Lie algebroids are curved Lie algebras

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Abstract

We show that there is an equivalence of ∞-categories between Lie algebroids and certain kinds of curved Lie algebras. For this we develop a method to study the ∞-category of curved Lie algebras using the homotopy theory of algebras over a complete operad.

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1 Introduction

Differential graded (dg) Lie algebras have shown to be of great importance in deformation theory and rational homotopy theory [SS12, Qui69]. On the deformation theory side, this culminated with a theorem of Lurie and Pridham [Lur11a, Pri10] stating an equivalence (of ∞-categories) between dg-Lie algebras and pointed formal deformation problems (also known as formal moduli problems, or FMPs).

On the side of rational homotopy theory, (reduced) dg-Lie algebras are models for rational 1-connected pointed spaces. In both of these cases, dg-Lie algebras arise from the same procedure: given a pointed space or pointed formal moduli problem, the loop space at the basepoint has the structure of group and the corresponding dg-Lie algebra is the 'tangent space' of this group (this perspective on rational homotopy theory is made more precise in [Lur11b]). In particular, the datum of a basepoint plays a crucial role in the appearance of dg-Lie algebras.

Dg-Lie algebras sit inside the larger category of *curved Lie algebras*, which are graded Lie algebras with a "differential" that does not square to zero, but whose square is controlled by a curvature element: $d^2 = [\theta, -]$. It is a well-accepted idea that such curved Lie algebras, or more generally curved L_∞-algebras, correspond to geometric objects without a fixed basepoint. Indeed, let us point out that in the seminal paper [Kon03] Kontsevich considers formal graded pointed Q-manifolds, which are equivalent to L_∞-algebras; the unpointed version is known to correspond to curved L_∞-algebras. Based on this idea, there have been attempts to approach unbased rational homotopy theory by using curved dg-Lie algebras [Mau15, Mau17, CLM16].

A similar philosophy has been used on the deformation theory side, where people encountered the need for a version of deformation theory under or over a given space [CG18]. For instance, Costello [Cos11] uses curved Lie algebras, or rather curved L_∞-algebras, as models for certain formal derived (differentiable) stacks that appear in field theories, which he calls L_∞-spaces. These formal derived stacks can be seen as formal thickenings of a given manifold X, and the corresponding curved L_∞-algebras live over the de Rham algebra $\Omega^*(X)$. In the setting of Costello's formal derived geometry, Grady and Gwilliam [GG20] have shown that every *Lie algebroid* over a manifold X can be viewed as an L_∞-space. Recall that Lie algebroids are to Lie groupoids what Lie algebras are to Lie groups. It is therefore not so surprising to see them appearing in the context of "unbased" deformation theory.

In the context of derived *algebraic* geometry, the L_∞-spaces of Costello roughly correspond to the so-called perfect families of affine formal derived stacks over X_{dR}, as defined and studied in [CPT$^+$17]. Indeed, these are formal thickenings of X sitting between X itself and its de Rham stack as

$$X \longrightarrow Y \longrightarrow X_{\mathrm{dR}}.$$

From the perspective of X_{dR}, one can view Y as a formal thickening that does not quite come equipped with a basepoint (i.e. a section of the second map). One therefore expects Y to give rise to a curved L_∞-algebra over $\Omega^*(X)$, i.e. to an L_∞-space.

From the point of view of X, one can view Y as the quotient of X by the formal groupoid $X \times_Y X \rightrightarrows X$. Consequently, Y should give rise to a Lie algebroid on X (see e.g. [CCT14, GR17, Yu17]). In the algebraic context, the third author proved [Nui19b] that dg-Lie algebroids are indeed equivalent to formal moduli problems under $X = \mathrm{Spec}(A)$ (see also [CG18]), for A a connective commutative differential graded algebra (cdga). Together with previous works on formal derived geometry [GR17, CPT$^+$17], this suggests that dg-Lie algebroids over X do not just give rise to curved Lie algebras over X_{dR}, but that their ∞-categories should be very closely related. Indeed, the main objective of this paper is to show that there is an equivalence (of ∞-categories) between dg-Lie algebroids over A and certain curved Lie algebras over the complete filtered de Rham algebra of A:

Theorem A (See Theorem 5.1)**.** *Let k be a field of characteristic zero and suppose that A is a smooth algebra or a cofibrant cdga over k, locally of finite presentation. Then there is an equivalence of ∞-categories*

$$\mathrm{curv}\colon \text{Lie algebroids}(A/k) \longrightarrow \text{Curved Lie algebras}_{\mathrm{dR}(A)},$$

between Lie algebroids over A and curved Lie algebras over the de Rham algebra $\mathrm{dR}(A)$ equipped with the Hodge filtration, satisfying a normalizing assumption. This equivalence sends a Lie algebroid $L \overset{\rho}{\twoheadrightarrow} T_A$ to $\mathrm{curv}(L) = \ker(\rho) \otimes_A \mathrm{dR}(A)$, see Section 6.1.

This result relies on having a well-behaved homotopy theory for Lie algebroids and curved Lie algebras. The ∞-category of Lie algebroids can be described efficiently in terms of model categories, but for curved Lie algebras this issue is more subtle.

Indeed, even though curved algebraic structures have already appeared in many areas (matrix factorizations [CT13], deformation quantization [CF07], Floer theory [Fuk03, FOOO09], ...), their homotopy theory is still a subject of ongoing study. In particular, curved Lie algebras have no underlying cochain complexes and hence do not admit an obvious homotopy theory in terms of quasi-isomorphisms. To make sense of Theorem A, the first question that needs to be answered is therefore: "what is a good homotopy theory for curved Lie algebras?".

Various approaches to the homotopy theory of curved objects have been presented in the literature, each suiting different purposes [AT20, BMDC20, DSV18, HM12, Pos18]. A secondary purpose of this paper is to develop a homotopy theory for curved algebras (including homotopy transfer theorem) which is suitable for the study of derived deformation theory and in particular for Theorem A. The basic idea will be to endow objects with a complete filtration and control their homotopy theory by the associated graded.

The curved Lie algebras appearing in Theorem A will then come with a complete filtration; geometrically, this means that they correspond to formal stacks sitting in between X and its Hodge stack X_{Hodge}; the latter is a stack over $\hat{\mathbb{A}}^1/\mathbb{G}_m$ controlling the Hodge filtration on de Rham cohomology, whose special fiber is the shifted tangent bundle $T[1]X$. This geometric picture is substantiated by Theorems D and E below.

We point out that Theorem A does not apply to the situations usually considered in differential geometry: the algebra of functions on a smooth manifold $\mathcal{C}^\infty(M)$ is not in the conditions of Theorem 5.1 and the notion of Lie and L_∞-algebroids varies slightly, as Lie algebroids are typically required to arise from vector bundles [Mac87, Pra67]. Nevertheless the same methods can be adapted to prove a differential-geometric version of the main theorem:

Theorem B (See Theorem 6.14)**.** *Let M be a differentiable manifold. The* curv *construction establishes an equivalence of ∞-categories*

$$\mathrm{curv}\colon \text{Lie algebroids}(M) \longrightarrow \text{Vector bundle curved Lie algebras}_{\Omega^*(M)},$$

between (differential-geometric) L_∞-algebroids over M and those curved L_∞-algebras $\mathfrak{g}$ over the de Rham complex $\Omega^*(M)$ of M that are of the form $\mathfrak{g} \simeq \Omega^*(M) \otimes_{\mathcal{C}^\infty(M)} E$, with E a bounded above graded vector bundle on M.

Outline and main results

Contrary to dg-Lie algebras, curved Lie algebras and curved L_∞-algebras do not directly form a model category. This can be explained by the fact that curved Lie (or curved L_∞-)algebras do not arise as algebras in cochain complexes over a 'curved Lie' operad.

Nevertheless, part of the theory of curved Lie algebras works as if such an operad of curved Lie algebras existed, and its Koszul dual cooperad were the linear dual of the operad governing unital commutative algebras. For example, there is a "bar construction" sending a curved Lie algebra $\mathfrak{g}$ to the cocommutative coalgebra $\mathrm{Sym}^c(\mathfrak{g}[1])$, as well as a natural notion of ∞-morphisms. Furthermore, while curved Lie algebras are not cochain complexes but instead have a pre-differential that does not square to zero, the ones "appearing in nature" do typically carry a natural complete filtration such that the pre-differential squares to zero on the associated graded.

▷ The approach carried out in **Section 2** aims to formalize the heuristic above. The first step is to work in the underlying category of *complete filtered complexes*, i.e. cochain complexes equipped with a decreasing complete filtration. In this category one can define obvious notions of complete operads and their algebras; we show that algebras over a complete filtered operad form a model category such that weak equivalences are maps inducing quasi-isomorphisms at the level of the associated graded (Theorem 2.15). Equivalently, one can think of such objects as graded mixed complexes as appearing in [CPT+17], see Section 2.2.1. Most of the operadic calculus from [LV12], such as the (co)operadic bar-cobar constructions, ∞-morphisms and homotopy transfer, generalises to the complete filtered setting.

In particular, applying this machinery to a filtered version of the counital cocommutative cooperad ucoCom, we obtain a complete operad $\mathsf{cLie}_\infty := \Omega(\mathsf{ucoCom}\{1\})$. The algebras over this operad, which we call *mixed-curved L_∞-algebras*, differ from ordinary curved L_∞-algebras in the sense that their pre-differential comes with a splitting as $d + \ell_1$ where d squares strictly to zero and ℓ_1 is filtration increasing. Still, the model category of mixed-curved L_∞-algebras can be fruitfully used to study the ∞-category of curved L_∞-algebras and ∞-morphisms between them (Definition 2.58). In short, we prove the following result:

Theorem C (See Section 2.5.1 and Corollary 2.64)**.** *The complete operad* cLie_∞ *admits a model* cLie*, whose algebras are mixed-curved Lie algebras. The ∞-category* $\mathbf{cLie}^{\mathrm{mix}}$ *associated to the model category of mixed-curved Lie algebras is equivalent to the ∞-category of mixed-curved L_∞-algebras with ∞-morphisms between them.*

The (simplicial) ∞-category $\mathbf{cLie}$ *of curved Lie algebras is then given by a pullback of ∞-categories, each of which arises from a model category*

$$\mathbf{cLie} \simeq \mathbf{cLie}^{\mathrm{mix}} \times_{\mathbf{Mod}_k^{\mathrm{cpl}}} \mathbf{Mod}_k^{\mathrm{gr}}.$$

Here $\mathbf{Mod}_k^{\mathrm{cpl}}$ *denotes the ∞-category of complete complexes and* $\mathbf{Mod}_k^{\mathrm{gr}}$ *denotes the ∞-category of graded complexes. Consequently,* $\mathbf{cLie}$ *is a presentable ∞-category.*

While we focus on curved Lie algebras in this paper, our framework can equally well handle other types of curved algebras. For example, it applies to curved associative algebras which arise, for instance, from vector bundles with non-flat connections, see Section 2.5.4.

