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Deformations and derived geometry

In 2018 Joost Nuiten has been awarded the Stieltjes Prize for one of the two best PhD theses in mathematics in the Netherlands. The prize was awarded for his thesis entitled *Lie Algebroids in Derived Differential Geometry*, which he completed at Utrecht University. After getting his PhD he became a postdoc at Université de Montpellier. In this article he describes his research on deformation theory and derived geometry.

Deformations and perturbations are studied in many different branches of mathematics.

- In dynamical systems, one may be interested in the various ways to perturb an orbit and how this affects its periodicity. On the other hand, one can also perturb the dynamical system itself, e.g. by adding higher order corrections to its Hamiltonian.
- In geometry, one can consider families of complex or algebraic varieties deforming a fixed variety X_0.
- In a more algebraic setting, one can try to deform an associative product $\mu_0 : A \times A \longrightarrow A$ on a vector space into another associative product of the form

$$\mu_\hbar(a,b) = \mu_0(a,b) + \hbar\mu_1(a,b) + \hbar^2\mu_2(a,b) + \cdots.$$

In all these different situations, one poses a similar kind of question: given a certain mathematical object X_0, how many families $\{X_\hbar\}$ of 'nearby objects' around X_0 are there? In particular, can X_0 be deformed into something else or is it *rigid*, i.e. unchanged under any deformation?

To address such questions, a first step is to study the *formal* deformations of X_0. Concretely, this means that we allow formal power series in $\hbar$, for example in the formula for the deformed product $\mu_\hbar$.

One can study such formal deformations inductively, by working up to terms of order $\hbar^{n+1}$. More precisely, any formal deformation arises as the limit

$$\lim_{n\to\infty} \{X_\hbar\}_{\hbar^{n+1}=0}$$

of a compatible sequence of n-th order infinitesimal deformations, i.e. deformations modulo $\hbar^{n+1}$. To produce formal deformations, one is therefore lead to the following question:

Question. Suppose we have found an infinitesimal deformation $\{X_\hbar\}_{\hbar^{n+1}=0}$ of order n, can we extend it to a deformation of order $n+1$? If yes, how many possible extensions are there?

When deforming an associative product μ_0, this comes down to inductively finding the terms μ_n for which the resulting operation $\mu_\hbar$ is associative up to terms of order $\hbar^{n+1}$. The infinitesimal deformations of a variety are best described using the language of schemes: one deforms X_0 into a scheme over Spec$(\mathbb{C}[\hbar]/(\hbar^{n+1}))$.

In geometric situations like the last one, infinitesimal deformations have an additional local-to-global property: we can first deform X_0 *locally* (deform various small open subsets of X_0) and then try to glue all these local deformations together. We will come back to such 'local' deformation problems at the end of the text, where we discuss how they can be studied in terms of *Lie algebroids*.

Obstructions

In many different situations, the answers to the above question turn out to exhibit the same general pattern: one can write down a sequence of vector spaces $H^0, H^1, H^2, \ldots$ that controls the infinitesimal deformations of X_0 in the following way:

1. There is a bijection between first order deformations of X_0 and the vector space H^1.
2. There is a bijection between first order *automorphisms* of X_0 and H^0.
3. Suppose that X_n is an n-th order deformation of X_0. Then one can construct a canonical element

$$\mathrm{ob}(X_n) \in H^2$$

called its *obstruction class*, with the following property: X_n can be extended to a deformation of order $n+1$ if and only if

$$\mathrm{ob}(X_n) = 0.$$

In this case, there is a bijection between the set of possible $(n+1)$-st order extensions and the cohomology group H^1.

In fact, these vector spaces arise most naturally from a cochain complex of vector spaces

$$\cdots \xrightarrow{d} \mathfrak{g}^{-1} \xrightarrow{d} \mathfrak{g}^0 \xrightarrow{d} \mathfrak{g}^1 \xrightarrow{d} \mathfrak{g}^2 \xrightarrow{d} \cdots$$

(where $d \circ d = 0$) by taking cohomology groups

$$H^i(\mathfrak{g}) = \frac{\ker(d\colon \mathfrak{g}^i \to \mathfrak{g}^{i+1})}{\mathrm{im}(d\colon \mathfrak{g}^{i-1} \to \mathfrak{g}^i)}.$$

For example, Kodaira and Spencer [3] have shown that the infinitesimal deformations of a complex manifold are controlled by the Dolbeault complex

$$\Omega^{0,0}(X_0,T_{X_0}) \xrightarrow{\bar\partial} \Omega^{0,1}(X_0,T_{X_0}) \xrightarrow{\bar\partial} \Omega^{0,2}(X_0,T_{X_0}) \xrightarrow{\bar\partial} \dots$$

The zeroth cohomology group of this complex consists of holomorphic vector fields: these are precisely the infinitesimal automorphisms of X_0.

