HIGHER STACKS AS A CATEGORY OF FRACTIONS

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1. Introduction

A well-known principle in the theory of Lie groupoids asserts that the tranverse geometry of a Lie groupoid models the geometry of the associated differentiable stack. This statement (which applies in many geometric contexts) has been made more precise from a categorical perspective in the the work of Pronk [21]: the 2-category of differentiable stacks can be obtained, up to equivalence, from the category of Lie groupoids by formally identifying Lie groupoids that are Morita equivalent, i.e. whose transverse geometry is the same. The identification of Morita equivalent Lie groupoids is achieved by taking the 2-categorical localization of the category of Lie groupoids at a class of maps known as Morita equivalences. In turn, various descriptions of this 2-categorical localization exist in the literature: for example, its mapping groupoids can be described in terms of bibundles and intertwiners between those [9], or in terms of zig-zags $\cdot \stackrel{\sim}{\leftarrow} \cdot \to \cdot$ of maps, where the left map is either a Morita equivalence or is restricted to be a hypercover.

The aim of this paper is to extend these results to higher stacks and higher groupoids, i.e. simplicial objects satisfying certain horn-filling conditions. In particular, we will show how higher categories of stacks can be obtained from categories of higher groupoids by formally inverting a set of Morita equivalences between them. In the differential-geometric setting, these higher groupoids are known as Lie n-groupoids, which are simplicial manifolds X for which the manifold X_k of k-simplices maps to the manifold of $\Lambda^i[k]$-horns by a surjective submersion and a diffeomorphism for $k > n$ (see eg. [1, 8, 26]). The category $\mathrm{Gpd}_n^{\mathrm{Lie}}$ of Lie n-groupoids comes equipped with classes of hypercovers and Morita equivalences, which reduce to the usual notions of hypercovers and Morita equivalences between Lie groupoids when $n = 1$.

For higher n, the 2-categorical localization of $\mathrm{Gpd}_n^{\mathrm{Lie}}$ at the Morita equivalences is only a rough approximation to a richer (and better behaved) object: its simplicial (or ∞-categorical) localization $\mathrm{Gpd}_n^{\mathrm{Lie}}[W_{\mathrm{Mor}}^{-1}]$, which in fact turns out to be an $(n+1)$-category, agreeing with the 2-categorical localization when $n = 1$. The mapping spaces of this simplicial localization $\mathrm{Gpd}_n^{\mathrm{Lie}}[W_{\mathrm{Mor}}^{-1}]$ have analogous descriptions as in the case of ordinary groupoids: on the one hand, the space of maps between two Lie n-groupoids X and Y can be described as the classifying space of the category of spans

$$X \xleftarrow{\sim} \tilde{X} \longrightarrow Y$$

where the left map is a hypercover (see e.g. [13]). Given two spans $X \stackrel{\sim}{\leftarrow} \tilde{X} \to Y$ and $Y \stackrel{\sim}{\leftarrow} \tilde{Y} \to Z$, there is a composed span $X \stackrel{\sim}{\leftarrow} \tilde{X} \times_Y \tilde{Y} \to Z$, and one can use this to define a (coherently associative) composition structure on these mapping spaces, yielding a model for the ∞-categorical localization $\mathrm{Gpd}_n^{\mathrm{Lie}}[W_{\mathrm{Mor}}^{-1}]$ (see e.g. [11], [19]). On the other hand, the mapping space between two Lie n-groupoids X and Y has a description in terms of X–Y-bibundles (see Section 6). However, contrary to the classical case, the bibundles we obtain are more general objects than ordinary smooth manifolds: they are certain

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$(n-1)$-stacks, carrying homotopy coherent (and coherently commuting) actions of X and Y.

Such higher-categorical analogues of stacks of groupoids (over the site of smooth manifolds) can be organized into an ∞-category $\mathrm{Sh}_\infty(\mathrm{Mfd})$ of stacks (of spaces) over smooth manifolds. Explicitly, this ∞-category $\mathrm{Sh}_\infty(\mathrm{Mfd})$ can be modeled by the simplicially enriched category $\mathrm{sSh}(\mathrm{Mfd})$ of simplicial sheaves on the site of smooth manifolds, endowed with a model structure in which the weak equivalences are the maps that induce isomorphisms on sheaves of homotopy groups [15]. There is an obvious functor

$$\mathrm{Gpd}_n^{\mathrm{Lie}} \subseteq \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Mfd}) \longrightarrow \mathrm{sSh}(\mathrm{Mfd}) \tag{1}$$

sending each Lie n-groupoid X to the simplicial sheaf that is represented in degree k by the manifold X_k. This functor sends Morita equivalences to weak equivalences in this model structure and therefore induces a functor of ∞-categories

$$\mathrm{Gpd}_n^{\mathrm{Lie}}[W_{\mathrm{Mor}}^{-1}] \longrightarrow \mathrm{Sh}_\infty(\mathrm{Mfd}). \tag{2}$$

Our main result can then be formulated as follows:

Theorem. *The above functor* $\mathrm{Gpd}_n^{\mathrm{Lie}}[W_{\mathrm{Mor}}^{-1}] \to \mathrm{Sh}_\infty(\mathrm{Mfd})$ *is fully faithful (i.e. induces weak equivalences on mapping spaces), with essential image consisting of the* n-geometric stacks *in the sense of Simpson* [24]. *In particular, given two Lie n-groupoids X and Y, the space of maps* $\mathrm{Map}_{\mathrm{Gpd}_n^{\mathrm{Lie}}[W_{\mathrm{Mor}}^{-1}]}(X, Y)$ *admits the following three models:*

(1) the nerve of the category of spans of Lie n-groupoids $X \xleftarrow{\sim} \tilde{X} \to Y$ where the left map is a hypercover (or a Morita equivalence).
(2) the space of maps $\mathrm{Map}_{\mathrm{sSh}(\mathrm{Mfd})}(X^\wedge, Y^\wedge)$ beween the fibrant replacements of X and Y in the Joyal model structure on simplicial sheaves.
(3) the quasicategory (in fact, the Kan complex) of X–Y-bibundles as defined in Definition 6.1.

This theorem is very much of a (higher) categorical, rather than a differential-geometric nature. As such, it naturally generalizes to various other contexts, some of which are of a more homotopy-theoretic flavour. For example, in the setting of derived algebraic geometry, one can consider 'derived Artin n-groupoids', i.e. (smooth) n-groupoid objects taking values in derived schemes. In this case, derived schemes already have a homotopy theory of their own, which means that derived Artin n-groupoids already form a higher category *before* localizing at the Morita equivalences. In particular, a derived scheme does not necessarily determine a sheaf of sets, so that the functor (1) has to be refined to a functor of the form

$$\mathrm{Gpd}_n^{\mathrm{dArt}} \subseteq \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{\mathrm{op}}, \mathrm{Sch}_{\mathrm{d}}) \longrightarrow \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{\mathrm{op}}, \mathrm{Sh}_\infty(\mathrm{Sch}_{\mathrm{d}})) \xrightarrow{\mathrm{colim}} \mathrm{Sh}_\infty(\mathrm{Sch}_{\mathrm{d}})$$

from the ∞-category of derived Artin n-groupoids to the ∞-category of stacks (of spaces) over derived schemes. This functor sends Morita equivalences to equivalences of stacks, and therefore determines a functor

$$\mathrm{Gpd}_n^{\mathrm{dArt}}[W_{\mathrm{Mor}}^{-1}] \longrightarrow \mathrm{Sh}_\infty(\mathrm{Sch}_{\mathrm{d}}) \tag{3}$$

In fact, the above functor factors over a certain simplicial category of 'higher (derived) Artin groupoids', constructed by Pridham [20], which is shown to be equivalent to the full sub-∞-category of $\mathrm{Sh}_\infty(\mathrm{Sch}_{\mathrm{d}})$ on the higher Artin stacks. Although this simplicial category has the property that Morita equivalences between higher groupoids are indeed homotopy equivalences, it is not entirely clear whether it is *universal* with this property, i.e. whether it is also equivalent to $\mathrm{Gpd}_n^{\mathrm{dArt}}[W_{\mathrm{Mor}}^{-1}]$.

For this reason, we will take a different route and use an explicit description of $\mathrm{Gpd}_n^{\mathrm{dArt}}[W_{\mathrm{Mor}}^{-1}]$ in terms of ∞-categories of spans (discussed in [19]), similar to the case of Lie n-groupoids. Using this description of localizations of categories of 'geometric' n-groupoids, we give a proof of our main result, which in particular implies that the functors (2) and (3) are fully faithful:

Theorem (5.1)**.** *Let $\mathcal{X}$ be an ∞-topos together with a subcategory $\mathcal{X}_{0-\mathrm{geom}}$ of '0-geometric stacks' and a set P of 'smooth' maps between them (see Assumption 3.6). The colimit functor $|-|\colon \mathrm{Gpd}_n^{\mathrm{geom}} \subseteq \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{\mathrm{op}}, \mathcal{X}) \to \mathcal{X}$ induces a functor*

$$\mathrm{Gpd}_n^{\mathrm{geom}}[W_{\mathrm{Mor}}^{-1}] \longrightarrow \mathcal{X}$$

which is fully faithful with essential image given by the n-geometric stacks.

In the end, this result mostly involves categorical and simplicial methods, instead of geometric ones. For example, the description of $\mathrm{Gpd}_n^{\mathrm{geom}}[W_{\mathrm{Mor}}^{-1}]$ in terms of spans reduces the proof to a repeated application of Quillen's Theorem A, using a certain 'refinement of hypercovers' construction (appearing in [20]) in each step. Even if the original category of n-groupoid objects is just an ordinary category (as in the case of Lie n-groupoids), the intermediate stages appearing in this procedure involve ∞-categories of spans. To deal with these objects, we decided to work with quasicategories instead of simplicial categories or model categories. The use of quasicategories also allows us to compare the mapping spaces of $\mathrm{Gpd}_n^{\mathrm{geom}}[W_{\mathrm{Mor}}^{-1}]$ (or its essential image in $\mathcal{X}$) with certain categories of bibundles. More precisely, we show the following:

Proposition (6.3)**.** *Let X and Y be geometric n-groupoids in $\mathcal{X}$. Then the mapping space $\mathrm{Map}_{\mathrm{Gpd}_n^{\mathrm{geom}}[W_{\mathrm{Mor}}^{-1}]}(X,Y) \simeq \mathrm{Map}_{\mathcal{X}}(|X|,|Y|)$ can be identified with the ∞-category (which is in fact a Kan complex) of X–Y-bibundles in the sense of Definition 6.1.*

Outline. We will use Section 2 to elaborate a bit on its content in the setting of Lie n-groupoids discussed above, which may hopefully serve as a guiding example for the rest of the paper. Section 3 is a recollection of the basic notions and results involving stacks and geometric stacks (following e.g. [24, 25]). Section 4 recalls the notions of n-groupoid objects, geometric n-groupoid objects (e.g. Lie n-groupoids) and hypercovers between them (based on e.g. [1, 8, 20]); its two main results are topos-theoretic variations of the well-known fact that taking the realization of a simplicial space sends hypercovers to equivalences and preserves homotopy pullbacks along Kan fibrations (see Proposition 4.9 and 4.12). Section 5 is devoted to a proof of our main result stated above (Theorem 5.1). We included a brief discussion of bibundles in Section 6, where we prove Proposition 6.3, and conclude some examples of this theorem in Section 7.

Conventions. Throughout, we will use quasicategories as our chosen model for ∞-categories. We will use $\mathcal{S}$ to denote the ∞-category of spaces (i.e. the coherent nerve of the simplicial category of Kan complexes), $\mathcal{C}/c$ to denote the usual over-∞-category of $\mathcal{C}$ in the sense of Joyal and $\mathcal{C}^{/c}$ to denote the 'alternative slice construction' of [16, Section 4.2.1]. We will denote by $\Delta\{i_1, ..., i_k\} \subseteq \Delta[n]$ the subface of the n-simplex whose vertices are $i_1, ..., i_k$.

2. Example: Lie n-groupoids

In this section we will summarize the main results and definitions that appear in the rest of the text, in the familiar setting of differential topology. Let Mfd be the category of (Hausdorff, second countable) smooth manifolds and let sSh(Mfd) be the category of simplicial sheaves with respect to the usual open cover topology. A simplicial sheaf $F\colon \mathrm{Mfd}^{\mathrm{op}} \to \mathrm{sSet}$ induces a sheaf on $\mathbb{R}^n$, for each $n \geq 0$; we will say that a map of simplicial sheaves $F \to G$ is a weak equivalence if the map $F_x \to G_x$ of stalks at each $x \in \mathbb{R}^n$ is a weak equivalence of simplicial sets, for each $n \geq 0$ (or equivalently, if $F \to G$ induces isomorphisms on homotopy sheaves). These are the weak equivalences in a model structure due to Joyal [15], in which the cofibrations are the monomorphisms.

Although any simplicial sheaf is weakly equivalent to a simplicial sheaf which is represented in each simplicial degree by a coproduct of manifolds (cf. [6]), there is a particular class of more 'geometric' simplicial sheaves, which are represented by *Lie n-groupoids*. Recall that a Lie n-groupoid is a simplicial manifold $X\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Mfd}$ satisfying the following version of the Kan condition: for each horn inclusion $\Lambda^i[k] \to \Delta[k]$, the map of smooth manifolds

$$X_k = X\big(\Delta[k]\big) \longrightarrow X\big(\Lambda^i[k]\big)$$

is a surjective submersion and a diffeomorphism when $k > n$, where for any simplicial set K, we define

$$X(K) := \lim_{[j]\in\mathbf{\Delta}/K} X_j \in \mathrm{Sh}(\mathrm{Mfd}).$$

For $K = \Lambda^i[k]$, this limit is representable by a manifold because of the Kan conditions for lower-dimensional horn inclusions (cf. [8, 26]). Let $\mathrm{Gpd}_n^{\mathrm{Lie}} \subseteq \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Mfd})$ be the category of Lie n-groupoids and maps of simplicial manifolds between them, and note that we can (and will) think of $\mathrm{Gpd}_n^{\mathrm{Lie}}$ as a full subcategory of the category $\mathrm{sSh}(\mathrm{Mfd})$. Let us say that a map of Lie n-groupoids $X \to Y$ is a *Morita equivalence* if it induces a weak equivalence of simplicial sheaves.

In addition to the Morita equivalences, there are obvious analogues of the usual notions of Kan fibrations and trivial Kan fibrations for Lie n-groupoids:

Definition 2.1 ([8, 26])**.** A map between Lie n-groupoids $Y \to X$ is said to be a *Kan fibration* (*of height m*) if for each horn inclusion $\Lambda^i[k] \to \Delta[k]$, the map

$$Y\big(\Delta[k]\big) \longrightarrow X\big(\Delta[k]\big) \times_{X(\Lambda^i[k]} Y\big(\Lambda^i[k]\big)$$

is a surjective submersion (and a diffeomorphism when $k > m$). Similarly, it is said to be a trivial Kan fibration, or a *hypercover* (*of height m*) if the map

$$Y\big(\Delta[k]\big) \longrightarrow X\big(\Delta[k]\big) \times_{X(\partial\Delta[k])} Y\big(\Delta[k]\big) \tag{4}$$

is a surjective submersion between smooth manifolds (and a diffeomorphism if $k > m$).

Any map $f\colon Y \to X$ between Lie n-groupoids factors as a section of a hypercover (of height $n-1$), followed by a Kan fibration (of height n): indeed, such a factorization is provided by the usual *path fibration* $Y \times_X X^{\Delta[1]} \to X$, where $X^{\Delta[1]}$ is the simplicial manifold given by $\big(X^{\Delta[1]}\big)_n = X\big(\Delta[n] \times \Delta[1]\big)$ (see the discussion above Definition 4.26).

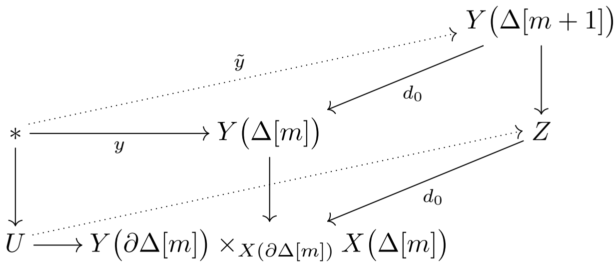
Lemma 2.2. *Let $f\colon Y \to X$ be a map of Lie n-groupoids. Then the following are equivalent:*

(1) f is a Morita equivalence.
(2) the path fibration $Y \times_X X^{\Delta[1]} \to X$ is a hypercover.

Proof. Since a hypercover is clearly a Morita equivalence and the Morita equivalences satisfy the 2-out-of-3 property, it follows that (2) implies (1). For the converse, it suffices to prove that a Kan fibration and Morita equivalence $Y \to X$ between Lie n-groupoids is a hypercover. Assuming that the map (4) is a surjective submersion between smooth manifolds for each boundary inclusion $\partial\Delta[k] \to \Delta[k]$ with $k < m$, we will show that the map

$$p\colon Y\big(\Delta[m]\big) \longrightarrow X\big(\Delta[m]\big) \times_{X(\partial\Delta[m])} Y\big(\Delta[m]\big)$$

is a surjective submersion between smooth manifolds. Since the target is a manifold by [26, Lemma 2.4], it suffices to check that for every point y in the domain, there exists a local section of p whose value at $p(y)$ is y. To do this, let U be a generic (small) open neighbourhood of $p(y)$ and consider the diagram



(5)

where $Z = Y\big(\Lambda^0[m+1]\big) \times_{X(\Lambda^0[m+1])} X\big(\Delta[m+1]\big)$. To find a diagonal for the front square (possibly after shrinking U), observe that there exists a dotted local section $U \to Z$ of the map d_0, since the map $Y \to X$ induces trivial Kan fibrations on stalks. The resulting maps $* \to Z$ and $y\colon * \to Y\big(\Delta[m]\big)$ together determine a map

$$* \longrightarrow Y\big(\partial\Delta[m+1]\big) \times_{X(\partial\Delta[m+1])} X\big(\Delta[m+1]\big)$$

which admits a dotted lift $\tilde{y}\colon * \to Y\big(\Delta[m+1]\big)$. Since $Y \to X$ was a Kan fibration, the back face of (5) admits a diagonal lift, which yields a diagonal for the front face after composing with $d_0\colon Y\big(\Delta[m+1]\big) \to Y\big(\Delta[m+1]\big)$. □

Remark 2.3. Since condition (2) can be formulated purely in terms of simplicial manifolds, it will be more convenient to take this as the definition of a Morita equivalence in the rest of the paper. As a third alternative definition of Morita equivalences, one can consider the following construction: associated to any Lie n-groupoid X is a Lie $(n-1)$-groupoid $\mathrm{Map}^{\mathrm{R}}_X$ whose manifold of k-simplices is given by the submanifold of $X\big(\Delta[k+1]\big)$ on those $(k+1)$-simplices whose restriction to the face $\Delta\{0, ..., k\}$ are full degenerate. This Lie $(n-1)$-groupoid comes with an obvious map $\mathrm{Map}^{\mathrm{R}}_X \to X_0 \times X_0$ which one can think of as realizing Map^{R}_X as the union of all mapping spaces $\mathrm{Map}_X(x, y)$ for $x, y \in X_0$.

Using this, a map $f\colon Y \to X$ of Lie n-groupoids is a Morita equivalence if and only if it satisfies the following two conditions:

(3a) f is essentially surjective, i.e. the map $Y\big(\{0\}\big) \times_{X(\{0\})} X\big(\Delta[1]\big) \to X(\{1\})$ is a surjective submersion.

(3b) f is fully faithful, i.e. the map over $Y_0 \times Y_0$

$$\mathrm{Map}^{\mathrm{R}}_Y \longrightarrow \mathrm{Map}^{\mathrm{R}}_X \times_{X_0\times X_0} Y_0 \times Y_0$$

is a Morita equivalence of Lie $(n-1)$-groupoids (the codomain is a Lie $(n-1)$-groupoid by (3a)).