Perhaps surprisingly, our framework produces non-trivial results even when restricted to uncurved Lie (or L_∞) algebras, since there can be non-trivial curved morphisms between uncurved objets. In particular, by expressing the Maurer–Cartan space of an L_∞-algebra $\mathfrak{g}$ as the space of curved maps $0 \rightsquigarrow \mathfrak{g}$, in Section 6.2 we recover some results due to Dolgushev and Rogers.

We point out that our approach is not directly comparable to [BMDC20]. While Bellier-Millès and Drummond-Cole also consider filtered objects, their operads *themselves* have a curvature and a pre-differential not squaring to zero.

Many of the results above extend naturally if we replace the ground field k by a cdga A. However, the curved Lie algebras appearing in Theorem A do not quite fit into this framework, because their pre-differential is required to interact with the de Rham differential on $\mathrm{dR}(A)$. In other words, it becomes important to view $\mathrm{dR}(A)$ as a *graded mixed* cdga (with weight-grading given by the form degree, as in [CPT+17]), and to consider curved Lie algebras that interact with the graded mixed structure.

▷ The goal of **Section 3** is then to carry out a similar analysis as before, but for curved algebras in modules over a graded mixed cdga B. The main insight, spelled out in Lemma 3.7, is that mixed-curved L_∞-algebras over such B are also governed by a complete filtered B-operad $\mathsf{cLie}_{\infty,B}$, constructed as a distributive law $\mathsf{cLie}_{\infty,B} \cong B \circ \mathsf{cLie}_\infty$.

While $\mathsf{cLie}_{\infty,B}$ is not obtainable as a cobar construction (it is not even augmented), the upshot of Section 3 is that we still have (somewhat *ad hoc*) bar-cobar resolutions, ∞-morphisms and crucially, a version of the Homotopy Transfer Theorem 3.12. This opens the way to a generalization of Theorem C (Theorem 3.24), which allows us to study classical curved Lie algebras over B via a pullback of ∞-categories:

$$\mathbf{cLie}_B \simeq \mathbf{cLie}_B^{\mathrm{mix}} \times_{\mathbf{Mod}_B^{\mathrm{cpl}}} \mathbf{Mod}_{B_{\mathrm{gr}}}^{\mathrm{gr}}.$$

▷ Starting from **Section 4**, our goal is to study the homotopy theory of Lie algebroids over a cdga A. Notice that Lie algebroids over a fixed base are not algebras over an operad, so that the usual methods of constructing a model structure on them do not quite work. In [Nui19a], the third author showed that Lie (or equivalently L_∞) algebroids carry a semi-model structure for which the weak equivalences are quasi-isomorphisms.

In fact, we can go further than [Nui19a] and study Lie algebroids which are themselves curved. Such type of objects have been considered for instance in [Baa]. Similar to the previous sections, the ∞-category $\mathbf{cLie}(A/k)$ of curved L_∞-algebroids over A can be conveniently studied using a mixed variant of curved L_∞-algebroids, which can be organized into a (semi) model category. Most results of Section 4 are extensions of the results of the previous section to Lie algebroids, and can be summarized as follows.

(1) The category of mixed-curved L_∞-algebroids over A carries a semi-model structure whose weak equivalences are A-module maps inducing quasi-isomorphisms on the associated graded (Theorem 4.12).

(2) While there are no bar or cobar constructions for curved L_∞-algebroids, there is a "bar-cobar" resolution $\mathfrak{L} \mapsto Q(\mathfrak{L})$ on mixed-curved L_∞-algebroids such that structure preserving maps of mixed-curved L_∞-algebroids $Q(\mathfrak{L}) \to \mathfrak{H}$ correspond to ∞-morphisms $\mathfrak{L} \rightsquigarrow \mathfrak{H}$ (Proposition 4.16).

(3) The association $(d, \ell_1) \mapsto d + \ell_1$ induces an equivalence $\mathbf{cLie}(A/k)^{\mathrm{gr-mix}} \simeq \mathbf{cLie}(A/k)$ from the ∞-category of graded mixed-curved Lie algebroids to the ∞-category of curved Lie algebroids (Proposition 4.20).

Finally, we also give a description of curved L_∞-algebroids using uncurved objects (which is already interesting for curved L_∞-algebras over the base field k):

Theorem D (See Theorem 4.23)**.** *There is an equivalence of ∞-categories*

$$\mathbf{cLie}(A/k) \simeq \mathbf{Lie}(A/k)^{\mathrm{gr}}/\mathcal{R}(T_A)$$

between curved L_∞-algebroids and graded uncurved L_∞-algebroids over a certain graded Lie algebroid $\mathcal{R}(T_A)$, whose Chevalley–Eilenberg complex is the Rees algebra of the de Rham complex of A.

▷ Finally, in **Section 5** we prove the main theorems. In fact, we deduce them from a more general result characterizing the category of all curved L_∞-algebroids over complete filtered algebras of the form $C^*(\mathfrak{t})$, where $\mathfrak{t} \to T_A$ is a complete L_∞-algebroid over A and $C^*(\mathfrak{t})$ denotes its Chevalley–Eilenberg complex (with the Hodge filtration).

Theorem E (See Theorem 5.2)**.** *Let A be a nonpositively graded cdga and let $\mathfrak{t}$ be a complete L_∞-algebroid over A such that $F^0(\mathfrak{t}) = 0$ and each $F^i(\mathfrak{t})$ is finitely generated quasiprojective as an A-module. Then there is an equivalence of ∞-categories*

$$\mathrm{curv}\colon \mathbf{cLie}(A/k)_{/\mathfrak{t}} \xrightarrow{\sim} \mathbf{cLie}_{C^*(\mathfrak{t})}.$$

Taking $\mathfrak{t} = T_A$ the terminal Lie algebroid on A and restricting to uncurved L_∞ algebroids we recover precisely the statement of our main Theorem A.

In light of Theorem D and the relation between Lie algebroids and formal stacks, this suggests a more geometric interpretation of the ∞-category of all curved L_∞-algebras over $\mathrm{dR}(A)$ in terms of formal stacks over the Hodge stack.

Notations and conventions

Throughout, differentials have degree 1 and filtrations are decreasing. In the body of the text, in the absence of additional adjectives, all objects are assumed differential graded (dg) by default i.e., they live over the category of cochain complexes over a field k of characteristic zero. So for instance, when we refer to a Lie algebra, this is synonymous to a dgla, whereas a classical curved Lie algebra will be described as a graded Lie algebra with a degree $+1$ endomorphism and a degree 2 curvature element satisfying some properties.

By default our operads are unital. The equivalence from unital augmented operads to non-unital operads sending an operad to the kernel of the augmentation map is denoted by $\mathcal{P} \mapsto \overline{\mathcal{P}}$. On the other hand, cooperads are by default assumed to be non-counital. Given a cooperad $\mathcal{C}$, we denote the corresponding counital coaugmented cooperad by $\mathcal{C}_+ = \mathcal{C} \oplus I$, see Convention 2.14. All other operadic terminology and conventions are in line with [LV12].

In line with the unitality assumptions, Sym denotes the free unital commutative algebra, i.e. $\mathrm{Sym}V = k \oplus V \oplus V \otimes V \oplus \dots$.

Everywhere in the paper, A will denote a cdga over k (over which Lie algebroids live), while B will be a graded mixed cdga, whose main example is the de Rham algebra $B = \mathrm{dR}(A)$ equipped with the Hodge filtration.

Finally, we use a roman typestyle for ordinary categories and a bold font for ∞-categories, while we reserve a sans serif typestyle for named (co)operads. For instance, Lie will denote the Lie operad, while $\mathrm{Alg}_{\mathsf{Lie}}$ will denote the model category of Lie algebras and $\mathbf{Alg}_{\mathsf{Lie}} \simeq \mathbf{Alg}_{\mathsf{Lie}_\infty}$ the corresponding ∞-category. We will not distinguish between a simplicially enriched category and the corresponding ∞-category.

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2 Complete filtered operadic homotopy theory

The goal of this section is to develop the appropriate homotopical framework in which to consider curved Lie algebras. We will do this by studying operads and their algebras in the complete filtered setting: we show that given a filtered operad $\mathcal{P}$, $\mathcal{P}$-algebras form a model category and satisfy a form of the Homotopy Transfer Theorem in such a way that the associated ∞-category is equivalent to the one of $\mathcal{P}_\infty$-algebras and ∞-morphisms. We then discuss the complete operads cLie and cLie_∞ governing respectively mixed-curved Lie algebras and mixed-curved L_∞-algebras, which can be used to study curved Lie algebras in the usual sense.

2.1 Recollections on filtered complexes

Given a field k of characteristic zero, a *filtered complex* is a $\mathbb{Z}$-indexed sequence of cochain complexes of k-vector spaces and inclusions between them

$$\dots \hookrightarrow F^1V \hookrightarrow F^0V \hookrightarrow F^{-1}V \hookrightarrow \dots$$

We will typically denote a filtered complex by its colimit $V \coloneqq \operatorname{colim}_{n\to-\infty} F^nV$ and think of each F^nV as a subcomplex of V. A filtered complex V is said to be *nonnegatively filtered* if $F^0V = V$. Given a filtered complex, one can shift its filtration weight by p to get another filtered complex $V\langle p\rangle$

$$F^qV\langle p\rangle \coloneqq F^{p+q}V.$$

We will denote by $\mathrm{Mod}_k^{\mathrm{filt}}$ the category of filtered complexes, with maps between them given by maps of cochain complexes preserving filtration weights.

