricts a $\mathcal{C}^\infty$-derivation $v\colon \mathcal{O}(U_n) \longrightarrow \mathcal{O}(U_n)$ over $\mathcal{O}(d_0(U_n))$ to a derivation in

  $$\mathrm{Der}_{\mathcal{O}(d_0(V_m))}\big(\mathcal{O}(V_m), \mathcal{O}(U_n)\big) \cong \mathcal{O}(U_n) \otimes_{\mathcal{O}(V_m)} \mathrm{Der}_{\mathcal{O}(d_0(V_m))}(\mathcal{O}(V_m)).$$

  If E is a representation of T_{V_m/X_m}, then $f^*E = \mathcal{O}(U_n) \otimes_{\mathcal{O}(V_m)} E$ carries a natural representation of T_{U_n/X_n}: if $b \otimes w$ denotes the image of v under (7.3.8), then v acts by

  $$\nabla_v(a \otimes e) = v(a) \otimes e + ab \otimes \nabla_w(e).$$

Lemma 7.3.9. *Let $X_\bullet$ be a (smooth) Lie n-groupoid. There is a sequence of functors $\mathfrak{U}^{\mathrm{op}} \longrightarrow \mathrm{Cat}_\infty$*

$$\mathrm{Perf}(d_0(-)) \xrightarrow{d_0^*} \mathrm{Perf}(T_{(-)/X_\bullet}) \xrightarrow{\text{forget}} \mathrm{Perf}(-)$$

where the middle functor sends $([n], U_n)$ to the ∞-category of perfect representations of T_{U_n/X_n}.

Proof. The commuting diagram (7.3.7) provides a sequence of diagrams of model categories $\mathfrak{U}^{\mathrm{op}} \longrightarrow \mathrm{ModCat}^{\mathrm{L}}$

$$\mathrm{Mod}^{\mathrm{dg}}_{\mathcal{O}(d_0(-))} \xrightarrow{d_0^*} \mathrm{Rep}^{\mathrm{dg}}_{T_{(-)/X_\bullet}} \xrightarrow{\text{forget}} \mathrm{Mod}^{\mathrm{dg}}_{\mathcal{O}(-)}.$$

The categories of dg-modules are endowed with the projective model structure and the categories of dg-representations are endowed with the A-model structure from Variant 3.3.9. After inverting the quasi-isomorphisms, this yields a $\mathfrak{U}^{\mathrm{op}}$-indexed diagram of ∞-categories of modules and Lie algebroid representations. Since all functors involved are given by restriction and therefore preserve perfect complexes, the result follows. $\square$

The map $d_0^*\colon \mathrm{Perf}(d_0(-)) \longrightarrow \mathrm{Perf}(T_{(-)/X_\bullet})$ is a local equivalence between $\mathcal{U}^{\mathrm{op}}$-diagrams of ∞-categories. Indeed, when evaluated an open neighbourhood $U_n \subseteq X_{1+n}$ of the form $U_n \simeq d_0(U_n) \times \mathbb{R}^n$, the functor d_0^* is an equivalence by Corollary 7.3.2. On the other hand, the functor $\mathrm{Perf}(T_{(-)/X_\bullet})$ is a sheaf, since Lie algebroid representations can be glued at the level of the underlying modules. In other words, $\mathrm{Perf}(T_{(-)/X_\bullet})$ is the associated sheaf of $\mathrm{Perf}(d_0(-))$ over $\mathcal{U}$.

On the other hand, there is a natural sequence of functors $\mathfrak{U}^{\mathrm{op}} \longrightarrow \mathrm{Cat}_\infty$

$$\mathrm{Perf}(d_0(-)) \xrightarrow{d_0^*} \mathrm{Perf}\big((X_\bullet)^\wedge_{(-)}\big) \longrightarrow \mathrm{Perf}(-)$$

where the middle functor sends an open $U_n \subseteq X_{1+n}$ to the ∞-category of perfect complexes on the formal completion $(X_n)^\wedge_{U_n}$ of X_n along $d_0\colon U_n \subseteq X_{1+n} \longrightarrow X_n$. By the Poincaré lemma (Example 7.2.8), the map d_0^* is a local equivalence. On the other hand, the functor $\mathrm{Perf}\big((X_\bullet)^\wedge_{(-)}\big)$ is a sheaf because for any covering sieve S of U_n, there is an equivalence

$$\operatorname*{colim}_{U_{n,\alpha}\in S} (X_n)^\wedge_{U_{n,\alpha}} \simeq (X_n)^\wedge_{U_n}.$$

It follows that $\mathrm{Perf}\big((X_\bullet)^\wedge_{(-)}\big)$ is the associated sheaf of $\mathrm{Perf}(d_0(-))$ as well. We therefore obtain the following:

Proposition 7.3.10. *Let $X_\bullet$ be a Lie n-groupoid and let $p\colon M = X_0 \longrightarrow X$ be the canonical atlas. Then the functor $p^*\colon \mathrm{Perf}(X) \longrightarrow \mathrm{Perf}(T_{M/X})$ can be modeled by the limit of the cosimplicial diagram of functors*

$$\mathrm{Perf}(X_\bullet) \xrightarrow{d_0^*} \mathrm{Perf}(T_{X_{\bullet+1}/X_\bullet}).$$

Proof. The previous discussion shows that the natural transformations

$$d_0^*\colon \mathrm{Perf}(X_\bullet) \longrightarrow \mathrm{Perf}(T_{X_{\bullet+1}/X_\bullet}) \qquad \text{and} \qquad d_0^*\colon \mathrm{Perf}(X_\bullet) \longrightarrow \mathrm{Perf}((X_\bullet)^\wedge_{X_{1+\bullet}})$$

are naturally equivalent. By Example 7.1.5, the map $d_0\colon (X_{1+\bullet})_{\mathrm{dR}} \longrightarrow (X_\bullet)_{\mathrm{dR}}$ is a Kan fibration between n-groupoids in Sh(Aff), so that

$$\operatorname*{colim}_{\mathbf{\Delta}^{\mathrm{op}}} (X_\bullet)^\wedge_{X_{\bullet+1}} = \operatorname*{colim}_{\mathbf{\Delta}^{\mathrm{op}}} \Big(X_\bullet \times_{(X_\bullet)_{\mathrm{dR}}} (X_{1+\bullet})_{\mathrm{dR}}\Big) \simeq X \times_{X_{\mathrm{dR}}} M_{\mathrm{dR}} = X^\wedge_M.$$

It follows that the limit of $d_0^*\colon \mathrm{Perf}(X_\bullet) \longrightarrow \mathrm{Perf}((X_\bullet)^\wedge_{X_{1+\bullet}})$ can be identified with the canonical restriction functor $p^*\colon \mathrm{Perf}(X) \longrightarrow \mathrm{Perf}(X^\wedge_M) \simeq \mathrm{Perf}(T_{M/X})$ of Corollary 7.3.2. $\square$

In other words, when $X_\bullet$ is a Lie n-groupoid, then $\mathrm{Perf}(X) \longrightarrow \mathrm{Perf}(T_{M/X})$ can be identified with the functor sending a matching family $E_n \in \mathrm{Perf}(X_n)$ to the matching family $d_0^*E_n \in \mathrm{Perf}(T_{X_{1+n}/X_n})$ of representations of the tangent bundle to the fibers of d_0.

Example 7.3.11. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and let $X_\bullet$ be its nerve. Unwinding the definitions, an object in

$$\lim\Big(\mathrm{Perf}(T_{X_{1+n}/X_n})\Big)$$

is given by the datum of a quasi-coherent sheaf E on $\mathcal{G}$ with a flat connection along the fibers of the target map $t = d_0 \colon \mathcal{G} \longrightarrow M$, together with (coherent) equivariance data for the principal right action of $\mathcal{G}$ on itself.

When E is just an ordinary $\mathcal{G}$-equivariant vector bundle on $\mathcal{G}$, such an equivariant flat connection on E is determined uniquely by a representation of the $\mathcal{G}$-invariant vector fields on the $\mathcal{G}$-equivariant sections of E. These equivariant sections can be identified with the sections of the restriction of E along the unit map $M \longrightarrow \mathcal{G}$. We therefore retrieve the usual notion of a representation of the Lie algebroid of $\mathcal{G}$ on a vector bundle $E\big|M$.

Example 7.3.12. Let $X_\bullet$ be a (smooth) Lie n-groupoid and let $p\colon M = X_0 \longrightarrow X$ be the canonical atlas. Suppose that $E \in \mathrm{Perf}(X)$ is modeled by a family E_n of cofibrant dg-modules over $\mathcal{O}(X_n)$, together with a coherent family of local quasi-isomorphisms $(\alpha^*)^*E_n \xrightarrow{\sim} E_m$ for any injection $\alpha\colon [n] \longrightarrow [m]$. The family of dg-representations

$$d_0^*E_\bullet \in \mathrm{Rep}^{\mathrm{dg}}_{T_{X_{1+\bullet}/X_\bullet}}$$

then represents the image $p^*E \in \mathrm{Perf}(T_{M/X})$ of E under the functor (7.3.3).

The map $\Gamma(X, E) \longrightarrow C^*(T_{M/X}, p^*E)$ of Example 7.3.5 can then be computed at follows: taking the Chevalley-Eilenberg complex of each $d_0^*E_n$ yields a natural transformation of $\mathcal{O}(X_\bullet)$-modules

$$E_\bullet \longrightarrow C^*(T_{X_{1+\bullet}/X_\bullet}, d_0^*E_\bullet) \longrightarrow d_0^*E_\bullet.$$

The first map sends an element $e \in E_n$ to the constant element $1 \otimes e \in d_0^*E_n$. The induced map on total complexes

$$\mathrm{Tot}(E_\bullet) \longrightarrow \mathrm{Tot}\Big(C^*(T_{X_{1+\bullet}/X_\bullet}, d_0^*E_\bullet)\Big)$$

is a model for the map $\Gamma(X, E) \longrightarrow C^*(T_{M/X}, p^*E)$. In particular, this induces a van Est-type spectral sequence

$$E_1^{p,q} = \Omega^p(X_{1+q}/X_q, d_0^*E_q) \Longrightarrow H^{p-q}(T_{M/X}, p^*E)$$

from the fiberwise de Rham cohomology of X_{1+q} with coefficients in $d_0^*E_q$, converging to the Lie algebroid cohomology of $T_{M/X}$ with coefficients in p^*E.

Suppose that $X_\bullet$ is the nerve of a Lie groupoid $\mathcal{G} \rightrightarrows M$ and that E is a vector bundle over M equipped with a (strict) $\mathcal{G}$-representation. It follows that the map $\Gamma(X, E) \longrightarrow C^*(T_{M/X}, p^*E)$ of Example 7.3.5 can be identified with the classical Van Est homomorphism from Lie groupoid cohomology to Lie algebroid cohomology [20].

7.3.2 Differential forms. As a special case of Example 7.3.5, we find that a smooth map $M \longrightarrow X$ to a derived stack induces a map $\mathcal{O}(X) \longrightarrow C^*(T_{M/X})$ from the function algebra of X to the Chevalley-Eilenberg complex of the Lie algebroid $T_{M/X}$. Similarly, the de Rham complex $\mathrm{dR}(X)$ can be described at the infinitesimal level by the *Weil algebra* of $T_{M/X}$.