Similarly, the deformations of an associative algebra A are controlled by its Hochschild complex [1]: this cochain complex is given in degree i by the vector space of multilinear maps

$$A^{\otimes i-1} \longrightarrow A.$$

The differentials are built by composing with the multiplication μ_0 on A (using the Gerstenhaber bracket discussed below). The first Hochschild cohomology group consists of bilinear maps μ_1 such that $\mu_0(a,b)+\hbar\cdot\mu_1(a,b)$ is an associative product up to $\hbar^2$.

Lie algebras

The various complexes $\mathfrak{g}$ that thus appear in deformation theory tend to carry some additional algebraic structure. For instance, the obstruction to extending a first order deformation to a second order one gives rise to an additional operation

$$\mathrm{ob}_2 : H^1(\mathfrak{g}) \to H^2(\mathfrak{g}).$$

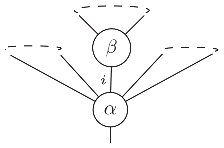
In practice, it turns out that the extra algebraic structure appearing naturally on $\mathfrak{g}$ is a binary operation

$$[-,-] : \mathfrak{g}^i \otimes \mathfrak{g}^j \to \mathfrak{g}^{i+j}$$

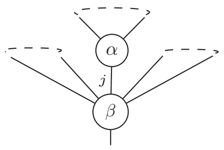
that makes it a differential graded *Lie algebra*.

For example, the Dolbeault complex $\Omega^{0,*}(X_0,T_{X_0})$ carries a Lie bracket coming from the commutator bracket of vector fields. The Hochschild complex carries the so-called Gerstenhaber bracket, a version of the commutator where we sum over all ways of inserting one map into the other:

$$[\alpha,\beta] = \sum_i \pm$$



$$-\sum_j \pm$$



The additional structure of the Lie bracket on $\mathfrak{g}$ can be used to explicitly compute the obstruction classes mentioned above. For example, the map ob_2 is simply given by

$$\mathrm{ob}_2(x) = \frac{1}{2}[x,x].$$

There are more complicated formulas for the n-th order obstructions, in terms of operations resembling the Massey products from algebraic topology (and division by $n!$). This natural appearance of Lie brackets in deformation theory has lead to the following:

Principle (Deligne). *Every deformation problem over a field of characteristic zero is controlled by a differential graded Lie algebra.*

In fact, there is a very explicit mechanism by which a dg-Lie algebra $\mathfrak{g}$ controls deformations. Indeed, infinitesimal deformations of X_0 correspond bijectively to infinitesimal deformations of the element 0 in the space of elements $x \in \mathfrak{g}^1$ that satisfy the Maurer–Cartan equation

$$d(x) + \frac{1}{2}[x,x] = 0.$$

(One furthermore has to take the quotient by a certain equivalence relation; see [2] for more details.)

In this way, dg-Lie algebras provide an efficient tool to study deformation problems, sometimes with striking results. A famous example is Kontsevich' construction of a deformed (noncommutative) star-product on the algebra of functions on a Poisson manifold: this relies on the (difficult) algebraic fact that its Hochschild complex is formal [4].

On the other hand, the above heuristic by no means gives a concrete recipe to find the relevant dg-Lie algebra. Instead, one typically needs some creativity and skill to come up with the dg-Lie algebra controlling the deformation problem at hand. Deligne's principle therefore confronts us with the following challenge:

Problem. Given a deformation problem, construct in a natural way the dg-Lie algebra that controls it.

Geometric perspective

There has been a lot of work on giving a systematic solution to the above problem, in terms of geometry. The starting point of these works is the following geometric idea: let us think about the object X_0 that we want to deform as a single point inside some space $\mathcal{M}$. From this perspective, a deformation of X_0 is simply a path inside the space $\mathcal{M}$.

Likewise, a first order deformation of X_0 should be an infinitesimal path in $\mathcal{M}$, i.e. a tangent vector. This gives a geometric interpretation of the vector space $H^1(\mathfrak{g})$, as the *tangent space*

$$H^1(\mathfrak{g}) = T_{X_0}\mathcal{M}.$$

This line of thought suggests a possible solution to the above problem: we may try to reconstruct the dg-Lie algebra controlling deformations of X_0 from the infinitesimal geometry of a putative *moduli space* $\mathcal{M}$.