A filtered complex is *complete* if the filtration is complete and Hausdorff, i.e. if the map $V \to \lim_{\leftarrow n} V/F^nV$ is an isomorphism. The inclusion of the complete complexes into all filtered complexes admits a left adjoint

$$\widehat{(-)}\colon \mathrm{Mod}_k^{\mathrm{filt}} \rightleftarrows \mathrm{Mod}_k^{\mathrm{cpl}} \colon \iota$$

sending a filtered complex to its completion. In particular, the category of complete complexes admits all colimits, which are computed in the category of filtered complexes and then completed. For example, infinite coproducts of complete complexes are given by *completed direct sums.*

A *(weight-)graded complex* is simply a $\mathbb{Z}$-indexed family of complexes $\{V\langle p\rangle\}_{p\in\mathbb{Z}}$. There are functors between complete complexes and graded complexes

$$\mathrm{Mod}_k^{\mathrm{gr}} \xrightarrow{\mathrm{Tot}} \mathrm{Mod}_k^{\mathrm{cpl}} \xrightarrow{\mathrm{Gr}} \mathrm{Mod}_k^{\mathrm{gr}}.$$

The functor Tot sends a graded complex to its *total complex*, i.e. to

$$\mathrm{Tot}(V) \coloneqq \widehat{\bigoplus_p} V\langle p\rangle, \qquad\qquad F^q\,\mathrm{Tot}(V) = \prod_{n\leq 0} V\langle n-q\rangle.$$

The second functor takes the associated graded $\mathrm{Gr}^p(V) \coloneqq F^pV\big/F^{p+1}V$. Note that for any filtered complex V, the map to its completion $V \longrightarrow \widehat{V}$ induces an isomorphism on the associated graded.

Definition 2.1. A map of (complete) filtered complexes $V \longrightarrow W$ is called a *surjection* if it is a surjection is every filtration weight. It is called an *admissible monomorphism* if each $F^pV \longrightarrow F^pW$ is a monomorphism and furthermore, each map

$$F^pV \oplus_{F^{p+1}V} F^{p+1}W \longrightarrow F^pW$$

is a monomorphism.

Remark 2.2. Let us say that $0 \longrightarrow V \longrightarrow W \longrightarrow Z \longrightarrow 0$ is a *short exact sequence* of complete complexes if it is short exact in each filtration weight. A map is an admissible monomorphism (a surjection) precisely if it is the first (second) map in such a short exact sequence. With this notion of short exact sequence, the category of complete complexes becomes an exact category in the sense of Quillen [Qui73].

Remark 2.3. Note that, even though we are working over a field, not every inclusion is admissible, i.e. fits into a short exact sequence: for example, take $k \longrightarrow k'$ where the codomain is just k, in filtration degree 1 instead of 0.

Remark 2.4. Suppose that $p \colon W \longrightarrow Z$ is a surjection of complete complexes. Without differentials, p admits a section: indeed, without differentials we can simply choose a basis for Z. Each basis vector has a certain (maximal) filtration weight, and choosing inverse images with the same weight provides a filtration-preserving section.

Lemma 2.5. *The functor* $\mathrm{Gr} \colon \mathrm{Mod}_k^{\mathrm{cpl}} \longrightarrow \mathrm{Mod}_k^{\mathrm{gr}}$ *preserves (infinite) direct sums and products, filtered colimits and is exact (for the exact structure as in Remark 2.2). In particular, it preserves pushouts along admissible monomorphisms and pullbacks along maps that are surjective in each filtration weight.*

Proof. The first part is readily verified and the second part is true for any functor between Quillen exact categories preserving exact sequences. □

The category of filtered complexes is a closed symmetric monoidal category via the tensor product

$$F^r(V \otimes W) \coloneqq \sum_{p+q=r} F^p V \otimes F^q W.$$

One easily sees that the internal mapping object is the filtered complex given in weight r by maps that increase filtration weight by (at most) r:

$$F^r \mathrm{Hom}(V, W) = \big\{\text{filtration preserving maps } V \longrightarrow W\langle r\rangle\big\}.$$

We will also refer $\mathrm{Hom}(V, W)$ (with the above filtration) as the *filtered mapping complex.*

Proposition 2.6. *The category of complete cochain complexes carries a closed symmetric monoidal structure* $\hat{\otimes}$ *such that the completion functor* $\widehat{(-)} \colon \mathrm{Mod}_k^{\mathrm{filt}} \longrightarrow \mathrm{Mod}_k^{\mathrm{cpl}}$ *is symmetric monoidal.*

Proof. This follows from the fact that for any filtered complex V and any complete complex W, the filtered mapping complex $\mathrm{Hom}(V, W)$ is itself already complete; indeed, it is a limit

$$\lim_{m \to -\infty} \lim_{n \to \infty} \mathrm{Hom}(F^m V, W/F^n W)$$

of complexes with (complete) filtrations vanishing in sufficiently high degrees. □

Remark 2.7. The category of complexes is a full monoidal subcategory of complete complexes, by endowing a complex V with the trivial filtration $F^1 V = 0$ and $F^r V = V$ for $r \leq 0$. We tend to tacitly view complexes as complete complexes in this way. In particular, $\mathrm{Mod}_k^{\mathrm{cpl}}$ is tensored and enriched over cochain complexes; the complex of maps $V \longrightarrow W$ is simply $F^0 \mathrm{Hom}(V, W)$.

Remark 2.8. The category of (weight-)graded cochain complexes has a similar closed symmetric monoidal structure, where

$$(V \otimes W)\langle r\rangle = \bigoplus_{p+q=r} V\langle p\rangle \otimes W\langle q\rangle \quad \text{and} \quad \mathrm{Hom}(V, W)\langle p\rangle = \prod_q \mathrm{Hom}(V\langle q\rangle, W\langle q+p\rangle).$$

The functors $\mathrm{Tot}\colon \mathrm{Mod}_k^{\mathrm{gr}} \longrightarrow \mathrm{Mod}_k^{\mathrm{cpl}}$ and $\mathrm{Gr}\colon \mathrm{Mod}_k^{\mathrm{cpl}} \longrightarrow \mathrm{Mod}_k^{\mathrm{gr}}$ are both symmetric monoidal and furthermore preserve internal mapping objects, i.e.

$$\mathrm{Tot}\left(\mathrm{Hom}(V,W)\right) \cong \mathrm{Hom}\big(\mathrm{Tot}(V),\mathrm{Tot}(W)\big), \qquad \mathrm{Gr}\left(\mathrm{Hom}(V,W)\right) \cong \mathrm{Hom}\big(\mathrm{Gr}(V),\mathrm{Gr}(W)\big).$$

To see the second isomorphism, note that without differential one can decompose $V = \widehat{\bigoplus}_\alpha k\langle p_\alpha\rangle[n_\alpha]$ as a completed sum of copies of k, in various degrees and filtration weights. Indeed, to do this one simply has to choose a basis for the associated graded of V and lift all basis vectors to V itself. The above isomorphism then takes the form

$$\mathrm{Gr}\Big(\prod_\alpha W\langle -p_\alpha\rangle[-n_\alpha]\Big) \cong \prod_\alpha \Big(\mathrm{Gr}(W)\langle -p_\alpha\rangle[-n_\alpha]\Big)$$

which holds because taking the associated graded preserves products.

We can then consider operads over this symmetric monoidal category.

Definition 2.9 (Complete operads)**.** A *complete operad* $\mathcal{P}$ is an (by default *unital*, symmetric) operad in the category of complete complexes, i.e. a unital algebra in symmetric sequences of complete complexes, with respect to the composition product $\hat{\circ}$. Explicitly, $\mathcal{P}$ comes with composition maps

$$\gamma\colon \big(\mathcal{P}\,\hat{\circ}\,\mathcal{P}\big)(n) = \widehat{\bigoplus_k}\mathcal{P}(k)\,\hat{\otimes}_{\Sigma_k}\left(\widehat{\bigoplus_{i_1+\cdots+i_k=n}}\mathrm{Ind}_{\Sigma_{i_1}\times\cdots\times\Sigma_{i_k}}^{\Sigma_n}\Big(\mathcal{P}(i_1)\,\hat{\otimes}\ldots\hat{\otimes}\,\mathcal{P}(i_k)\Big)\right) \longrightarrow \mathcal{P}(n)$$

from a completed direct sum of completed tensor products. Given a complete operad $\mathcal{P}$, a *complete* $\mathcal{P}$*-algebra* is a complete complex A equipped with a map $\mathcal{P}\,\hat{\circ}\,A \longrightarrow A$ satisfying the usual associativity and unitality conditions.

Remark 2.10 (Filtered operads)**.** In a similar way, one can define (not necessarily complete) filtered operads and algebras over them, using the category of symmetric sequences of filtered complexes, equipped with the (non-completed) composition product $\circ$. A complete operad is then equivalently a filtered operad whose underlying symmetric sequence is complete: the structure map $\mathcal{P}\circ\mathcal{P} \longrightarrow \mathcal{P}$ then extends uniquely to the completion.

Likewise, if $\mathcal{P}$ is a filtered operad, then there is an equivalence between filtered $\mathcal{P}$-algebras A whose underlying filtered complex is complete, and complete algebras over the completion $\widehat{\mathcal{P}}$: indeed, the structure map $\mathcal{P}\circ A \longrightarrow A$ extends uniquely to the completion $\widehat{\mathcal{P}}\,\hat{\circ}\,A \longrightarrow A$.

Example 2.11. It will be important (particularly in Section 2.4) that due to Proposition 2.6 we can define the *endomorphisms operad* of a filtered complex A to be the filtered operad End_A given by $\mathrm{End}_A(n) \coloneqq \mathrm{Hom}(A^{\otimes n}, A)$, such that filtered algebras over $\mathcal{P}$ can be identified with filtered operad maps $\mathcal{P} \longrightarrow \mathrm{End}_A$. When A is complete, this is a complete operad, which is isomorphic to the complete endomorphism operad with operations $A\,\hat{\otimes}\ldots\hat{\otimes}\,A \longrightarrow A$.

Since we consider objects with non-trivial arity 0 pieces, we need to be precise about the version of cooperads we consider, namely in what concerns partial vs total cocompositions and conilpotence.