To see this, let us start by briefly recalling from [73] the description of differential forms on derived stacks in terms of graded mixed complexes. A *mixed complex* is a chain complex E together with a chain map

$$d\colon E \longrightarrow E[-1]$$

which squares to zero. If the chain complex E admits a grading $E = \bigoplus_{p\in\mathbb{Z}} E(p)$ (where we refer to p as the *weight*) and d increases the weight by 1, then E is called a *graded mixed complex*. The category of (graded) mixed complexes has a natural symmetric monoidal structure given by

$$(E \otimes F)(p) = \bigoplus_i E(i) \otimes F(p-i) \qquad d_{E\otimes F} = d_E \otimes 1 + 1 \otimes d_F.$$

A *graded mixed cdga* is a commutative monoid for this monoidal structure, i.e. a cdga A equipped with a graded mixed structure, such that $d\colon A \longrightarrow A[-1]$ is a derivation and $A(p)\cdot A(q) \subseteq A(p+q)$. Similarly, a *(graded) mixed dg-$\mathcal{C}^\infty$-ring* is a dg-$\mathcal{C}^\infty$-ring A together with a $\mathcal{C}^\infty$-derivation $d\colon A \longrightarrow A[-1]$ which squares to zero (and a grading making it a graded mixed cdga).

Example 7.3.13. Any dg-$\mathcal{C}^\infty$-ring A determines a dg-$\mathcal{C}^\infty$-ring $\mathrm{Sym}_A\big(\Omega^1_A[1]\big)$, with A-linear differential induced by the differential on A and Ω^1_A. The universal derivation $d\colon A \longrightarrow \Omega^1_A$ determines a $\mathcal{C}^\infty$-derivation

$$d\colon \mathrm{Sym}_A\big(\Omega^1_A[1]\big) \longrightarrow \mathrm{Sym}_A\big(\Omega^1_A[1]\big)[-1]$$

which squares to zero. Together with the obvious grading by the polynomial degree in $\Omega^1_A[1]$, this makes $\mathrm{Sym}_A\big(\Omega^1_A[1]\big)$ a graded mixed dg-$\mathcal{C}^\infty$-ring, which we will denote by $\mathrm{dR}(A)$.

Remark 7.3.14. Graded mixed complexes and algebras all arise as algebras over certain dg-operads. Consequently, they can be organized into model categories, whose weak equivalences (fibrations) are the maps inducing quasi-isomorphisms (surjections) on the underlying chain complexes.

Lemma 7.3.15 (c.f. [98])**.** *The forgetful functor $U\colon \mathcal{C}^\infty\mathrm{Alg}^{\mathrm{dg,mix}} \longrightarrow \mathcal{C}^\infty\mathrm{Alg}^{\mathrm{dg}}$ from mixed dg-$\mathcal{C}^\infty$-rings admits a left adjoint, sending $A \mapsto \mathrm{dR}(A)$, and a right adjoint, sending A to $(A, d=0)$. Furthermore, the category $\mathcal{C}^\infty\mathrm{Alg}^{\mathrm{dg,mix}}$ of mixed dg-$\mathcal{C}^\infty$-rings admits a model structure whose weak equivalences (fibrations) are detected by U.*

Proof. One immediately verifies that U has the given left and right adjoint. To see that the model structure on $\mathcal{C}^\infty\mathrm{Alg}^{\mathrm{dg}}$ transfers along U, it suffices to verify that dR preserves trivial cofibrations. This follows from the fact that $\Omega^1_A \otimes_A B \longrightarrow \Omega^1_B$ is a trivial cofibration whenever $A \longrightarrow B$ is a trivial cofibration. □

Let A be a complete (cofibrant) $\mathcal{C}^\infty$-ring and let $X = \mathrm{Spec}(A)$. The $\mathcal{C}^\infty$-ring $\mathrm{dR}(A)$ can be thought of geometrically as the function algebra on the shifted tangent bundle

$$T[-1]X = \mathrm{Spec}_A\big(L_A[1]\big) \simeq \mathrm{Map}\big(\mathrm{Spec}(\mathbb{R}[\epsilon_1]), X\big)$$

consisting of maps $\mathrm{Spec}(\mathbb{R}[\epsilon_1]) \longrightarrow X$ (see Example 5.2.35).

Lemma 7.3.16 ([73, Lemma 1.15])**.** *Let X be a derived stack and consider its shifted tangent stack*

$$T[-1]X = \mathrm{Spec}(L_X[1]).$$

Then the (unbounded) function ring $\mathcal{O}(T[-1]X) \in \mathrm{CAlg}$ has the natural structure of a mixed graded cdga, denoted $\mathrm{dR}(X) \in \mathrm{CAlg}^{\mathrm{gr,mix}}$ and called the de Rham complex *of X.*

Proof. If $p\colon Y \longrightarrow X$ is a map between locally finitely presented derived stacks, there is an induced square

$$\begin{tikzcd} T[-1]Y \arrow[r] \arrow[d, "T[-1]p"'] & Y \arrow[d, "p"] \\ T[-1]X \arrow[r] & X. \end{tikzcd}$$

The map from $T[-1]Y$ into the pullback is given by the map

$$\mathrm{Spec}_Y(L_Y[1]) \longrightarrow \mathrm{Spec}_Y(p^*L_X[1])$$

induced by the map $p^*L_X[1] \longrightarrow L_Y[1]$ of perfect complexes over Y. When p is a smooth surjection, $L_{Y/X}$ is locally free and the map $p^*L_X[1] \longrightarrow L_Y[1]$ locally admits a retraction.

A smooth surjection therefore induces a surjection $T[-1]Y \longrightarrow T[-1]X$. This implies that a derived Lie n-groupoid $X_\bullet \longrightarrow X$ induces an equivalence of stacks

$$T[-1]X \simeq \operatorname*{colim}_{n\in\mathbf{\Delta}^{\mathrm{op}}} T[-1]X_n.$$

By Example 7.3.13, there is a functor $\mathrm{dR}\colon \mathrm{Aff}^{\mathrm{op}} \longrightarrow \mathrm{CAlg}^{\mathrm{gr,mix}}$ sending an affine X to a natural graded mixed enhancement of the function algebra $\mathcal{O}(T[-1]X)$. This functor extends to a functor from the ∞-category of derived Lie n-groupoids

$$\mathrm{dR}\colon \mathrm{Lie}_n^{\mathrm{op}} \longrightarrow \mathrm{CAlg}^{\mathrm{mix,gr}};\ \ X_\bullet \longmapsto \lim_{[n]\in\mathbf{\Delta}} \mathrm{dR}(X_n).$$

The underlying functor to CAlg is just the functor sending $X_\bullet$ to $\mathcal{O}(T[-1]X)$ and thus sends Morita equivalences to equivalences of commutative algebras. It follows that the above functor descends to a functor on the ∞-category of derived stacks and equips $\mathcal{O}(T[-1]X)$ with a natural graded mixed structure. ☐

There is a variant of the complex of differential forms for Lie algebroids, given by their Weil algebra (see [2]). For simplicity, we will only provide a point-set model for the Weil algebra that applies to *fibrant-cofibrant* dg-Lie algebroids.

Assumption 7.3.17. Let A be an Ω-cofibrant dg-$\mathcal{C}^\infty$-ring and let $\mathbb{T}_A \xrightarrow{\sim} T_A$ be a replacement of its tangent Lie algebroid by a dg-Lie algebroid whose underlying dg-A-module is cofibrant. The Quillen pair $\mathrm{LieAlgd}_A^{\mathrm{dg}} \leftrightarrows \mathrm{LieAlgd}_A^{\mathrm{dg}}/\mathbb{T}_A$ is a Quillen equivalence.

We will assume that all dg-Lie algebroids under consideration come with a map $\mathfrak{g} \longrightarrow \mathbb{T}_A$ to the resolved tangent bundle. When A is the function ring of a smooth manifold or $A = \mathbb{R}\{x_i\}$ is finitely presented and cofibrant, one can simply take $\mathbb{T}_A = T_A$.

Construction 7.3.18. Let $\rho\colon \mathfrak{g} \longrightarrow \mathbb{T}_A$ be a fibration of dg-Lie algebroids whose domain is A-cofibrant. Then the kernel $\mathfrak{n} := \ker(\rho) \subseteq \mathfrak{g}$ is a cofibrant dg-A-module which models the fiber of the anchor map of $\mathfrak{g}$ (this is the main reason for replacing T_A by $\mathbb{T}_A$). Let $\mathfrak{g} \oplus \mathfrak{n}[-1]$ be the square zero extension of $\mathfrak{g}$ by the shifted adjoint representation of $\mathfrak{g}$ on $\mathfrak{n}$.

The Chevalley-Eilenberg differential on $C^*(\mathfrak{g}\oplus\mathfrak{n}[-1])$ restricts to the subspaces of maps that are polynomial of degree q in $\mathfrak{n}$

$$C^*(\mathfrak{g}\oplus\mathfrak{n}[-1])^q := \mathrm{Hom}_A\Big(\mathrm{Sym}_A\mathfrak{g}[1] \otimes \mathrm{Sym}_A^q\mathfrak{n}, A\Big) \subseteq C^*(\mathfrak{g}\oplus\mathfrak{n}[-1]).$$

There is a natural derivation $d\colon C^*(\mathfrak{g}\oplus\mathfrak{n}[-1])^q \longrightarrow C^*(\mathfrak{g}\oplus\mathfrak{n}[-1])^{q+1}$ given by

$$(d\alpha)(X_1,\dots,X_n,\xi_1,\dots,\xi_q) = \sum_{j=1}^{q} \alpha(X_1,\dots,X_n,\sigma(\xi_j),\xi_1,\dots,\xi_q)$$

where $X_i \in \mathfrak{g}[1]$, $\xi_j \in \mathfrak{n}$ and $\sigma\colon \mathfrak{n} \longrightarrow \mathfrak{g}[1]$ the inclusion, followed by a degree shift. This defines a graded mixed cdga

$$W(\mathfrak{g}) := \bigoplus_q C^*(\mathfrak{g} \oplus \mathfrak{n}[-1])^q \subseteq C^*(\mathfrak{g} \oplus \mathfrak{n}[-1])$$

whose connective cover is a mixed graded dg-$\mathcal{C}^\infty$-ring.

Definition 7.3.19. The *Weil algebra* of a fibrant, A-cofibrant dg-Lie algebroid $\mathfrak{g} \longrightarrow \mathbb{T}_A$ is the mixed graded cdga $W(\mathfrak{g})$ from Construction 7.3.18.

Remark 7.3.20. When $\mathfrak{g} \longrightarrow \mathbb{T}_A$ is not a fibration, one can use an alternative model for the Weil algebra, due to Arias Abad and Crainic [2]: instead of using the kernel of the map $\mathfrak{g} \longrightarrow \mathbb{T}_A$, this uses its mapping fiber $\tilde{\mathfrak{n}}$. In loc. cit. the authors construct a graded mixed cdga whose weight q part is given by

$$\mathrm{Hom}_A\Big(\mathrm{Sym}_A \mathfrak{g}[1] \otimes \mathrm{Sym}^q_A \tilde{\mathfrak{n}}, A\Big).$$

When $\mathfrak{g} \longrightarrow \mathbb{T}_A$ is a fibration, restriction along the canonical map $\mathfrak{n} \longrightarrow \tilde{\mathfrak{n}}$ from the kernel to the mapping fiber induces a weak equivalence between the Weil algebra from the above definition and the one defined in [2] (see [71] for more details).