Indeed, this follows from the corresponding result (see e.g. for Kan complexes by passing to stalks. Inductively, this reduces being a Morita equivalence to $n+2$ essential surjectivity conditions (on arrows in degree $\leq n+1$).

By construction, the inclusion $\mathrm{Gpd}^{\mathrm{Lie}}_n \to \mathrm{sSh(Mfd)}$ sends Morita equivalences to weak equivalences and therefore induces a functor on localizations

$$\mathrm{Gpd}^{\mathrm{Lie}}_n[W^{-1}_{\mathrm{Mor}}] \longrightarrow \mathrm{sSh(Mfd)}[W^{-1}].$$

As discussed in the introduction, we are more interested in the simplicial (or ∞-categorical) localizations of these two relative categories, rather than their 1-categorical localizations.

To describe the simplicial localization of the category of Lie n-groupoids, note that in light of Lemma 2.2 the localization of $\mathrm{Gpd}_n^{\mathrm{Lie}}$ at the hypercovers is equivalent to its localization at the Morita equivalences. The fact that the hypercovers between Lie n-groupoids are stable under base change results in the following description of the mapping spaces of $\mathrm{Gpd}_n^{\mathrm{Lie}}[W_{\mathrm{Mor}}^{-1}]$ (see e.g. [13]): for any two Lie n-groupoids X and Y, the space of maps between them is the classifying space of the category whose objects are spans $X \xleftarrow{\sim} \tilde{X} \to Y$, where $\tilde{X} \to X$ is a hypercover, and whose morphisms are commuting diagrams of the form

$$\begin{array}{ccccc} & & \tilde{X} & & \\ & \overset{\sim}{\swarrow} & \downarrow & \searrow & \\ X & & & & Y. \\ & \underset{\sim}{\nwarrow} & & \nearrow & \\ & & \tilde{X}' & & \end{array}$$

There is a similar description of the mapping spaces in $\mathrm{sSh}(\mathrm{Mfd})[W^{-1}]$, at least between locally fibrant simplicial sheaves (in the sense of [3, 12]), of which Lie n-groupoids are particular examples. The resulting mapping spaces tend to be much larger than the mapping spaces in $\mathrm{Gpd}_n^{\mathrm{Lie}}[W_{\mathrm{Mor}}^{-1}]$, as they also include spans whose tip need not be representable by a Lie n-groupoid. Nonetheless, Theorem 5.1 asserts that the functor $\mathrm{Gpd}_n^{\mathrm{Lie}}[W_{\mathrm{Mor}}^{-1}] \to \mathrm{sSh}(\mathrm{Mfd})[W^{-1}]$ induces weak equivalences on mapping spaces.

The fact that sSh(Mfd) is a simplicial model category allows for various other descriptions of its simplicial localization $\mathrm{sSh}(\mathrm{Mfd})[W^{-1}]$. For example, one may also realize $\mathrm{sSh}(\mathrm{Mfd})[W^{-1}]$ as the full simplicially enriched subcategory of sSh(Mfd) on the fibrant simplicial sheaves (in the Joyal model structure). Consequently, the functor $\mathrm{Gpd}_n^{\mathrm{Lie}} \to \mathrm{sSh}(\mathrm{Mfd})$ sending each Lie n-groupoid to a fibrant replacement $X^\wedge$ of X induces weak equivalences on mapping spaces

$$\mathrm{Map}_{\mathrm{Gpd}_n^{\mathrm{Lie}}[W^{-1}]}(X, Y) \simeq \mathrm{Map}_{\mathrm{sSh}(\mathrm{Mfd})}(X^\wedge, Y^\wedge)$$

so that $\mathrm{Gpd}_n^{\mathrm{Lie}}[W^{-1}]$ can be identified with the full simplicial subcategory of sSh(Mfd) on those fibrant simplicial sheaves that can be represented by Lie n-groupoids, up to weak equivalence.

The simplicial category $\mathrm{sSh}(\mathrm{Mfd})[W^{-1}]$ is a model for the ∞-category $\mathrm{Sh}_\infty(\mathrm{Mfd})$ of stacks on the site of smooth manifolds, which is a prototypical example of an ∞-topos. For each Lie n-groupoid, the simplicial sheaf $X^\wedge$ is a model for the *associated stack* of X, which can be thought of a the (homotopy) quotient of the Lie n-groupoid in the ∞-category $\mathrm{Sh}_\infty(\mathrm{Mfd})$. More precisely, it will be useful to think of a Lie n-groupoid X as a certain simplicial diagram in $\mathrm{Sh}_\infty(\mathrm{Mfd})$, i.e. as a bisimplicial sheaf which happens to be constant in one simplicial direction, and to think of the functor $X \mapsto X^\wedge$ as the composite functor

$$\mathrm{Gpd}_n^{\mathrm{Lie}} \subseteq \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{\mathrm{op}}, \mathrm{Sh}_\infty(\mathrm{Mfd})) \xrightarrow{\mathrm{colim}} \mathrm{Sh}_\infty(\mathrm{Mfd}).$$

For every Lie n-groupoid X, there is a canonical map $q\colon X_0 \to \mathrm{colim}\, X$, which given an *atlas* for the stack $\mathrm{colim}\, X$. There is a canonical groupoid object in $\mathrm{Sh}_\infty(\mathrm{Mfd})$ associated to this map q (its Čech nerve), whose stack of arrows is given by $X_0 \times_{\mathrm{colim}\, X} X_0$. In turn, this stack arises as the colimit of the Lie $(n-1)$-groupoid $\mathrm{Map}_X^{\mathrm{R}}$ from Remark 2.3 (see Lemma 5.8).

This observation can be reversed to provide the following alternative (inductive) description of the essential image of the above functor (see Proposition 5.4): let us say that a stack is $(n-1)$-geometric if it arises as the colimit of a Lie $(n-1)$-groupoid. Then a stack F is n-geometric if and only if it arises as the quotient of a (homotopy coherent) groupoid object $\mathcal{G}$ in $(n-1)$-geometric stacks, where the source and target map $d_0, d_1\colon \mathcal{G}_1 \to \mathcal{G}_0$ are surjective submersions (see Definition 3.9). In fact, Theorem 5.1 implies that formally inverting the Morita equivalences between such groupoid objects *also* results in the full sub-∞-category of $\mathrm{Sh}_\infty(\mathrm{Mfd})$ on the n-geometric stacks. We can therefore summarize the situation as follows:

Corollary 2.4. *The following three ∞-categories (or simplicial categories) are equivalent:*

(1) the full sub-∞-category of $\mathrm{Sh}_\infty(\mathrm{Mfd})$ on the n-geometric stacks, i.e. the full simplicial subcategory of $\mathrm{sSh}(\mathrm{Mfd})$ on those fibrant simplicial sheaves that are weakly equivalent to Lie n-groupoids.

(2) the simplicial (or ∞-categorical) localization of $\mathrm{Gpd}_n^{\mathrm{Lie}}$ at the Morita equivalences (or equivalently, at the hypercovers).

(3) the simplicial (or ∞-categorical) localization at the Morita equivalences of the category of (homotopy coherent) groupoid objects in the ∞-category of $(n-1)$-geometric stacks, whose source and target map are surjective submersions.

Remark 2.5. The ∞-categorical localization of the ∞-category of groupoid objects in $(n-1)$-geometric stacks can be constructed in the same way as the localization $\mathrm{Gpd}_n^{\mathrm{Lie}}[W_{\mathrm{Mor}}^{-1}]$ and has mapping spaces given by classifying spaces of the ∞-categories of spans $X \leftarrow \tilde{X} \rightarrow Y$, where the left map is a hypercover (see Definition 4.17). Alternatively, the mapping space between two groupoids (and more generally, between n-groupoids) X and Y can also be described as the space of *bibundles* between them; we will come back to this in Section 6.

3. Preliminaries on geometric stacks

In this section we will recall the basic homotopy theory of (higher) stacks (tracing back at least to the seventies [3, 15]) and geometric stacks (due to Simpson [24]), both of which are natural generalizations of the classical notion of a (geometric) stack in groupoids [5]. For simplicity, we will formulate the theory of geometric stacks at the level of toposes, rather than sheaves and sites.

3.1. Toposes and stacks.

Definition 3.1. An ∞-topos $\mathcal{X}$ is an ∞-category $\mathcal{X}$ which arises as a left exact reflective localization of a presheaf ∞-category. In other words, an ∞-category $\mathcal{X}$ is an ∞-topos if there exists a small ∞-category $\mathcal{C}$, together with an adjunction

$$L\colon \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \rightleftarrows \mathcal{X} \colon i$$

so that the right adjoint i is fully faithful and the left adjoint L preserves finite limits.

Remark 3.2. The presentation of an ∞-topos $\mathcal{X}$ as a reflective subcategory of some $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ can be modeled directly at the model-categorical level: one simply takes a suitable Bousfield localization of the covariant model structure on $\mathrm{sSet}/\mathcal{C}^{\mathrm{op}}$ or the projective (or injective) model structure on the category $\mathrm{Fun}(\mathfrak{C}[\mathcal{C}]^{\mathrm{op}}, \mathrm{sSet})$ of simplicial presheaves on the simplicial category associated to $\mathcal{C}$. The resulting model category is usually called a *model topos* [23].

The main examples of ∞-toposes arise by considering stacks on sites (or a homotopical variant thereof, like simplicial sites and model sites).

Example 3.3. Let $\mathcal{C}$ be a site and consider a functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ to the ∞-category of spaces. If $\{U_i \to V\}$ is a cover of an object V in $\mathcal{C}$, then there is an associated augmented cosimplicial diagram of spaces

$$F(V) \dashrightarrow \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_{ij}) \rightrightarrows \prod_{i,j,k} F(U_{ijk}) \;\cdots$$

where $U_{ij} = U_i \times_V U_j$, and similarly for n-fold fiber products. The functor F is a *stack* (or satisfies *descent*) if the above diagram realizes $F(V)$ as the (homotopy) limit of the solid cosimplicial diagram of spaces. If F is a sheaf of sets, we can think of F as a stack by considering each set $F(V)$ as a discrete space and if F is a stack of groupoids, we can think of F as a stack by taking the classifying space of each groupoid $F(V)$.

The full sub-∞-category $\mathrm{Sh}_\infty(\mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ on those functors that satisfy descent is an ∞-topos. Indeed, the inclusion $\mathrm{Sh}_\infty(\mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ admits a left adjoint L preserving finite limits, which sends a functor F to its associated stack. Under certain finite dimensionality conditions on the site $\mathcal{C}$ (see e.g. [16, Theorem 7.2.3.6]), the category $\mathrm{Sh}_\infty(\mathcal{C})$ can be modeled by the Joyal model structure on simplicial sheaves [15], or by the *local* model structure on simplicial presheaves [12]; in general, these model categories describe the further subcategory $\mathrm{Sh}^\wedge_\infty(\mathcal{C}) \subseteq \mathrm{Sh}_\infty(\mathcal{C})$ (which is also an ∞-topos) of functors satisfying descent with respect to hypercovers.

For some of the (defining) properties of ∞-toposes, see [23] or [16, Section 6.1]. One such property is the fact that ∞-toposes have a good notion of 'surjections' (more precisely, *effective epimorphisms*). Since such maps will appear frequently later on, let us quickly recall their basic theory. Let $\mathbf{\Delta}_+$ be the category of (possibly empty) finite linear ordinals, i.e. the augmented simplex category which can be depicted as

$$[-1] \xrightarrow{d^0} [0] \underset{d^1}{\overset{d^0}{\rightrightarrows}} [1] \quad \overset{d^0,\, d^1,\, d^2}{\underset{s^0,\, s^1}{\leftrightarrows}} \quad [2] \ \cdots$$

Alternatively, one can identify $\mathbf{\Delta}_+$ with the cone of the simplex category. There is an obvious inclusion $i\colon \mathbf{\Delta} \to \mathbf{\Delta}_+$, as well as the inclusion $j\colon [1] \to \mathbf{\Delta}_+$ classifying the map $[-1] \to [0]$. Consider the associated composite of adjunctions

$$\mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{\mathrm{op}}, \mathcal{X}) \underset{i^*}{\overset{i_!}{\rightleftarrows}} \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta}_+)^{\mathrm{op}}, \mathcal{X}) \underset{j_*}{\overset{j^*}{\rightleftarrows}} \mathrm{Fun}(\Delta[1]^{\mathrm{op}}, \mathcal{X}) \tag{6}$$

where $i_!$ takes the left Kan extension along (the opposite of) i and j_* takes the right Kan extension along (the opposite of) j. Since $i\colon \mathbf{\Delta} \to \mathbf{\Delta}_+$ realizes $\mathbf{\Delta}_+$ as the cone of $\mathbf{\Delta}$, the left Kan extension of a simplicial object X along (the opposite of) i is simply the associated colimiting cocone of X. We will usually denote the object $(i_!X)_{-1}$ (i.e. the colimit of X) by $|X|$.

The composite right adjoint sends an arrow $X_0 \to X_{-1}$ in $\mathcal{H}$ to its *Čech nerve* $\check{C}_\bullet(f)$ (see [16, 6.1.2.11]), which is the simplicial object given in degree n by

$$\check{C}_n(f) \simeq X_0 \times_{X_{-1}} X_0 \times_{X_{-1}} \cdots \times_{X_{-1}} X_0 \qquad (n \text{ times}).$$

Upon applying the composite left adjoint to this simplicial object, we obtain the natural map $X_0 \to |\check{C}_\bullet(f)|$ from X_0 to the colimit of the Čech nerve of f in $\mathcal{X}$.

Definition 3.4 ([16, Section 6.2.3])**.** A map $f\colon X \to Y$ is called an *effective epimorphism* if the counit map $|\check{C}_\bullet(f)| \to Y$ is an equivalence.

Example 3.5.

(1) Consider a diagram $\Delta[2] \to \mathcal{X}$ of the form

$$\begin{array}{ccc} & Y & \\ {\scriptstyle f}\nearrow & & \searrow{\scriptstyle g} \\ X & \xrightarrow[h]{} & Z. \end{array}$$

If g and f are effective epimorphisms, then so is h and when h is an effective epimorphism, then so is g [16, Corollary 6.2.3.12]. In particular, any morphism admitting a section (up to homotopy) is an effective epimorphism.

(2) The class of effective epimorphisms is stable under base change [16, Remark 6.2.3.7].

(3) If X is a simplicial object in $\mathcal{X}$, then the canonical map $f\colon X_0 \to |X|$ is an effective epimorphism: indeed, by the triangle identities for the composite adjunction of (6), the counit map from $X_0 \to |\check{C}_\bullet(f)|$ to $f\colon X_0 \to |X|$ admits a section (up to homotopy).

(4) When $\mathcal{X} = \mathcal{S}$ is the ∞-category of spaces, an effective epimorphism is just a map inducing a surjection on π_0. When $\mathcal{X} = \mathrm{Sh}_\infty(\mathcal{C})$ is the category of stacks on a site, a map is an effective epimorphism if and only if the induced map on π_0-sheaves is an epimorphism.

3.2. **Geometric stacks.** We will now recall the notion of an n-geometric stack, introduced by Simpson [24]. Informally, one defines n-geometric stacks inductively, starting with a class of 0-geometric stacks (often given by objects like manifolds, schemes or derived schemes) and defining an n-geometric stack to be an object X that can be covered by a 0-geometric stack M, in such a way that the fibers of $M \to X$ vary smoothly over M. In particular, it requires an a priori notion of 0-geometric stacks and smooth maps between them, subject to the following conditions (see also [25]):

Assumption 3.6. Let $\mathcal{X}$ be an ∞-topos. Fix a full subcategory $\mathcal{X}_{0-\mathrm{geom}} \subseteq \mathcal{X}$ of 0-*geometric* stacks in $\mathcal{X}$, as well as a class P of *smooth* maps between them, such that the following conditions hold:

(1) the smooth maps are stable under homotopy and finite composition.

(2) the full subcategory $\mathcal{X}_{0-\mathrm{geom}} \subseteq \mathcal{X}$ is closed under all pullbacks along smooth maps. The base change of a smooth map is again smooth.

(3) Given maps $f\colon Z \to Y$ and $g\colon Y \to X$ in $\mathcal{X}_{0-\mathrm{geom}}$, the map g is smooth if the composite gf is smooth and f is both smooth and an effective epimorphism.

(4) If $p\colon Y \to X$ is an effective epimorphism whose target is 0-geometric, then there exists a triangle in $\mathcal{X}$ of the form

$$\begin{array}{ccc} U & \longrightarrow & Y \\ & {\scriptstyle q}\searrow \quad \swarrow{\scriptstyle p} & \\ & X & \end{array}$$

where the map q is a smooth effective epimorphism between 0-geometric stacks.

Example 3.7. When $\mathcal{X} = \mathrm{Sh}_\infty(\mathrm{Mfd})$, one can take $\mathcal{X}_{0-\mathrm{geom}} = \mathrm{Mfd}$ and take the smooth maps to be either the submersions or the etale maps. Note that the open inclusions do not satisfy condition (3) and that the set of all $\mathcal{C}^\infty$-maps does not satisfy condition (2). Some more examples are discussed in Section 7.

While the first three conditions of Assumption 3.6 concern the behaviour of the 0-geometric objects and smooth maps themselves, condition (4) relates arbitrary effective epimorphisms in $\mathcal{X}$ to smooth maps. For certain applications it will be useful to consider the following relative version of condition (4):

Variant 3.8. Let $\mathcal{A} \subseteq \mathcal{X}$ be a full sub-∞-category of $\mathcal{X}$ which is closed under limits and contains $\mathcal{X}_{0-\mathrm{geom}}$. We will say that condition (4) holds *relative* to $\mathcal{A}$ if it holds for any effective epimorphism $Y \to X$ in $\mathcal{A}$ whose codomain is 0-geometric.

Definition 3.9 (Simpson, [24])**.** Let $(\mathcal{X}, \mathcal{X}_{0-\mathrm{geom}}, P)$ be as in Assumption 3.6.

(a_0) We call a morphism $X \to Y$ in $\mathcal{X}$ *0-smooth* if for any 0-geometric stack U with a map to Y, the base change $X \times_Y U \to U$ is equivalent to a smooth map between 0-geometric stacks.

(b_n) We say an object $X \in \mathcal{X}$ is *n-geometric* if it admits an *atlas*: there is an object $U \in \mathcal{X}_{0-\mathrm{geom}}$ and a map $U \to X$ which is (i) an effective epimorphism and (ii) $(n-1)$-smooth.

(a_n) We call a morphism $X \to Y$ *n-smooth* if for any $U \in \mathcal{X}_{0-\mathrm{geom}}$ with a map to Y, the pullback $X \times_Y U$ is an n-geometric stack which admits an atlas by a 0-geometric stack V, so that the composite map $V \to X \times_Y U \to U$ is smooth.

Remark 3.10. The above list is traditionally extended to include the notion of *n-representable* maps [25]. In the case where the 0-geometric stacks are not closed under pullbacks in $\mathcal{X}$ (e.g. when $\mathcal{X}_{0-\mathrm{geom}} = \mathrm{Mfd}$), this notion is not so useful because maps between 0-geometric stacks need not be 0-geometric.

Remark 3.11. Let $p\colon Y \to U$ be a map from an n-geometric stack to a 0-geometric stack. If there exists an atlas $V \to Y$ so that the composite $V \to Y \to U$ is smooth, then for *every* atlas $W \to Y$, the composite $W \to Y \to U$ is smooth (this follows from part (3) of Assumption 3.6).

The following lemma summarizes the basic properties of smooth maps:

Lemma 3.12 ([25]). *Under the assumptions from 3.6, the following hold:*

(1) The n-smooth maps are stable under homotopy, composition and base change. If the codomain of an n-smooth map is n-geometric, then its domain is n-geometric as well.