Recall that given a symmetric sequence $\mathcal{C}$ with partial cocompositions $\Delta_i\colon \mathcal{C}(k) \to \mathcal{C}(m)\otimes\mathcal{C}(k-n+1), i = 1,\ldots,m$ (appropriately Σ-equivariant and coassociative), one can iterate them to obtain a total cocomposition $\Delta\colon \mathcal{C}(k) \to \prod_{i_1+\cdots+i_n=k}\mathcal{C}(n)\otimes\mathcal{C}(i_1)\otimes\cdots\otimes\mathcal{C}(i_n)$, making $\mathcal{C}$ a coalgebra with respect to $\circ$, as the argument dual to [LV12, Proposition 5.3.8] shows. Notice however that in the absence of a unit one cannot obtain the partial cocomposition from the total one.

Definition 2.12. A *complete cooperad* $\mathcal{C}$ is a symmetric sequence of complete complexes equipped with (by default *non-counital*) Σ-equivariant coassociative partial cocompositions $\Delta_i \colon \mathcal{C}(k) \to \mathcal{C}(m) \otimes \mathcal{C}(k-n+1), i = 1, \dots, m$. In particular, $\mathcal{C}$ is a coalgebra in symmetric sequences of complete complexes with respect to the completed composition product $\hat{\circ}$.

A *(conilpotent)* $\mathcal{C}$*-coalgebra* is a complete complex C together with a map $C \longrightarrow \mathcal{C} \,\hat{\circ}\, C$ satisfying the usual associativity constraint.

A complete cooperad $\mathcal{C}$ is said to be *complete conilpotent* if the n-fold cocomposition determines a map

$$(\Delta, \Delta^2, \Delta^3, \dots) \colon \mathcal{C} \longrightarrow \widehat{\bigoplus_{n \geq 2}} \mathcal{C} \,\hat{\circ} \dots \hat{\circ}\, \mathcal{C}$$

into the *completed sum* of completed composition products. In other words, an operation in $\mathcal{C}(n)$ can be decomposed into infinitely many trees, but their sum converges with respect to the filtration.

Warning 2.13. Recall that (strict) conilpotency in the usual sense is the requirement that the coradical filtration be exhaustive, $\operatorname{colim}_n \operatorname{corad}^n(\mathcal{C}) = \mathcal{C}$. The condition of complete conilpotency is *weaker* than conilpotency in the usual sense: for each element $c \in \mathcal{C}$ and $r \geq 0$, there exists an n such that $\Delta^n(c)$ is of filtration degree r, but $\Delta^n(c)$ need not vanish for large n. However, note that the associated graded $\mathrm{Gr}(\mathcal{C})$ is a cooperad in weight-graded complexes which *is* conilpotent in the usual sense.

Convention 2.14. [Augmented and non-unital operads] There is an equivalence of categories between unital augmented operads and non-unital operads, given by quotienting out the unit in arity one. We will denote this construction by $\mathcal{P} \mapsto \overline{\mathcal{P}}$. We take the convention that our operads are unital unless otherwise specified. On the other hand, we take the convention that cooperads are non-counital. Given a cooperad $\mathcal{C}$, we denote the corresponding counital coaugmented cooperad by $\mathcal{C}_+ = \mathcal{C} \oplus I$.

The reason for this choice is that non-unitaly is convenient to define conilpotent cooperads, and for the constructions in Section 2.3, but slightly inconvenient when talking about *algebras* over operads: indeed, when $\mathcal{P}$ is a non-unital complete operad and A is a complete complex, the free $\mathcal{P}$-algebra on A is given by $\mathcal{P} \circ A$, not $\overline{\mathcal{P}} \circ A$.

2.2 Model structures on filtered complexes and algebras

In this section we show that the category of complete complexes can be endowed with a model structure whose weak equivalences are maps inducing a quasi-isomorphism at the level of the associated graded. Furthermore, this model structure transfers to a model structure on algebras over operads:

Theorem 2.15. *Let $\mathcal{P}$ be a complete operad. The category of complete algebras over $\mathcal{P}$ admits a cofibrantly generated model category structure such that:*

- *Weak equivalences are maps inducing quasi-isomorphisms on the associated graded.*
- *Fibrations are maps that induce surjections in each filtration weight p.*

In particular every $\mathcal{P}$-algebra is fibrant.

In particular, taking $\mathcal{P}$ to be the trivial operad gives a model structure on complete complexes.

Proof. The category of (not necessarily complete) filtered complexes admits a cofibrantly generated model structure in which weak equivalences (resp. fibrations) are graded quasi-isomorphisms (resp. surjections in each filtration degree): this is the special case of [CSLW19, Theorem 3.14] where $r = 0$. Now consider the adjoint pair

$$\widehat{\mathrm{Free}}\colon \mathrm{Mod}_k^{\mathrm{filt}} \rightleftarrows \mathrm{Alg}_{\mathcal{P}}^{\mathrm{cpl}}\colon \mathrm{forget}$$

between complete $\mathcal{P}$-algebras and (not necessarily complete) filtered complexes. To check that the model structure transfers along this adjunction, it suffices to provide a functorial path object in complete $\mathcal{P}$-algebras. This is just the classical argument from Hinich [Hin97]: if A is a complete $\mathcal{P}$-algebra, then $A \mathbin{\hat{\otimes}} \Omega[\Delta^1]$ is a complete $\mathcal{P}$-algebra as well, where $\Omega[\Delta^1]$ is considered as a commutative algebra with the trivial filtration. This factors the diagonal as

$$A \longrightarrow A \mathbin{\hat{\otimes}} \Omega[\Delta^1] \longrightarrow A \times A.$$

Since the associated graded functor from complete complexes to graded complexes is symmetric monoidal, one finds a factorization of graded algebras $\mathrm{Gr}(A) \longrightarrow \mathrm{Gr}(A) \otimes \Omega[\Delta^1] \longrightarrow \mathrm{Gr}(A) \times \mathrm{Gr}(A)$, where $\Omega[\Delta^1]$ is in weight 0. This is clearly a weak equivalence, followed by a surjection. □

Lemma 2.16. *Let $f\colon V \longrightarrow W$ be a map of complete complexes.*

(1) f induces a quasi-isomorphism on the associated graded if and only if it induces a quasi-isomorphism in each filtration degree.

(2) f induces a surjection on the associated graded if and only if it induces a surjection in each filtration degree.

Proof. In both cases, the 'if' part is immediate. For (1), let C denote the mapping cone of f. Since taking the associated graded commutes with taking mapping cones, the associated graded of C is acyclic. In particular, the sequence $\cdots \hookrightarrow F^pC \hookrightarrow F^{p-1}C \hookrightarrow \dots$ consists of acyclic cofibrations, so that each inclusion $F^pC \hookrightarrow C$ into the colimit is an acyclic cofibration. Consequently, each C/F^pC is acyclic, so the completion $C = \mathrm{holim}\, C/F^pC$ is acyclic as well. This implies that each F^pC is acyclic, so that f is a quasi-isomorphism in each degree.

For (2), by induction on $q \geq 1$ using the snake lemma, one sees that f fits into short exact sequences $0 \to K^q \longrightarrow F^pV/F^{p+q}V \longrightarrow F^pW/F^{p+q}W \to 0$ such that the natural map $K^{q+1} \longrightarrow K^q$ is surjective. Taking the limit $q \to \infty$ and using that F^pV and F^pW are complete, one then obtains an exact sequence $F^pV \longrightarrow F^pW \longrightarrow \varprojlim_q{}^1 K^q = 0$. This implies that f is surjective in each filtration degree. □

Lemma 2.17. *A map of complete complexes (over a field k) $V \longrightarrow W$ is a cofibration if and only if it is an admissible monomorphism (Definition 2.1).*

Proof. Suppose that $V \longrightarrow W$ is a cofibration and let $V[0,1]$ denote the mapping cone of V. Using the lifting property against the trivial fibration $V[0,1] \longrightarrow 0$, one sees that without differential, $V \longrightarrow W$ is given by a summand inclusion $W \cong V \oplus W/V$. Note that by Remark 2.4, such summand inclusions are exactly the admissible monomorphisms.

Conversely, suppose that $V \longrightarrow W$ is an admissible monomorphism, i.e. a summand inclusion without the differential. Let $p\colon Y \longrightarrow X$ be an acyclic fibration of complete complexes, with fiber Z. We then have a short exact sequence of filtered mapping complexes

$$0 \longrightarrow \mathrm{Hom}(W/V, Z) \longrightarrow \mathrm{Hom}(W, Y) \longrightarrow \mathrm{Hom}(V, Y) \times_{\mathrm{Hom}(V,X)} \mathrm{Hom}(W, X) \longrightarrow 0$$

and we have to check that the right map is a trivial fibration (in filtration degree 0). Using Lemma 2.16, it suffices to verify that the associated graded of $\mathrm{Hom}(W/V, Z)$ is acyclic. By

Remark 2.8, it suffices to verify that $\mathrm{Hom}(\mathrm{Gr}(W/V), \mathrm{Gr}(Z))$ is acyclic, which is immediate because Z was graded-acyclic. $\square$

Remark 2.18. Consider the category $\mathrm{Fun}(\mathbb{Z}, \mathrm{Mod}_k)$ of sequences of cochain complexes $\ldots \longrightarrow F^1V \longrightarrow F^0V \longrightarrow \ldots$, equipped with the projective model structure. One can verify that an object is cofibrant in this model structure if and only if it is a filtered complex. The obvious fully faithful inclusion $\mathrm{Mod}_k^{\mathrm{cpl}} \hookrightarrow \mathrm{Fun}(\mathbb{Z}, \mathrm{Mod}_k)$ is a right Quillen functor, which furthermore preserves cofibrant objects. In particular, it induces a fully faithful functor of ∞-categories. The essential image can be seen to consist of those sequences of complexes such that $\mathrm{holim}_{i \longrightarrow \infty} F^iV \simeq 0$. Consequently, $\mathrm{Mod}_k^{\mathrm{cpl}}$ is a model for the ∞-category of (derived) complete complexes.