Remark 7.3.21. A map $f\colon \mathfrak{g} \longrightarrow \mathfrak{h}$ of fibrant-cofibrant dg-Lie algebroids over $\mathbb{T}_A$ induces a map $W(\mathfrak{h}) \longrightarrow W(\mathfrak{g})$ of mixed graded cdgas, which is a weak equivalence if f is. It follows that taking Weil algebras induces a functor of ∞-categories

$$W\colon \mathrm{LieAlgd}_A^{\mathrm{op}} \longrightarrow \mathrm{CAlg}^{\mathrm{gr,mix}}.$$

The functor sending $\rho\colon \mathfrak{g} \longrightarrow \mathbb{T}_A$ to $\mathfrak{g} \oplus \ker(\rho)[-1]$ preserves sifted homotopy colimits, which are computed at the level of the underlying complexes. This implies that $W\colon \mathrm{LieAlgd}_A^{\mathrm{op}} \longrightarrow \mathrm{CAlg}^{\mathrm{gr,mix}}$ preserves sifted colimits as well.

Suppose that $\phi_0\colon B \longrightarrow A$ is a map between Ω-cofibrant dg-$\mathcal{C}^\infty$-rings and let $\mathfrak{g} \longrightarrow \mathbb{T}_A$ be a dg-Lie algebroid over A. By Lemma 7.3.15, a map of dg-$\mathcal{C}^\infty$-rings $\phi\colon B \longrightarrow \mathrm{c}^*(\mathfrak{g})$ over A induces a map of graded mixed cdgas over A

$$\mathrm{dR}(B) = \mathrm{Sym}_B\big(\Omega^1_B[1]\big) \longrightarrow W(\mathfrak{g}) \subseteq C^*(\mathfrak{g} \oplus \mathfrak{n}[-1]). \tag{7.3.22}$$

At the level of connective covers, this map is a map of graded mixed dg-$\mathcal{C}^\infty$-rings.

Lemma 7.3.23. *Suppose that the map $\phi\colon B \longrightarrow \mathrm{c}^*(\mathfrak{g})$ induces an equivalence $\mathfrak{g} \longrightarrow \mathfrak{D}(B)$ of dg-Lie algebroids over A. Then the map (7.3.22) is adjoint to an equivalence $\mathfrak{g} \oplus \mathfrak{n}[-1] \longrightarrow \mathfrak{D}\big(\mathrm{Sym}_B L_B[1]\big)$.*

Proof. Recall from Proposition 4.1.26 that $\mathfrak{D}(B) \longrightarrow \mathfrak{g}$ can be described as follows: the map ϕ induces a map

$$(\phi_0, \phi_1)\colon B \longrightarrow C^*(\mathfrak{g}) \longrightarrow A \oplus_{\rho^*} \mathfrak{g}[1]^\vee.$$

over A, which is classified by a map to the mapping fiber of $\rho^*\colon L_A \longrightarrow \mathfrak{g}^\vee$

$$L_B \otimes_B A \longrightarrow L_A \oplus_{\rho^*} \mathfrak{g}[1]^\vee;\ db \longmapsto (d\phi_0(b), \phi_1(b)).$$

The adjoint of this map is a map $\beta\colon \mathrm{cof}(\rho) \longrightarrow \mathrm{Der}(B, A)$ from the cofiber of the anchor $\rho\colon \mathfrak{g} \longrightarrow T_A$. The map $\mathfrak{g} \longrightarrow \mathfrak{D}(B)$ induces a map between the cofibers of the anchor maps, which is precisely given by β.

Similarly, the map (7.3.22) induces a map $\mathrm{Sym}_B(L_B[1]) \longrightarrow A \oplus_{\rho^*} \mathfrak{g}[1]^\vee \oplus \mathfrak{n}^\vee$ which is classified by

$$\begin{aligned}\big(L_B \otimes_B A\big) \oplus \big(L_B[1] \otimes_B A\big) &\longrightarrow \big(L_A \oplus_{\rho^*} \mathfrak{g}[1]^\vee\big) \oplus \mathfrak{n}^\vee; \\ \Big(db_1, db_2\Big) &\longmapsto \Big(d\phi_0(b_1), \phi_1(b_1), \phi_1(b_2)\Big).\end{aligned}$$

The map $\mathfrak{g} \oplus \mathfrak{n}[-1] \longrightarrow \mathfrak{D}(\mathrm{Sym}_B L_B[1])$ induces a map on cofibers, which is the adjoint of this above map. This adjoint map is a direct sum of the map β, together with the composition

$$\mathfrak{n}[-1] \xrightarrow{\sim} \mathrm{fib}(\rho) \simeq \mathrm{cof}(\rho)[-1] \xrightarrow{\beta[-1]} \mathrm{Der}(B, A)[-1]$$

where the first map is the natural inclusion of the kernel into the mapping fiber. This is a weak equivalence, which implies that $\mathfrak{g} \oplus \mathfrak{n}[-1] \longrightarrow \mathfrak{D}(\mathrm{Sym}_B L_B[1])$ is an equivalence. □

When $M = \mathrm{Spec}(A) \longrightarrow \mathrm{Spec}(B) = X$ is a map between affines, Lemma 7.3.23 provides a natural map of graded mixed cdgas $\mathrm{dR}(X) \longrightarrow W(T_{M/X})$. Furthermore, postcomposition with the natural map of commutative algebras $\bigoplus_q W^q(\mathfrak{g}) \longrightarrow \prod_q W^q(\mathfrak{g})$ yields a map of commutative algebras

$$\mathcal{O}(T[-1]X) \simeq \mathrm{dR}(X) \longrightarrow W(T_{M/X}) \longrightarrow C^*(T_{M/T[-1]X})$$

which agrees with the canonical (restriction) map $\mathcal{O}(T[-1]X) \longrightarrow C^*(T_{M/T[-1]X})$ from Example 7.3.5.

Corollary 7.3.24. *Let $f\colon M = \mathrm{Spec}(A) \longrightarrow X$ be a map from an affine to a derived stack and consider the composite map $M \longrightarrow X \xrightarrow{0} T[-1]X$ into the shifted tangent stack. Then there is a map of graded mixed commutative algebras*

$$\mathrm{dR}(X) \longrightarrow W(T_{M/X})$$

whose underlying map of commutative algebras can be identified with the canonical map

$$\mathcal{O}(T[-1]X) \longrightarrow W(T_{M/X}) \longrightarrow C^*(T_{M/T[-1]X}).$$

Proof. It suffices to work locally on M, so that we may assume that $M \longrightarrow X$ arises from a map $M \longrightarrow X_\bullet$ of derived Lie n-groupoids and that the map $M \longrightarrow T[-1]X$ arises from the map $M \longrightarrow X_\bullet \longrightarrow T[-1]X_\bullet$. The result now follows from the fact that dR and $W(-)$ send the colimits of $X_\bullet$ and $T_{M/X_\bullet}$ to limits of graded mixed algebras, by Remark 7.3.21 and (the proof of) Lemma 7.3.16. □

Corollary 7.3.25. *Let $f\colon M \longrightarrow X$ be a smooth surjection from a smooth manifold to a smooth m-stack, whose fibers are n-connected. Then the map of graded-mixed complexes*

$$\mathrm{dR}(X) \longrightarrow W(T_{M/X})$$

has $(m-n-2)$-truncated fibers.

Proof. Since X is a smooth stack, hence locally finitely presented, the cotangent complex L_X is a perfect complex. For each weight q, the map $\mathrm{dR}(X)^q \longrightarrow W(T_{M/X})^q$ can be identified with the map

$$\Gamma\Big(X, \mathrm{Sym}^q L_X[1]\Big) \longrightarrow C^*\Big(T_{M/X}, \mathrm{Sym}^q_{\mathcal{O}(M)} L_X[1]\big|M\Big)$$

from Example 7.3.5. Each of these maps has $(m-n-2)$-truncated fibers by Corollary 7.3.2, so that the direct sum $\mathrm{dR}(X) \longrightarrow W(T_{M/X})$ has $(m-n-2)$-truncated fibers as well. □

Remark 7.3.26. Recall from [73] that the *weighted negative cyclic complex* of a graded mixed complex E is the graded complex

$$\mathrm{NC}^w(E) := \bigoplus_p \mathrm{NC}^w(E)(p) \qquad \mathrm{NC}^w(E)(p) = \prod_{q\geq 0} E(p+q)[-2q]$$

where the differential on $\mathrm{NC}^w(E)(p)$ sends a tuple $\mathbf{e} = (e_q)_{q\geq 0}$ to the tuple

$$(\partial \mathbf{e})_q = \partial_E(e_q) + d(e_{q-1}).$$

In particular, for any derived stack X one can think of $\mathrm{NC}^w(\mathrm{dR}(X))(0)$ as the complex describing the (derived) de Rham cohomology of X. If X is a smooth stack arising from a Lie n-groupoid $X_\bullet$, this complex is simply given by the totalization of the cosimplicial chain complex of differential forms on $X_\bullet$. If $M \longrightarrow X$ is an atlas for this smooth stack with n-connected fibers, Corollary 7.3.25 implies that the map

$$\mathrm{NC}^w(\mathrm{dR}(X))(0) \longrightarrow \mathrm{NC}^w(W(T_{M/X}))$$

induces isomorphisms on homotopy groups in degrees $\geq -n$ (both chain complexes are concentrated in homologically nonpositive degrees when X is smooth).

Example 7.3.27. Let $p\colon M \longrightarrow X$ smooth atlas of a smooth m-stack. An element $\omega \in \mathrm{NC}^w(\mathrm{dR}(X))(2)[q-2]$ describes a closed 2-form on X of degree q. Such a 2-form determines an element $\omega_0 \in \big(\bigwedge^2 L_X\big)[q]$, which determines a map $\omega_0\colon L_X^\vee \longrightarrow L_X[q]$. The closed 2-form ω defines a q-shifted symplectic structure on X when this map is an equivalence (see [73]).

The closed 2-form ω determines a closed 2-form

$$\omega_{\mathrm{inf}} \in \mathrm{NC}^w(W(T_{M/X}))(2)[q-2].$$

Using that $\mathrm{fib}(T_{M/X} \longrightarrow T_M) \simeq \big(p^*L_X[1]\big)^\vee$, such a 2-form determines a map

$$\omega_{\mathrm{inf},0}\colon p^*(L_X^\vee) \longrightarrow p^*L_X[q].$$

In fact, this map is simply the restriction of ω_0 along p. Since p is a smooth surjection, it follows that ω_0 is an equivalence iff $\omega_{\mathrm{inf},0}$ is an equivalence. In the latter case, ω_{inf} defines a q-shifted symplectic structure on the Lie algebroid $T_{M/X}$ (see [79]).

In other words, each shifted symplectic structure on an m-stack X gives rise to a shifted symplectic structure on $T_{M/X}$. Conversely, any such shifted symplectic structure on $T_{M/X}$ can be integrated to X when the fibers of $p\colon M \longrightarrow X$ are m-connected.

The Weil algebra $W(\mathfrak{g})$ admits a natural interpretation in terms of loop spaces. Recall that for any Lie algebroid $\mathfrak{g}$, the free loop space $\mathcal{L}(\mathfrak{g})$ is the limit of the constant S^1-diagram with value $\mathfrak{g}$. When $\mathfrak{g} \longrightarrow \mathbb{T}_A$ be a fibrant dg-Lie algebroid over $\mathbb{T}_A$, there is a simple way to compute $\mathcal{L}\mathfrak{g}$ using the cotensoring of dg-Lie algebroids over (unbounded) cdgas from Construction 3.1.31. Indeed, $\mathcal{L}\mathfrak{g}$ can simply be computed as $\mathfrak{g}\boxtimes\Omega[S^1]$, where S^1 is some finite simplicial model for the circle. In fact, recall that $H^*(S^1) = k[\epsilon_{-1}]$ is the free graded algebra on a generator of (homological) degree -1. We can therefore choose a weak equivalence of cdgas $k[\epsilon_{-1}] \longrightarrow \Omega[S^1]$ and identify

$$\mathcal{L}\mathfrak{g} = \mathfrak{g} \boxtimes k[\epsilon_{-1}].$$

Unwinding the definitions, $\mathcal{L}\mathfrak{g}$ is exactly the dg-Lie algebroid $\mathfrak{g}\oplus\mathfrak{n}[-1]$ used in Construction 7.3.18. This dg-Lie algebroid has itself a graded mixed structure, given by the obvious inclusion

$$d\colon \mathcal{L}\mathfrak{g}(1) = \mathfrak{n}[-1] \longrightarrow \mathcal{L}\mathfrak{g}(0)[-1] = \mathfrak{g}[-1]$$

The inclusion d is a derivation for the Lie bracket and the Lie bracket respects the weights. The Weil algebra $W(\mathfrak{g})$ can then be identified with the Chevalley-Eilenberg complex of $\mathcal{L}\mathfrak{g}$, computed internally to graded mixed complexes (see [71] for more details).