(2) Let $f\colon Y \to X$ be a map of stacks and let $p\colon Z \to X$ be an effective epimorphism. If the base change $f^*Z \to Z$ is n-smooth, then f is n-smooth.

(3) Let $f\colon Y \to X$ be a k-smooth map between n-geometric stacks. Then f is n-smooth.

Furthermore, in the situation of Variant 3.8 assertions (2) and (3) hold when all objects involved are contained in $\mathcal{A} \subseteq \mathcal{X}$.

Proof. It is straightforward to verify (1) (see also [25]). For (2), let $U \in \mathcal{X}_{0-\mathrm{geom}}$ and let $g\colon U \to X$ be a map. Since $p\colon Z \to X$ is an effective epimorphism (in $\mathcal{A}$), by part (4) of Assumption 3.6 (relative to $\mathcal{A}$) there exists a commuting diagram in $\mathcal{X}$

$$\begin{array}{ccc} V & \xrightarrow{g'} & Z \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ U & \xrightarrow[g]{} & X \end{array}$$

in which p' is a smooth effective epimorphism in $\mathcal{X}_{0-\mathrm{geom}}$. The base change of the above commuting diagram along the map $f\colon Y \to X$ now yields a commuting diagram

$$\begin{array}{ccc} f^*V & \xrightarrow{f^*(g')} & f^*Z \\ {\scriptstyle f^*(p')}\downarrow & & \downarrow{\scriptstyle f^*(p)} \\ f^*U & \xrightarrow[f^*(g)]{} & Y \end{array}$$

The object f^*V is equivalent to the pullback $V \times_Z f^*Z$, which is n-geometric since $f^*Z \to Z$ was assumed n-smooth. The map $f^*(p')\colon f^*V \to f^*U$ is a base change of p' and hence a 0-smooth effective epimorphism. The composition of this map with an atlas $W \to f^*V$ provides an atlas $W \to f^*U$ for f^*U, which shows that f^*U is n-geometric.

The composite map $W \to f^*U \to U$ is equivalent to the composition $W \to f^*V \to V \to U$, which is the composition of three smooth maps.

For (3), let $p\colon U \to X$ be an atlas for the n-geometric stack U (in $\mathcal{A}$). By (2), it suffices to show that the k-smooth map $U \times_X Y \to U$ is in fact n-smooth. But the map $U \times_X Y \to Y$ is $(n-1)$-smooth with an n-geometric target, so that $U \times_X Y$ is n-geometric (hence $U \times_X Y \to U$ is also n-smooth). □

Remark 3.13. Using the above properties, one easily sees that the full subcategory of n-geometric stacks, together with the class of n-smooth maps between them, also satisfies the conditions of Assumption 3.6.

Let $X \in \mathcal{X}$ be an n-geometric stack and let $p\colon U \to X$ be an atlas for X. The Čech nerve of the map p yields a groupoid object $\mathcal{G}\colon \mathrm{N}(\mathbf{\Delta})^{\mathrm{op}} \to \mathcal{X}$ with the property that $\mathcal{G}_0 = U$ and that the two maps $d_0, d_1\colon \mathcal{G}_1 \to U$ are $(n-1)$-smooth (in particular, $\mathcal{G}_1$ is $(n-1)$-geometric). Conversely, consider any groupoid object $\mathcal{G}\colon \mathrm{N}(\mathbf{\Delta})^{\mathrm{op}} \to \mathcal{X}$ with the property that $d_0, d_1\colon \mathcal{G}_1 \to \mathcal{G}_0$ are smooth maps between $(n-1)$-geometric stacks. Then the associated map $p\colon \mathcal{G}_0 \to |\mathcal{G}|$ is effective epimorphism by Example 3.5. Furthermore, since any groupoid object $\mathcal{G}$ in an ∞-topos $\mathcal{X}$ is equivalent to the Čech nerve of $\mathcal{G}_0 \to |\mathcal{G}|$ [16, Theorem 6.1.0.6], the map p is $(n-1)$-smooth: indeed, the base change of p along itself is equivalent to the $(n-1)$-smooth map $d_0\colon \mathcal{G}_1 \to \mathcal{G}_0$. Composing p with an atlas of $\mathcal{G}_0$ now provides an atlas for $|\mathcal{G}|$, so that $|\mathcal{G}|$ is an n-geometric stack.

In other words, the n-geometric stacks are precisely those stacks that arise as the quotients of smooth groupoid objects in the category of $(n-1)$-geometric stacks. In Section 5 we will give an iteration of the above procedure, which realizes each n-geometric stack as the colimit of a k-groupoid object with values in $(n-k)$-geometric stacks.

4. Higher groupoids

In this section we will recall the notion of an n-groupoid object (in an ∞-topos) and the notion of a *geometric* n-groupoid object, defined in terms of the data 3.6.

4.1. Simplicial homotopy in toposes.

Definition 4.1. Let $X\colon \mathrm{N}(\mathbf{\Delta})^{\mathrm{op}} \to \mathcal{C}$ be a simplicial object in an ∞-category $\mathcal{C}$ and let K be a simplicial set. If it exists, we denote by $X(K)$ the *matching object* of X associated to K, obtained as the limit in $\mathcal{C}$ of the diagram

$$\mathrm{N}(\mathbf{\Delta}_{\mathrm{nd}}/K)^{\mathrm{op}} \longrightarrow \mathrm{N}(\mathbf{\Delta})^{\mathrm{op}} \xrightarrow{X} \mathcal{C}$$

where $\mathbf{\Delta}_{\mathrm{nd}}/K$ is the full subcategory of $\mathbf{\Delta}/K$ on the nondegenerate simplices of K.

Example 4.2. The category $\mathbf{\Delta}_{\mathrm{nd}}/\Delta[k]$ has the identity map as a terminal object, so that $X(\Delta[k]) \simeq X_k$. For any injective map of simplicial sets $i\colon K \to L$, there is a natural functor $\mathbf{\Delta}_{\mathrm{nd}}/K \to \mathbf{\Delta}_{\mathrm{nd}}/L$ (over $\mathbf{\Delta}$) which induces (if it exists) a natural map $X(L) \to X(K)$ called the *matching map* associated to i. More generally, if $p\colon Y \to X$ is a map of simplicial objects in $\mathcal{C}$ and $i\colon K \to L$ is an injective map of simplicial sets, then the *relative matching map* of p with respect to i is (if it exists) the map associated to the square

$$\begin{array}{ccc} Y(L) & \longrightarrow & Y(K) \\ \downarrow & & \downarrow \\ X(L) & \longrightarrow & X(K) \end{array}$$

from $Y(L)$ to the pullback.

Remark 4.3. Let $\mathcal{C}$ be an ∞-category with all small limits. Then restriction along the opposite of the Yoneda embedding yields an equivalence of ∞-categories

$$\mathrm{Fun}^{\mathrm{R}}(\mathrm{PSh}_\infty(\mathrm{N}(\mathbf{\Delta}))^{\mathrm{op}}, \mathcal{C}) \xrightarrow{\simeq} \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{\mathrm{op}}, \mathcal{C})$$

between the category of limit-preserving functors $\mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{\mathrm{op}}, \mathcal{S})^{\mathrm{op}} \to \mathcal{C}$ and the category of simplicial objects $\mathcal{C}$ [16, Theorem 5.1.5.6]. If $X\colon \mathrm{N}(\mathbf{\Delta})^{\mathrm{op}} \to \mathcal{C}$ is a simplicial object in $\mathcal{C}$, denote the resulting limit preserving functor by $X(-)\colon \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{S})^{\mathrm{op}} \to \mathcal{C}$. The value of this functor on a simplicial set K (viewed as a discrete simplicial space) is precisely $X(K)$, since any simplicial set is the colimit of its category of nondegenerate elements.

Remark 4.4. If $K \to K'$ and $K \to L$ are two inclusions of simplicial sets with pushout L', then the square of inclusions of posets

$$\begin{array}{ccc} \mathbf{\Delta}_{\mathrm{nd}}/K & \longrightarrow & \mathbf{\Delta}_{\mathrm{nd}}/K' \\ \downarrow & & \downarrow \\ \mathbf{\Delta}_{\mathrm{nd}}/L & \longrightarrow & \mathbf{\Delta}_{\mathrm{nd}}/L' \end{array}$$

is cocartesian. It follows from [16, Proposition 4.2.3.8, Remark 4.2.3.9] that the corresponding square of matching objects

$$\begin{array}{ccc} X(L') & \longrightarrow & X(K') \\ \downarrow & & \downarrow \\ X(L) & \longrightarrow & X(K) \end{array}$$

is cartesian (more precisely, if $X(K), X(K')$ and $X(L)$ exist, then $X(L')$ exists and fits into the above pullback square).

Remark 4.5. Let $\mathcal{C} = \mathrm{N}(\mathbf{A}^\circ)$ be the ∞-category associated to a simplicial combinatorial model category $\mathbf{A}$. Then there is an equivalence of ∞-categories $\mathrm{N}(\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathbf{A})^\circ) \to \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{\mathrm{op}}, \mathcal{C})$ where $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathbf{A})$ carries the Reedy model structure [16, Proposition 4.2.4.4]. If $X\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{A}$ is a Reedy fibrant (and cofibrant) diagram with associated diagram $X'\colon \mathrm{N}(\mathbf{\Delta})^{\mathrm{op}} \to \mathcal{C}$, and K is a simplicial set, then $X'(K)$ can be modeled by the realization $X^K = \lim_{[n]\in(\mathbf{\Delta}_{\mathrm{nd}}/K)^{\mathrm{op}}} X_n$ in the sense of [10, §16.3].

Definition 4.6. Let $\mathcal{X}$ be an ∞-topos and $0 \leq n \leq \infty$. A map of simplicial objects $Y \to X$ in $\mathcal{X}$ is said to be a *Kan fibration of height* n if for each horn inclusion $\Lambda^i[k] \to \Delta[k]$, the relative matching map

$$Y(\Delta[k]) \simeq Y_k \longrightarrow X(\Delta[k]) \times_{X(\Lambda^i[k])} Y(\Lambda^i[k])$$

is an effective epimorphism if $k \leq n$ and an equivalence if $k > n$. A simplicial object X is called an n*-groupoid* in $\mathcal{X}$ if $X \to *$ is a Kan fibration of height n.

Similarly, a map of simplicial objects $Y \to X$ in $\mathcal{X}$ is said to be a *hypercover of height* n if for each boundary inclusion $\partial\Delta[k] \to \Delta[k]$, the map

$$Y(\Delta[k]) \simeq Y_k \longrightarrow X(\Delta[k]) \times_{X(\partial\Delta[k])} Y(\partial\Delta[k])$$

is an effective epimorphism if $k \leq n$ and an equivalence if $k > n$.

Remark 4.7. In light of Remark 3.2 and 4.5, the above definition admits an analogous formulation in terms of model categories: one simply picks a model category $\mathbf{A}$ presenting $\mathcal{X}$ and considers Reedy fibrations between Reedy fibrant diagrams for which the relevant matching maps are effective epimorphisms (which is a property invariant under weak equivalences) or weak equivalences. All results in this section can be proven in terms of model categories, at the cost of having to add Reedy fibrancy hypotheses at various points.

Remark 4.8. If $F, G\colon \mathcal{I} \to \mathcal{C}$ are two diagrams in an ∞-category $\mathcal{C}$, recall that a natural transformation $\phi\colon F \to G$ between them is said to be *cartesian* (or *equifibered*) if each map $j \to i$ in $\mathcal{I}$ induces a cartesian square

$$\begin{array}{ccc} F(j) & \longrightarrow & F(i) \\ {\scriptstyle\phi_j}\downarrow & & \downarrow{\scriptstyle\phi_i} \\ G(j) & \longrightarrow & G(i). \end{array}$$

A natural transformation of simplicial objects $p\colon Y \to X$ is a Kan fibration of height 0 if and only if it is cartesian.

The next two propositions give the basic properties of Kan fibrations and hypercovers (of finite height) that we will need:

Proposition 4.9. *Let $p\colon Y \to X$ be a hypercover of height $n < \infty$ in an ∞-topos $\mathcal{X}$. Then the induced map of colimits $|Y| \to |X|$ is an equivalence.*

This result is elementary when $\mathcal{H} = \mathcal{S}$ is the category of spaces and follows, using Boolean localization [14], when $\mathcal{X}$ arises from the Joyal model structure on simplicial objects in an ordinary topos [15]. Before going into the proof, let us recall the following simple construction:

Construction 4.10. Let $X\colon \mathrm{N}(\mathbf{\Delta}^{\mathrm{op}}) \to \mathcal{X}$ be a simplicial object in $\mathcal{X}$. For each k, restriction along the inclusion $\tau\colon \mathrm{N}(\mathbf{\Delta}_{\leq k}) \to \mathrm{N}(\mathbf{\Delta})$ induces a functor

$$\tau^*\colon \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta}^{\mathrm{op}}), \mathcal{X})/X \longrightarrow \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta}^{\mathrm{op}}_{\leq k}), \mathcal{X})/\tau^*X$$

which admits a right adjoint, sending $V \to \tau^*X$ to the base change $\tau_*V \times_{\tau_*\tau^*X} X$ of the right Kan extension along τ. Let

$$\mathrm{cosk}_k\colon \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{\mathrm{op}}, \mathcal{X})/X \longrightarrow \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{\mathrm{op}}, \mathcal{X})/X$$

be the composition of τ^* with this right adjoint. For a map of simplicial objects $p\colon Y \to X$, the induced map $q\colon Y \to \mathrm{cosk}_k(p)$ (over X) is given in degree j by the map

$$Y(\Delta[j]) \longrightarrow Y(\mathrm{sk}_k\Delta[j]) \times_{X(\mathrm{sk}_k\Delta[j])} X(\Delta[j]).$$

This can be checked directly, or follows immediately by picking a model-categorical presentation $\mathbf{A}$ of $\mathcal{X}$, in which case the above coskeleton construction for simplicial objects in $\mathbf{A}$ is well-known .

In any case, note that the map $q\colon Y \to \mathrm{cosk}_k(p)$ is an equivalence in degrees $\leq k$. If p is a hypercover, then q is a degreewise effective epimorphism and the map $\mathrm{cosk}_k(p) \to X$ is a hypercover of height k. If p is itself a hypercover of height k, then q is a degreewise equivalence (in other words, p is *k-coskeletal*).

Proof (of Proposition 4.9). We will mimic the proof of [7, A.4] (and the variant thereof in [16, 6.5.3.9]). We proceed by induction on the height n, the case $n = -1$ being trivial since in that case $p\colon Y \to X$ is a natural equivalence. Let us suppose that the proposition holds for hypercovers of height $n-1$ and let $p\colon Y \to X$ be a hypercover of height n. Taking the $(n-1)$-coskeleton yields a map $q\colon Y \to \mathrm{cosk}_{n-1}(p) =: U$ in $\mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{\mathrm{op}}, \mathcal{X})/X$. As mentioned above, the map q is an equivalence in degrees $\leq n-1$ and an effective epimorphism in degrees $\geq n$. For the rest of the proof, it will be useful to identify the map q in $\mathrm{Fun}(\mathrm{N}(\boldsymbol{\Delta})^{\mathrm{op}}, \mathcal{X})/X$ with a diagram

$$\alpha\colon \mathrm{N}(\boldsymbol{\Delta}^{\mathrm{op}} \times [2]) \longrightarrow \mathcal{X}$$

classifying $Y \to U \to X$. Let $\overline{\alpha}\colon \mathrm{N}(\boldsymbol{\Delta}_+^{\mathrm{op}} \times [2]) \to \mathcal{X}$ be the left Kan extension of α along the obvious inclusion. It follows from [16, Proposition 4.3.2.9] that $\overline{\alpha}$ classifies a diagram of augmented simplicial objects $\overline{Y} \to \overline{U} \to \overline{X}$, each of which is a colimit diagram (i.e. $\overline{Y}_{-1} = |Y|$ is the colimit of Y).

We need to prove that the composite map $\overline{Y}_{-1} \to \overline{U}_{-1} \to \overline{X}_{-1}$ is an equivalence. Since $U \to X$ is a hypercover of height $n-1$ (Construction 4.10), the map $\overline{U}_{-1} \to \overline{X}_{-1}$ is an equivalence. To prove that $\overline{Y}_{-1} \to \overline{X}_{-1}$ is an equivalence, we will show that it is a retract of $\overline{U}_{-1} \to \overline{X}_{-1}$ in $\mathcal{X}/\overline{X}_{-1}$.

To this end, let us denote by $\boldsymbol{\Delta}_{++}$ the cone of $\boldsymbol{\Delta}_+$ and denote extra initial object by $[-2]$ (so that $\boldsymbol{\Delta}_{++}$ looks like $[-2] \to [-1] \to [0] \cdots$, proceeding as $\boldsymbol{\Delta}$). There is an obvious functor $[2] \to \boldsymbol{\Delta}_{++}^{\mathrm{op}}$ sending i to $[-i]$. Let

$$\overline{W}\colon \mathrm{N}(\boldsymbol{\Delta}_+^{\mathrm{op}} \times \boldsymbol{\Delta}_{++}^{\mathrm{op}}) \longrightarrow \mathcal{X}$$

be the right Kan extension of $\overline{\alpha}$ along $\mathrm{N}(\boldsymbol{\Delta}_+^{\mathrm{op}} \times [2]) \to \mathrm{N}(\boldsymbol{\Delta}_+^{\mathrm{op}} \times \boldsymbol{\Delta}_{++}^{\mathrm{op}})$. For each $n \geq -1$, a simple cofinality argument shows that the object $\overline{W}_{n,\bullet}$ is given by the Čech nerve of the map $Y_n \to U_n$

$$\cdots \; Y_n \times_{U_n} Y_n \rightrightarrows Y_n \longrightarrow U_n \longrightarrow X_n \tag{7}$$

together with the additional map $U_n \to X_n$ which witnesses that the entire augmented simplicial object sits over X_n. Now recall that there are functors

$$\delta\colon \boldsymbol{\Delta}_+ \longrightarrow \boldsymbol{\Delta}_+ \times \boldsymbol{\Delta}_{++}; \quad [m] \longmapsto ([m],[m])$$

$$\iota_k\colon \boldsymbol{\Delta}_+ \longrightarrow \boldsymbol{\Delta}_+ \times \boldsymbol{\Delta}_{++}; \quad [m] \longmapsto ([m],[k])$$

together with natural transformations $\iota_{-2} \to \delta \to \iota_0$. Restricting $\overline{W}$ along the opposite of these functors yields a $\Delta[2]$-diagram of augmented simplicial objects

$$\iota_0^*\overline{W} = \overline{Y} \longrightarrow \delta^*\overline{W} \longrightarrow \iota_{-2}\overline{W} = \overline{X}$$

which in degree -1 is our original diagram $\overline{Y}_{-1} \to \overline{U}_{-1} \to \overline{X}_{-1}$. Each of the above three augmented simplicial objects is a colimit diagram. This holds by construction for $\overline{Y}$ and $\overline{X}$, and for $\delta^*\overline{W}$, this follows from the following argument. By construction, $\overline{U}_{-1}$ is the colimit of $\overline{W}\big|\mathrm{N}(\boldsymbol{\Delta}^{\mathrm{op}} \times \{-1\})$. Since the inclusion $\mathrm{N}(\boldsymbol{\Delta}^{\mathrm{op}} \times \{-1\}) \subseteq \mathrm{N}(\boldsymbol{\Delta}^{\mathrm{op}} \times \boldsymbol{\Delta}_+^{\mathrm{op}})$ is cofinal, it follows that $\overline{U}_{-1}$ is the colimit of the entire diagram $\overline{W}\big|\mathrm{N}(\boldsymbol{\Delta}^{\mathrm{op}} \times \boldsymbol{\Delta}_+^{\mathrm{op}})$. However, for each $k \geq 0$ the diagram $\overline{W}\big|\mathrm{N}(\{k\} \times \boldsymbol{\Delta}_+^{\mathrm{op}})$ is given by the left part of (7), which is a colimit diagram since $Y_n \to U_n$ was an effective epimorphism. It follows from [16, 4.3.3.9] that $\overline{U}_{-1}$ is the colimit of the bisimplicial object $\overline{W}\big|\ mmN(\boldsymbol{\Delta}^{\mathrm{op}} \times \boldsymbol{\Delta}^{\mathrm{op}})$. But the the diagonal map $\delta\colon \mathrm{N}(\boldsymbol{\Delta}^{\mathrm{op}}) \to \mathrm{N}(\boldsymbol{\Delta}^{\mathrm{op}} \times \boldsymbol{\Delta}^{\mathrm{op}})$ is cofinal, so we conclude that $\delta^*\overline{W}_{-1} = \overline{U}_{-1}$ is the colimit of $\delta^*\overline{W}\big|\mathrm{N}(\boldsymbol{\Delta}^{\mathrm{op}})$.