Corollary 2.19. *The model structure on complete cochain complexes is monoidal model with respect to the completed tensor product of Proposition 2.6. Furthermore, the functor $V \hat{\otimes} (-)$ preserves graded-quasi isomorphisms for any object V.*

Proof. It suffices to verify the pushout-product axiom. Let $V_1 \longrightarrow W_1$ and $V_2 \longrightarrow W_2$ be two cofibrations. Using Lemma 2.17, their pushout-product is a summand inclusion without differential, hence a cofibration, whose cokernel is just $W_1/V_1 \otimes W_2/V_2$. In particular, if one of the two maps is furthermore a graded quasi-isomorphism, then this cokernel is graded acyclic. $\square$

Remark 2.20. As a consequence of Corollary 2.19, one finds that for any complete dg-algebra B, the model category of complete B-modules is tensored over complete complexes (via the tensor product $\hat{\otimes}_k$). In particular, for any two complete B-modules M, N, there is complete mapping complex $\mathrm{Hom}_B(M, N)$, given in filtration degree p by the maps of B-modules increasing filtration weight by (at most) p.

In fact, Lemma 2.17 has an analogue for complete modules over a complete dg-algebra B. To this end, let us recall the following terminology:

Definition 2.21. Let B be a complete dg-algebra over k. A complete B-module M is called *quasiprojective* if without differential, it is the retract of a free complete B-module $B \hat{\otimes}_k V$.

Since taking the associated graded is symmetric monoidal (Remark 2.8), every complete B-module has an underlying weight-graded module over the weight-graded dg-algebra $\mathrm{Gr}(B)$. The category of such weight-graded modules admits a model structure, whose weak equivalences (fibrations) are quasi-isomorphisms (surjections) of complexes in each weight. Using this we have the following characterization of cofibrations of complete B-modules in terms of their associated graded:

Proposition 2.22. *Let B be a complete dg-algebra and $f: M \longrightarrow N$ a map of complete B-modules. Then f is a cofibration if and only if it is an admissible monomorphism with quasiprojective cokernel N/M, such that $\mathrm{Gr}(N/M)$ is a cofibrant module over $\mathrm{Gr}(B)$.*

Proof. Let us first prove that all cofibrations indeed have the listed properties. By the small object argument, $f: M \longrightarrow N$ is a retract of a transfinite composition of pushouts of maps of the form $B \hat{\otimes}_k V \longrightarrow B \hat{\otimes}_k W$, where $V \longrightarrow W$ is an admissible monomorphism of filtered complexes. Such maps are themselves admissible monomorphisms, with quasiprojective cokernel. Furthermore, Lemma 2.5 and Remark 2.8 imply that taking the associated graded sends this to the retract of a transfinite composition of pushouts of maps $\mathrm{Gr}(B) \otimes \mathrm{Gr}(V) \longrightarrow \mathrm{Gr}(B) \otimes \mathrm{Gr}(W)$; such a map is a cofibration of weight-graded modules over $\mathrm{Gr}(B)$. It follows that the cokernel $\mathrm{Gr}(N/M) \cong \mathrm{Gr}(N) / \mathrm{Gr}(M)$ is a cofibrant module over $\mathrm{Gr}(B)$.

Conversely, suppose that f has the listed properties and let $p: Y \longrightarrow X$ be an acyclic fibration of complete B-modules, with kernel B. Since the cokernel of f is quasiprojective, there exists a splitting $N \cong M \oplus N/M$ without differentials. This implies that there is a short exact sequence of B-linear filtered mapping complexes

$$0 \longrightarrow \mathrm{Hom}_B(N/M, Z) \longrightarrow \mathrm{Hom}_B(N, Y) \longrightarrow \mathrm{Hom}_B(M, Y) \times_{\mathrm{Hom}_B(M,X)} \mathrm{Hom}_B(N, X) \longrightarrow 0.$$

We have to check that the right map is a trivial fibration in filtration degree 0. By Lemma 2.16, it suffices to verify that the associated graded of $\mathrm{Hom}_B(N/M, Z)$ is acyclic. An argument very similar to Remark 2.8, using that N/M is quasiprojective to write it as a retract of a completed sum of shifted copies of B, shows that there is an isomorphism

$$\mathrm{Gr}\Big(\mathrm{Hom}_B\big(N/M, Z\big)\Big) \cong \mathrm{Hom}_{\mathrm{Gr}(B)}\big(\mathrm{Gr}(N/M), \mathrm{Gr}(Z)\big).$$

Since $\mathrm{Gr}(Z)$ is acyclic and $\mathrm{Gr}(N/M)$ is a cofibrant module over $\mathrm{Gr}(B)$ by assumption, we conclude that $\mathrm{Hom}_B(N/M, Z)$ is indeed graded acyclic. □

2.2.1 Complete filtered complexes versus graded mixed complexes

Recall that by definition, the associated graded functor $\mathrm{Gr}\colon \mathrm{Mod}_k^{\mathrm{cpl}} \longrightarrow \mathrm{Mod}_k^{\mathrm{gr}}$ preserves and detects weak equivalences. Since it preserves exact sequences and all direct sums and direct products, the induced functor between (stable) ∞-categories preserves limits and colimits and detects equivalences. It follows formally from this that one can identify the ∞-category of complete complexes with algebras in weight-graded complexes over a certain monad. In fact, there is a well-known way to describe complete filtered complexes concretely in terms of weight-graded complexes with additional algebraic structure:

Definition 2.23. A *graded mixed complex* is a weight graded complex V, equipped with operations $\delta_k\colon V \longrightarrow V$ for $k \geq 1$ of weight k and degree 1, such that

$$d \circ \delta_k + \delta_k \circ d + \sum_{i+j=k} \delta_i \delta_j = 0.$$

We will write $\mathrm{Mod}_k^{\mathrm{gr-mix}}$ for the category of graded mixed complexes and maps between them that preserve the weights and strictly commute with the operations δ_k.

Remark 2.24. The above definition of a graded mixed complex is also known as a (shifted) multicomplex, and differs from the graded mixed complexes appearing in e.g. [CPT+17], where instead the *strict* notion of graded mixed complex is used, corresponding to the situation where $\delta_k = 0$ for $k \geq 2$. In fact, the inclusion of the category of strict graded mixed complexes into the one of graded mixed complexes becomes an equivalence of ∞-categories after localizing at the weak equivalences (i.e. weightwise quasi-isomorphisms). Indeed, strict graded mixed complexes are weight graded modules over $k\langle\delta_1\rangle/\delta_1^2$, where δ_1 has both degree 1 and weight 1, while graded complexes are weight graded modules over its quasi-free resolution $k\langle\delta_i | i \leq 1\rangle$, where δ_i has degree 1 and weight i, and

$$d(\delta_k) = -\sum_{i+j=k} \delta_i \delta_j.$$

Definition 2.25. An ∞-morphism between two graded mixed complexes, denoted by a wiggly arrow $V \rightsquigarrow W$, is a collection of maps $\varphi_k\colon V \longrightarrow W$ for $k \geq 0$ of weight k and degree 0, such that

$$\sum_{i+j=k} (\delta_i \varphi_j + \varphi_i \delta_j) = 0,$$

where we use the convention that $\delta_0 = d$. We will write $\mathrm{Mod}_k^{\mathrm{gr-mix},\infty}$ for the category of graded mixed complexes and ∞-morphisms between them.

Remark 2.26. The category $\mathrm{Mod}_k^{\mathrm{gr-mix},\infty}$ is almost a model category (see e.g. [Val20, §4.1]), in the sense that all axioms but the bicompleteness one are satistfied, though finite products and pullbacks of fibrations exist: an ∞-morphism $\varphi = (\varphi_k)_{k\geq 0}$ is a (co)fibration (resp. a weak equivalence) if φ_0 is a (co)fibration (resp. a weak equivalence). One easily sees that every object is then both fibrant and cofibrant. Moreover, one can actually prove that the faithful (but not fully faithful) functor $\mathrm{Mod}_k^{\mathrm{gr-mix}} \to \mathrm{Mod}_k^{\mathrm{gr-mix},\infty}$ given by the identity on objects induces an equivalence of ∞-categories after localizing at weak equivalences.

Led by the above Remark 2.26, we define the ∞-category $\mathbf{Mod}_k^{\mathrm{gr-mix}}$ as the simplicial category whose objects are graded mixed complexes, and with n-simplices in the space of morphisms from V to W being ∞-morphisms $V \rightsquigarrow W \otimes \Omega[\Delta^n]$. It then follows that the ∞-functors $\mathrm{Mod}_k^{\mathrm{gr-mix}}[\mathrm{w.e.}^{-1}] \to \mathrm{Mod}_k^{\mathrm{gr-mix},\infty}[\mathrm{w.e.}^{-1}] \to \mathbf{Mod}_k^{\mathrm{gr-mix}}$ are equivalences.

If V is a graded mixed complex, then the total complex $\mathrm{Tot}(V)$ comes equipped with a differential $d_{\mathrm{tot}} = d + \sum_{k\geq 1}\delta_k$. Moreover, every n-simplex of ∞-morphisms $\varphi = (\varphi_k)_{k\geq 0}\colon V \rightsquigarrow W \otimes \Omega[\Delta^n]$ leads to an n-simplex of filtered morphisms $\varphi_{\mathrm{tot}} = \sum_{k\geq 0}\varphi_k\colon \mathrm{Tot}(V) \rightsquigarrow \mathrm{Tot}(W)\otimes\Omega[\Delta^n]$. Hence we have a functor

$$\mathrm{Tot}\colon \mathbf{Mod}_k^{\mathrm{gr-mix}} \longrightarrow \mathbf{Mod}_k^{\mathrm{cpl}}.$$

between simplicial categories.

Proposition 2.27. *The functor* $\mathrm{Tot}\colon \mathbf{Mod}_k^{\mathrm{gr-mix}} \longrightarrow \mathbf{Mod}_k^{\mathrm{cpl}}$ *is an equivalence of ∞-categories.*

This result, well-known to experts, provides the blueprint for our discussion in Section 2.5, where we show how curved (filtered) L_∞-algebras are equivalent to a kind of 'graded mixed-curved L_∞-algebra'. We will also recover it in Example 2.71 from an operadic perspective.

Proof. The ∞-functor is fully faithful by definition, hence we just have to prove that it is essentially surjective. For every complete filtered complex W, we can choose a splitting of the filtration on W (without differential) and obtain an isomorphism of complete filtered graded vector spaces $W \cong \mathrm{Tot}(\mathrm{Gr}(W))$. Decomposing the differential on W into its homogeneous components of weight k as $d_W = d + \sum_{k\geq 1}\delta_k$, this determines a graded mixed structure on $\mathrm{Gr}(W)$, which is such that $W \cong \mathrm{Tot}\big(\mathrm{Gr}(W), \delta_1, \dots\big)$ as complete filtered complexes. ☐

2.3 Bar, cobar, and twisting morphisms

In this section we will see that the classical bar and cobar constructions between operads and conilpotent cooperads, as well as the twisting morphism yoga generalize in a fairly straightforward way to the complete setting. A closely related discussion appears in [DSV18, Chapter 2]. Recall from 2.14 that cooperads are not assumed to be counital.