Remark 7.3.28. The graded-mixed structure on $\mathcal{L}\mathfrak{g}$ can also be interpreted at follows. Recall (see e.g. [9], [73]) that graded mixed complexes can be viewed as dg-comodules over the cohomology Hopf algebra

$$\mathcal{H} = H^*(\mathbb{G}_m \ltimes B\mathbb{G}_a, \mathcal{O}) = k[t, t^{-1}] \otimes_k k[\epsilon_{-1}].$$

Here t has degree 0, ϵ_{-1} has (homological) degree -1 and the comultiplication is given by $\Delta(t) = t \otimes t$ and $\Delta(\epsilon_{-1}) = t \otimes \epsilon_{-1}$. The action of $\mathbb{G}_m \ltimes B\mathbb{G}_a$ on $B\mathbb{G}_a$ (by rescaling and translation) induces a coaction

$$H^*(B\mathbb{G}_a, \mathcal{O}) = k[\epsilon_{-1}] \longrightarrow k[\epsilon_{-1}] \otimes_k \mathcal{H} = H^*(\mathbb{G}_m \ltimes B\mathbb{G}_a \times B\mathbb{G}_a, \mathcal{O}).$$

This induces a coaction of $\mathcal{H}$ on the free loop space $\mathcal{L}\mathfrak{g}$ of the form

$$\mathcal{L}\mathfrak{g} = \mathfrak{g} \boxtimes k[\epsilon_{-1}] \longrightarrow \mathfrak{g} \boxtimes \big(k[\epsilon_{-1}] \otimes_k \mathcal{H}\big) \cong \mathcal{L}\mathfrak{g} \boxtimes \mathcal{H},$$

simply by restricting the canonical $\mathcal{H}$-comodule structure on $\mathfrak{g} \otimes k[\epsilon_{-1}]$. Unwinding the definitions, this coaction of $\mathcal{H}$ on $\mathcal{L}\mathfrak{g}$ corresponds to the graded mixed structure on $\mathcal{L}\mathfrak{g}$ described above.

Remark 7.3.29. The coaction of $\mathcal{H}$ on $H^*(S^1) = H^*(B\mathbb{G}_a, \mathcal{O}) = k[\epsilon_{-1}]$ restricts to a coaction of $H^*(S^1)$ on itself. In terms of graded mixed structures, this coaction encodes just the mixed structure on $k[\epsilon_{-1}]$. On the other hand, this coaction arises topologically from the rotation action $\mu\colon S^1 \times S^1 \longrightarrow S^1$, by passing to cohomology. In fact, it provides a rational model for the rotation action by [98].

Similarly, the mixed structure on $\mathcal{L}\mathfrak{g}$ is encoded by the coaction

$$\mathcal{L}\mathfrak{g} = \mathfrak{g} \boxtimes H^*(S^1) \xrightarrow{\mathfrak{g}\boxtimes H^*(\mu)} \mathfrak{g} \boxtimes H^*(S^1 \times S^1) \cong \mathcal{L}\mathfrak{g} \boxtimes H^*(S^1).$$

One can therefore think of the mixed structure on $\mathcal{L}\mathfrak{g}$ as a (rational) algebraic incarnation of the S^1-action on $\mathcal{L}\mathfrak{g}$ by rotation of loops. The mixed structure on $W(\mathfrak{g})$ arises from the associated S^1-action on $C^*(\mathcal{L}(\mathfrak{g}))$.

Remark 7.3.30. The above description of the Weil algebra $W(\mathfrak{g})$ as $C^*(\mathcal{L}\mathfrak{g})$ admits a very well-known global counterpart (see [98, 9, 73]). Consider a map $M \longrightarrow X$ from a locally finitely presented affine to a locally finitely presented derived stack and let

$$M \longrightarrow X \longrightarrow \mathcal{L}X$$

be the canonical map into the loop stack, taking values in constant loops. As shown in [9, Lemma 6.7], there is a canonical map of stacks $T[-1]X \longrightarrow \mathcal{L}X$ which induces an equivalence on cotangent complexes at M (where M takes values in the zero section of $T[-1]X$). Consequently, the map $T[-1]X^\wedge_M \longrightarrow \mathcal{L}X^\wedge_M$ is an equivalence by Proposition 7.1.13. One can then identify $W(T_{M/X})$ with the function algebra of the formal completion $\mathcal{L}X^\wedge_M$. Informally, one can then think of its mixed structure as arising from the S^1-action on $\mathcal{L}X^\wedge_M$ by rotation of loops.

7.3.3 Further examples. So far, we have seen two types of maps $M \longrightarrow X$ from a (derived) manifold to a (derived) stack whose Lie algebroids can be described explicity. For any (derived) Lie groupoid $\mathcal{G} \rightrightarrows M$, the map to the quotient $M \longrightarrow X = M/\mathcal{G}$ has Lie algebroid $T_{M/X}$ given by the usual Lie algebroid of $\mathcal{G}$ (Proposition 6.4.30). Furthermore, if

$$M = \operatorname{Spec}(A) \longrightarrow \operatorname{Map}(*, \operatorname{Perf}^{[a,b]}) = X$$

classifies a perfect complex E over M, then the associated formal moduli problem over A is the formal moduli problem Def_E describing the deformations of E (Example 2.3.35). Theorem 4.4.1 asserts that $T_{M/X}$ is simply the Atiyah Lie algebroid $\mathrm{At}(E)$ of E.

Various other examples can be deduced from these two examples.

Example 7.3.31. Suppose that $\mathcal{G}_\bullet \longrightarrow M$ is a smooth (strict) groupoid object in the category of (smooth) Lie n-groupoids, where M is a manifold. This determines natural maps of stacks $M \longrightarrow X_\bullet = M/\mathcal{G}_\bullet$, whose colimit is a smooth surjection $M \longrightarrow X$ to a smooth $(n+1)$-stack. Explicitly, one can describe X by the $\overline{W}$-construction of the simplicial groupoid $\mathcal{G}_\bullet$ [26, 104].

The augmented simplicial diagram $X_\bullet \longrightarrow X$ (under M) is a smooth hypercover (of finite height) of X. It follows that the induced diagram of formal moduli problems $\widehat{X}_\bullet \longrightarrow \widehat{X}$ is a formally smooth hypercover, in the sense of Lemma 4.2.29. In particular, the Lie algebroid $T_{M/X}$ is simply the colimit of the simplicial diagram of Lie algebroids $T_{M/X_\bullet}$.

This colimit can be computed at the level of the underlying complexes (Theorem 3.1.15) and each Lie algebroid T_{M/X_n} is simply the (discrete) Lie algebroid of the Lie groupoid $\mathcal{G}_n \rightrightarrows M$. It follows that $T_{M/X}$ is just the normalized complex of the simplicial Lie algebroid $T_{M/X_\bullet}$, with its canonical dg-Lie algebroid structure.

Example 7.3.32. Let $E \in \mathrm{Perf}(X)$ be a perfect complex over a derived stack and consider the derived stack $\mathrm{Spec}_X(E^\vee) \longrightarrow X$ over X (see Lemma 5.2.34). Given a map $M \longrightarrow X$ from an affine to X, one obtains a diagram of stacks

$$\begin{tikzcd} & M \arrow[dl] \arrow[d] \arrow[dr] & \\ X \arrow[r, "0"'] & \mathrm{Spec}_X(E^\vee) \arrow[r] & X \end{tikzcd}$$

which induces a retract diagram of Lie algebroids over M

$$T_{M/X} \longrightarrow T_{M/\mathrm{Spec}_X(E^\vee)} \longrightarrow T_{M/X}.$$

Let E_{inf} denote the image of E under the functor $\mathrm{Perf}(X) \longrightarrow \mathrm{Perf}(T_{M/X})$ of Corollary 7.3.2. Then $T_{M/\mathrm{Spec}_X(E^\vee)}$ is equivalent to the split square zero extension $T_{M/X} \oplus E_M[-1]$.

To see this, let us first treat the universal case where $\mathrm{Spec}_X(E^\vee) \longrightarrow X$ is the universal bundle

$$\mathrm{Spec}(E^\vee_{\mathrm{uni}}) \longrightarrow \mathrm{Map}(*, \mathrm{Perf}).$$

The stack $\mathrm{Spec}(E^\vee_{\mathrm{uni}})$ is the classifying stack for perfect complexes E, together with an element $e \in E$. The zero section sends a perfect complex E to the pair $(E, e = 0)$.

When $M \longrightarrow \mathrm{Map}(*, \mathrm{Perf})$ classifies a perfect complex E_0 over M, the formal moduli problem associated to the zero section $M \longrightarrow \mathrm{Spec}(E^\vee_{\mathrm{uni}})$ simply describes deformations of the pointed $\mathcal{O}(M)$-module $(E_0, 0)$. It follows from Theorem 4.4.1 that the associated Lie algebroid is the Atiyah Lie algebroid $\mathrm{At}(E_0, 0)$ in the sense of Example 3.1.3, which fits into a retract diagram

$$\mathrm{At}(E_0) \longrightarrow \mathrm{At}(E_0, 0) \longrightarrow \mathrm{At}(E_0).$$

To compute the Atiyah Lie algebroid of $(E_0, 0)$, one has to replace $(E_0, 0)$ by a *cofibrant* pointed $\mathcal{O}(M)$-module. In other words, one has to replace the map $0\colon \mathcal{O}(M) \longrightarrow E_0$ by a cofibration, e.g. by the canonical map $\tilde{0}\colon \mathcal{O}(M) \longrightarrow E_0 \oplus \mathcal{O}(M)[0,1]$ into the mapping cylinder. One can then easily verify that

$$\begin{array}{rcl} \mathrm{At}(E_0)\oplus E_0[-1] & \longrightarrow & \mathrm{At}\Big(E_0\oplus \mathcal{O}(M)[0,1],\tilde{0}\Big) \\ (\nabla_v, e) & \longmapsto & \begin{pmatrix} \nabla_v & 0 & e \\ 0 & v & 0 \\ 0 & 0 & v \end{pmatrix} \end{array}$$

is a quasi-isomorphism of dg-Lie algebroids, which is compatible with the projection to $\mathrm{At}(E_0)$.

The general case follows from the universal case. Indeed, if E is a perfect complex over a derived stack X, then $\mathrm{Spec}_X(E)$ fits into a pullback square

$$\begin{tikzcd} \mathrm{Spec}_X(E^\vee) \arrow[r] \arrow[d] & \mathrm{Spec}(E^\vee_{\mathrm{uni}}) \arrow[d] \\ X \arrow[r] & \mathrm{Perf}^{[a,b]} \end{tikzcd}$$

Since the formation of Lie algebroids preserves limits, one obtains a pullback diagram of Lie algebroids over M

$$\begin{tikzcd} T_{M/\mathrm{Spec}_X(E)} \arrow[r] \arrow[d] & \mathrm{At}(E_0)\oplus E_0[-1] \arrow[d] \\ T_{M/X} \arrow[r] & \mathrm{At}(E_0). \end{tikzcd}$$

The bottom map describes the $T_{M/X}$-representation E_{inf}, whose underlying module is the restriction $E_0 = E\big|M$. The result follows immediately from this.