To show that $\overline{Y}_{-1}$ is a retract of $\overline{U}_{-1}$ over $\overline{X}_{-1}$, it therefore suffices to prove that the simplicial object Y is a retract of $\delta^*\overline{W}\big|\mathrm{N}(\boldsymbol{\Delta}^{\mathrm{op}})$ over X. Unfortunately, such a retraction may not exist at the level of simplicial objects. However, it will exist at the level of *semisimplicial* objects (i.e. without the degeneracies). Since the inclusion $\mathrm{N}(\boldsymbol{\Delta}_s^{\mathrm{op}}) \to \mathrm{N}(\boldsymbol{\Delta}^{\mathrm{op}})$ of the semisimplicial category is cofinal [16, 6.5.3.7], this will be enough to prove that $\overline{Y}_{-1}$ is a retract of $\overline{U}_{-1}$ over $\overline{X}_{-1}$.

Let us denote the underlying semisimplicial object of a simplicial object Y by Y^s. It follows from [16, 6.5.3.8] that the map $Y^s \to X^s$ is n-coskeletal whenever $Y \to X$ is a hypercover of height n. In particular, to construct a retraction $\delta^*\overline{W}\big|\mathrm{N}(\mathbf{\Delta}_s^{\mathrm{op}}) \to Y^s$ over X^s, it will suffice to construct a retraction (up to homotopy)

$$\begin{array}{ccccc} \iota_0^*\overline{W} & \longrightarrow & \delta^*\overline{W} & \dashrightarrow & \iota_0^*\overline{W} \\ & \searrow & \downarrow & \swarrow & \\ & & \iota_{-2}^*\overline{W} & & \end{array}$$

where *all of the above objects are restricted to* $\mathrm{N}(\mathbf{\Delta}^{\mathrm{op}}_{s,\leq n})$. To construct this retraction (up to homotopy), let us consider the functor

$$f\colon \mathbf{\Delta}_{s,\leq n} \longrightarrow \mathbf{\Delta}_+ \times \mathbf{\Delta}_{++}$$

given by $f([n]) = ([n],[n])$ and $f([m]) = ([m],[-1])$ for all $m < n$. One easily sees that this is (only) functorial with respect to the face maps in degrees $\leq n$ (using that $[-1]$ is initial in $\mathbf{\Delta}_+$). The functor f comes equipped with natural transformations $\iota_n \leftarrow f \to \delta$ (with ι_n and δ restricted to $\mathrm{N}(\mathbf{\Delta}_{s,\leq n})$). Unwinding the definitions, one obtains a commuting diagram of $\mathrm{N}(\mathbf{\Delta}^{\mathrm{op}}_{s,\leq n})$-indexed diagrams over $X = \iota_{-2}^*\overline{W}$ of the form

$$\begin{array}{ccccc} & & \delta^*\overline{W} & & \\ & \nearrow & \uparrow{\scriptstyle\simeq} & & \\ \iota_0^*\overline{W} & \longrightarrow & f^*\overline{W} & & \\ & \searrow & \downarrow{\scriptstyle\simeq} & & \\ & & \iota_n^*\overline{W} & \dashrightarrow & \iota_0^*\overline{W}. \end{array}$$

The vertical two arrows are levelwise equivalences since the map $Y_k \to U_k$ is an equivalence in degrees $k \leq n-1$, so that its Čech nerve is essentially constant. On the other hand, the canonical map $\iota_0^*\overline{W} \to \iota_n^*\overline{W}$ associated to the map $[n] \to [0]$ in $\mathbf{\Delta}$ has a retraction (over $\iota_{-2}^*\overline{W}$), which is obtained by picking a section of the map $[n] \to [0]$. It follows that $\delta^*\overline{W}$ is a retract of $Y = \iota_0^*\overline{W}$ over $\iota_{-2}^*\overline{W} = X$, as we needed to show. □

Remark 4.11. The analogue of this lemma when $n = \infty$ holds if and only if the ∞-topos is hypercomplete (see [16, Section 6.5.2]): indeed, suppose that $p\colon Y \to X$ is a hypercover of infinite height and factor p as $Y \to \mathrm{cosk}_n(p) \to X$. The second map induces an equivalence on colimits by the previous proposition and the map $|Y| \to |\mathrm{cosk}_n(p)|$ is $(n-1)$-connective [16, Lemma 6.5.3.10] (i.e. its $(n-2)$-truncation is terminal in $\mathcal{X}/|\mathrm{cosk}_n(p)|$). It follows that $|Y| \to |X|$ is $(n-1)$-connective for all $n \geq 0$, which means that it is an equivalence if $\mathcal{X}$ is hypercomplete.

Conversely, if f is n-connective for all n, then (essentially by definition of n-connectivity [16, Definition 6.5.1.10]) the map of constant simplicial diagrams $\mathrm{cst}(f)\colon \mathrm{cst}(Y) \to \mathrm{cst}(X)$ is a hypercover, whose induced map of colimits is f. It follows that any ∞-connective map is an equivalence, which means precisely that $\mathcal{X}$ is hypercomplete.

Proposition 4.12. *Let $p\colon Y \to X$ be a Kan fibration of height $n < \infty$ between two simplicial objects in an ∞-topos $\mathcal{X}$. Then p is a* realization fibration, *i.e. for any map of simplicial objects $X' \to X$, the natural map of colimits $|Y \times_X X'| \to |Y| \times_{|X|} |X'|$ is an equivalence.*

Example 4.13. Recall from Remark 4.8 that Kan fibrations of height 0 are cartesian natural transformations. Such Cartesian natural transformations are always realization fibrations: indeed, if $F \to G$ is a cartesian natural transformation of $\mathcal{J}$-diagrams in $\mathcal{X}$ and $G' \to G$ is another natural transformation, then the base change $F' = F \times_G G' \to G'$ is a cartesian transformation as well. By (Rezk) descent (see [22] or [16, Theorem 6.1.3.9(4)]), the natural transformation

$$F'(j) \xrightarrow{\simeq} G'(j) \times_{G(j)} F(j) \xrightarrow{\simeq} G'(j) \times_{G(j)} \big(G(j) \times_{\operatorname{colim} G} \operatorname{colim} F\big)$$

is an equivalence. Since the target is naturally equivalent to $G'(j) \times_{\operatorname{colim} G} \operatorname{colim} F$ and base change in an ∞-topos $\mathcal{X}$ preserves all small colimits, the result follows.

The proof of Proposition 4.12 consists of a descent argument (as in [22]) to reduce to a simpler statement about Kan fibrations over simplices, which is proven by a well-known simplicial argument.

Notation 4.14. If $S \in \mathcal{X}$ is an object, let us denote by $[k]_S \colon \mathrm{N}(\mathbf{\Delta})^{\mathrm{op}} \to \mathcal{X}$ the image of S under the functor $\mathcal{X} \to \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{\mathrm{op}}, \mathcal{X})$ taking the left Kan extension along the inclusion $\{[k]\} \to \mathrm{N}(\mathbf{\Delta})^{\mathrm{op}}$. A map $S \to T$ in $\mathcal{X}$ induces a map $[k]_S \to [k]_T$ and a map $[k] \to [k']$ in $\mathbf{\Delta}$ induces a map $[k]_S \to [k']_S$. Observe that the colimit $|[k]_S|$ is equivalent to S (being the composition of two left Kan extensions).

Lemma 4.15. *The following statements are equivalent for $0 \leq n < \infty$:*

(1) for any ∞-topos $\mathcal{X}$, a Kan fibration $p \colon Y \to X$ of height n is a realization fibration.
(2) for any ∞-topos $\mathcal{X}$, a Kan fibration $p \colon Y \to [k]_S$ of height n is a realization fibration.
(3) for any ∞-topos $\mathcal{X}$, a Kan fibration $p \colon Y \to [k]_S$ of height n and maps $[m] \to [k]$ in $\mathbf{\Delta}$ and $T \to S$ in $\mathcal{X}$, the map $|Y \times_{[k]_S} [m]_T| \to |Y| \times_S T$ is an equivalence.
(4) for any ∞-topos $\mathcal{X}$ and a Kan fibration $p \colon Y \to [k]_*$ of height n, the maps $\alpha \colon \{0\} \to [k]$ and $\beta \colon \{k\} \to [k]$ in $\mathbf{\Delta}$ induce equivalences $|\alpha^* Y| \to |Y|$ and $|\beta^* Y| \to |Y|$ in $\mathcal{X}$.

Proof. It is easy to see that (1) ⇒ (2) ⇒ (3) ⇒ (4). To see that (2) ⇒ (1), let $p \colon Y \to X$ be a Kan fibration of height n, let $f \colon X' \to X$ be a map of simplicial objects and let Y' denote the levelwise pullback $X' \times_X Y$. Observe that for any $X \in \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{\mathrm{op}}, \mathcal{X})$, there exists a colimit diagram $V \colon \mathcal{J}^{\triangleright} \to \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{\mathrm{op}}, \mathcal{X})$ with cone object $V(*) = X$, such that each $V(j)$ with $j \in \mathcal{J}$ is of the form $[k]_S$ for some $[k] \in \mathbf{\Delta}, S \in \mathcal{X}$. Taking the base change of the diagram V over X with X', Y and Y' yields a diagram $\mathcal{J} \times \Delta[1] \times \Delta[1] \to \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{\mathrm{op}}, \mathcal{X})$ whose value at fixed $j \in \mathcal{J}$ is given by the cartesian square

$$\begin{array}{ccc} Y' \times_X V(j) & \longrightarrow & Y \times_X V(j) \\ \downarrow & & \downarrow \\ X' \times_X V(j) & \longrightarrow & V(j). \end{array}$$

The natural transformation $Y \times_X V(-) \to V(-)$ is cartesian (see Remark 4.8), which means that $\operatorname{colim}_{\mathcal{J}} Y \times_X V(-) \simeq Y$ by descent, and similarly for $X' \times_X V(-)$ and $Y' \times_X V(-)$. Furthermore, the right vertical map in the above square is a Kan fibration of height n over an object $[k]_S$ and therefore a realization fibration by (2).

This implies (a) that the above square of simplicial diagrams in $\mathcal{X}$ remains cartesian after taking the colimit and (b) that the natural transformation $|Y \times_X V(-)| \to |V(-)|$ of $\mathcal{J}$-diagrams in $\mathcal{X}$ is cartesian as well. Example 4.13 (for the cartesian map of $\mathcal{J}$-indexed diagrams $|Y \times_X V(-)| \to |V(-)|$) shows that taking the colimit over $\mathcal{J}$ gives the desired equivalence $|Y'| \to |X'| \times_{|X|} |Y|$.

To see that (3) ⇒ (2), let $p \colon Y \to [k]_S$ be a Kan fibration of height n and let $Z \to [k]_S$ be a map. We can realize $Z \to [k]_S$ as the colimit of a diagram $U \colon \mathcal{J} \to \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{\mathrm{op}}, \mathcal{X})/[k]_S$ such that each $U(j)$ is a map $[m]_T \to [k]_S$ induced by maps $[m] \to [k]$ in $\mathbf{\Delta}$ and $T \to S$ in

$\mathcal{X}$. By assumption (3) it follows that the natural transformation of $\mathcal{J}$-diagrams in $\mathcal{X}$

$$\left|U(-) \times_{[k_S]} Y\right| \longrightarrow \left|U(-)\right| \times_{|[k]_S|} |Y|$$

is a levelwise equivalence. Taking the colimit over $\mathcal{J}$ and using that base change in $\mathcal{X}$ preserves colimits, one finds that the map $|Z \times_{[k]_S} Y| \to |Z| \times_{|[k]_S|} |Y|$ is an equivalence.

Finally, to see that (4) $\Rightarrow$ (3), suppose that $p\colon Y \to [k]_S$ is a map and $[m] \to [k]$ and $T \to S$ are maps in $\mathbf{\Delta}$ and $\mathcal{X}$. Since $[k]_S\colon \mathrm{N}(\mathbf{\Delta})^{\mathrm{op}} \to \mathcal{X}$ naturally sits over its colimit S, we may replace $\mathcal{X}$ by the ∞-topos $\mathcal{X}/S$ and assume that $S = *$. The map $[m]_T \to [k]_*$ factors as $[m]_T \to [k]_T \to [k]_*$ and gives a composite pullback diagram

$$\begin{array}{ccccc} Y'' & \longrightarrow & Y' & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ [m]_T & \longrightarrow & [k]_T & \longrightarrow & [k]_* \end{array}$$

Because taking the product with T preserves coproducts, the map $[k]_T \to [k]_*$ can be identified with the projection map $T \times [k]_* \to [k]_*$. Consequently, it is a Kan fibration of height 0 and a realization fibration by Example 4.13. The right square therefore remains cartesian after taking the colimit over $\mathrm{N}(\mathbf{\Delta})^{\mathrm{op}}$, so it suffices to show that the map $Y'' \to Y'$ induces an equivalence of colimits (as the map $[m]_T \to [k]_T$ does too). Replacing $\mathcal{X}$ by $\mathcal{X}/T$ and T by $*$, this follows from assumption (4) and a simple 2-out-of-3 argument. □

Proof (of Proposition 4.12). It remains to prove part (4) of the previous lemma. Let $Y \to [k]_*$ be a Kan fibration of height n. We will show that base change $Y_0 \to Y$ of the map $\{0\} \to [k]_*$ induces an equivalence on colimits; the other case proceeds similarly.

Let $h\colon [k] \times [1] \to [k]$ be the homotopy from the constant map with value 0 to the identity map on $[k]$ and consider the base change $h^*Y \to [k]_* \times [1]_*$. Note that the base change to $[k]_* \times \{0\}$ yields the map $Y_0 \times [k]_* \to [k]_*$ and the base change to $[k]_* \times \{1\}$ yields $p\colon Y \to [k]_*$.

Consider the Kan fibration $\Gamma(\Delta[1], h^*Y) \to [k]_*$ given by

$$(h^*Y)^{\Delta[1]} \times_{([1]_* \times [k]_*)^{\Delta[1]}} \{\mathrm{id}_{[1]}\} \times [k]_* \to [k]_*$$

(see Example 4.22 for the definition of the cotensor of a simplicial object in $\mathcal{X}$ and a simplicial set). The map $Y_0 \to Y$ now fits into a commuting diagram

$$\begin{array}{ccccc} & \swarrow & Y_0 & \searrow & \\ & & \downarrow & & \\ Y_0 \times [k]_* & \xleftarrow{\mathrm{ev}_0} & \Gamma(\Delta[1], h^*Y) & \xrightarrow{\mathrm{ev}_1} & Y \end{array}$$

A simple exercise in simplicial combinatorics shows that the bottom two maps are hypercovers of height n and therefore induce equivalences upon taking colimits over $\mathrm{N}(\mathbf{\Delta})^{\mathrm{op}}$, by Proposition 4.9. Since the map $Y_0 \to Y_0 \times [k]_*$ induces an equivalence upon taking colimits as well, the result follows. □

Remark 4.16. If $\mathcal{X}$ is a hypercomplete ∞-topos, then Proposition 4.12 also holds for $n = \infty$. Indeed, in that case $\mathcal{X}/S$ is hypercomplete for any $S \in \mathcal{X}$, as long as $\mathcal{X}$ is hypercomplete (this follows e.g. from the characterization of hypercompleteness in Remark 4.11). The proof of Lemma 4.15 then shows that the four given statements are equivalent for $n = \infty$ and all hypercomplete ∞-toposes. Invoking Remark 4.11 rather than Proposition 4.9 in the above proof then yields the desired result.

4.2. **Geometric groupoids.** Let $\mathcal{X}$ be an ∞-topos, together with a full subcategory $\mathcal{X}_{0-\mathrm{geom}} \subseteq \mathcal{X}$ and a class P of smooth maps as in Assumption 3.6. In this situation, there is an obvious 'geometric' analogue of the notion of an n-groupoid object.

Definition 4.17. Let $p\colon Y \to X$ be a map of simplicial objects in $\mathcal{X}$ and $0 \leq n \leq \infty$. We will say that p is a *geometric Kan fibration* of height n if it is a Kan fibration of height n such that the relative matching map

$$Y(\Delta[k]) \simeq Y_k \longrightarrow X(\Delta[k]) \times_{X(\Lambda^i[k])} Y(\Lambda^i[k])$$

is 0-smooth for all $k \leq n$. If in addition the map $Y_0 \to X_0$ in $\mathcal{X}$ is 0-smooth, we will say that the map p is a *smooth* Kan fibration (of height n).

We will say that a simplicial object X in $\mathcal{X}$ is a *geometric n-groupoid* if it is an n-groupoid object in $\mathcal{X}_{0-\mathrm{geom}} \subseteq \mathcal{X}$ and the maps $X_k \to X(\Lambda^i[k])$ are smooth for all $k \leq n$. Let $\mathrm{Gpd}_n^{\mathrm{geom}} \subseteq \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{\mathrm{op}}, \mathcal{X})$ be the full sub-∞-category on the geometric n-groupoids.

A map $p\colon Y \to X$ of simplicial objects in $\mathcal{X}$ is a *smooth hypercover* of height n if it is a hypercover of height n and all maps

$$Y(\Delta[k]) \simeq Y_k \longrightarrow X(\Delta[k]) \times_{X(\partial\Delta[k])} Y(\partial\Delta[k])$$

are 0-smooth.

Variant 4.18. Recall from Remark 3.13 that the full subcategory $\mathcal{X}_{m-\mathrm{geom}} \subseteq \mathcal{X}$ of m-geometric stacks, together with the class P_m of m-smooth maps between them also satisfies the conditions of Assumption 3.6. Taking this as input of the above definition yields a notion of *m-geometric n-groupoid* (i.e. a simplicial object in $\mathcal{X}_{m-\mathrm{geom}}$ whose matching maps for horn inclusions are m-smooth maps), *m-geometric* (resp. *m-smooth*) *Kan fibration* and *m-smooth hypercover.* For $m = 0$ these notions reproduce the above notions. Furthermore observe that a map $Y \to X$ of simplicial objects in $\mathcal{X}_{0-\mathrm{geom}}$ is an m-geometric Kan fibration if and only if it is a (0-) geometric Kan fibration (see Lemma 3.12), and similarly for the other two classes of maps.

Example 4.19. When $\mathcal{X} = \mathrm{Sh}_\infty(\mathrm{Mfd})$, $\mathcal{X}_{0-\mathrm{geom}}$ is the class of smooth manifolds and P is the class of submersions, this retrieves the notions of Lie m-groupoids, Kan fibrations and hypercovers from Section 2.

Remark 4.20. If X is a geometric n-groupoid in $\mathcal{X}$, then for any horn $\Lambda^i[k]$, the object $X(\Lambda^i[k])$ is 0-geometric. Indeed, this follows from the usual inductive argument (see e.g. [8, Lemma 2.4], [26, Lemma 2.1]), by realizing the inclusion $\{i\} \to \Lambda^i[k]$ as an iterated pushout of lower-dimensional horn inclusions and invoking Remark 4.4.