Most algebro-geometric objects can indeed be organized into such moduli spaces. Probably the most classical example of a moduli space is projective space $\mathbb{P}^n(\mathbb{C})$, whose points are lines in $\mathbb{C}^{n+1}$. Likewise, an orbit of a dynamical system can itself be considered as a point in the *orbit space*, and an associative algebra can be thought of as a point in the 'moduli space of associative algebras'.

We will be rather open-minded about the precise meaning of 'moduli space' (technically, it is best described as a functor of points). Instead of providing more details, let us just informally describe two ways to get examples:

a. As solution sets of (polynomial) equations.
b. As quotients where we (smoothly) glue together points.

For example, there is a moduli space of associative algebra structures on a vector space V, constructed as follows. We start with the vector space of all bilinear maps

$$\mu : V \otimes V \to V$$

and consider the (quadratic) function $\mu \mapsto \mathrm{As}(\mu)$ sending each such μ to the trilinear map

$$\mu(\mu(a,b),c) - \mu(a,\mu(b,c)).$$

The solution set $\{\mu : \mathrm{As}(\mu) = 0\}$ is precisely the subspace of associative multiplications. This is not quite the correct moduli space yet: in addition, we should identify two associative multiplications if they are related by a change of coordinates, i.e. by conjugating with some $T \in \mathrm{GL}(V)$.

Tangent complex

In these more geometric terms, formal deformation theory then concerns the following kind of question about the structure of moduli spaces:

Question. Given a moduli space $\mathcal{M}$ and a point $x \in \mathcal{M}$, what does the formal neighbourhood $\mathcal{M}_x^\wedge$ of x inside $\mathcal{M}$ look like?

One method to address such a question is by a linear approximation, in terms of the tangent space $T_x\mathcal{M}$. This works particularly well when $\mathcal{M}$ is *smooth* at the point x: in this case we can identify $\mathcal{M}_x^\wedge \cong T_x\mathcal{M}$.

However, moduli spaces typically have many singular points at which the tangent space is ill-behaved. For example, some tangent vectors to $\mathcal{M}$ may not actually extend to small paths within $\mathcal{M}$, or the dimension of the tangent space might be larger than expected.

These problems can be conveniently dealt with using the language of *derived algebraic geometry*. Informally, this theory keeps track of possible degeneracies appearing in the procedures (a) and (b). At a technical level, this is implemented by systematically replacing equalities by simplicial homotopies.

For example, in (b) we can glue together points for *different reasons*. This happens for instance when we can take the quotient by a group action which is not free. Similarly, the equations we impose in (a) need not all be independent and may themselves satisfy some further constraints. One sees this in the moduli space of associative algebra structures on V: the associative rule gives rise to various routes by which one can reorder the parentheses in a fourfold product

$$\begin{tikzcd}[column sep=small] & a\cdot(b\cdot(c\cdot d)) \arrow[dl, dash, double] \arrow[dr, dash, double] & \\ a\cdot((b\cdot c)\cdot d) \arrow[d, dash, double] & & (a\cdot b)\otimes(c\cdot d) \arrow[d, dash, double] \\ (a\cdot(b\cdot c))\cdot d \arrow[rr, dash, double] & & ((a\cdot b)\cdot c)\cdot d \end{tikzcd}$$

The fact that any two ways of going around the pentagon must be 'the same' leads to a constraint on the associativity equation itself. Algebraically, this constraint can be written using the Gerstenhaber bracket as $[\mu, \mathrm{As}(\mu)] = 0$.

We will not provide further details on derived algebraic geometry (see, e.g., [11] for an overview). However, we should mention that the development of derived geometry has been strongly motivated and inspired by its relevance to deformation theory, building on ideas of Drinfel'd (and expanded by many others). In particular, derived algebraic geometry associates to each point $x \in \mathcal{M}$ a *tangent complex*

$$\begin{tikzcd} \cdots \arrow[r, "d"] & (\mathbb{T}_x\mathcal{M})^{-1} \arrow[r, "d"] & (\mathbb{T}_x\mathcal{M})^{0} \arrow[dll, "d", rounded corners, to path={[pos=0.5] -- ([xshift=2ex]\tikztostart.east) |- ([yshift=1ex]\tikztotarget.north) \tikztonodes -- (\tikztotarget)}] \\ (\mathbb{T}_x\mathcal{M})^{1} \arrow[r, "d"] & \cdots \end{tikzcd}$$