Example 7.3.33. Let $\mathcal{G}\rightrightarrows M$ be a Lie groupoid with Lie algebroid $\mathfrak{g}$ and let E be a chain complex of $\mathcal{G}$-representations, i.e. E is given in each degree by a vector bundle over M with an action of $\mathcal{G}$. Then E determines a perfect complex over the quotient stack $X = M/\mathcal{G}$ and the Lie algebroid $T_{M/\mathrm{Spec}(E)}$ is the split square zero extension $\mathfrak{g}\oplus E[-1]$, where $E[-1]$ is the shifted complex of associated Lie algebroid representations.

Example 7.3.34. Given a derived stack X and a perfect complex E over X, consider a pullback diagram

$$\begin{tikzcd} X_\eta \arrow[r] \arrow[d, "p"'] & X \arrow[d, "0"] \\ X \arrow[r, "\eta"'] & \mathrm{Spec}_X(E[1]^\vee) \end{tikzcd}$$

where η is a section of the canonical map $\mathrm{Spec}_X(E[1]^\vee)\longrightarrow X$. A map $M\longrightarrow X_\eta$ from a derived manifold induces a pullback square of Lie algebroids

$$\begin{tikzcd} T_{M/X_\eta} \arrow[r] \arrow[d] & T_{M/X} \arrow[d, "0"] \\ T_{M/X} \arrow[r, "\eta_{\mathrm{inf}}"'] & T_{M/X}\oplus E_{\mathrm{inf}} \end{tikzcd}$$

which realizes T_{M/X_η} as a square zero extension of $T_{M/X}$ by $E_{\mathrm{inf}}[-1]$, classified by η_{inf}. To identify η_{inf}, note that Example 7.3.5 provides a map $\Gamma(X, E[1])\longrightarrow C^*(T_{M/X}, E_{\mathrm{inf}}[1])$ over $\Gamma(M, E[1])\simeq E_{\mathrm{inf}}[1]$.

Since the section $\eta\colon X\longrightarrow \mathrm{Spec}_X(E[1]^\vee)$ is null-homotopic when restricted to M, it corresponds to an element η in the fiber of $\Gamma(X, E[1])\longrightarrow\Gamma(M, E[1])$. The map η_{inf} then corresponds to the image of η in the reduced Chevalley-Eilenberg complex $\overline{C}^*(T_{M/X}, E_{\mathrm{inf}}[1])$.

Example 7.3.35. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and let $X = M/\mathcal{G}$. If E is an ordinary $\mathcal{G}$-representation, then the map $\Gamma(X, E[1]) \longrightarrow C^*(T_{M/X}, E_{\mathrm{inf}}[1])$ agrees with the usual van Est homomorphism from [20] (Example 7.3.12). If $\widetilde{X} \longrightarrow X$ is an extension of X classified by a class η in the (reduced) Lie groupoid cohomology of $\mathcal{G}$ with coefficients in E, then the Lie algebroid $T_{M/X}$ is the square zero extension of the Lie algebroid $T_{M/X}$ classified by the image η_{inf} under the van Est map. This square zero extension admits a point-set description in terms of L_∞-algebroids, using the following observation:

Lemma 7.3.36. *Let $\mathfrak{g}$ be an L_∞-algebroid whose underlying dg-A-module is cofibrant and let E be a $\mathfrak{g}$-representation. Let $\alpha \in \overline{C}^*(\mathfrak{g}, E[1])$ be a (degree 0) cocycle in the reduced Chevalley-Eilenberg complex. The square zero extension $\mathfrak{g}_\alpha \longrightarrow \mathfrak{g}$ classified by α can then be described explicitly as a map of L_∞-algebroids*

$$\mathfrak{g} \oplus_\alpha E \longrightarrow \mathfrak{g}$$

whose domain has L_∞-structure of Example 3.1.8: elements in E square to zero and brackets of elements in $\mathfrak{g}$ with a single element $e \in E$ are given by the representation of $\mathfrak{g}$ on E. Furthermore, the brackets of elements $X_i \in \mathfrak{g}$ are given by

$$[X_1, \ldots, X_n] = \Big([X_1, \ldots, X_n]_{\mathfrak{g}}, \alpha(X_1, \ldots, X_n)\Big).$$

The resulting L_∞-algebroid over $\mathfrak{g}$ is a model for the square zero extension $\mathfrak{g}_\alpha \longrightarrow \mathfrak{g}$ classified by α.

Proof. A tedious, but straightforward computation shows that the above brackets determine an L_∞-algebroid structure on $\mathfrak{g} \oplus_\alpha E$. To see that $\mathfrak{g} \oplus_\alpha E$ represents the square zero extension $\mathfrak{g}_\alpha$, let $f\colon \mathfrak{h} \longrightarrow \mathfrak{g}$ be a map from a cofibrant L_∞-algebroid and consider the simplicial set of dotted arrows (defined using the simplicial cotensoring from Construction 3.1.31)

$$\begin{tikzcd}
\mathrm{cobar}(\mathfrak{h}) \arrow[r, dotted] \arrow[d, "\sim"'] & \mathfrak{g} \oplus_\alpha E \arrow[d] \\
\mathfrak{h} \arrow[r, "f"'] & \mathfrak{g}.
\end{tikzcd}$$

Since the left vertical map is a weak equivalence between cofibrant objects, the right map is a fibration and $\mathrm{L}_\infty\mathrm{Algd}$ is right proper, it follows that this simplicial set is a model for the space of sections $\mathfrak{h} \longrightarrow \mathfrak{g} \oplus_\alpha E$ over $\mathfrak{g}$.

A dotted lift $\mathrm{cobar}(\mathfrak{h}) \longrightarrow \mathfrak{g} \oplus_\alpha E$ is uniquely determined by its second component $\mathrm{cobar}(\mathfrak{h}) \longrightarrow E$. Unwinding the definitions of the cobar construction and the Chevalley-Eilenberg complex, one sees that this second component describes a degree 1 element $\beta \in \overline{C}^*(\mathfrak{h}, f^*E[1])$ with the property that $\partial\beta = f^*\alpha$. It follows that there is a natural equivalence between the space of lifts $\mathfrak{h} \longrightarrow \mathfrak{g} \oplus_\alpha E$ and the space of null-homotopies of $f^*\alpha$.

On the other hand, the space of null-homotopies of $f^*\alpha$ is naturally equivalent to the space of sections of $\mathfrak{g}_\alpha \longrightarrow \mathfrak{g}$, by definition of $\mathfrak{g}_\alpha$. It follows that $\mathfrak{g} \oplus_\alpha E$ indeed represents the square zero extension $\mathfrak{g}_\alpha$. □

Example 7.3.37. Let $\theta\colon M \longrightarrow K(U(1), n)$ be a map from a smooth manifold, classifying a higher $U(1)$-bundle gerbe. Such a map fits into a diagram of derived stacks (where $U(1)$ and $\mathbb{R}$ are considered as smooth manifolds)

$$\begin{tikzcd}
M \arrow[r, "\theta"] \arrow[dr, "{[\theta]}"'] & K(U(1), n) \arrow[r] \arrow[d] & * \arrow[d] \\
& K(\mathbb{Z}, n+1) \arrow[r] & K(\mathbb{R}, n+1)
\end{tikzcd}$$

where $[\theta] \in H^{n+1}(M, \mathbb{Z})$ is the integral cohomology class classifying θ. The right square is cartesian and induces a cartesian diagram of Lie algebroids over M

$$\begin{tikzcd} T_{M/K(U(1),n)} \arrow[r] \arrow[d] & T_M \arrow[d, "0"] \\ T_M \arrow[r, "\theta_{\mathrm{inf}}"'] & T_M \oplus \mathcal{O}_M[n+1]. \end{tikzcd}$$

The map θ_{inf} is classified by a class $\theta_{\mathrm{inf}} \in \pi_{-n-1}C^*(T_M)$ in the $(n+1)$-st de Rham cohomology group of M. To identify this class, note that it is induced by the canonical map (under M)

$$\mathrm{Sing}(M) \xrightarrow{[\theta]} K(\mathbb{Z}, n+1) \longrightarrow K(\mathbb{R}, n+1).$$

Using Proposition 7.2.11, one sees that θ_{inf} is simply the image of $[\theta]$ in de Rham cohomology. If we choose a closed $(n+1)$-form F_θ presenting $[\theta]$, then Lemma 7.3.36 shows that $T_{M/K(U(1),n)}$ is modeled by the L_∞-algebroid

$$T_M \oplus \mathcal{O}_M[n] \qquad\qquad [X_1, \dots, X_{n+1}] = F_\theta(X_1, \dots, X_n).$$

7.3.4 Integrability of regular L_∞-algebroids. We conclude with a simple observation about the integrability of (finite dimensional) L_∞-algebroids over a smooth manifold M, essentially due to van Est [28] (and due to Crainic [20] for the case of Lie algebroids).

Definition 7.3.38. Let $\mathfrak{g} \in \mathrm{LieAlgd}_M$ be a (connective) Lie algebroid over a smooth manifold M. We will say that $\mathfrak{g}$ *integrates* to an n-stack if there exists a smooth n-stack X and a smooth surjection $M \longrightarrow X$ whose associated Lie algebroid is (equivalent to) $\mathfrak{g}$.

Remark 7.3.39. If $\mathfrak{g}$ integrates to an n-stack, then the underlying module of $\mathfrak{g}$ is perfect, with Tor-amplitude contained in $[0, n-1]$.

Remark 7.3.40. Theorem 7.2.1 shows that any Lie algebroid admits a *canonical* integration, assuming it can be integrated at all. Indeed, let $\mathfrak{g}$ be a Lie algebroid over M which arises from a smooth map $p\colon M \longrightarrow X$ to an n-stack. The map p factors over the 'source n-connected cover' $\tilde{p}\colon M \longrightarrow \widetilde{X}$ of p (Section 5.3). Since the map $\widetilde{X} \longrightarrow X$ is étale, one finds that the Lie algebroid associated to $\tilde{p}$ is $\mathfrak{g}$ as well. It follows that $\mathfrak{g}$ can be integrated by a smooth map $\tilde{p}\colon M \longrightarrow X$ to a smooth n-stack whose fibers are n-connected.

By Theorem 7.2.1, such an integration is unique (up to a contractible space of choices): for any other integration $q\colon M \longrightarrow Y$ with n-connected fibers, there is a unique map $\widetilde{X} \longrightarrow Y$ (under M) integrating the identity map on $\mathfrak{g}$. Reversing the rôles of $\widetilde{X}$ and Y, one finds that this unique map is an equivalence.

Proposition 7.3.41. *Let $\mathfrak{g}$ be a Lie algebroid over M which integrates to a smooth n-stack and let $E \in \mathrm{Perf}^{[0,n-1]}(\mathfrak{g})$. For any square zero extension*

$$E \longrightarrow \tilde{\mathfrak{g}} \longrightarrow \mathfrak{g}$$

the Lie algebroid $\tilde{\mathfrak{g}}$ integrates to a smooth $(n+1)$-stack. Furthermore, it integrates to a smooth n-stack if the n-connected integration of $\mathfrak{g}$ is already $(n+1)$-connected.