Proposition/definition 2.28. Let $\mathcal{P}$ be a complete augmented operad and $\mathcal{C}$ be a complete cooperad.

(1) The *convolution Lie algebra* of $\mathcal{C}$ and $\mathcal{P}$ is the complete Lie algebra

$$\mathfrak{g} = \prod_{n\geq 0}\mathrm{Hom}(\mathcal{C}(n), \overline{\mathcal{P}}(n))$$

with filtration induced by the internal Hom. The bracket is induced from the pre-Lie product $f \star g = \gamma_{(1)}^{\mathcal{P}} \circ (f \hat{\otimes} g) \circ \Delta_{(1)}^{\mathcal{C}}$, where the indices $_{(1)}$ indicate infinitesimal (co)composition.

(2) A *twisting morphism* $\phi\colon \mathcal{C} \to \mathcal{P}$ is a Maurer–Cartan element of the Lie algebra $\mathfrak{g}$ (in particular, it takes values in $\overline{\mathcal{P}}$). The set of twisting morphisms is denoted $\mathrm{Tw}(\mathcal{C}, \mathcal{P}) = \{f \in F^0\mathfrak{g}_1 \mid \partial f + \frac{1}{2}[f, f] = 0\}$.

Proof. It is easy to see that the filtrations are preserved by the Lie bracket and $\mathfrak{g}$ is complete since the direct product preserves completeness. See [LV12, Section 6.4] for a treatment of twisting morphisms in the unfiltered case. □

Let E be a complete symmetric sequence. The free operad generated by E, denoted $\mathcal{T}(E)$ is the completion of the vector space spanned by trees labeled by elements of E, where the filtration level of an E-labeled tree the sum of the filtration levels of each vertex. Composition is given by grafting trees.

Similarly, the cofree complete conilpotent cooperad on E is denoted by $\mathcal{T}^c(E)$; it differs only from $\mathcal{T}(E)$ by the unit in arity 1, and has cocomposition given by ungrafting trees.

Proposition/definition 2.29. The *bar construction* of a complete operad $\mathcal{P}$ is the complete (non-unital) conilpotent cooperad

$$\mathrm{Bar}\,\mathcal{P} = (\mathcal{T}^c(\overline{\mathcal{P}}[1]), d_{\mathcal{P}} + d_{\gamma^{(1)}_{\mathcal{P}}})$$

with the filtration induced by the cofree conilpotent cooperad functor and differential arising from the differential on $\mathcal{P}$ and the bar differential, contracting edges of trees.

Similarly, the *cobar construction* of a complete cooperad $\mathcal{C}$ is the complete (unital augmented) operad

$$\Omega\mathcal{C} = (\mathcal{T}(\mathcal{C}[-1]), d_{\mathcal{C}} + d_{\Delta^{(1)}_{\mathcal{C}}}).$$

These functors form an adjoint pair $\Omega\colon \mathrm{Coop}^{\mathrm{conil}} \leftrightarrows \mathrm{Op}^{\mathrm{aug}} \colon \mathrm{Bar}$ between complete augmented operads and *complete conilpotent* cooperads. Furthermore the counit of the adjunction $\Omega\,\mathrm{Bar} \overset{\sim}{\Rightarrow} \mathrm{id}_{\mathrm{Op}^{\mathrm{aug}}}$ is a weak equivalence.

Proof. The proof of the adjunction follows from showing that there are natural bijections $\mathrm{Hom}_{\mathrm{Op}^{\mathrm{aug}}}(\Omega\mathcal{C}, \mathcal{P}) \cong \mathrm{Tw}(\mathcal{C}, \mathcal{P}) \cong \mathrm{Hom}_{\mathrm{Coop}^{\mathrm{conil}}}(\mathcal{C}, \mathrm{Bar}\,\mathcal{P})$, which is a straightforward adaptation from the unfiltered case [LV12, Theorem 6.5.10] (in fact, the left bijection also exists when $\mathcal{C}$ is not conilpotent).

For the second part, notice that the functor associated graded commutes with the bar and cobar constructions (since it preserves tensor products). Ignoring degrees, elements of $\Omega\,\mathrm{Bar}(\mathrm{Gr}(\mathcal{P}))$ can be seen as trees whose vertices are themselves ("inner") trees whose vertices are labeled by $\mathcal{P}$. Taking a second filtration by the number of inner edges (which is the bar filtration) we recover at the level of the associated graded only the piece of the differential corresponding to the one from $\mathcal{P}$ and a second one making an inner edge into an outer edge. One checks that the associated graded retracts into $\mathcal{P}$ by constructing a homotopy that makes an outer edge into an inner edge (cf. [Fre04, Proposition 3.1.12] for the unfiltered case). □

Notice that, in addition to the complete filtration coming from $\mathcal{C}$, $\Omega\mathcal{C}$ admits another (decreasing) filtration given by the number of vertices in $\mathcal{T}\mathcal{C}$. This filtration will be referred to as the *cobar filtration*. Dually, the number of vertices in $\mathcal{T}^c\mathcal{P}$ induces an exhausting increasing filtration on $\mathrm{Bar}\,\mathcal{P}$ that will be called the *bar filtration*. Taking the associated spectral sequence one can show that Bar preserves weak equivalences of complete operads.

Definition 2.30. A twisting morphism $\mathcal{C} \to \mathcal{P}$ is said to be *Koszul* if the induced map $\Omega\mathcal{C} \to \mathcal{P}$ is a weak equivalence of complete operads.

In particular, the projection $\mathrm{Bar}\,\mathcal{P} \to \mathcal{P}$ is a Koszul twisting morphism by Proposition/definition 2.29. Recall that a twisting morphism $\mathcal{C} \to \mathcal{P}$ gives rise to a bar and cobar construction at the level of (co)algebras:

Proposition/definition 2.31. Let $\phi\colon \mathcal{C} \to \mathcal{P}$ be a Koszul twisting morphism between complete (co)operads. The *bar construction* of a $\mathcal{P}$-algebra A, denoted $\mathrm{Bar}_\phi A$ and the *cobar construction* of a conilpotent $\mathcal{C}$-coalgebra C, denoted $\Omega_\phi C$ are the quasi-free (co)algebras

$$\mathrm{Bar}_\phi A = (\mathcal{C}_+ \circ A, d_A + d_\phi) \qquad\qquad \Omega_\phi C = (\mathcal{P} \circ C, d_C + d^\phi).$$

See Convention 2.14 about free algebras and our convention conerning unitality. The differentials are induced by those on A and C, together with the (co)bar differentials as in [LV12, Section 11.2]. An ∞-*morphism* between two $\mathcal{P}$-algebras, denoted by a wiggly arrow $\rightsquigarrow$, is by definition a morphism of $\mathcal{C}$-coalgebras between the respective bar constructions.

These (co)bar constructions define an adjoint pair $\Omega_\phi\colon \mathrm{Coalg}_{\mathcal{C}} \leftrightarrows \mathrm{Alg}_{\mathcal{P}}\colon \mathrm{Bar}_\phi$, and the counit of the adjunction $\Omega_\phi \mathrm{Bar}_\phi A \to A$ is a weak equivalence if $\mathcal{C}$ is a complete conilpotent cooperad.

Proof. The functors are adjoint by the same argument as [LV12, Proposition 11.3.2]. For $\Omega_\phi \mathrm{Bar}_\phi A \to A$ being a weak equivalence, we argue as in the unfiltered case [LV12, Theorem 11.3.6], but unlike there we require a proof that does not used that $\mathcal{C}$ and $\mathcal{P}$ are connected weight graded. Since the functor $\mathrm{Gr}\colon \mathrm{Mod}_k^{\mathrm{filt}} \longrightarrow \mathrm{Mod}_k^{\mathrm{gr}}$ preserves colimits and tensor products and detects weak equivalences, it suffices to prove this at the graded level (so we can forget about filtrations, while the weight-grading will play no role); in particular, $\mathcal{C}$ is now conilpotent in the usual sense (see Warning 2.13).

We start by showing that for the universal Koszul twisting morphism $\iota\colon \mathcal{C} \to \Omega\mathcal{C}$, the map $\Omega_\iota \mathrm{Bar}_\iota A \to A$ is a weak equivalence. Ignoring differentials, $\Omega_\iota \mathrm{Bar}_\iota A$ takes the form $\Omega\mathcal{C}\circ\mathcal{C}_+\circ A$. One can take a filtration on $\Omega_\iota \mathrm{Bar}_\iota A$ given by the sum of the coradical filtrations on all $\mathcal{C}_+$ and $\mathcal{C}[-1]$ pieces appearing. On the associated graded, the only non-internal piece of the differential that survives is the counital part that takes an element p in one of the $\mathcal{C}_+$ pieces, replaces it by $p \circ 1$ and "moves" p to the $\Omega\mathcal{C}$ while increasing its degree by 1. There is a natural contracting homotopy to this differential that takes any rightmost $c \in \mathcal{C}[-1] \subset \Omega\mathcal{C}$ connected only to units $1 \in \mathcal{C}_+$ and moves it to the $\mathcal{C}_+$ side. It follows that the only surviving piece corresponds to $k \circ k \circ A = A$.

Secondly, let us show that the map $\Omega_\iota \mathrm{Bar}_\iota A \to \Omega_\phi \mathrm{Bar}_\phi A$ is a weak equivalence. Ignoring differentials, this corresponds to showing that the map $\Omega\mathcal{C} \to \mathcal{P}$ induces an equivalence $\Omega\mathcal{C} \circ \mathcal{C}_+ \circ A \xrightarrow{\sim} \mathcal{P} \circ \mathcal{C}_+ \circ A$. This time, one takes a filtration consisting of the total coradical filtration on the $\mathcal{C}_+$ part (ignoring the $\Omega\mathcal{C}$ and $\mathcal{P}$ pieces). Now, on the associated graded, we obtain precisely $\Omega\mathcal{C} \circ \mathcal{C}_+ \circ A \xrightarrow{\sim} \mathcal{P} \circ \mathcal{C}_+ \circ A$ with only the internal differentials, which is a quasi-isomorphism since ϕ is Koszul.