We record the following basic properties of the above classes of maps:

Lemma 4.21. *The following statements hold:*

1. *The classes of of geometric (resp. smooth) Kan fibrations and smooth hypercovers of height n are stable under base change.*
2. *Let $p\colon Y \to X$ be a geometric Kan fibration of height ∞ between two geometric n-groupoids. Then p is a geometric Kan fibration of height n.*
3. *Let $p\colon Y \to X$ be a smooth hypercover of height ∞. Then p is a smooth Kan fibration, which is of height $n < \infty$ if and only if p is a hypercover of height $n-1$.*

Proof. To avoid having to go in the ∞-categorical details involving the functoriality of the relative matching objects, let us invoke Remark 4.5 and pick a model-categorical presentation $\mathbf{A}$ for $\mathcal{X}$, so that the Reedy model structure on $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathbf{A})$ presents $\mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{\mathrm{op}}, \mathcal{X})$ and the usual relative matching maps present the ∞-categorical relative matching maps (at least for Reedy fibrations). Let $p\colon Y \to X$ and $q\colon Z \to X$ be Reedy fibrations between Reedy fibrant objects.

For (1), observe that the relative matching maps of the base change $q^*Y \to Z$ are (homotopy) base changes of the relative matching maps of p. If the latter are equivalences or smooth maps, so are their homotopy base changes.

For (2), let $k > n$ and consider an inclusion $\Lambda^i[k] \to \Delta[k]$. Since Y and X are n-groupoid objects, the horizontal maps in the square

$$\begin{array}{ccc} Y(\Delta[k]) & \longrightarrow & Y(\Lambda^i[k]) \\ \downarrow & & \downarrow \\ X(\Delta[k]) & \longrightarrow & X(\Lambda^i[k]) \end{array}$$

are trivial fibrations, so that the matching map is an equivalence. This implies that p is a (geometric) Kan fibration of height n.

For (3), consider the inclusions of simplicial sets $\Lambda^i[k] \subseteq \partial\Delta[k] \subseteq \Delta[k]$ and observe that the first map is the pushout of an inclusion $\partial\Delta[k-1] \to \Delta[k-1]$. Associated to these maps of simplicial sets is a diagram

$$\begin{array}{ccccc} Y(\Delta[k]) & \xrightarrow{\mu_k} & X(\Delta[k]) \times_{X(\partial\Delta[k])} Y(\partial\Delta[k]) & \xrightarrow{\mu'} & X(\Delta[k]) \times_{X(\Lambda^i[k])} Y(\Lambda^i[k]) \\ & & \downarrow & & \downarrow \\ & & Y(\Delta[k-1]) & \xrightarrow[\mu_{k-1}]{} & X(\Delta[k-1]) \times_{X(\partial\Delta[k-1])} Y(\partial\Delta[k-1]) \end{array}$$

in which the vertical maps restrict to the face opposite i. Using Remark 4.4, one sees that the right square is (homotopy) cartesian. Since p is a hypercover and 0-smooth maps and effective epimorphisms are stable under homotopy base change and composition, it follows that the top horizontal composite is a 0-smooth effective epimorphism, so that p is a Kan fibration.

If p is a hypercover of height $n-1$, then the horizontal maps are equivalences for all $k > n$, so that p is a Kan fibration of height n. Conversely, if p is a Kan fibration of height n, then the top horizontal map is an equivalence for all $k > n$. Since the map μ_k is an effective epimorphism, it follows that both μ_k and μ' are equivalences [16, Example 5.2.8.16]. The right vertical map in the above diagram is an effective epimorphism (the total composite of the above diagram is an effective epimorphism), so it follows that the homotopy base change of μ_{k-1} along an effective epimorphism is an equivalence. But then μ_{k-1} is itself an equivalence [16, Proposition 6.2.3.14] and we conclude that p is a hypercover of height $n-1$. □

Example 4.22. Let $\times\colon \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{\mathrm{op}}, \mathcal{S})^{\mathrm{op}} \times \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{\mathrm{op}}, \mathcal{S})^{\mathrm{op}} \to \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{\mathrm{op}}, \mathcal{S})^{\mathrm{op}}$ be the opposite of the product functor and observe that this functor preserves small limits in each of its variables. It follows that for each limit preserving functor $X\colon \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{\mathrm{op}}, \mathcal{S})^{\mathrm{op}} \to \mathcal{X}$ with values in an ∞-topos, the composite $\mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{\mathrm{op}}, \mathcal{S})^{\mathrm{op}} \times \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{\mathrm{op}}, \mathcal{S})^{\mathrm{op}} \to \mathcal{X}$ preserves limits in each variable. Precomposition with $\times$ therefore yields a functor

$$\mathrm{Fun}^{\mathrm{R}}(\mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{\mathrm{op}}, \mathcal{S})^{\mathrm{op}}, \mathcal{X}) \times \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{\mathrm{op}}, \mathcal{S})^{\mathrm{op}} \longrightarrow \mathrm{Fun}^{\mathrm{R}}(\mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{\mathrm{op}}, \mathcal{S})^{\mathrm{op}}, \mathcal{X})$$

sending a pair (X, K) to $X^K(-) := X(K \times -)$. When $X\colon \mathrm{N}(\mathbf{\Delta})^{\mathrm{op}} \to \mathcal{X}$ is a simplicial object in $\mathcal{X}$ and K is a simplicial set (viewed as a discrete simplicial space), we denote by X^K the corresponding simplicial object under the equivalence of Remark 4.3. Note that this construction is functorial in X and K.

When X is a geometric n-groupoid object in $\mathcal{X}$, the map $X^{\Delta[m]} \to X^{\partial\Delta[m]}$ is a geometric Kan fibration of height n for all $m \geq 0$ and of height 0 when $m \geq n$. Indeed, for any horn inclusion $\Lambda^i[k] \to \Delta[k]$, the map

$$X^{\Delta[m]}(\Delta[k]) \longrightarrow X^{\partial\Delta[m]}(\Delta[k]) \times_{X^{\partial\Delta[m]}(\Lambda^i[k])} X^{\partial\Delta[m]}(\Delta[k])$$

is equivalent (by construction) to the map $X(\Delta[m] \times \Delta[k]) \to X(K)$, where K is the subcomplex of $\Delta[m] \times \Delta[k]$ given by

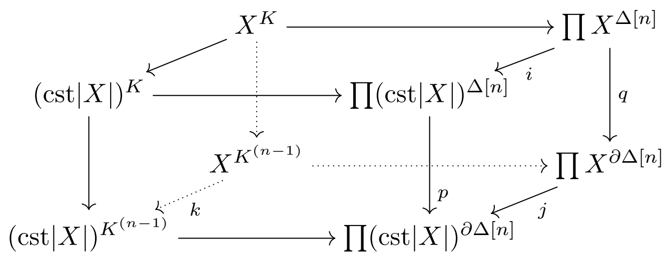
$$K = \partial\Delta[m] \times \Delta[k] \cup \Delta[m] \times \Lambda^i[k].$$

The inclusion of K into $\Delta[m] \times \Delta[k]$ is a finite composition of pushouts of horn inclusions, each of which is in dimension $\geq \max(k, m+1)$ since $K \subseteq \Delta[m] \times \Delta[k]$ induces an isomorphism of skeleta in dimension $\max(k-2, m-1)$. From this the assertion follows.

A similar argument shows that for any horn inclusion $\Lambda^i[m] \to \Delta[m]$, the map $X^{\Delta[m]} \to X^{\Lambda^i[m]}$ is a smooth hypercover of height $n-1$.

Corollary 4.23. *Let X be a geometric n-groupoid and let $X \to \mathrm{cst}|X|$ be the natural map to the constant diagram whose value is the colimit of X. For each finite simplicial set K, the map $X^K \to (\mathrm{cst}|X|)^K$ induces an equivalence on colimits.*

Proof. The proof is by induction on the dimension of K: if K is of dimension 0, then the result follows from the fact that $\mathrm{N}(\mathbf{\Delta})^{\mathrm{op}}$ is sifted, so that colimits over it commute with finite products. Now assume that the result holds for all simplicial sets of dimension $n-1$ and let K be of dimension n, with $(n-1)$-skeleton $K^{(n-1)}$. Then there is a cube of simplicial objects in $\mathcal{X}$ of the form



where the products run over the (finite) set of nondegenerate n-simplices of K. The front and back face are cartesian squares and the vertical maps p and q are Kan fibrations in $\mathcal{X}$ (of finite height). It follows from Proposition 4.12 that the front and back face remain cartesian after taking the colimit. But by assumption the maps i, j and k induce equivalences on colimits (the map i since the natural map $X^{\Delta[n]} \to X$ induces an equivalence on colimits), so that the map on pullbacks is an equivalence as well. $\square$

Example 4.24. Recall that there exists a functor $f\colon \mathbf{\Delta} \to \mathbf{\Delta}$ sending each $[n]$ to $[0] \star [n] = [1+n]$, together with natural transformations of endofunctors of $\mathbf{\Delta}$

$$\mathrm{id}_{\mathbf{\Delta}} \xrightarrow{\partial^0} f \underset{\partial^{\{0\}}}{\overset{\sigma^{\{0\}}}{\rightleftarrows}} \mathrm{cst}_{[0]}.$$

The restriction of a simplicial object X along f is usually denoted by $\mathrm{Dec}_0(X)$, and fits into a diagram

$$X \xleftarrow{d_0} \mathrm{Dec}_0(X) \underset{d_{\{0\}}}{\overset{s_{\{0\}}}{\leftrightarrows}} \mathrm{cst}(X_0).$$

For any map $p\colon Y \to X$ of simplicial objects in $\mathcal{X}$, let $p'\colon \mathrm{Dec}_0(Y) \to \mathrm{Dec}_0(X) \times_X Y$ be the natural map associated to the square

$$\begin{array}{ccc} \mathrm{Dec}_0(Y) & \xrightarrow{\mathrm{Dec}_0(p)} & \mathrm{Dec}_0(X) \\ {\scriptstyle d_0}\downarrow & & \downarrow{\scriptstyle d_0} \\ Y & \xrightarrow[p]{} & X. \end{array}$$

If p is a geometric Kan fibration of height n, then p' is a *smooth* Kan fibration of height $n-1$. Indeed, one easily checks that p' is smooth in degree 0 and that the relative matching map of p' with respect to $\Lambda^i[k] \to \Delta[k]$ agrees with the relative matching map of p with respect to $\Lambda^{1+i}[1+k]$.

Remark 4.25. Let $X\colon \mathrm{N}(\mathbf{\Delta})^{\mathrm{op}} \to \mathcal{X}$ be an n-groupoid in $\mathcal{X}$. Then the degeneracy map $s_{\{0\}}\colon \mathrm{cst}(X_0) \to \mathrm{Dec}_0(X)$ induces an equivalence on colimits. Indeed, the map $s_{\{0\}}$ has a retraction, given by the map $d_{\{0\}}$ restricting to the initial vertex of each $[0] \star [n]$. Furthermore, there is a functor

$$h\colon ([0] \star [n]) \times [1] \longrightarrow [0] \star [n] \qquad h\big|_{([0]\star[n])\times\{0\}} = 0 \qquad h\big|_{([0]\star[n])\times\{1\}} = \mathrm{id}$$

depending naturally on $[n]$.

The natural transformation h induces a map $h^*\colon X \to X^{\Delta[1]}$, such that the composition $d_1h^*\colon X \to X^{\Delta[1]} \to X$ is given by $s_{\{0\}}d_{\{0\}}$ and d_0h^* is the identity map. Since both $d_0, d_1\colon X^{\Delta[1]} \to X$ are hypercovers of height n (Example 4.22), it follows that h^* induces an equivalence on colimits. This implies that the composite $s_{\{0\}}d_{\{0\}}$ induces an equivalence on colimits, so at the level of colimits, $d_{\{0\}}$ is a two-sided homotopy inverse for $s_{\{0\}}$.

Let $f\colon X \to Y$ be a map of geometric n-groupoids. Using the (functorial) exponential construction of Example 4.22, we obtain a commuting diagram

$$\begin{array}{ccccccc} X & \xrightarrow{s_0} & X^{\Delta[1]} & \xrightarrow{(d_1,d_0)} & X \times X & \xrightarrow{\pi_1} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^{\Delta[1]}} & & \downarrow{\scriptstyle f\times f} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[s_0]{} & Y^{\Delta[1]} & \xrightarrow[(d_1,d_0)]{} & Y \times Y & \xrightarrow[\pi_1]{} & Y. \end{array}$$

Let $P_f = X \times_Y Y^{\Delta[1]}$ be the pullback of the right vertical map and the last two bottom horizontal maps. Then the top horizontal composition (which is the identity on X) factors (up to homotopy) as a map $\sigma\colon X \to P_f$, followed by the base change $P_f \to X \times Y \to X$ of the smooth hypercover $d_1 = \pi_1(d_1, d_0)\colon Y^{\Delta[1]} \to Y \times Y \to Y$. Let p be the composite map $P_f \to X \times Y \to Y$ where the first map is the base change of $(d_1, d_0)\colon Y^{\Delta[1]} \to Y \times Y$ and the second map is the projection. Then p is the composition of two geometric Kan fibrations and unraveling the definitions, one sees that the composition $p\sigma$ is homotopic to f. This provides a factorization (up to homotopy) of any map $f\colon X \to Y$ between geometric n-groupoids as a section (up to homotopy) of a smooth hypercover, followed by a geometric Kan fibration $P_f \to Y$ of height n (the *path fibration*).

Definition 4.26. A map $f\colon X \to Y$ between geometric n-groupoids is called a *Morita equivalence* if the induced path fibration $P_f = X \times_Y Y^{\Delta[1]} \to Y$ is a smooth hypercover (of height $n-1$ by Lemma 4.21).

Remark 4.27. The ∞-category $\mathrm{Gpd}_n^{\mathrm{geom}}$, together with the classes of Morita equivalences, geometric Kan fibrations (of height n) and smooth hypercovers (of height $n-1$) forms something rather close to a category of fibrant objects in the sense of Brown [3] (see also [1]). However, $\mathrm{Gpd}_n^{\mathrm{geom}}$ need not be closed under pullbacks along geometric Kan fibrations, for the simple reason that $\mathcal{X}_{0-\mathrm{geom}}$ is not necessarily closed under pullbacks. Instead, $\mathrm{Gpd}_n^{\mathrm{geom}}$ does admit pullbacks along *smooth* Kan fibrations and in particular along hypercovers, which is all we need. Note that one can replace any map by a geometric Kan fibration, but not necessarily by a smooth Kan fibration.

5. Geometric stacks as a category of fractions

Let $(\mathcal{X}, \mathcal{X}_{0-\mathrm{geom}}, P)$ be as in Assuption 3.6 and consider the composite functor

$$\mathrm{Gpd}_n^{\mathrm{geom}} \longrightarrow \mathrm{Fun}(\mathrm{N}(\boldsymbol{\Delta})^{\mathrm{op}}, \mathcal{X}) \xrightarrow{|-|} \mathcal{X} \tag{8}$$

sending a geometric n-groupoid object to its colimit in $\mathcal{X}$. It follows from Proposition 4.9 that this functor sends smooth hypercovers to equivalences in $\mathcal{X}$, so by the 2-out-of-3 property it sends the Morita equivalences to equivalences in $\mathcal{X}$ as well. Consequently, the above functor induces a functor

$$\mathrm{Gpd}_n^{\mathrm{geom}}[W_{\mathrm{Mor}}^{-1}] \longrightarrow \mathcal{X}$$

from the ∞-categorical localization of $\mathrm{Gpd}_n^{\mathrm{geom}}$ at the Morita equivalences. The aim of this section is to give a proof of the following result:

Theorem 5.1. *Let $(\mathcal{X}, \mathcal{X}_{0-\mathrm{geom}}, P)$ be as in Assumption 3.6 and let $0 \le n < \infty$. The colimit functor $|-|\colon \mathrm{Gpd}_n^{\mathrm{geom}} \to \mathcal{X}$ induces a functor*

$$\mathrm{Gpd}_n^{\mathrm{geom}}[W_{\mathrm{Mor}}^{-1}] \longrightarrow \mathcal{X} \tag{9}$$

which is fully faithful with essential image given by the n-geometric stacks.

Remark 5.2. In the setting of Variant 3.8, where Condition (4) of 3.6 holds reltive to a full subcategory $\mathcal{A} \subseteq \mathcal{X}$, this result still holds when $\mathcal{A}$ contains all colimits of geometric n-groupoid objects (see Remark 5.9).

To prove this theorem, we will use the explicit model for the localization $\mathrm{Gpd}_n^{\mathrm{geom}}[W_{\mathrm{Mor}}^{-1}]$ constructed in [19], using that the localization of $\mathrm{Gpd}_n^{\mathrm{geom}}$ at the Morita equivalences is equivalent to its localization at the set W of hypercovers (by the 2-out-of-3 property). Since hypercovers are stable under base change, this localization can be decribed explicitly in terms of spans: in particular, the space of maps from X to Y in $\mathrm{Gpd}_n^{\mathrm{geom}}[W_{\mathrm{Mor}}^{-1}]$ is equivalent to the groupoid completion of the ∞-category $\mathrm{Span}^W_{\mathrm{Gpd}_n^{\mathrm{geom}}}(X, Y)$ of spans $X \leftarrow \tilde{X} \to Y$, where the left map is a hypercover (of height $(n-1)$):

Definition 5.3. Let $\mathcal{C}$ be a quasicategory and let W be a class of maps in $\mathcal{C}$, closed under homotopy and composition. For any two objects $c, d \in \mathcal{C}$, the quasicategory $\mathrm{Span}^W_{\mathcal{C}}(c, d)$ is the full sub-∞-category

$$\mathrm{Span}^W_{\mathcal{C}}(c, d) \subseteq \mathrm{Fun}(\Lambda^0[2], \mathcal{C}) \times_{\mathrm{Fun}(\{1,2\},\mathcal{C})} \{(c, d)\}$$

consisting of those spans $c \leftarrow \tilde{c} \to d$ for which the left map is contained in W. If W contains all maps in $\mathcal{C}$ (resp. only the equivalences), we will denote this ∞-category simply by $\mathrm{Span}_{\mathcal{C}}(c, d)$ (resp. $\mathrm{Span}^{\mathrm{eq}}_{\mathcal{C}}(c, d)$).

The functor (8) then induces a functor

$$\mathrm{Span}^W_{\mathrm{Gpd}_n^{\mathrm{geom}}}(X, Y) \longrightarrow \mathrm{Span}^{\mathrm{eq}}_{\mathcal{X}}(|X|, |Y|)$$

whose target is a Kan complex, equivalent to the mapping space $\mathrm{Map}_{\mathcal{X}}(|X|, |Y|)$. In light of [19, Corollary 3.13], to show that (9) is fully faithful, it suffices to show that each of these functors is a Kan-Quillen equivalence. This is proven in Section 5.3 by a repeated use of Quillen's Theorem A. The main technical ingredient that goes into this proof is a result about refinements of hypercovers (and n-groupoid objects) with values in m-geometric stacks to hypercovers (and $(n+1)$-groupoid objects) in $(m-1)$-geometric stacks, which is essentially due to Pridham [20] (although some extra care is needed when $\mathcal{X}_{0-\mathrm{geom}}$ does not have pullbacks). We will start by recalling this refinement of hypercovers.

5.1. **Refinements of hypercovers.** In this section we will recall how hypercovers by simplicial objects in $\mathcal{X}_{m-\mathrm{geom}}$ can be refined to hypercovers in $\mathcal{X}_{0-\mathrm{geom}}$ and that n-groupoid objects in $\mathcal{X}_{m-\mathrm{geom}}$ can be refined to $(n+1)$-groupoid objects in $\mathcal{X}_{(m-1)-\mathrm{geom}}$, following the discussion in [20].