Proof. Let $M \longrightarrow X$ be the source $(n+1)$-connected cover of the n-stack integrating $\mathfrak{g}$, which is an $(n+1)$-stack in general. The square zero extension $\tilde{\mathfrak{g}}$ is classified by a map of Lie algebroids

$$\mathfrak{g} \xrightarrow{\eta} \mathfrak{g} \oplus E[1]$$

to the split square zero extension of $\mathfrak{g}$ by $E[1]$. The functor $\mathrm{Perf}^{[0,n-1]}(X) \longrightarrow \mathrm{Perf}^{[0,n-1]}(\mathfrak{g})$ is an equivalence by Corollary 7.3.2, so that E integrates to a perfect complex $\widetilde{E}$ over X, with Tor-amplitude contained in $[0, n-1]$.

The split square zero extension $\mathfrak{g} \oplus E[1]$ is the Lie algebroid of the $(n+1)$-stack $\mathrm{Spec}_X(\widetilde{E}[2]^\vee) \longrightarrow X$, by Example 7.3.32. Since X is $(n+1)$-connected, Theorem 7.2.1 implies that the section η integrates to a section

$$\tilde{\eta}\colon X \longrightarrow \mathrm{Spec}(\widetilde{E}[2]^\vee)$$

under M. By Example 7.3.34, the extension $\tilde{\mathfrak{g}}$ arises as the Lie algebroid of the pullback

$$\begin{tikzcd} \widetilde{X} \arrow[r] \arrow[d, "p"'] & X \arrow[d, "0"] \\ X \arrow[r, "\tilde{\eta}"'] & \mathrm{Spec}_X(\widetilde{E}[2]^\vee). \end{tikzcd}$$

Since $\widetilde{E}[2]^\vee$ has Tor-amplitude contained in $(-\infty, -1]$, the zero section

$$X \longrightarrow \mathrm{Spec}(E^\vee[2])$$

is n-smooth (Lemma 5.2.34) and the map $p\colon \widetilde{X} \longrightarrow X$ is n-smooth.

It remains to check that the map $M \longrightarrow \widetilde{X}$ is a smooth surjection. To see that it induces a surjection on π_0, it suffices to observe that the map p induces an isomorphism on π_0-sheaves. This follows from the fac that the map $0\colon X \longrightarrow \mathrm{Spec}(\widetilde{E}[2]^\vee)$ induces isomorphisms on π_0-sheaves. Since the dual of $L_{M/\tilde{X}}$ is the connective module $\tilde{\mathfrak{g}}$, it follows that $M \longrightarrow \widetilde{X}$ is smooth. □

Corollary 7.3.42. *Let $\mathfrak{g}$ be a connective Lie algebroid on a smooth manifold M with the property that each $\pi_i(\mathfrak{g})$ is a (finite rank) vector bundle and $\pi_i(\mathfrak{g}) = 0$ for $i > n$. Then $\mathfrak{g}$ integrates to an $(n+1)$-stack.*

Proof. If $\mathfrak{g}$ is an ordinary Lie algebroid, then the integrability to a 2-stack is proven in [99]. For $n > 0$, we proceed by induction along the Postnikov tower

$$\mathfrak{g} = \tau_{\leq n}\mathfrak{g} \longrightarrow \tau_{\leq n-1}\mathfrak{g} \longrightarrow \dots \longrightarrow \tau_{\leq 0}\mathfrak{g}.$$

Each stage $\tau_{\leq k}\mathfrak{g}$ is a regular Lie algebroid and $\tau_{\leq k}\mathfrak{g} \longrightarrow \tau_{\leq k-1}\mathfrak{g}$ is a square zero extension by the $\tau_{\leq k-1}\mathfrak{g}$-representation $\pi_k(\mathfrak{g})[k]$. Since $\mathfrak{g}$ is assumed regular, this representation is contained in $\mathrm{Perf}^{[0,k]}(\tau_{\leq k-1}\mathfrak{g})$, so that the result follows from Proposition 7.3.41. □

Example 7.3.43. Let $\mathfrak{g}$ be an L_∞-algebroid over M whose underlying complex

$$0 \longrightarrow E_n \longrightarrow E_{n-1} \longrightarrow \dots \longrightarrow E_0$$

is a bounded of vector bundles whose differential is (locally) of constant rank. Then $\mathfrak{g}$ satisfies the conditions of Corollary 7.3.42 and can be integrated to an $(n+1)$-stack. In particular, any finite dimensional L_∞-algebra admits such an integration (cf. [39, 35] for similar results by completely different methods).

Bibliography

[1] M. Alexandrov, M. Kontsevich, A. Schwarz, and O. Zaboronsky, *The geometry of the master equation and topological quantum field theory*. Internat. J. Modern Phys. A, 12:1405–1429, 1997.

[2] C. Arias Abad and M. Crainic, *The Weil algebra and the Van Est isomorphism*. Ann. Inst. Fourier (Grenoble), 61:927–970, 2011.

[3] C. Arias Abad and M. Crainic, *Representations up to homotopy of Lie algebroids*. J. Reine Angew. Math., 663:91–126, 2012.

[4] C. Arias Abad and M. Crainic, *Representations up to homotopy and Bott's spectral sequence for Lie groupoids*. Adv. Math., 248:416–452, 2013.

[5] D. Ayala and J. Francis, *Fibrations of ∞-categories*. arXiv:1702.02681, 2017.

[6] C. Barwick and D. M. Kan, *Relative categories: another model for the homotopy theory of homotopy theories*. Indag. Math. (N.S.), 23:42–68, 2012.

[7] M. Bayeh et al. *Left-induced model structures and diagram categories*. In: *Women in topology: collaborations in homotopy theory*. Vol. 641. Contemp. Math. Amer. Math. Soc., Providence, RI, 2015, 49–81.

[8] K. Behrend and E. Getzler, *Geometric higher groupoids and categories*. arXiv:1508.02069, 2015.

[9] D. Ben-Zvi and D. Nadler, *Loop spaces and connections*. J. Topol., 5:377–430, 2012.

[10] C. Berger and I. Moerdijk, *On the derived category of an algebra over an operad*. Georgian Math. J., 16:13–28, 2009.

[11] J. Block and A. M. Smith, *The higher Riemann-Hilbert correspondence*. Adv. Math., 252:382–405, 2014.

[12] A. J. Blumberg and E. Riehl, *Homotopical resolutions associated to deformable adjunctions*. Algebr. Geom. Topol., 14:3021–3048, 2014.

[13] J. M. Boardman and R. M. Vogt, *Homotopy invariant algebraic structures on topological spaces*. Springer-Verlag, Berlin-New York. 1973.

[14] D. Borisov and J. Noel, *Simplicial approach to derived differential manifolds*. arXiv:1112.0033, 2011.

[15] A. K. Bousfield and V. K. A. M. Gugenheim, *On* PL *de Rham theory and rational homotopy type*. Mem. Amer. Math. Soc., 8:ix+94, 1976.

[16] K. S. Brown, *Abstract homotopy theory and generalized sheaf cohomology*. Trans. Amer. Math. Soc., 186:419–458, 1973.

[17] D. Calaque and J. Grivaux, *Formal moduli problems and formal derived stacks*. arXiv:1802.09556, 2018.

[18] D. Carchedi and D. Roytenberg, *Homological Algebra for Superalgebras of Differentiable Functions*. arXiv:1212.3745, 2012.

[19] D. Carchedi and D. Roytenberg, *On theories of superalgebras of differentiable functions*. Theory Appl. Categ., 28:30, 1022–1098, 2013.

[20] M. Crainic, *Differentiable and algebroid cohomology, van Est isomorphisms, and characteristic classes*. Comment. Math. Helv., 78:681–721, 2003.

[21] E. J. Dubuc and A. Kock, *On 1-form classifiers*. Comm. Algebra, 12:1471–1531, 1984.

[22] E. J. Dubuc, *C^∞-schemes*. Amer. J. Math., 103:683–690, 1981.

[23] D. Dugger, *Combinatorial model categories have presentations*. Adv. Math., 164:177–201, 2001.

[24] D. Dugger, *Universal homotopy theories*. Adv. Math., 164:144–176, 2001.

[25] W. G. Dwyer and D. M. Kan, *Simplicial localizations of categories*. J. Pure Appl. Algebra, 17:267–284, 1980.

[26] W. G. Dwyer and D. M. Kan, *Homotopy theory and simplicial groupoids*. Nederl. Akad. Wetensch. Indag. Math., 46:379–385, 1984.

[27] W. G. Dwyer, D. M. Kan, and J. H. Smith, *Homotopy commutative diagrams and their realizations*. J. Pure Appl. Algebra, 57:5–24, 1989.

[28] W. T. van Est, *Group cohomology and Lie algebra cohomology in Lie groups. I, II*. Nederl. Akad. Wetensch. Proc. Ser. A., 56:484–492, 493–504, 1953.

[29] B. Fresse, *Modules over operads and functors*. Springer-Verlag, Berlin. 2009.

[30] B. Fresse, *Homotopy of Operads and Grothendieck-Teichmüller Groups: Part 2: The Applications of (Rational) Homotopy Theory Methods*. American Mathematical Society, Providence, RI. 2017.

[31] A. Frölicher and A. Nijenhuis, *A theorem on stability of complex structures*. Proc. Nat. Acad. Sci. U.S.A., 43:239–241, 1957.

[32] D. Gaitsgory, *Ind-coherent sheaves*. Mosc. Math. J., 13:399–528, 553, 2013.

[33] D. Gaitsgory and N. Rozenblyum, *A study in derived algebraic geometry*. American Mathematical Society, Providence, RI. 2017.

[34] D. Gepner, R. Haugseng, and T. Nikolaus, *Lax colimits and free fibrations in ∞-categories*. arXiv:1501.02161, 2015.

[35] E. Getzler, *Lie theory for nilpotent L_∞-algebras*. Ann. of Math. (2), 170:271–301, 2009.

[36] R. Godement, *Topologie algébrique et théorie des faisceaux*. Hermann, Paris. 1958.

[37] Y. Harpaz and M. Prasma, *The Grothendieck construction for model categories*. Adv. Math., 281:1306–1363, 2015.

[38] B. Hennion, *Tangent Lie algebra of derived Artin stacks*. Journal für die reine und angewandte Mathematik (Crelles Journal), 2015.

[39] A. Henriques, *Integrating L_∞-algebras*. Compos. Math., 144:1017–1045, 2008.

[40] V. Hinich, *DG coalgebras as formal stacks*. J. Pure Appl. Algebra, 162:209–250, 2001.

[41] V. Hinich, *Deformations of homotopy algebras*. Comm. Algebra, 32:473–494, 2004.

[42] V. Hinich, *Dwyer-Kan localization revisited*. Homology Homotopy Appl., 18:27–48, 2016.

[43] V. Hinich and V. Schechtman. *Homotopy Lie algebras*. In: *I. M. Gel'fand Seminar*. Vol. 16. Adv. Soviet Math. Amer. Math. Soc., Providence, RI, 1993, 1–28.

[44] M. J. Hopkins. *Topological modular forms, the Witten genus, and the theorem of the cube*. In: *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*. Birkhäuser, Basel, 1995, 554–565.

[45] J. Huebschmann, *Multi derivation Maurer-Cartan algebras and sh Lie-Rinehart algebras*. J. Algebra, 472:437–479, 2017.

[46] L. Illusie, *Complexe cotangent et déformations. I, II*. Springer-Verlag, Berlin-New York. 1971/1972.

[47] A. Joyal, *The Theory of Quasi-Categories and its Applications*. Preprint, 2008.

[48] A. Joyal and I. Moerdijk, *A completeness theorem for open maps*. Ann. Pure Appl. Logic, 70:51–86, 1994.