Finally the result follows from the 2-out-of-3 property since the map $\Omega_\iota \mathrm{Bar}_\iota A \to A$ is precisely the composite $\Omega_\iota \mathrm{Bar}_\iota A \to \Omega_\phi \mathrm{Bar}_\phi A \to A$. $\square$

2.4 Homotopy Transfer Theorem

In this section we prove a version of the Homotopy Transfer Theorem for complete complexes. The proof of the theorem itself is fairly standard and does not actually make use of the completeness of the filtered complexes. However, to obtain some classical consequences such as the construction of higher Massey products on the homology of an algebra there are some obstructions coming from the underlying category of filtered vector spaces. As we will see, essentially all obstructions vanish under the assumption that the filtered complexes involved are complete.

We start by noticing that the usual notion of homotopy equivalence of cochain complexes extends naturally to the filtered setting.

Definition 2.32. A map $f\colon W \longrightarrow V$ between complete complexes is a *filtered homotopy equivalence* if there exists a map of filtered complexes $g\colon V \to W$ and filtration-preserving homotopies $h_V\colon V \longrightarrow V$, $h_W\colon W \longrightarrow W$ of degree -1 such that

$$dh_V + h_V d = \mathrm{id}_V - f \circ g \qquad \text{and} \qquad dh_W + h_W d = \mathrm{id}_W - g \circ f.$$

A *homotopy retract* consists of filtered maps i, p of degree 0 and h of degree 1

$$\begin{tikzcd} V \arrow[r, shift right, "i"'] & W \arrow[l, shift right, "p"'] \arrow[loop right, "h"] \end{tikzcd}$$

such that $ip - \mathrm{id}_W = [d_W, h]$. It is called a *deformation retract* if furthermore $pi = \mathrm{id}_V$.

Lemma 2.33. *Let $p\colon W \twoheadrightarrow V$ be an acyclic fibration of complete complexes. Then p is part of a deformation retract. Furthermore, the homotopy h can be chosen to satisfy the side conditions $ph = 0$, $hi = 0$ and $h^2 = 0$. Dually, if $i\colon V \hookrightarrow W$ is an acyclic cofibration of complete complexes, then it is part of a similar deformation retract.*

Proof. We will only prove assertiona about p. By Lemma 2.17, every complete complex is cofibrant. It follows from the model category axioms that p admits a section i, which decomposes $W \cong V \simeq C$ with C weakly contractible. It then suffices to provide a contracting homotopy h on C such that $h^2 = 0$. Let $j\colon C \longrightarrow \mathrm{Cone}(C)$ be the inclusion of C into its cone. Since C is cofibrant and acyclic, this is a trivial cofibration between fibrant objects; it therefore admits a retraction. Writing $\mathrm{Cone}(C) = C \oplus C[1]$, this retraction takes the form $(\mathrm{id}, h)\colon C \oplus C[1] \longrightarrow C$, where h is a contracting homotopy. One can now define $h' = -hdh$ and check that h' is the desired contracting homotopy satisfying $(h')^2 = 0$. ☐

Proposition 2.34. *Let $f\colon W \longrightarrow V$ be a weak equivalence of complete filtered complexes. Then f is a filtered homotopy equivalence.*

Proof. We can decompose f as $W \longrightarrow W \oplus V[0,-1] \xrightarrow{p} V$, where $V[0,-1]$ is the (contractible) path space of V. The first map is the obvious summand inclusion (hence a homotopy equivalence) and p is given on W by f, while on $V[0,-1]$ it is determined uniquely by the fact that on $V[0]$ it is the identity. Note that p is both surjective in every filtration degree and a filtered quasi-isomorphism (since f was). Consequently, p is part of a deformation retract by Lemma 2.33, so that the composite f is a homotopy equivalence as well. ☐

Theorem 2.35 (Homotopy Transfer Theorem)**.** *Let $\mathcal{C}$ be a complete conilpotent operad. Suppose W is a complete $\Omega\mathcal{C}$-algebra that homotopy retracts (as a filtered complex) to a complete complex V, as above. Then there is a* transferred *$\Omega\mathcal{C}$-algebra structure on V such that i extends to an ∞-morphism of $\Omega\mathcal{C}$-algebras.*

Proof. The proof is identical to the one in [LV12, Section 10.3]. In loc. cit. Loday and Vallette consider the case where $\mathcal{P}$ is a Koszul operad (and $\mathcal{C} = \mathcal{P}^{\text{i}}$) for simplicity, but the proof carries through. Indeed, the transferred structure is constructed by establishing a universal map $\mathrm{Bar}\,\mathrm{End}_W \to \mathrm{Bar}\,\mathrm{End}_V$, obtained by composing incoming edges with i, outgoing edges with p and adding a copy of h to every internal edge. Since we require our homotopy retract to be made up of filtered maps, the map $\mathrm{Bar}\,\mathrm{End}_W \to \mathrm{Bar}\,\mathrm{End}_V$ is compatible with the filtrations. The extension of i to an ∞-morphism i_∞ involve similar formulas and is therefore compatible with the filtrations. ☐

2.4.1 Minimal models

Definition 2.36. A complete complex V is said to be *minimal* if for all $n \in \mathbb{Z}$, $dF^nV \subseteq F^{n+1}V$. In other words, if the differential vanishes on the associated graded.

Proposition 2.37. *Every complete filtered complex (over a field k) admits a deformation retract to a minimal complete filtered complex, with side conditions $ph = 0$, $hi = 0$ and $h^2 = 0$. This minimal complete complex is unique up to non-canonical isomorphism.*

Lemma 2.38. *Consider a diagram of complete complexes*

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r, dotted] \arrow[d, hook, "\sim"'] & M \arrow[r, dotted] \arrow[d, dotted, "\sim"] & M'' \arrow[r] \arrow[d, hook, "\sim"] & 0 \\ 0 \arrow[r] & V' \arrow[r] & V \arrow[r] & V'' \arrow[r] & 0 \end{tikzcd}$$

in which the bottom row is short exact and the vertical maps are acyclic cofibrations. Then there exists a complete module M and a dotted extension of the diagram as indicated, such that the top row is exact and the map $M \longrightarrow V$ is an acyclic cofibration as well.

Proof. Note that a map $W' \longrightarrow W$ is an acyclic cofibration if and only if it is of the form $W \longrightarrow W \oplus C$, where C is acyclic (since it admits a retraction for model-categorical reasons). Let $N = V \times_{V''} M''$, which fits into a short exact sequence $0 \to V' \to N \to M'' \to 0$. The map $N \longrightarrow V$ is the pullback of an acyclic cofibration along a fibration, and is hence easily seen to be an acyclic cofibration as well (using the above observation).

We can write $V' \cong M' \oplus C$, for some contractible complex. Then the inclusion $C \hookrightarrow V' \hookrightarrow N$ is a cofibration whose domain is contractible. This implies that $N \longrightarrow N/C$ is an acyclic fibration and hence a deformation retract by Lemma 2.33. In particular, the short exact sequence $0 \to C \to N \to N/C \to 0$ splits and we can identify $C \longrightarrow N$ with a summand inclusion $C \subseteq N = C \oplus M$. We therefore obtain a commuting diagram

$$\begin{tikzcd} M \arrow[r, hook, "\sim"] \arrow[dr, hook, "\sim"'] & N \arrow[r] \arrow[d, hook, "\sim"] & M'' \arrow[d, hook, "\sim"] \\ & V \arrow[r] & V'' \end{tikzcd}$$

where all downwards pointing arrows are acyclic cofibrations. Since the projection $N \longrightarrow M''$ sent $C \subseteq V' = M' \oplus C$ to zero, we see that the map $M \longrightarrow M''$ is surjective in each filtration degree, with kernel given precisely by M'. We therefore obtain the desired short exact sequence $M' \to M \to M''$ mapping to $V' \to V \to V''$ by acyclic cofibrations. $\square$

Proof of Proposition 2.37. To see uniqueness, suppose that V and W are weakly equivalent minimal complete complexes. By Proposition 2.34 every weak equivalence of complete complexes has a homotopy inverse, so we may assume that there exists a weak equivalence $f\colon V \to W$, as opposed to a zig-zag of weak equivalences. The induced map $\mathrm{Gr}(f)\colon \mathrm{Gr}(V) \to \mathrm{Gr}(W)$ is a quasi-isomorphism of complexes with trivial differential, hence an isomorphism. It follows that f is itself an isomorphism.

We will prove existence of minimal models in two steps, first dealing with the positive part of the filtration and then with the negative part.

Positive filtration. Assume that $V = F^0V$. We will inductively construct a tower of acyclic cofibrations $M^q \xrightarrow{\sim} V/F^qV$ (where V/F^qV has the induced filtration), where each M^q is a minimal complete complete and each $M^{q+1} \twoheadrightarrow M^q$ is a quotient map.

For $q = 0$, one simply sets $M^0 = 0$. For the inductive step, suppose we have already constructed $i\colon M^q \xrightarrow{\sim} V/F^qV$. The complex $\mathrm{Gr}^q(V)$ can then be written as $F^qV/F^{q+1}V \cong$

$H(F^qV/F^{q+1}V) \oplus C$, where C is an acyclic complex and $H(F^qV/F^{q+1}V)$ is the homology. Let us now consider the following diagram, in which the rows are exact

$$\begin{tikzcd} 0 \arrow[r] & H(F^qV/F^{q+1}V) \arrow[r, dotted] \arrow[d, hook, "\sim"'] & M^{q+1} \arrow[r, dotted] \arrow[d, dotted, "\sim"] & M^q \arrow[r] \arrow[d, "i"] & 0 \\ 0 \arrow[r] & F^qV/F^{q+1}V \arrow[r] & V/F^{q+1}V \arrow[r] & V/F^qV \arrow[r] & 0. \end{tikzcd}$$

By Lemma 2.38, there exists a complete complex M^{q+1} making the top row exact, together with an acyclic cofibration $M^{q+1} \xrightarrow{\sim} V/F^{q+1}V$. Since the top row is short exact, it induces a short exact sequence on the associated graded (Lemma 2.5). Since $\mathrm{Gr}(M^q)$ is concentrated in weight $< q$ and $\mathrm{Gr}(H(F^qV/F^{q+1}V))$ is concentrated in weight q, we see that the differential on $\mathrm{Gr}(M^{q+1})$ vanishes, so that M^{q+1} is minimal.