Proposition 5.4. *Let $X\colon \mathrm{N}(\mathbf{\Delta})^{\mathrm{op}} \to \mathcal{X}_{m-\mathrm{geom}}$ be an m-geometric n-groupoid (see Variant 4.18), with $m \geq 1$. Then there exists an $(m-1)$-geometric $(n+1)$-groupoid object Y, together with an $(m-1)$-smooth hypercover $p\colon Y \to X$ of height n. In particular, the map p induces an equivalence on colimits $|Y| \to |X|$.*

The proof of the proposition uses the following construction: associated to an object $U \in \mathcal{X}$ and a natural number j is a simplicial diagram

$$\mathrm{Ran}_j(U)\colon \mathrm{N}(\mathbf{\Delta})^{\mathrm{op}} \longrightarrow \mathcal{X}$$

which is the right Kan extension of the constant diagram along $\{[j]\} \to \mathbf{\Delta}^{\mathrm{op}}$. Since for every $[k] \in \mathbf{\Delta}^{\mathrm{op}}$, the comma category $[k]/\{[j]\}$ is just the discrete set of maps $[j] \to [k]$ in $\mathbf{\Delta}$, it follows (see [16, Definition 4.3.2.2]) that the value of $\mathrm{Ran}_j(U)$ on an object $[k] \in \mathbf{\Delta}$ is naturally equivalent to the product $\prod_{[j]\to[k]} U$.

Lemma 5.5. *If $f\colon U \to V$ is a map of stacks, then the induced map $\mathrm{Ran}_j(U) \to \mathrm{Ran}_j(V)$ has the following two properties:*

- *(a) the relative (homotopy) matching map with respect to $\partial\Delta[k] \to \Delta[k]$ is an equivalence for $k > j$ and the base change of a finite product of copies of f if $k \leq j+1$.*
- *(b) the relative (homotopy) matching map with respect to a horn inclusion $\Lambda^i[k] \to \Delta[k]$ is an equivalence if $k > j+1$ and the base change of a finite product of copies of f if $k \leq j+1$.*

Proof. For any k we can decompose $\mathrm{Hom}([j],[k])$ as the disjoint union of $\mathrm{Hom}([j], \partial\Delta[k])$ and the set of surjective maps $[j] \twoheadrightarrow [k]$ (which is empty if $k > j$). The relative matching map with respect to the boundary inclusion $\partial\Delta[k] \to \Delta[k]$ can then be identified with the map

$$\textstyle\prod_{[j]\to[k]} U \simeq \prod_{[j]\to\partial\Delta[k]} U \times \prod_{[j]\twoheadrightarrow[k]} U \longrightarrow \prod_{[j]\to\partial\Delta[k]} U \times \prod_{[j]\twoheadrightarrow[k]} V.$$

Assertion (a) follows immediately from this. For assertion (b), one applies the same argument, using that $\mathrm{Hom}([j],[k])$ is the disjoint union of $\mathrm{Hom}([j], \Lambda^i[k])$ together with the set of all maps $[j] \to [k]$ whose image contains the set $\{0, ..., \hat{i}, ..., k\}$ (which is empty if $k > j+1$). □

Proof (of Proposition 5.4). We will prove by increasing induction on $-1 \leq j \leq n$ that there exists a $(m-1)$-smooth hypercover $Y^{(j)} \to X$ (of height n) such that

- $(\star_j)$ $Y^{(j)}_k$ is an $(m-1)$-geometric stack for all $k \leq j$. Furthermore, $Y^{(j)}$ is an m-geometric n-groupoid if $j < k$ and an $(m-1)$-geometric $(n+1)$-groupoid if $j = n$.

Clearly one can take $Y^{(-1)} = X$ and $Y^{(n)}$ is the desired $(m-1)$-geometric $(n+1)$-groupoid.

Fix $0 \leq j \leq n$ and suppose that $Y^{(j-1)}$ has been constructed. By replacing X with $Y^{(j-1)}$, we may assume that X consists of $(m-1)$-geometric stacks in degrees $k \leq j-1$.

Now let $\phi\colon U \to X_j$ be an atlas for the m-geometric stack appearing in degree j and consider the map of simplicial objects

$$\pi\colon Y^{(j)} := Y := X \times_{\mathrm{Ran}_j(X_j)} \mathrm{Ran}_j(U) \longrightarrow X$$

The relative matching maps of π are base changes of the relative matching maps of $\mathrm{Ran}_j(U) \to \mathrm{Ran}_j(X_j)$ and the matching maps of Y are compositions of relative matching maps and the matching maps of X. It follows from part (a) of Lemma 5.5 that π is a hypercover of height $j \leq n$, whose relative matching maps are $(m-1)$-smooth. In particular, Y_0 is an $(m-1)$-geometric stack. Furthermore, it follows from part (b) of Lemma 5.5 that the matching maps

$$Y_k \longrightarrow Y(\Lambda^i[k])$$

remain $(m-1)$-smooth effective epimorphisms and remain equivalences if both $k > n$ (so that the matching maps in X were equivalences) and $k > j+1$ (so that the relative matching maps of $Y \to X$ were equivalences). In particular, Y remains an m-geometric n-groupoid when $j < n$ and becomes an $(m-1)$-geometric $(n+1)$-groupoid in the final stage where $j = n$.

It remains to show that Y_k is an $(m-1)$-geometric stack for all $k \leq j$. Observe that each Y_k fits into a pullback diagram

$$\begin{array}{ccc} Y_k & \longrightarrow & U^{\times \mathrm{Hom}([j],[k])} \\ \downarrow & & \downarrow \\ X_k & \xrightarrow[\delta]{} & X_j^{\times \mathrm{Hom}([j],[k])} \end{array}$$

where the bottom map is the diagonal. Since the right map is $(m-1)$-smooth, it follows that Y_k is an $(m-1)$-geometric stack for all $k < j$. When $k = j$, one can form the composite pullback

$$\begin{array}{ccccc} P & \longrightarrow & Y_j & \longrightarrow & U^{\times \mathrm{Hom}([j],[j])} \\ \downarrow & & \downarrow & & \downarrow \\ U & \longrightarrow & X_j & \xrightarrow[\delta]{} & X_j^{\times \mathrm{Hom}([j],[j])} \end{array}$$

along the atlas of X_j. Since the right vertical map is $(m-1)$-smooth, the stack P is an $(m-1)$-geometric stack. Since $U \to X_j$ is an $(m-1)$-smooth effective epimorphism, so is the map $P \to Y_j$. This implies that Y_j is an $(m-1)$-geometric stack as well. □

Lemma 5.6 ([20, Proposition 4.5])**.** *Let $Z \to X$ be a height n hypercover of a simplicial object X in $\mathcal{X}_{0-\mathrm{geom}}$ and suppose that the relative matching maps are $(m+1)$-smooth in degrees $\leq n$. Then there exists a map $V \to Z$ such that*

- *the map $V \to Z$ is a hypercover of height n whose relative matching maps are m-smooth.*
- *the composite $V \to Z \to X$ is a hypercover of height n whose relative matching maps are m-smooth.*

Proof. The proof is similar to the previous proof: we show by induction on $-1 \leq j \leq n$ that there exists a height n hypercover $V^{(j)} \to Z$, whose relative matching maps are m-smooth, such that the composite map $V^{(j)} \to Z \to X$ induces maps

$$V^{(j)}(\Delta[k]) \longrightarrow V^{(j)}(\partial\Delta[k]) \times_{X(\partial\Delta[k])} X(\Delta[k])$$

which are m-smooth for all $k \leq j$. Clearly one can take $V^{(-1)} = X$ and $V^{(n)}$ is the desired refinement $V \to Z$.

Fix $0 \leq j \leq n$ and suppose that $V^{(j)}$ has been constructed. Replacing Z by $V^{(j)}$, we may assume that $Z \to X$ has the property that the map

$$Z(\Delta[k]) \longrightarrow Z(\partial\Delta[k]) \times_{X(\partial\Delta[k])} X(\Delta[k])$$

is m-smooth for all $k \leq j-1$. Let $\phi\colon U \to Z_j$ be an atlas for the $(m+1)$-geometric stack Z_j appearing in degree j and consider the map of simplicial objects

$$\pi\colon V^{(j)} := V := Z \times_{\mathrm{Ran}_j(Z_j)} \mathrm{Ran}_j(U) \longrightarrow Z$$

The relative matching maps of π are (homotopy) base changes of the relative matching maps of $\mathrm{Ran}_j(U) \to \mathrm{Ran}_j(X_j)$ and the matching maps of V are compositions of relative matching maps and the matching maps of Z. It follows from part (a) of Lemma 5.5 that π is a height n hypercover whose relative matching maps are all m-smooth. Furthermore, the composite $V \to Z \to X$ is a height n hypercover for which the maps

$$V_k \longrightarrow V(\partial\Delta[k]) \times_{X(\partial\Delta[k])} X_k$$

are m-smooth in degrees $< j$ and $(m+1)$-smooth in degrees $> j$. In degree j, this map is the base change along $V(\partial\Delta[j]) \to Z(\partial\Delta[j])$ of the composite map

$$U \longrightarrow Z_j \longrightarrow Z(\partial\Delta[j]) \times_{X(\partial\Delta[j])} X_j.$$

This is an $(m+1)$-smooth map between m-geometric stacks (note that $Z(\partial\Delta[j]) \to X(\partial\Delta[j])$ is m-smooth by inductive assumption, while X_j is 0-geometric). By Lemma 3.12, it is m-smooth. □

5.2. **The essential image.** As an immediate consequence of Proposition 5.4, we obtain:

Corollary 5.7. *For any n-geometric stack X, there is a geometric n-groupoid $\mathcal{G}\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{X}_{0-\mathrm{geom}}$ such that $|\mathcal{G}| \simeq X$.*

In other words, any n-geometric stack can be presented as the colimit of a geometric n-groupoid object with values in 0-geometric stacks. Conversely, we have:

Lemma 5.8. *Let $p\colon Y \to X$ be a smooth Kan fibration of height n between geometric m-groupoids in $\mathcal{X}_{0-\mathrm{geom}}$. Then the induced map $|Y| \to |X|$ on colimits is an n-smooth map. If X is a geometric n-groupoid, then $|X|$ is an n-geometric stack.*

Proof. If $p\colon Y \to X$ is a smooth Kan fibration of height n, to see that $|p|$ is n-smooth it suffices to prove that the base change along the effective epimorphism $X_0 \to |X|$ is n-smooth (see Lemma 3.12(2)). Using Proposition 4.12, this base change is the colimit of the map of simplicial objects $p'\colon Y \times_X \mathrm{cst}(X_0) \to \mathrm{cst}(X_0)$, which is a smooth Kan fibration of height n. In other words, we may reduce to the case where X is the constant simplicial diagram on a 0-geometric stack X_0.

We now proceed by induction on the number n. For $n = 0$, if $p\colon Y \to X = \mathrm{cst}(X_0)$ is of height 0, then the domain Y is essentially constant with value Y_0. The map $|p|$ then agrees with the map $Y_0 \to X_0$ in $\mathcal{X}_{0-\mathrm{geom}}$, which was smooth by assumption.

Now suppose that a smooth Kan fibration of height $n-1$ determines an $(n-1)$-smooth map. We will first show that any geometric n-groupoid Z determines an n-geometric stack: indeed, it follows from Example 4.24 that the map $\mathrm{Dec}_0(Z) \to Z$ is a smooth Kan fibration of height $n-1$. This implies that the induced map $Z_0 \simeq |\mathrm{Dec}_0(Z)| \to |Z|$ is $(n-1)$-smooth. Since it is clearly an effective epimorphism (Example 3.5), it follows that $|Z|$ is n-geometric.

Finally, let $p\colon Y \to X = \mathrm{cst}(X_0)$ be a smooth Kan fibration of height n. Since X is constant, the object Y is a geometric n-groupoid and $|Y|$ is n-geometric, with atlas given by the map $Y_0 \to |Y|$. The composite map $Y_0 \to |Y| \to X_0$ is smooth by assumption, which shows that $|p|$ is n-smooth. $\square$

Remark 5.9. From the way Lemma 3.12 is used in the proofs of Lemma 5.6 and Lemma 5.8, it follows that their proofs apply in the relative setting of Variant 3.8 as well, assuming that all colimits of geometric n-groupoids (and therefore all n-geometric stacks) are contained in $\mathcal{A} \subseteq \mathcal{X}$. Under this assumption on $\mathcal{A}$, Theorem 5.1 also applies when condition (4) of 3.6 holds only *relative* to $\mathcal{A}$. See Section 7 for an example of this situation.

Let us also record the following corollary, which will be used later in the proof.

Corollary 5.10. *Let X be a geometric n-groupoid in $\mathcal{X}$ and let $X \to \mathrm{cst}|X|$ be the induced map to the constant diagram whose value is the colimit of X. This map is a hypercover of height $n-1$, each of whose matching maps are $(n-1)$-smooth.*

Proof. The geometric Kan fibration $p\colon X^{\Delta[m]} \to X^{\partial\Delta[m]}$ of height n (Example 4.22) induces a smooth Kan fibration $p'\colon \mathrm{Dec}_0(X^{\Delta[m]}) \to \mathrm{Dec}_0(X^{\partial\Delta[m]}) \times_{X^{\partial\Delta[m]}} X^{\Delta[m]}$ of height $n-1$ (Example 4.24). Since p is in particular a realization fibration and the colimit of $\mathrm{Dec}_0(Y)$ is simply Y_0 (Remark 4.25), it follows that the colimit of p' is given by

$$X_m \longrightarrow X(\partial\Delta[m]) \times_{|X^{\partial\Delta[m]}|} |X^{\Delta[m]}|.$$

Using that the map $|X^K| \to |X|^K$ is an equivalence for any finite simplicial set K (Corollary 4.23), the above map can be identified with the relative matching of $X \to \mathrm{cst}|X|$ corresponding to the boundary inclusion $\partial\Delta[m] \to \Delta[m]$. But the map p' is a smooth Kan fibration of height $n-1$, so its colimit is $(n-1)$-smooth by the previous lemma. Similarly, when $m \geq n$ the map p is a Kan fibration of height 0, so that the map p' is an equivalence. This implies that the above maps are equivalences for $m \geq n$, so that $X \to \mathrm{cst}|X|$ is a hypercover of height $n-1$. $\square$

5.3. **Fully faithfulness.** Since $\mathrm{Gpd}_n^{\mathrm{geom}}$ is an ∞-category with hypercovers in the sense of [19], proving that the functor $\mathrm{Gpd}_n^{\mathrm{geom}}[W_{\mathrm{Mor}}^{-1}] \to \mathcal{X}$ is fully faithful reduces to verifying the hypothesis of [19, Corollary 3.13]: we have to show that for every two geometric n-groupoids, the functor

$$\mathrm{Span}^W_{\mathrm{Gpd}_n}(X,Y) \xrightarrow{|-|} \mathrm{Span}^{\mathrm{eq}}_{\mathcal{X}}(|X|,|Y|);\ \Big(X \leftarrow \tilde{X} \to Y\Big) \longmapsto \Big(|X| \xleftarrow{\simeq} |\tilde{X}| \to |Y|\Big)$$

is a Kan-Quillen equivalence (where the map $\tilde{X} \to X$ is a hypercover of height $(n-1)$). To see this, it will be useful to decompose the above functor as follows: for each $0 \leq m \leq n-1$, let

$$\mathcal{D}_m \subseteq \mathrm{Span}_{\mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{\mathrm{op}},\mathcal{X})}(X,Y)$$

be the full subcategory of the category of spans $X \leftarrow Z \to Y$ between X and Y in $\mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{\mathrm{op}},\mathcal{X})$ on those spans for which the map $Z \to X$ is a hypercover of height $n-1$, whose relative matching maps are m-smooth. Since $\mathcal{D}_0$ agrees with $\mathrm{Span}^W_{\mathrm{Gpd}_n}(X,Y)$, the functor $|-|\colon \mathrm{Span}^W_{\mathrm{Gpd}_n}(X,Y) \to \mathrm{Span}^{\mathrm{eq}}_{\mathcal{X}}(|X|,|Y|)$ factors over the sequence of inclusions $\mathcal{D}_m \subseteq \mathcal{D}_{m+1}$ as

$$\mathrm{Span}^W_{\mathrm{Gpd}_n}(X,Y) = \mathcal{D}_0 \longrightarrow \mathcal{D}_1 \longrightarrow \cdots \longrightarrow \mathcal{D}_{n-1} \xrightarrow{|-|} \mathrm{Span}^{\mathrm{eq}}_{\mathcal{X}}(|X|,|Y|). \tag{10}$$

We will show that each of the above functors is a Kan-Quillen equivalence.

Lemma 5.11. *For each $0 \leq m < n-1$, the inclusion $\mathcal{D}_m \to \mathcal{D}_{m+1}$ is a Kan-Quillen equivalence.*

Proof. Fix a span $\alpha = \big[X \leftarrow Z \to Y\big]$ in $\mathcal{D}_{m+1}$. By Quillen's Theorem A [16, Theorem 4.1.3.1], it suffices to prove that $\mathcal{D}_m/\alpha = \mathcal{D}_m \times_{\mathcal{D}_{m+1}} \mathcal{D}_{m+1}/\alpha$ is weakly contractible. By Lemma 5.12, it suffices to prove that there is an object π in the category $\mathcal{D}_m/\alpha$ such that all products with π exist.

Our first aim will be to argue that objects in $\mathcal{D}_m/\alpha$ are essentially determined by an object over Z, the 'tip of the span'. Indeed, let $\mathrm{s}\mathcal{X} := \mathrm{Fun}(\mathrm{N}(\boldsymbol{\Delta})^{\mathrm{op}}, \mathcal{X})$ be the ∞-category of simplicial objects in $\mathcal{X}$ and consider the composite functor

$$\mathrm{Span}_{\mathrm{s}\mathcal{X}}(X, Y) \longrightarrow \mathrm{s}\mathcal{X}^{/Y} = \mathrm{Fun}(\Delta[1], \mathrm{s}\mathcal{X}) \times_{\mathrm{Fun}(\{1\}, \mathrm{s}\mathcal{X})} \{Y\} \xrightarrow{\mathrm{dom}} \mathrm{s}\mathcal{X}.$$

Passing to the over-category of the span α, this induces a composite functor

$$\mathrm{Span}_{\mathrm{s}\mathcal{X}}(X, Y)/\alpha \longrightarrow \big(\mathrm{s}\mathcal{X}^{/Y}\big)/[Z \to Y] \longrightarrow \mathrm{s}\mathcal{X}/Z. \tag{11}$$

The first map is a base change of the map given in [19, Lemma 3.11(1)] and therefore a trivial fibration. The second map fits into a diagram

$$\big(\mathrm{s}\mathcal{X}/Y\big)/[Z \to Y] \longrightarrow \big(\mathrm{s}\mathcal{X}^{/Y}\big)/[Z \to Y] \longrightarrow \mathrm{s}\mathcal{X}/Z$$

where the first map is a categorical equivalence [16, 4.2.1.5] and the composite map is a trivial fibration. It follows that (11) is a categorical equivalence. We may therefore replace the full subcategory $\mathcal{D}_m/\alpha \subseteq \mathrm{Span}_{\mathrm{s}\mathcal{X}}(X, Y)/\alpha$ with its essential image under this equivalence, which is the full subcategory of $\mathrm{s}\mathcal{X}/Z$ on those maps $U \to Z$ such that the composite $U \to Z \to X$ is a height $n-1$ hypercover whose matching maps are m-smooth. The construction of Lemma 5.6 provides an object $\pi\colon V \to Z$ in this full subcategory $\mathcal{D}_m/\alpha \subseteq \mathrm{s}\mathcal{X}/Z$.