[49] A. Joyal and M. Tierney. *Quasi-categories vs Segal spaces*. In: *Categories in algebra, geometry and mathematical physics*. Vol. 431. Contemp. Math. Amer. Math. Soc., Providence, RI, 2007, 277–326.

[50] D. Joyce, *Algebraic Geometry over $\mathcal{C}^\infty$-rings*. arXiv:1001.0023, 2016.

[51] D. Joyce. *An introduction to d-manifolds and derived differential geometry*. In: *Moduli spaces*. Vol. 411. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 2014, 230–281.

[52] M. M. Kapranov. *On DG-modules over the de Rham complex and the vanishing cycles functor*. In: *Algebraic geometry (Chicago, IL, 1989)*. Vol. 1479. Lecture Notes in Math. Springer, Berlin, 1991, 57–86.

[53] M. Kapranov, *Free Lie algebroids and the space of paths*. Selecta Math. (N.S.), 13:277–319, 2007.

[54] L. Kjeseth, *Homotopy Rinehart cohomology of homotopy Lie-Rinehart pairs*. Homology Homotopy Appl., 3:139–163, 2001.

[55] K. Kodaira and D. C. Spencer, *On deformations of complex analytic structures. I, II*. Ann. of Math. (2), 67:328–466, 1958.

[56] M. Kontsevich, *Deformation quantization of Poisson manifolds*. Lett. Math. Phys., 66:157–216, 2003.

[57] D. Li, *Higher Groupoid Actions, Bibundles, and Differentiation*. arXiv:1512.0420, 2015.

[58] J.-L. Loday and B. Vallette, *Algebraic operads*. Springer, Heidelberg. 2012.

[59] J. Lurie, *Higher topos theory*. Princeton University Press. Princeton, NJ. 2009.

[60] J. Lurie, *Derived Algebraic Geometry V: Structured Spaces*. Available at author's webpage: `http://www.math.harvard.edu/~lurie/`, 2011.

[61] J. Lurie, *Derived Algebraic Geometry X: Formal Moduli Problems*. Available at author's webpage: `http://www.math.harvard.edu/~lurie/`, 2011.

[62] J. Lurie, *Higher Algebra*. Available at author's webpage: `http://www.math.harvard.edu/~lurie/`, 2016.

[63] J. Lurie, *Spectral Algebraic Geometry*. Available at author's webpage: `http://www.math.harvard.edu/~lurie/`, 2017.

[64] K. C. H. Mackenzie and P. Xu, *Integration of Lie bialgebroids*. Topology, 39:445–467, 2000.

[65] M. Manetti, *Extended deformation functors*. Int. Math. Res. Not., 719–756, 2002.

[66] I. Moerdijk, N. V. Quê, and G. E. Reyes. *Rings of smooth functions and their localizations. II*. In: *Mathematical logic and theoretical computer science (College Park, Md., 1984–1985)*. Vol. 106. Lecture Notes in Pure and Appl. Math. Dekker, New York, 1987, 277–300.

[67] I. Moerdijk and J. Mrčun, *On integrability of infinitesimal actions*. Amer. J. Math., 124:567–593, 2002.

[68] I. Moerdijk and G. E. Reyes, *Models for smooth infinitesimal analysis*. Springer-Verlag, New York. 1991.

[69] J. Muñoz and J. Ortega, *Sobre las Algebras localmente convexas*. Collectanea Mathematica, 20:128–148, 1969.

[70] J. Nuiten, *Localizing ∞-categories with hypercovers*. arXiv:1612.03800, 2016.

[71] J. Nuiten, *Homotopical algebra for Lie algebroids*. arXiv:1712.03441, 2017.

[72] J. Nuiten, *Koszul duality for Lie algebroids*. arXiv:1712.03442, 2017.

[73] T. Pantev, B. Toën, M. Vaquié, and G. Vezzosi, *Shifted symplectic structures*. Publ. Math. Inst. Hautes Études Sci., 117:271–328, 2013.

[74] D. Pavlov and J. Scholbach, *Admissibility and rectification of colored symmetric operads*. arXiv:1410.5675, 2014.

[75] J. P. Pridham, *Unifying derived deformation theories*. Adv. Math., 224:772–826, 2010.

[76] J. P. Pridham, *Derived moduli of schemes and sheaves*. J. K-Theory, 10:41–85, 2012.

[77] J. P. Pridham, *Presenting higher stacks as simplicial schemes*. Adv. Math., 238:184–245, 2013.

[78] J. P. Pridham, *Shifted Poisson and symplectic structures on derived N-stacks*. J. Topol., 10:178–210, 2017.

[79] B. Pym and P. Safronov, *Shifted symplectic Lie algebroids*. arXiv:1612.09446, 2016.

[80] D. G. Quillen, *Homotopical algebra*. Springer-Verlag, Berlin-New York. 1967.

[81] D. G. Quillen, *Rational homotopy theory*. Ann. of Math. (2), 90:205–295, 1969.

[82] G. S. Rinehart, *Differential forms on general commutative algebras*. Trans. Amer. Math. Soc., 108:195–222, 1963.

[83] M. Schlessinger, *Functors of Artin rings*. Trans. Amer. Math. Soc., 130:208–222, 1968.

[84] C. J. Schommer-Pries, *Central extensions of smooth 2-groups and a finite-dimensional string 2-group*. Geom. Topol., 15:609–676, 2011.

[85] T. Schürg, B. Toën, and G. Vezzosi, *Derived algebraic geometry, determinants of perfect complexes, and applications to obstruction theories for maps and complexes*. J. Reine Angew. Math., 702:1–40, 2015.

[86] G. Segal, *Categories and cohomology theories*. Topology, 13:293–312, 1974.

[87] P. Ševera, *L-infinity algebras as 1-jets of simplicial manifolds (and a bit beyond)*. arXiv:math/0612349, 2006.

[88] *Théorie des intersections et théorème de Riemann-Roch*. Springer-Verlag, Berlin-New York. 1971.

[89] C. Simpson, *Algebraic (geometric) n-stacks*. arXiv:alg-geom/9609014, 1998.

[90] C. Simpson. *Homotopy over the complex numbers and generalized de Rham cohomology*. In: *Moduli of vector bundles (Sanda, 1994; Kyoto, 1994)*. Vol. 179. Lecture Notes in Pure and Appl. Math. Dekker, New York, 1996, 229–263.

[91] M. Spitzweck, *Operads, Algebras and Modules in General Model Categories*. arXiv:math/0101102, 2001.

[92] D. I. Spivak, *Derived smooth manifolds*. Duke Math. J., 153:55–128, 2010.

[93] S. Stolz and P. Teichner. *What is an elliptic object?* In: *Topology, geometry and quantum field theory*. Vol. 308. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 2004, 247–343.

[94] R. W. Thomason, *Algebraic K-theory and étale cohomology*. Ann. Sci. École Norm. Sup. (4), 18:437–552, 1985.

[95] B. Toën, *Problèmes de modules formels*. Séminaire Bourbaki, no. 1111:2016.

[96] B. Toën and M. Vaquié, *Moduli of objects in dg-categories*. Ann. Sci. École Norm. Sup. (4), 40:387–444, 2007.

[97] B. Toën and G. Vezzosi, *Homotopical algebraic geometry. II. Geometric stacks and applications*. Mem. Amer. Math. Soc., 193:x+224, 2008.

[98] B. Toën and G. Vezzosi, *Algèbres simpliciales S^1-équivariantes, théorie de de Rham et théorèmes HKR multiplicatifs*. Compos. Math., 147:1979–2000, 2011.

[99] H.-H. Tseng and C. Zhu, *Integrating Lie algebroids via stacks*. Compos. Math., 142:251–270, 2006.

[100] B. Vallette, *Homotopy theory of homotopy algebras*. arXiv:1411.5533, 2014.

[101] G. Vezzosi. *A model structure on relative dg-Lie algebroids*. In: *Stacks and categories in geometry, topology, and algebra*. Vol. 643. Contemp. Math. Amer. Math. Soc., Providence, RI, 2015, 111–118.

[102] G. Vezzosi, *Quadratic forms and Clifford algebras on derived stacks*. Adv. Math., 301:161–203, 2016.

[103] L. Vitagliano, *On the strong homotopy Lie-Rinehart algebra of a foliation*. Commun. Contemp. Math., 16:1450007, 49, 2014.

[104] J. Wolfson, *Descent for n-bundles*. Adv. Math., 288:527–575, 2016.

[105] C. Zhu, *n-groupoids and stacky groupoids*. Int. Math. Res. Not., 4087–4141, 2009.

Glossary of notation

$A\{a^{-1}\}$	localization of $\mathcal{C}^\infty$-ring, 101
$A[\epsilon_n]$	first order infinitesimal, 35
$\mathrm{At}(E)$	Atiyah Lie algebroid of a module, 38
$\mathrm{At}_{\mathcal{P}}(R)$	of a $\mathcal{P}$-algebra, 39
$\mathcal{B}(A)$	elementary open subsets of $\mathrm{Spec}(A)$, 102
$C^*(\mathfrak{g}, E)$	Chevalley-Eilenberg complex, 67
$c^*(\mathfrak{g}, E)$	connective cover, 68
$\overline{C}^*(\mathfrak{g}, E)$	reduced, 66
$\mathfrak{D}$	Koszul duality functor, 65, 69
Def_R	deformation functor of a $\mathcal{P}$-algebra, 33, 36, 88
$\mathrm{Def}_{X_\bullet/M}$	of a diagram of stacks over M, 154
$\overline{\mathrm{Env}}_{\mathfrak{g}}$	reduced enveloping operad, 52
$\mathrm{et}_!(F)$	etale space of a sheaf F, 120
$(f_*, f^!)$	induction/restriction of representations, 61
$\mathfrak{g} \boxtimes B$	tensoring of dg-Lie algebroids, 47
Inf_M	infinitesimal site, 142
$K(\mathfrak{g})$	Koszul complex, 67
$L_{X/S}$	cotangent complex, 133, 136
$L_{A/B}$	of a map of $\mathcal{C}^\infty$-rings, 22
$L_{\mathfrak{g}}$	of a Lie algebroid, 65
$L_{X/S,x}$	relative cotangent space, 132
$\mathrm{MC}_{\mathfrak{g}}$	formal moduli problem classified by $\mathfrak{g}$, 75
$Q(\mathfrak{g})$	cofibrant replacement of dg-Lie algebroid, 45
$\mathrm{dR}(X)$	de Rham complex, 179
$\mathrm{Sing}(Y/X)$	fiberwise singular complex, 123
$\mathfrak{s}_n$	free Lie algebroid on one generator, 76
$\mathrm{Spec}(A)$	spectrum of $\mathcal{C}^\infty$-ring, 103
$\mathrm{Spec}_X(E)$	total space of a quasi-coherent sheaf, 117
Stack_n	moduli of derived n-stacks, 111, 147
$T_{M/X}$	relative tangent bundle, 129, 146
$T_{A/B}$	of a map of $\mathcal{C}^\infty$-rings, 38, 70
$T_{A/X}$	of a formal moduli problem, 36, 75, 79
$\tau_{\geq 0}$	connective cover, 13
$\tau_{\leq n}$	n-truncation
$\mathcal{U}(\mathfrak{g})$	universal enveloping algebra, 60
$W(\mathfrak{g})$	Weil algebra, 181
X_{dR}	de Rham space, 160
X_E	square zero extension of X by a quasi-coherent sheaf E, 130
$\widehat{X}$	deformation functor associated to X, 129
$X^\wedge_Y$	formal completion of X at Y, 160
Ω_A	Kähler differentials, 22