The zeroth cohomology group of this complex reproduces the usual tangent space. Explicitly, $\mathbb{T}_x\mathcal{M}$ takes the following form when $\mathcal{M}$ is built using constructions (a) and (b):

- Every variable x_i in (a) determines a basis vector for $(\mathbb{T}_x\mathcal{M})^0$.
- Every equation $f_j(x_i) = y_j$ in (a) determines a basis vector in $(\mathbb{T}_x\mathcal{M})^1$.
- For every path $\gamma_k(t)$ consisting of points that we glue together in (b), there is a basis vector in $(\mathbb{T}_x\mathcal{M})^{-1}$.

The differential $\mathbb{T}_x\mathcal{M}^0 \to \mathbb{T}_x\mathcal{M}^1$ is then given by the Jacobi matrix

$$\left(\frac{\partial f_j}{\partial x_i}\right)$$

and $d : \mathbb{T}_x\mathcal{M}^{-1} \to \mathbb{T}_x\mathcal{M}^0$ is the matrix whose columns are the derivatives

$$\frac{\partial \gamma_k}{\partial t}.$$

The composite of these two maps is zero because the paths γ_k are contained in the space of solutions $\{(x_i) : f_j(x_i) = y_j\}$. In addition, there can be higher terms in the tangent complex, which account for equations between equations and relations between symmetries.

At the regular points $x \in \mathcal{M}$, the tangent complex agrees (up to chain homotopy) with the usual tangent space. However, it is usually much more descriptive at the singular points. For example, its Euler characteristic reproduces the expected dimension of $\mathcal{M}$ at x. Most importantly, we can use the tangent complex to describe the formal neighbourhood of x in $\mathcal{M}$:

Theorem (Pridham [9], Lurie [6]). *The shifted tangent complex $\mathbb{T}_x\mathcal{M}^{*-1}$ has the canonical structure of a dg-Lie algebra. This dg-Lie algebra completely determines the formal neighbourhood $\mathcal{M}_x^\wedge$.*

In fact, Pridham and Lurie prove something much more precise: they establish an equivalence of homotopy categories between dg-Lie algebras and formal neighbourhoods of moduli spaces around points. The latter notion can be made precise in terms of functors of points, much in the spirit of the work of Schlessinger [10].

In particular, this theorem explains the main principle of deformation theory: the dg-Lie algebra controlling deformations of X_0 arises precisely as the tangent complex to the moduli space inside of which we are trying to deform X_0. Furthermore, all dg-Lie algebras arise in this way.

Varying the basepoint

The above theorem is not just useful for deformation theory, but also for derived algebraic geometry itself. Indeed, it gives us an algebraic tool to study the geometry of a given (derived) moduli space $\mathcal{M}$ around a point. For example, it can be used to give a Lie algebraic description of functions, vector fields and differential forms on $\mathcal{M}$ at a point $x \in \mathcal{M}$.

In many situations, notably related to enumerative geometry, one is not just interested in the infinitesimal geometry of $\mathcal{M}$, but also in its *global* structure. For instance, derived algebraic geometry allows one to globally construct symplectic and Poisson structures on moduli spaces with singularities, such as the moduli space of flat vector bundles on a Riemann surface.

To study such structures globally on $\mathcal{M}$, it becomes important to understand how the Lie algebras associated to different points in $\mathcal{M}$ are exactly related. One way to do so is the following. Instead of looking at a single point in $\mathcal{M}$, let us consider a family of points

$$Z \to \mathcal{M}$$

parametrized by some variety Z. Let us pose the question: how can we deform Z *within* $\mathcal{M}$? Phrased more geometrically: what does a formal neighbourhood of $\mathcal{M}$ around Z look like?

Note that this is exactly the type of 'local' deformation problem we saw in the first section: we can study deformations (or formal neighbourhoods) of small open subsets of Z and then glue these together.

When Z is a regular subvariety of $\mathcal{M}$, we can again try to use a linear approximation: we can approximate $\mathcal{M}$ by the *normal bundle* of Z inside $\mathcal{M}$. In the singular setting, the theory of derived algebraic geometry provides a certain cochain complex of sheaves on Z that refines the normal bundle.