To see that all products with π in $\mathcal{D}_m/\alpha$ exist, let $\phi\colon U \to Z$ be any object in $\mathcal{D}_m/\alpha \subseteq \mathrm{s}\mathcal{X}/Z$. It suffices to show that the product of π and ϕ in $\mathrm{s}\mathcal{X}/Z$ remains an object of $\mathcal{D}_m/\alpha$. In other words, we have to prove that the composite map $U \times_Z V \to U \to X$ is a height $n-1$ hypercover whose matching maps are m-smooth. But the map $\pi\colon V \to Z$ is such a hypercover by construction (see 5.6), so the base change $U \times_Z V \to U$ is such a hypercover as well. It follows that the map $U \times_Z V \to U \to X$ is the composition of two hypercovers of height $n-1$, whose matching maps are m-smooth. This implies that $\mathcal{D}_k/\alpha$ admits binary products with the object $\pi\colon V \to Z$, so that it is weakly contractible. □

Lemma 5.12. *Let $\mathcal{C}$ be an ∞-category and let $c \in \mathcal{C}$ be an object. If $\mathcal{C}$ admits products with c, then $\mathcal{C}$ is weakly contractible.*

Proof. The functor $c \times (-)\colon \mathcal{C} \to \mathcal{C}$ comes equipped with a natural projection map to the identity functor and with a natural transformation to the constant functor with value $\{c\}$. Since the identity functor is equivalent (via a zig-zag of natural transformations) to a contant functor, the ∞-category $\mathcal{C}$ is weakly contractible. □

Lemma 5.13. *The functor $|-|\colon \mathcal{D}_{n-1} \to \mathrm{Span}^{\mathrm{eq}}_{\mathcal{X}}(|X|, |Y|)$ is a Kan-Quillen equivalence.*

Proof. In fact, we will show that this functor is a left adjoint. To this end, note that the colimit functor $|-|\colon \mathrm{Fun}(\mathrm{N}(\boldsymbol{\Delta})^{\mathrm{op}}, \mathcal{X}) \to \mathcal{X}$ factors as

$$\mathrm{Fun}(\mathrm{N}(\boldsymbol{\Delta})^{\mathrm{op}}, \mathcal{X}) \xrightarrow{i_!} \mathrm{Fun}(\mathrm{N}(\boldsymbol{\Delta}_+)^{\mathrm{op}}, \mathcal{X}) \xrightarrow{\mathrm{ev}_{-1}} \mathcal{X}$$

where $i\colon \mathrm{N}(\boldsymbol{\Delta})^{\mathrm{op}} \to \mathrm{N}(\boldsymbol{\Delta}_+)^{\mathrm{op}}$ is the inclusion into the opposite of the augmented simplex category and $i_!$ takes the left Kan extension along i, so that $(i_!X)_{-1} \simeq |X|$ is the colimit of X. Let us denote by

$$\mathcal{E} \subseteq \mathrm{Span}_{\mathrm{Fun}(\mathrm{N}(\boldsymbol{\Delta}_+)^{\mathrm{op}}, \mathcal{X})}(i_!X, i_!Y)$$

the full subcategory consisting of spans $\overline{\alpha} := \big[i_!X \leftarrow \overline{Z} \to i_!Y\big]$ such that

(1) the span $i^*\overline{\alpha} = \big[X \leftarrow \overline{Z}\big|\mathrm{N}(\boldsymbol{\Delta})^{\mathrm{op}} \to Y\big]$ is contained in $\mathcal{D}_{n-1}$, i.e. the map $\overline{Z}\big|\mathrm{N}(\boldsymbol{\Delta})^{\mathrm{op}} \to X$ is a height n hypercover whose relative matching maps are $(n-1)$-smooth.
(2) $\overline{Z}$ is a colimit diagram.

In other words, an object in $\mathcal{E}$ can be depicted (slightly informally) as a diagram

$$\begin{array}{ccccc} X_\bullet & \xleftarrow{w} & Z_\bullet & \longrightarrow & Y_\bullet \\ \downarrow & & \downarrow & & \downarrow \\ |X| & \underset{\simeq}{\longleftarrow} & |Z| & \longrightarrow & |Y| \end{array}$$

where the map w is an $(n-1)$-smooth hypercover, which implies that $|Z| \to |X|$ is an equivalence. The functor $|-|\colon \mathcal{D}_{n-1} \to \mathrm{Span}^{\mathrm{eq}}_{\mathcal{X}}(|X|, |Y|)$ then factors as

$$\mathcal{D}_{n-1} \xrightarrow{i_!} \mathcal{E} \xrightarrow{\mathrm{ev}_{-1}} \mathrm{Span}^{\mathrm{eq}}_{\mathcal{X}}(|X|, |Y|).$$

The left Kan extension functor $i_!$ admits a right adjoint i^*, which simply restricts spans between $i_!X$ and $i_!Y$ in $\mathrm{Fun}(\mathrm{N}(\boldsymbol{\Delta}_+)^{\mathrm{op}}, \mathcal{X})$ to spans between X and Y in $\mathrm{Fun}(\mathrm{N}(\boldsymbol{\Delta})^{\mathrm{op}}, \mathcal{X})$. Under this functor, the full subcategory $\mathcal{E}$ is send to $\mathcal{D}_{n-1}$ by definition.

It remains to show that $\mathrm{ev}_{-1}\colon \mathcal{E} \to \mathrm{Span}^{\mathrm{eq}}_{\mathcal{X}}(|X|, |Y|)$ admits a right adjoint. This functor fits into a commuting diagram

$$\begin{array}{ccccc} \mathcal{E} & \xrightarrow{\mathrm{ev}_{-1}} & \mathrm{Span}^{\mathrm{eq}}_{\mathcal{X}}(|X|, |Y|) & \longrightarrow & \{(i_!X, i_!Y)\} \\ \downarrow & & \downarrow & & \downarrow \eta \\ \mathrm{Fun}(\mathrm{N}(\boldsymbol{\Delta}_+)^{\mathrm{op}} \times \Lambda^0[2], \mathcal{X}) & \xrightarrow[f^*]{} & \mathrm{Fun}(K, \mathcal{X}) & \longrightarrow & \mathrm{Fun}(\mathrm{N}(\boldsymbol{\Delta}_+)^{\mathrm{op}} \times \{1, 2\}, \mathcal{X}) \end{array} \tag{12}$$

where $K = \mathrm{N}(\boldsymbol{\Delta}_+)^{\mathrm{op}} \times \{1, 2\} \cup \{[-1]\} \times \Lambda^0[2]$. The middle vertical functor sends a diagram $\alpha = \big[|X| \leftarrow B \to |Y|\big]$ to the K-indexed diagram which can be depicted as

$$\begin{array}{ccccc} X_\bullet & & & & Y_\bullet \\ \downarrow & & & & \downarrow \\ |X| & \longleftarrow & B & \longrightarrow & |Y| \end{array}$$

i.e. whose value on $\{[-1]\} \times \Lambda^0[2]$ is α and whose value on $\mathrm{N}(\boldsymbol{\Delta}_+)^{\mathrm{op}} \times \{1, 2\}$ is given by $i_!X$ and $i_!Y$. The fibers of the bottom two categories in (12) over η are isomorphic to $\mathrm{Span}_{\mathrm{Fun}(\mathrm{N}(\boldsymbol{\Delta}_+)^{\mathrm{op}}, \mathcal{X})}(i_!X, i_!Y)$ and $\mathrm{Span}_{\mathcal{X}}(|X|, |Y|)$, of which the top two quasicategories are full subcategories.

The restriction functor f^* admits a right adjoint f_*, given by right Kan extension. Since the inclusion $K \subseteq \mathrm{N}(\boldsymbol{\Delta}_+)^{\mathrm{op}} \times \Lambda^0[2]$ is fully faithful, this right adjoint restricts to a right adjoint

$$f_*\colon \mathrm{Span}_{\mathcal{X}}(|X|, |Y|) \longrightarrow \mathrm{Span}_{\mathrm{Fun}(\mathrm{N}(\boldsymbol{\Delta}_+)^{\mathrm{op}}, \mathcal{X})}(i_!X, i_!Y)$$

between the fibers over η ([18, Proposition 7.3.2.5]). It remains to show that this restricted right adjoint sends the full subcategory $\mathrm{Span}^{\mathrm{eq}}_{\mathcal{X}}(|X|, |Y|) \subseteq \mathrm{Span}_{\mathcal{X}}(|X|, |Y|)$ to the full subcategory $\mathcal{E} \subseteq \mathrm{Span}_{\mathrm{Fun}(\mathrm{N}(\boldsymbol{\Delta}_+)^{\mathrm{op}}, \mathcal{X})}(i_!X, i_!Y)$. To this end, observe that under right Kan extension along $K \subseteq \mathrm{N}(\boldsymbol{\Delta}_+)^{\mathrm{op}} \times \Lambda^0[2]$, the span $|X| \leftarrow B \to |Y|$ is sent to the span of augmented simplicial objects

$$\begin{array}{ccccc} X_\bullet & \longleftarrow & X_\bullet \times Y_\bullet \times_{|Y|\times|X|} B & \longrightarrow & Y_\bullet \\ \downarrow & & \downarrow & & \downarrow \\ |X| & \longleftarrow & B & \longrightarrow & |Y|. \end{array}$$

To see that this is contained in $\mathcal{E}$ when the map $B \to |X|$ is an equivalence, observe that the colimit of $X_\bullet \times Y_\bullet \times_{|Y|\times|X|} B$ is B, since colimits are stable under base change. It remains to prove that the map of simplicial objects $X_\bullet \times Y_\bullet \times_{|Y|\times|X|} B \to X_\bullet$ is a height $n-1$ hypercover whose matching maps are given by $(n-1)$-smooth maps. Replacing B by the equivalent object $|X|$, we may assume that $B \to |X|$ is the identity map. In that case, the map $q\colon X_\bullet \times Y_\bullet \times_{|Y|\times|X|} |X| \to X_\bullet$ is the base change of the map of simplicial objects $\pi\colon Y_\bullet \to \mathrm{cst}|Y|$. But π was a hypercover of height $n-1$ whose relative matching maps were $(n-1)$-smooth by Corollary 5.10. It follows that q is a hypercover of height $n-1$ whose relative matching maps are $(n-1)$-smooth, so that the above diagram is contained in $\mathcal{E}$. □

Proof (of Theorem 5.1). The functor $\mathrm{Gpd}_n^{\mathrm{geom}}[W_{\mathrm{Mor}}^{-1}] \to \mathcal{X}$ is fully faithful by Lemma 5.11, Lemma 5.13 and [19, Corollary 3.13]. Its essential image is identified with the n-geometric stacks by Corollary 5.7 and Lemma 5.8. □

6. Bibundles

The manoeuvre used in Lemma 5.13 can be varied to yield a description of mapping spaces between stacks in terms of *bibundles*, which we will outline in this section. To describe this in more detail, let us consider the category $\mathbf{\Delta}_{[1]} := \mathbf{\Delta}/[1]$ of nonempty linear orders over $[1]$. There is a natural functor $j\colon \mathbf{\Delta}_{[1]} \to (\mathbf{\Delta}_+)^{\times 2}$ sending $\alpha\colon [n] \to [1]$ to $(\alpha^{-1}(0), \alpha^{-1}(1))$, one of whose components may be the empty linear order $[-1]$. This functor is fully faithful, with essential image given by the full subcategory of $(\mathbf{\Delta}_+)^{\times 2}$ on all objects except $([-1],[-1])$: an inverse is given by the join functor

$$([n],[m]) \longmapsto \Big([n]\star[m] \to [0]\star[0] = [1]\Big).$$

Throughout the rest of this section, we will tend to identify $\mathbf{\Delta}_{[1]}$ with its essential image in $(\mathbf{\Delta}_+)^{\times 2}$.

Definition 6.1. Let $X, Y\colon \mathrm{N}(\mathbf{\Delta})^{\mathrm{op}} \to \mathcal{X}$ be two simplicial objects in $\mathcal{X}$. We will say that a functor $P\colon \mathrm{N}(\mathbf{\Delta}_{[1]})^{\mathrm{op}} \to \mathcal{X}$ is an *X–Y-bibundle* if it satisfies the following conditions:

(1) the restriction $P\big|\{-1\} \times \mathrm{N}(\mathbf{\Delta})^{\mathrm{op}}$ agrees with Y and $P\big|\mathrm{N}(\mathbf{\Delta})^{\mathrm{op}} \times \{-1\}$ agrees with X.
(2) for each $n \geq 0$, the restriction $P\big|\big\{[-1] \to [n]\big\}^{\mathrm{op}} \times \mathrm{N}(\mathbf{\Delta})^{\mathrm{op}}$ induces a *cartesian* natural transformation $P_{n,\bullet} \to P_{-1,\bullet} = Y_\bullet$ of simplicial objects (see Remark 4.8) and the restriction $P\big|\mathrm{N}(\mathbf{\Delta})^{\mathrm{op}} \times \big\{[-1] \to [n]\big\}^{\mathrm{op}}$ induces a cartesian natural transformation $P_{\bullet,n} \to P_{\bullet,-1} = X_\bullet$ of simplicial objects in $\mathcal{X}$.
(3) the augmented simplicial object $P\big|\{0\} \times \mathrm{N}(\mathbf{\Delta}_+)^{\mathrm{op}}$ is a colimit diagram.

Let $\mathrm{Bib}(X,Y) \subseteq \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta}_{[1]})^{\mathrm{op}}, \mathcal{X}) \times_{\mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{\mathrm{op}},\mathcal{X})^{\times 2}} \{(X,Y)\}$ denote the full sub-∞-category on the bibundles.

Unwinding the above definitions, one sees that a bibundle is given in low degrees by a solid diagram of the form

$$\begin{array}{ccccc}
X_1 \times_{X_0} P_{0,0} \times_{Y_0} Y_1 & \overset{\pi_2\times 1}{\underset{\lambda\times 1}{\rightrightarrows}} & P_{0,0} \times_{Y_0} Y_1 & \longrightarrow & Y_1 \\
{\scriptstyle 1\times\pi_1}\downarrow\,\downarrow{\scriptstyle 1\times\rho} & & {\scriptstyle \pi_1}\downarrow\,\downarrow{\scriptstyle \rho} & & \downarrow\,\downarrow \\
X_1 \times_{X_0} P_{0,0} & \overset{\pi_2}{\underset{\lambda}{\rightrightarrows}} & P_{0,0} & \longrightarrow & Y_0 \\
\downarrow & & \downarrow & & \vdots \\
X_1 & \rightrightarrows & X_0 & \dashrightarrow & |Y|
\end{array} \tag{13}$$

in which the middle column realizes X_0 as the quotient of $P_{0,0}$ by the action of Y on it. To relate this to the usual definition of bibundles, note that when X and Y are groupoid objects, the map λ specifies an action of X on $P_{0,0}$ (over Y_0), while ρ specifies an action of Y on $P_{0,0}$ (over X_0). The fact that the action of Y on $P_{0,0}$ extends to an action of Y on the entire simplicial object $P_{\bullet,0}$ witnesses the fact that the action of Y commutes with the action of X. Finally, the fact that the augmented simplicial diagram $P_{0,\bullet}$ is a colimit diagram shows that the (*homotopy*) quotient of $P_{0,0}$ by the Y-action is X_0, which means that $P_{0,0} \to X_0$ is a principal Y-bundle.

Lemma 6.2. *The following assertions hold:*

- *(a) for each* $n \geq 0$, *the augmented simplicial object* $P\big|\{n\} \times \mathrm{N}(\mathbf{\Delta}_+)^{\mathrm{op}}$ *is a colimit diagram.*
- *(b) the* ∞*-category* $\mathrm{Bib}(X, Y)$ *of* X*–*Y*-bibundles is a Kan complex.*

Proof. For (a), note that the second half of part (2) of Definition 6.1 implies that for each $n \geq 0, m \geq -1$, the square

$$\begin{array}{ccc}
P_{n,m} & \longrightarrow & P_{0,m} \\
\downarrow & & \downarrow \\
P_{n,-1} & \longrightarrow & P_{0,-1}
\end{array}$$

is cartesian. It follows that the natural transformation of augmented simplicial objects $P_{n,\bullet} \to P_{0,\bullet}$ is a cartesian natural transformation whose codomain is a colimit diagram. It then follows from descent (see [23]) that $P_{n,\bullet}$ is a colimit diagram.

For (b), let $P \to Q$ be a map of bibundles. Since for all $n \geq 0$, the maps $P_{n,\bullet} \to Y_\bullet$ and $Q_{n,\bullet} \to Y_\bullet$ are cartesian, it follows from the pasting lemma for pullbacks that the natural transformation of simplicial diagrams $P_{n,\bullet} \to Q_{n,\bullet}$ is cartesian as well. This cartesian transformation extends to a natural transformation between augmented simplicial diagrams, each of which is a colimit diagram. It follows from descent that this natural transformation of augmented simplicial diagrams is cartesian as well, so that the natural map $P_{n,m} \to Q_{n,m} \times_{Q_{n,-1}} P_{n,-1}$ is an equivalence. But the map $P_{\bullet,-1} \to Q_{\bullet,-1}$ is just the identity map on the simplicial object $X_\bullet$, from which we conclude that $P \to Q$ is a natural equivalence. □

Proposition 6.3. *Let* X *and* Y *be two simplicial objects in an* ∞*-topos* $\mathcal{X}$. *Then there is an equivalence of spaces* $\mathrm{Map}_{\mathcal{X}}(|X|, |Y|) \simeq \mathrm{Bib}(X, Y)$.

At an informal level, this equivalence can be described as follows: given an X–Y-bibundle P as in Diagram (13), the left Kan extension of P along the inclusion $\mathrm{N}(\mathbf{\Delta}_[1])^{\mathrm{op}} \to \mathrm{N}(\mathbf{\Delta}_+ \times \mathbf{\Delta}_+)^{\mathrm{op}}$ produces the extended diagram in (13), with the dashed arrows to $|Y|$ added. The bottom row of this diagram then yields a map $|X| \to |Y|$ from the colimit of the simplicial diagram X to $|Y|$. Conversely, given $|X| \to |Y|$ one can reconstruct the bottom row and right column of the extended diagram (13), from which one can obtain P by right Kan extension.

Proof. Let $\mathrm{Bib} \subseteq \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta}_{[1]})^{\mathrm{op}}, \mathcal{X})$ be the full sub-∞-category consisting of diagrams $P\colon \mathrm{N}(\mathbf{\Delta}_{[1]})^{\mathrm{op}} \to \mathcal{X}$ satisfying conditions (2) and (3) of Definition 6.1. Let us denote by $K_+ \subseteq \mathrm{N}(\mathbf{\Delta}_+ \times \mathbf{\Delta}_+)^{\mathrm{op}}$ the full subcategory on the elements $(-1, n)$ and $(n, -1)$, for $n \geq -1$ and by K the intersection $K_+ \cap \mathrm{N}(\mathbf{\Delta}_{[1]})^{\mathrm{op}}$. Observe that K is the disjoint union of two copies of $\mathrm{N}(\mathbf{\Delta})^{\mathrm{op}}$, so that restriction along $K \subseteq \mathrm{N}(\mathbf{\Delta}_{[1]})^{\mathrm{op}}$ induces a categorical fibration

$$\mathrm{Bib} \longrightarrow \mathrm{Fun}(K, \mathcal{X}) \cong \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{\mathrm{op}}, \mathcal{X})^{\times 2}$$

whose fiber over (X, Y) is exactly $\mathrm{Bib}(X, Y)$. Now let $\mathcal{D} \subseteq \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta}_+ \times \mathbf{\Delta}_+)^{\mathrm{op}}, \mathcal{X})$ denote the full sub-∞-category of those diagrams $Z\colon \mathrm{N}(\mathbf{\Delta}_+ \times \mathbf{\Delta}_+)^{\mathrm{op}} \to \mathcal{X}$ for which

(i) Z is the right Kan extension of its restriction to K_+.
(ii) the restriction of Z to $\{-1\} \times \mathrm{N}(\mathbf{\Delta}_+)^{\mathrm{op}}$ is a colimit diagram.