∞-categories

Aff	affine derived manifolds, 100
CAlg_k	commutative k-algebras, 14, 19
$\mathrm{CAlg}_k^{\geq 0}$	connective, 14
Cat_∞	small ∞-categories, 13
$\mathcal{C}^\infty\mathrm{Alg}$	$\mathcal{C}^\infty$-rings, 18
$\mathcal{C}^\infty\mathrm{Alg}^{\mathrm{sm}}/A$	small extensions of A, 35
dMfd	derived manifolds, 100
FMP_A	formal moduli problems over A, 35
Lie_A	Lie algebras over A, 42
$\mathrm{LieAlgd}_A$	Lie algebroids over A, 41
$\mathrm{LieAlgd}_A^{\mathrm{good}}$	good, 76
Mod_A	A-modules, 13
$\mathrm{Mod}_A^{\geq 0}$	connective, 13
$\mathrm{Mod}(X)$	over a formal moduli problem, 86
$\mathrm{Mod}_{\mathcal{C}^\infty}$	$\mathcal{C}^\infty$-rings and modules, 105
$\mathcal{P}\mathrm{Alg}_A$	$\mathcal{P}$-algebras over A, 14, 30
$\mathcal{P}\mathrm{Alg}_A^{\geq 0}$	connective, 14, 30
$\mathcal{P}\mathrm{Rep}_{\mathfrak{g}}$	with representation of $\mathfrak{g}$, 62, 90
$\mathcal{P}\mathrm{Alg}(X)$	over a formal moduli problem, 94
$\mathrm{Perf}(X)$	perfect complexes, 115
$\mathrm{Perf}^{[a,b]}(X)$	with Tor-amplitude contained in $[a,b]$, 116
$\mathrm{Perf}(\mathfrak{g})$	with representation of $\mathfrak{g}$, 175
Pr^{L}	locally presentable ∞-categories, 15
$\mathrm{QC}(X)$	quasi-coherent sheaves, 106, 115
$\mathrm{Rep}_{\mathfrak{g}}$	$\mathfrak{g}$-representations, 62
$\mathcal{S}$	spaces, 13
$\mathrm{Sh}(\mathrm{Aff})$	sheaves $\mathrm{Aff}^{\mathrm{op}} \longrightarrow \mathcal{S}$, 107
$\mathrm{Sh}(X)$	sheaves on a stack X, 120
$\mathrm{Sh}_{\mathcal{O}}(X)$	$\mathcal{O}$-module sheaves, 106
Sp	spectra, 75
$\mathrm{Top}_{\mathcal{C}^\infty}^{\mathrm{loc}}$	locally $\mathcal{C}^\infty$-ringed spaces, 100

Categories

$\mathcal{M}^{\mathrm{cof}}$	cofibrant objects of model category, 12
$\mathcal{M}^{\mathrm{fib}}$	fibrant objects of model category, 12
$\mathrm{CAlg}_k^{\mathrm{dg}}$	commutative dg-k-algebras, 14
$\mathcal{C}^\infty\mathrm{Alg}^{\mathrm{dg}}$	dg-$\mathcal{C}^\infty$-rings, 18
$\mathrm{LieAlgd}_A^{\mathrm{dg}}$	dg-Lie algebroids over A, 38
$\mathrm{LieAlgd}_A^{A-\mathrm{cof}}$	cofibrant as dg-A-module, 45
$L_\infty\mathrm{Algd}_A^{\mathrm{dg}}$	L_∞-algebroids over A, 39
$L_\infty\mathrm{Algd}_A^{\mathrm{gr,dg}}$	graded, 51
$L_\infty\mathrm{Algd}_A^{\mathbb{N},\mathrm{dg}}$	weakly filtered, 51
$L_\infty\mathrm{Algd}_A^{\mathrm{nonlin}}$	with nonlinear maps, 44
$\mathrm{Mod}_A^{\mathrm{dg}}$	dg-A-modules, 13
$\mathrm{Mod}_A^{\geq 0,\mathrm{dg}}$	nonnegatively graded, 13
$\mathrm{Mod}_A^{\Sigma,\mathrm{dg}}$	symmetric sequences of, 53
$\mathrm{Mod}^{\mathrm{dg}}$	commutative dg-algebras with modules, 84
$\mathcal{P}\mathrm{Alg}_A^{\mathrm{dg}}$	dg-$\mathcal{P}$-algebras over A, 14, 30
$\mathcal{P}\mathrm{Rep}_{\mathfrak{g}}^{\mathrm{dg}}$	with representation of $\mathfrak{g}$, 88
$\mathrm{Rep}_{\mathfrak{g}}^{\mathrm{dg}}$	dg-$\mathfrak{g}$-representations, 59
$\mathrm{Rep}_{\mathfrak{g}}^{\geq 0,\mathrm{dg}}$	nonnegatively graded, 61
$\mathrm{Rep}^{\mathrm{dg}}$	dg-Lie algebroids with representations, 84

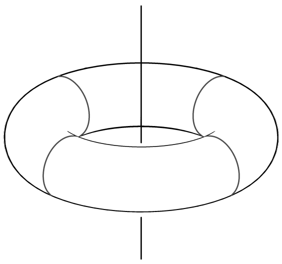
Index

Samenvatting

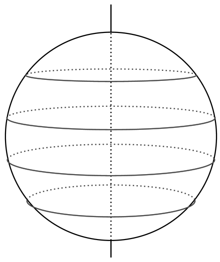
Het centrale thema van dit proefschrift is het verband tussen de meetkunde van moduliruimten, hun deformatietheorie en de theorie van Lie algebroïden. De twee hoofdresultaten geven een precieze beschrijving van dit verband, in termen van een equivalentie tussen Lie algebroïden en formele moduliruimten over een (mogelijk singuliere) differentieerbare variëteit, samen met een integreerbaarheidsstelling voor afbeeldingen tussen Lie algebroïden. Beide resultaten maken intensief gebruik van methodes uit de algebraïsche topologie en afgeleide differentiaaltopologie.

Differentiaaltopologie bestudeert meetkundige figuren, of ruimtes, met een goed begrip van ‘raakruimte’ dat kan worden gebruikt om functies herhaaldelijk te differentiëren. Voorbeelden van dit soort figuren zijn de rechte lijn $\mathbb{R}$ en de cirkel (beide hebben raaklijnen) en het platte vlak $\mathbb{R}^2$, de bolschil en de torus (met raakvlakken). Er zijn twee algemene methodes om meer voorbeelden van dergelijke ruimtes te produceren: door het opstellen van een stelsel vergelijkingen of door een symmetrie uit te delen.

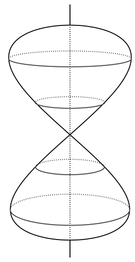
De genoemde voorbeelden kunnen allen worden verkregen via de eerste methode: de (eenheids)cirkel wordt beschreven door de vergelijking $x^2 + y^2 = 1$ in $\mathbb{R}^2$, en de torus en de bolschil door vergelijkingen (a) en (b) hieronder. De oplossingsruimte van een stelsel vergelijkingen hoeft echter niet altijd glad te zijn; zie bijvoorbeeld de zandloperfiguur (c).



(a): $8(x^2 + y^2) = (x^2 + y^2 + z^2 + 1)^2$



(b): $x^2 + y^2 + z^2 = 1$



(c): $x^2 + y^2 = z^2(1 - z^2)$

Deze drie figuren hebben een rotatiesymmetrie om de z-as. Door de punten op iedere cirkel om de z-as met elkaar te identificeren verkrijgt men een nieuwe ruimte die eruitziet als (a) de cirkel of (b), (c) het gesloten interval $[-1, 1]$.

Deformatietheorie onderzoekt kleine vervormingen in dit soort figuren. Zo zien kleine vervormingen van de torus en de bolschil er niet wezenlijk anders uit, maar kan Figuur (c) zowel worden gescheiden in twee bollen, als worden samengedrukt tot een samenhangend figuur. Een ander voorbeeld van een deformatieprobleem is de vraag in hoeverre een punt in een ruimte kan worden bewogen. Voor kleine bewegingen kan dit worden bestudeerd door middel van een lineaire benadering: kleine (eerste orde) verschuivingen van een punt worden precies beschreven door de raakruimte in dat punt.

Het opstellen van vergelijkingen of het uitdelen van symmetrieën resulteert echter niet altijd in een ruimte waarvan de raakruimtes direct duidelijk zijn. In Figuur (c) hebben bijvoorbeeld alle punten een zichtbaar raakvlak, behalve het (meest interessante!) middelpunt; daar blijkt de raakruimte niet 2-, maar 3-dimensionaal te zijn. Op dezelfde manier hebben bijna alle punten op het interval $[-1, 1]$ een raaklijn, behalve de twee randpunten ± 1; deze punten corresponderen met de rotatie-invariante maxima en minima van Figuur (b) en (c).

Om dit soort singuliere punten effectief te kunnen behandelen, maken we gebruik van de theorie van afgeleide differentiaaltopologie, uitgewerkt in Hoofdstuk 5. Deze theorie beschrijft ruimtes met singulariteiten in termen van zogenaamde ‘afgeleide variëteiten’ en ‘stacks’. Het uitgangspunt hierbij is dat men onthoudt *hoe* een meetkundig figuur precies is verkregen door het stellen van vergelijkingen en het uitdelen van symmetrie. Bovendien moet rekening worden gehouden met relaties tussen vergelijkingen en symmetrieën tussen symmetrieën.

Singuliere punten hebben nu een raakcomplex, dat bestaat uit meerdere lagen van raakruimtes: het middelpunt van Figuur (c) heeft bijvoorbeeld een raakcomplex dat 3-dimensionaal is in graad 0 en 1-dimensionaal in graad -1. Werk van Pridham en Lurie laat zien dat een formele omgeving van een punt, en daarmee de manieren om dit punt (infinitesimaal) te bewegen, volledig kan worden beschreven door de structuur van een Lie algebra op zijn raakcomplex.

Dit proefschrift behandelt een uitbreiding van dit resultaat tot deformaties van een *ruimte* van punten in plaats van één punt. Lie algebras worden nu vervangen door Lie algebroïden, waarvan de homotopietheorie wordt ontwikkeld in Hoofdstuk 3. Het eerste hoofdresultaat (Stelling I, bewezen in Hoofdstuk 4) is een equivalentie tussen de homotopietheorie van Lie algebroïden en de homotopietheorie van formele moduliruimten over een afgeleide variëteit M.

In het bijzonder geeft dit een beschrijving van een formele omgeving van een afgeleide variëteit M in een afgeleide stack X, en zodoende van de manieren om M te bewegen binnen X. Deze omgeving wordt volledig vastgelegd door de structuur van een Lie algebroïde op de raakbundel van M over X. Als M de oplossingsruimte is van een onafhankelijk stelsel vergelijkingen, dan is deze Lie algebroïde simpelweg de normaalbundel in graad -1, met triviale Lie haak.

Het tweede hoofdresultaat van dit proefschrift beschouwt de complementaire situatie waarin $M \longrightarrow X$ een gladde quotiëntafbeelding is. Dit is de situatie waarin X een afgeleide stack is, verkregen uit M door het uitdelen van symmetrieën en homotopieën daartussen. In dit geval beschrijft Stelling II een nauw verband tussen het quotiënt X en de bijbehorende Lie algebroïde over M: als de vezels van M over X voldoende samenhangend zijn, dan kan iedere afbeelding van Lie algebroïden worden geïntegreerd tot een afbeelding tussen stacks. Dit reproduceert in het bijzonder bekende resultaten van Van Est en Crainic over het verband tussen de cohomologie van Lie groepoïden en Lie algebroïden. Tevens volgt de integreerbaarheid van eindigdimensionale L_∞-algebras tot hogere Lie groepen.

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