Very informally, the derived normal bundle comes with a 'Lie algebra structure depending smoothly on the basepoint $z \in Z$'. More precisely, one can prove that it naturally carries the structure of a *dg-Lie algebroid*: its local sections come equipped with a Lie bracket that satisfies the Leibniz rule

$$[v, f(z) \cdot w] = f(z) \cdot [v,w] + \mathcal{L}_v(f)(z) \cdot w$$

for any function f on Z. One can think of Lie algebroids as generalizations of the tangent bundle, which allow for the usual calculus of vector fields. As such, they appear in many parts of geometry, e.g. in Poisson geometry and the study of foliations and Lie group actions.

Theorem (Nuiten [8]). *The formal neighbourhood $\mathcal{M}_Z^\wedge$ is completely determined by the derived normal bundle, together with its dg-Lie algebroid structure.*

For each point $x \in Z$, the dg-Lie algebra $\mathbb{T}_x\mathcal{M}$ can then be recovered from the fiber of the normal bundle at x. Consequently, the above theorem describes how these various Lie algebras are glued together.

On the other hand, the (derived) *global sections* of the dg-Lie algebroid also form a dg-Lie algebra: this is precisely the dg-Lie algebra controlling the deformations of Z within $\mathcal{M}$. We can therefore use Lie algebroids to simultaneously study deformations by algebraic and local-to-global methods.

As a final remark, let us point out one of the nice features of derived algebraic geometry: the construction of the derived normal bundle requires no assumptions on the map $Z \to \mathcal{M}$. In particular, we can take $\mathcal{M} = Z/\sim$ to be a (smooth) *quotient* of Z, as in (b). In this case, the dg-Lie algebroid $\mathfrak{g}$ provided by the theorem can be used to study the *global* geometry of $\mathcal{M}$, working over the (less singular) space Z. This is particularly useful in a differential-geometric setting, where the global geometry of $\mathcal{M}$ is related to the Lie algebroid $\mathfrak{g}$ by means of various integrability statements.

Example (Moduli of 2-dimensional algebras)

Let A be a 2-dimensional real unital algebra and pick a basis $1, x$. The multiplication on A is uniquely determined by the element $x^2 = a \cdot 1 + b \cdot x$. In fact, every pair of $a, b \in \mathbb{R}$ defines a unital associative product on A in this way.

We can also describe this same multiplication in a different basis $1, x'$. For example,

$$x' = x - \frac{b}{2} \cdot 1 \rightsquigarrow (x')^2 = a' \cdot 1$$

(where $a' = a + b^2/4$). Similarly,

$$x'' = \lambda \cdot x' \rightsquigarrow (x'')^2 = (\lambda^2 a') \cdot 1.$$

Since all bases (fixing 1) are related by these two transformations, the moduli space of 2-dimensional unital algebras is the quotient

$$\mathcal{M} = \mathbb{R}/\mathbb{R}^\times = \{a' \in \mathbb{R}\}/(a' \sim \lambda^2 a').$$

This has three points $a' = 0, 1, -1$. These correspond to the real algebras $\mathbb{R}[X]/X^2$, $\mathbb{R}[X]/(X^2-1)$ and $\mathbb{C}$.

The tangent complex at such a' is

$$\mathbb{T}^{-1}_{a'}\mathcal{M} = \mathbb{R} \xrightarrow{2a'} \mathbb{R} = \mathbb{T}^0_{a'}\mathcal{M}$$

where the differential is the derivative of $\lambda^2 \cdot a'$ at $\lambda = 1$. All Lie brackets are zero, except $\mathbb{T}^{-1} \times \mathbb{T}^0 \to \mathbb{T}^0$ which simply multiplies two real numbers. We invite the reader to compare this to the Hochschild complex mentioned before: the tangent complex is much smaller, but has the same cohomology.

When $a' \neq 0$, the tangent complex has zero cohomology. This expresses that the algebras $\mathbb{R}[X]/(X^2-1)$ and $\mathbb{C}$ are *rigid*: any formal deformation is isomorphic to the original algebra.

When $a' = 0$, the zeroth cohomology group is 1-dimensional. Indeed, the 1-parameter family of algebras $\mathbb{R}[X]/(X^2 - \hbar)$ gives a nontrivial deformation of $\mathbb{R}[X]/X^2$.

Further reading

The relation between deformation theory and dg-Lie algebras has a long history, of which we have omitted many chapters. The lecture notes of Manetti [7] provide an accessible account, discussing the work of Kodaira–Spencer in great detail. For treatments in terms of derived algebraic geometry, we recommend the talks of Lurie [5] and Toën [12].

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