It follows from condition (i) that restriction along $j\colon \mathrm{N}(\mathbf{\Delta}_{[1]})^{\mathrm{op}} \subseteq \mathrm{N}(\mathbf{\Delta}_+ \times \mathbf{\Delta}_+)^{\mathrm{op}}$ induces a functor $j^*\colon \mathcal{D} \to \mathrm{Bib}$. On the other hand, condition (ii), condition (3) from Definition 6.1 and [16, Proposition 4.3.2.9] imply that $j^*\colon \mathcal{D} \to \mathrm{Bib}$ realizes the domain as the ∞-category of all *left* Kan extensions of diagrams in Bib to all of $\mathrm{N}(\mathbf{\Delta}_+ \times \mathbf{\Delta}_+)^{\mathrm{op}}$. In particular, j^* is a trivial fibration.

Now let $\mathcal{E}$ be the full sub-∞-category of $\mathrm{Fun}(K_+, \mathcal{X})$ on those diagrams $Z\colon K_+ \to \mathcal{X}$ whose restriction to $\{-1\} \times \mathrm{N}(\mathbf{\Delta}_+)^{\mathrm{op}}$ is a colimit diagram. In light of condition (ii), restricting to K_+ induces a trivial fibration $\mathcal{D} \to \mathcal{E}$, so that we obtain a diagram of trivial fibrations over $\mathrm{Fun}(K, \mathcal{X}) \cong \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{\mathrm{op}}, \mathcal{X})^{\times 2}$ of the form

$$\begin{array}{ccccc} \mathrm{Bib} & \overset{\sim}{\twoheadleftarrow} & \mathcal{D} & \overset{\sim}{\twoheadrightarrow} & \mathcal{E} \\ & \searrow & \downarrow & \swarrow_{p} & \\ & & \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{\mathrm{op}}, \mathcal{X})^{\times 2} & & \end{array}$$

In particular, the space $\mathrm{Bib}(X, Y)$ of X–Y-bibundles is weakly equivalent to the fiber of the map $p\colon \mathcal{E} \subseteq \mathrm{Fun}(K_+, \mathcal{X}) \to \mathrm{Fun}(K, \mathcal{X})$ over the pair (X, Y). At an informal level, this map p sends a diagram $X \to |Y| \leftarrow Y$ to the pair (X, Y).

To identify the fiber $p^{-1}(X, Y)$ of the map p with $\mathrm{Map}_{\mathcal{X}}(|X|, |Y|)$, note that K_+ is given by the pushout $\mathrm{N}(\mathbf{\Delta}_+)^{\mathrm{op}} \coprod_{\{-1\}} \mathrm{N}(\mathbf{\Delta}_+)^{\mathrm{op}}$. It follows that the quasicategory $\mathcal{E}$ is given by the pullback

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & \mathrm{Fun}^{\mathrm{Lan}}(\mathrm{N}(\mathbf{\Delta}_+)^{\mathrm{op}}), \mathcal{X}) \\ \downarrow & & \downarrow \mathrm{ev}_{-1} \\ \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta}_+)^{\mathrm{op}}), \mathcal{X}) & \underset{\mathrm{ev}_{-1}}{\longrightarrow} & \mathcal{X} \end{array}$$

where $\mathrm{Fun}^{\mathrm{Lan}}(\mathrm{N}(\mathbf{\Delta}_+)^{\mathrm{op}}), \mathcal{X}) \subseteq \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta}_+)^{\mathrm{op}}), \mathcal{X})$ is the full sub-∞-category of colimit diagrams. The restriction functor $\mathrm{Fun}^{\mathrm{Lan}}(\mathrm{N}(\mathbf{\Delta}_+)^{\mathrm{op}}), \mathcal{X}) \to \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{\mathrm{op}}), \mathcal{X})$ is a trivial fibration. If $\overline{Y}$ is a lift of Y under this trivial fibration (so that $\overline{Y}$ gives a colimiting cocone $Y \to |Y|$), then the fiber of $\mathcal{E}$ over Y is weakly equivalent to its fiber over $\overline{Y}$. From this it follows that for any $X\colon \mathrm{N}(\mathbf{\Delta})^{\mathrm{op}} \to \mathcal{X}$, there is weak equivalence

$$\{X\} \times_{\mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{\mathrm{op}}, \mathcal{X})} \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta}_+)^{\mathrm{op}}, \mathcal{X}) \times_{\mathrm{Fun}(\{-1\}, \mathcal{X})} \{|Y|\} \overset{\sim}{\longrightarrow} p^{-1}(X, Y).$$

Finally, recall from [16, Proposition 4.2.1.2] that there is a natural categorical equivalence $\mathrm{N}(\mathbf{\Delta}) \times \Delta[1] \coprod_{\mathrm{N}(\mathbf{\Delta}) \times \{1\}} \{-1\} \to \mathrm{N}(\mathbf{\Delta}_+)$. Precomposition with this categorical equivalence yields a zig-zag of categorical equivalences

$$\mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{\mathrm{op}}, \mathcal{X})^{\Delta[1]} \times_{\mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{\mathrm{op}}, \mathcal{X})^{\times 2}} \{X, \mathrm{cst}(|Y|)\} \xleftarrow{\sim} \cdot \xrightarrow{\sim} p^{-1}(X, Y)$$

The domain of this categorical equivalence is a model for the mapping space in the functor category $\mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{\mathrm{op}}, \mathcal{X})$ from the diagram X to the constant diagram on $|Y|$. This space is equivalent to the mapping space $\mathrm{Map}_{\mathcal{X}}(|X|, |Y|)$ from the colimit of X into $|Y|$ by the universal property of the colimit. We conclude that $p^{-1}(X, Y)$, and therefore $\mathrm{Bib}(X, Y)$, is equivalent to $\mathrm{Map}_{\mathcal{X}}(|X|, |Y|)$. □

Remark 6.4. Let $f: |X| \to |Y|$ be a map and let $P: \mathrm{N}(\mathbf{\Delta}_{[1]})^{\mathrm{op}} \to \mathcal{X}$ be the associated bibundle under the equivalence of 6.3. The diagonal of P provides a simplicial object in $\mathcal{X}$ that maps both to X and to Y, and the resulting span of simplicial objects $X \leftarrow \delta^* P \to Y$ is precisely the value of the right adjoint of Lemma 5.13 on f.

Corollary 6.5. *Let X and Y be two geometric n-groupoid objects. If P is an X–Y-bibundle, then $P_{0,0}$ is $(n-1)$-geometric (and consequently, the entire diagram $P: \mathrm{N}(\mathbf{\Delta}_{[1]})^{\mathrm{op}} \to \mathcal{X}$ takes values in $(n-1)$-geometric stacks).*

Proof. Unwinding the equivalence of Proposition 6.3, one sees that for any map $|X| \to |Y|$, the associated bibundle P has $P_{0,0}$ given by the pullback $X_0 \times_{|Y|} Y_0$. Since the map $Y_0 \to |Y|$ is $(n-1)$-smooth by Lemma 5.8, it follows that $P_{0,0}$ is $(n-1)$-geometric. All other values of P are pullbacks of $P_{0,0}$ along maps $X_n \to X_0$ and $Y_n \to Y_0$ and are therefore $(n-1)$-geometric as well. □

This corollary is particularly interesting when $n = 1$, in which case it reproduces the description of mapping spaces between 1-geometric stacks in terms of bibundles between groupoid objects, which can be described completely in terms of 0-geometric objects.

7. Examples

In this section we give some more examples of the data of Assumption 3.6 and unpack Theorem 5.1 in these cases.

Example 7.1. We have already encountered the situation where $\mathcal{X} = \mathrm{Sh}_\infty(\mathrm{Mfd})$ is the category of sheaves on the site of smooth manifolds, where $\mathcal{X}_{0-\mathrm{geom}}$ is the subcategory of smooth manifolds and P is the class of submersions or étale maps. Theorem 5.1 asserts that higher differentiable (or étale) stacks can be modeled by higher (étale) Lie groupoids up to Morita equivalence. There are obvious variations of this theme, replacing smooth manifolds by real or complex analytic manifolds, topological spaces (with smooth maps given by maps that admit local sections through every point in the domain, or just the étale maps), or schemes (with smooth or étale maps).

In each of these cases, one can replace the input data by the full subcategory $\mathcal{X}_{m-\mathrm{geom}}$ of m-geometric stacks and the m-smooth maps (Remark 3.13). Theorem 5.1 then shows that one can model $(n+m)$-geometric stacks by smooth n-groupoids in m-geometric stacks, up to Morita equivalence. For example, one finds that the full subcategory of $\mathrm{Sh}_\infty(\mathrm{Mfd})$ on the 2-differentiable stacks can be modeled either by localizing the category of Lie 2-groupoids at the Morita equivalences, or by localizing the 2-category of 'stacky Lie groupoids' (i.e. groupoid objects in the 2-category of differentiable stacks, cf. [26]) at the Morita equivalences.

Example 7.2. Let $\mathcal{E}$ be a topos and let $\mathcal{X} = \mathcal{E}_\infty$ be the ∞-topos modeled by the Joyal model structure on $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{E})$. Take $\mathcal{X}_{0-\mathrm{geom}}$ to be the category of all 0-truncated objects in $\mathcal{X}$ (equivalent to the category $\mathcal{E}$ itself) and the smooth maps to simply be all maps. The only nontrivial part of Assumption 3.6 to check is part (4): but for this, observe that any fibrant object F in the Joyal model structure on simplicial sheaves admits an effective epimorphism from its sheaf of objects F_0.

The category of geometric n-groupoids can now be identified with the (ordinary) category of the locally fibrant simplicial sheaves (in the sense of Brown [3]) which are n-truncated, in the sense that horn inclusions in dimension $> n$ yield isomorphisms of sheaves. Theorem 5.1 reproduces the well-known result (see e.g. [14], where it is also shown when $n = \infty$) that the localization of this category at the hypercovers gives a model for n-truncated objects in $\mathcal{X}$. Indeed, let $U \to X$ be an effective epimorphism with U a 0-truncated object. If X in n-truncated, the pullback $U \times_X U$ is an $(n-1)$-truncated object, so that the map $U \to X$ is $(n-1)$-truncated. Using this, a simple inductive argument shows that the class of n-geometric objects in $\mathcal{X}$ is just the class of n-truncated objects in $\mathcal{X}$ and that the n-smooth maps are given by the n-truncated maps.

Example 7.3. Let $\mathrm{Mfd}_{\text{ét}}$ be the category of smooth manifolds and étale maps between them. The category $\mathcal{X} = \mathrm{Sh}_\infty(\mathrm{Mfd}_{\text{ét}})$ is studied in [4], where it is shown to admit a fully faithful left adjoint $\mathrm{Sh}_\infty(\mathrm{Mfd}_{\text{ét}}) \to \mathrm{Sh}_\infty(\mathrm{Mfd})$, whose image is defined to be the category of étale differentiable ∞-stacks (i.e. those stacks that can be obtained as colimits of diagrams in $\mathrm{Mfd}_{\text{ét}}$). To connect to Example 7.1, let us define $\mathcal{X}_{0-\mathrm{geom}} \subseteq \mathcal{X}$ to consist of all (small) disjoint unions of smooth manifolds, and let the smooth maps just be all maps between them (all maps are étale on each summand). Notice that $\mathcal{X}_{0-\mathrm{geom}}$ just consists of coproducts of representable sheaves of sets and that any subobject of an object in $\mathcal{X}_{0-\mathrm{geom}}$ is again in $\mathcal{X}_{0-\mathrm{geom}}$ (a subobject of a representable sheaf is representable by an open subset). It follows from this that all sheaves of sets on $\mathrm{Mfd}_{\text{ét}}$ are 1-geometric with respect to this datum and that all epimorphisms of sheaves are 1-smooth. The argument of Example 7.2 now shows that all n-truncated objects in $\mathrm{Sh}_\infty(\mathrm{Mfd}_{\text{ét}})$ are $(n+1)$-geometric (and conversely, an $(n+1)$-geometric object is $(n+1)$-truncated).

In particular, Theorem 5.1 asserts that all n-truncated objects in $\mathrm{Sh}_\infty(\mathrm{Mfd}_{\text{ét}})$ can be modeled by $(n+1)$-groupoids with values in coproducts of smooth manifolds and étale maps between them, up to Morita equivalence. The image of such an $(n+1)$-groupoid under the embedding $\mathrm{Sh}_\infty(\mathrm{Mfd}_{\text{ét}}) \to \mathrm{Sh}_\infty(\mathrm{Mfd})$ yields an étale $(n+1)$-groupoid in $\mathrm{Sh}_\infty(\mathrm{Mfd})$ in the sense of Example 7.1 (except that we now allow arbitrary coproducts of manifolds in $\mathcal{X}_{0-\mathrm{geom}}$). In other words, we find that all *truncated* étale differentiable ∞-stacks in the sense of [4] can be presented by higher étale Lie groupoids (which was anticipated in [4, Remark 6.1.11]).

Example 7.4. Example 7.1 has obvious analogues in derived geometry. For example, one can take $\mathcal{X} = \mathrm{Sh}_\infty(\mathrm{CAlg}_k^{\geq 0})$ to be the category of stacks on the simplicial (or model) site of (finitely presented) simplicial k-algebras, for k a commutive ring. Let $\mathcal{X}_{0-\mathrm{geom}}$ consist of derived schemes and P to be the set of smooth maps between those (see [25] for an extensive account). Then n-geometric stacks are 'derived n-Artin stacks' and Theorem 5.1 reproduces the result from [20] that such stacks can be presented by smooth n-groupoid objects with values in derived schemes. It also shows that the localization of the category of such n-groupoids constructed in [20] indeed presents the ∞-categorical localization.

Example 7.5. Let $\mathbf{T}$ be a (multi-sorted) algebraic theory and let $\mathcal{X} = \mathrm{Fun}(\mathrm{N}(\mathbf{T}), \mathcal{S})$ be the ∞-topos of space-valued functors on $\mathrm{N}(\mathbf{T})$. The ∞-category of $\mathbf{T}$-algebras in spaces is the (reflective) full subcategory of $\mathcal{X}$ on those functors $\mathrm{N}(\mathbf{T}) \to \mathcal{S}$ that preserve finite products. Define $\mathcal{X}_{0-\mathrm{geom}}$ to be the subcategory of all set-valued functors $\mathrm{N}(\mathbf{T}) \to \mathrm{Set}$ that preserve finite products (i.e. $\mathbf{T}$-algebras in sets), with all maps between them as the smooth maps. This only satisfies part (4) of Assumption 3.6 relative to the full subcategory $\mathrm{Alg}_{\mathbf{T}} \subseteq \mathcal{X}$ of $\mathbf{T}$-algebras in spaces (see Variant 3.8). Since this subcategory is closed under all $\mathrm{N}(\mathbf{\Delta})^{\mathrm{op}}$-indexed colimits, we may still apply Theorem5.1 (see Remark 5.9).

The category $\mathrm{Gpd}_n^{\mathrm{geom}}$ can now be identified with the full subcategory $\mathrm{Gpd}_n^{\mathrm{geom}} \subseteq \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Alg}_{\mathbf{T}})$ on those simplicial $\mathbf{T}$-algebras (in sets) whose underlying simplicial set is an n-truncated Kan complex. Furthermore, a map between two such simplicial $\mathbf{T}$-algebras is a smooth hypercover iff the underlying map of Kan complexes is a trivial Kan fibration. Theorem 5.1 shows that the localization of the n-truncated fibrant simplicial $\mathbf{T}$-algebras at the trivial Kan fibrations is a full subcategory of $\mathrm{Fun}(\mathrm{N}(\mathbf{T}), \mathcal{S})$. Since the subcategory of $\mathrm{Fun}(\mathrm{N}(\mathbf{T}), \mathcal{S})$ on the $\mathbf{T}$-algebras in spaces is closed under sifted colimits, it follows that $\mathrm{Gpd}_n^{\mathrm{geom}}[W^{-1}]$ is a full subcategory of the ∞-category of $\mathbf{T}$-algebras in n-truncated spaces. The essential image (i.e the n-geometric objects) can be identified along the lines of Example 7.2: one uses an inductive argument to prove that the n-geometric objects are precisely the $\mathbf{T}$-algebras in n-truncated spaces, while the n-smooth maps are those maps of $\mathbf{T}$-algebras whose underlying map of spaces is n-truncated. In particular, this reproduces the result of [2] that $\mathbf{T}$-algebras in spaces may be rigidified, at least for truncated $\mathbf{T}$-algebras. Similarly, one can combine this example with Example 7.2 to obtain a similar model for truncated $\mathbf{T}$-algebras in the ∞-topos $\mathcal{E}_\infty$ associated to a topos $\mathcal{E}$ in terms of truncated, locally fibrant simplicial $\mathbf{T}$-algebras in $\mathcal{E}$ (localized at the hypercovers between them).

Example 7.6. Let $\mathrm{CAlg}_k^{\mathrm{sm}}$ be the ∞-category of small augmented E_∞-algebras over a field k of characteristic zero (i.e. connective augmented dg-Artin algebras over k, see e.g. [17]) and let $\mathcal{X} = \mathrm{Fun}(\mathrm{CAlg}_k^{\mathrm{sm}}, \mathcal{S})$ be the ∞-category of space-valued functors on $\mathrm{Art}_k^{\geq 0}$. The category $\mathrm{CAlg}_k^{\mathrm{sm}}$ has finite products (the terminal object is k), but only admits pullbacks along *small maps*; such maps are finite compositions of base changes of the maps $k \to k \ltimes k[m]$ associated to the split square zero extension of k by the m-th suspension of the k-module k (for $m \geq 1$).

Let $\mathcal{X}_{0-\mathrm{geom}}$ be the class of all prorepresentable functors, i.e. all functors obtained as filtered colimits of corepresentable functors. Note that such pro-representable functors are examples of *formal moduli problems* (see [17, Definition 1.1.14]), which are functors F such that $F(k)$ is contractible and such that F preserves pullbacks along small maps. Define a map $F \to G$ between such formal moduli problems to be *formally smooth* if for each small map $A \to B$ in $\mathrm{CAlg}_k^{\mathrm{sm}}$, the induced map of spaces $F(A) \to F(B) \times_{G(B)} G(A)$ induces a surjection on path components [17, Definition 1.5.6].

The prorepresentable functors and formally smooth maps between them satisfy the conditions of Assumption 3.6, relative to the full subcategory $\mathrm{FormMod} \subseteq \mathcal{X}$ on the formal moduli problems (Variant 3.8): condition (2) follows from the fact that the prorepresentable functors are precisely those formal moduli problems whose value on $k[\epsilon] := k \ltimes k[0]$ is discrete [17, Corollary 2.3.6] and Condition (4) holds relative to FormMod by [17, Proposition 1.5.8]. Furthermore, the realization of a geometric n-groupoid X is always a formal moduli problem (so that Remark 5.9 applies). Indeed, this follows from the fact that for any small map $A \to B$ in $\mathrm{CAlg}_k^{\mathrm{sm}}$, the induced map of simplicial spaces $X(A) \to X(B)$ is a Kan fibration of height n and thus a realization fibration (Proposition 4.12).

Consequently, invoking Theorem 5.1, the category of geometric n-groupoids with values in prorepresentable functors yields – after localizing at the formally smooth hypercovers – a full subcategory of the ∞-category of formal moduli problems. To identify this subcategory (i.e. the n-geometric objects), one can again proceed as in Example 7.2: indeed, an inductive argument shows that the n-geometric objects are precisely those formal moduli problems whose value on $k[\epsilon]$ is n-truncated, while the n-smooth maps between them are the formally smooth maps $F \to G$ whose induced map of spaces $F(k[\epsilon]) \to G(k[\epsilon])$ is an n-truncated effective epimorphism.

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