Now, taking the limit of the tower of M^q provides a an acyclic cofibration $M = \lim_q M^q \longrightarrow \lim_q V/F^qV = V$ by the completeness of V. Since $\mathrm{Gr}(M) \longrightarrow \mathrm{Gr}(M^q)$ is an isomorphism in weight $\leq q$, we see that M is minimal.

Negative filtration. Let us now consider the general case where V need not agree with F^0V. We are going to inductively construct a compatible family of acyclic cofibrations $M^{(p)} \longrightarrow F^pV$, for $p \leq 0$, where each $M^{(p)}$ is minimal and $M^{(p)} \longrightarrow M^{(p-1)}$ is an isomorphism in filtration degree p.

For F^0V, we have constructed an acyclic cofibration $M^{(0)} \xrightarrow{\sim} F^0V$ in our previous argument. Next, assume we have constructed ${}^{(p)} \xrightarrow{\sim} F^pV$. We now apply Lemma 2.38 to the short exact sequence $0 \to F^pV \to F^{p-1}V \to F^{p-1}V/F^pV \to 0$ and the minimal models $M^{(p)} \xrightarrow{\sim} F^pV$ and $H(F^{p-1}V/F^pV) \xrightarrow{\sim} F^{p-1}V/F^pV$. This produces an acyclic cofibration $M^{(p-1)} \xrightarrow{\sim} F^{p-1}V$ whose domain fits into a short exact sequence

$$0 \longrightarrow M^{(p)} \longrightarrow M^{(p-1)} \longrightarrow H(F^{p-1}V/F^pV) \longrightarrow 0.$$

Passing to the associated graded, one sees that $M^{(p-1)}$ is minimal. Furthermore $M^{(p-1)}$ agrees with $M^{(p)}$ in filtration degree p because $F^p\big(H(F^{p-1}V/F^pV)\big) = 0$.

Finally, taking the colimit as $p \to -\infty$ yields an acyclic cofibration $M = \operatorname{colim} M^{(p)} \longrightarrow V$, where M is minimal (since it agrees with the minimal complex $M^{(p)}$ in filtration weight p). Note that the acyclic cofibration $M \longrightarrow V$ admits a retraction, which is then an acyclic fibration. The desired deformation retract with side conditions then follows from Lemma 2.33. ☐

Proposition 2.39. *Let V be a $\Omega\mathcal{C}$-algebra that deformation retracts into a minimal complex M satisfying the side conditions $ph = 0$, $hi = 0$ and $h^2 = 0$. Then, the map p extends to an ∞-morphism p_∞ between the transferred $\Omega\mathcal{C}$-structure on M given by Theorem 2.35 and V.*

Proof. The proof is the same as [LV12, Proposition 10.3.14], adapted to the complete filtered case. ☐

Corollary 2.40. *Restricting to the subcategory of complete filtered algebras, ∞-weak equivalences are ∞-quasi-invertible.*

Proof. Given an ∞-weak equivalence $f\colon V \rightsquigarrow W$ one constructs an ∞-quasi-inverse by taking the composite

$$W \overset{p_\infty}{\rightsquigarrow} H(W) \overset{[f]^{-1}}{\rightsquigarrow} H(V) \overset{i_\infty}{\rightsquigarrow} V$$

where $[f]^{-1}$ is the inverse ∞-morphism as in [LV12, Theorem 10.4.2]. ☐

2.4.2 The ∞-category of algebras

By definition, the ∞-category of complete algebras over a complete operad $\mathcal{P}$ is the ∞-categorical localization of the category of $\mathcal{P}$-algebras at the filtered weak equivalences. As an application of the Homotopy Transfer Theorem, we will describe this ∞-category more explicitly in terms of ∞-morphisms.

As should be expected, the bar-cobar construction provides a cofibrant replacement functor on $\mathcal{P}$-algebras. In fact, we will need a slightly stronger version of this.

Proposition 2.41. *Let $\phi\colon \mathcal{C} \longrightarrow \mathcal{P}$ be a twisting morphism and A a $\mathcal{P}$-algebra. Then the natural map of $\mathcal{P}$-algebras $\mathcal{P}\circ A \longrightarrow \Omega_\phi \operatorname{Bar}_\phi(A)$ is a cofibration. In particular, the bar-cobar construction $\Omega_\phi \operatorname{Bar}_\phi(A)$ is a cofibrant $\mathcal{P}$-algebra.*

The standard method for checking cofibrancy of the bar-cobar construction is to endow it with a filtration, coming from the coradical filtration on the cooperad $\mathcal{C}$. Since we do not have access to the coradical filtration in the complete setting (Warning 2.13), we will give a slightly different argument, using the homotopy transfer theorem as follows:

Lemma 2.42 (∞-sections from homotopy transfer)**.** *Let $p\colon B \twoheadrightarrow A$ be an acyclic fibration between $\Omega\mathcal{C}$-algebras and $i\colon A \longrightarrow B$ a linear section. Then i extends to an ∞-morphism i_∞ such that $pi_\infty = \mathrm{id}_A\colon A \rightsquigarrow A$.*

Proof. By (the proof of) Lemma 2.33, p and i are part of a deformation retract with homotopy h satisfying the side conditions. We can therefore apply the Homotopy Transfer Theorem 3.12 to obtain *another* $\Omega\mathcal{C}$-algebra structure on A for which i can be upgraded to an ∞-morphism.

Since p was already a (strict) map of $\Omega\mathcal{C}$-algebras from the start, the formula for the transferred structure on A in terms of trees with roots labeled by p [LV12, §10.3.3] shows that this transferred structure coincides with the original $\Omega\mathcal{C}$-algebra structure on A. Furthermore, the formula for i_∞ in terms of trees with roots labeled by h [LV12, §10.3.10] shows that $pi_\infty = \mathrm{id}_A$. ∎

Proof of Proposition 2.41. It suffices to verify this when $\mathcal{P} = \Omega\mathcal{C}$ and $\phi = \iota$ is the universal twisting morphism. Indeed, $\mathcal{P}(A) \longrightarrow \Omega_\phi \operatorname{Bar}_\phi(A)$ is the image of $\Omega\mathcal{C}\circ A \longrightarrow \Omega_\iota \operatorname{Bar}_\iota(A)$ (where A is viewed as a $\Omega\mathcal{C}$-algebra) under the left Quillen functor $\mathcal{P}\circ_{\Omega\mathcal{C}}(-)\colon \mathrm{Alg}_{\Omega\mathcal{C}} \longrightarrow \mathrm{Alg}_{\mathcal{P}}$.

In the case where $\mathcal{P} = \Omega\mathcal{C}$, since trivial fibrations are preserved under pullbacks, it suffices to verify that every acyclic fibration of $\Omega\mathcal{C}$-algebras

$$\begin{tikzcd}
\Omega\mathcal{C}\circ A \arrow[r, "s"] \arrow[d] & B \arrow[d, two heads, "p", "\sim"'] \\
\Omega_\iota \operatorname{Bar}_\iota(A) \arrow[r, equal] \arrow[ur, dotted] & \Omega_\iota \operatorname{Bar}_\iota(A)
\end{tikzcd}$$

there exists a dotted section as indicated. Since maps $\Omega_\iota \operatorname{Bar}_\iota(A) \longrightarrow B$ correspond bijectively to ∞-morphisms $A \rightsquigarrow B$, this is equivalent to finding an ∞-morphism $s_\infty\colon A \rightsquigarrow B$ whose linear part agrees with the map s, such that the composition $A \rightsquigarrow B \longrightarrow \Omega_\iota \operatorname{Bar}_\iota(A)$ agrees with the universal ∞-morphism v_∞ (adjoint to the identity on $\Omega_\iota \operatorname{Bar}_\iota(A)$).

Let us first observe that the linear map $A \longrightarrow \Omega_\iota \operatorname{Bar}_\iota(A)$ underlying v_∞ is the inclusion of a summand (induced by the inclusion $k \longrightarrow \Omega\mathcal{C}\circ_\iota \mathcal{C}_+$). In particular, it is a cofibration, so that we can find a linear section $i\colon \Omega_\iota \operatorname{Bar}_\iota(A) \longrightarrow B$ extending s. Lemma 2.42 shows that i can be extended to an ∞-morphism i_∞ such that $pi_\infty = \mathrm{id}_A$. Now the composite $s_\infty = i_\infty v_\infty$ provides the desired extension of s. ∎

Definition 2.43. Let $\phi\colon \mathcal{C} \longrightarrow \mathcal{P}$ be a Koszul twisting morphism. We denote by $\mathbf{Alg}^{\mathrm{cpl}}_{\mathcal{P}}$ the simplicially enriched category where:

(0) objects are complete $\mathcal{P}$-algebras.

(1) for any two objects A_0 and A_1, the simplicial set of morphisms between them is given in simplicial degree n by the set of ∞-morphisms

$$\mathrm{Map}_{\mathcal{P}}(A_0, A_1)_n = \big\{A_0 \rightsquigarrow A_1 \hat{\otimes} \Omega[\Delta^n]\big\}.$$

The composition of maps is given by

$$A_0 \overset{\phi}{\rightsquigarrow} A_1 \hat{\otimes} \Omega[\Delta^n] \overset{\psi\otimes\mathrm{id}}{\rightsquigarrow} A_2 \hat{\otimes} \Omega[\Delta^n] \hat{\otimes} \Omega[\Delta^n] \xrightarrow{\mathrm{id}\otimes\mu} A_2 \hat{\otimes} \Omega[\Delta^n]$$

where the last map arises from the multiplication on $\Omega[\Delta^n]$.

Note that $\mathbf{Alg}^{\mathrm{cpl}}_{\mathcal{P}}$ depends (implicitly) on a choice of Koszul twisting morphism.

Lemma 2.44. *For any two objects A_0, A_1, the simplicial set $\mathrm{Map}_{\mathcal{P}}(A_0, A_1)$ of ∞-morphisms is a Kan complex. Furthermore, every (strict) weak equivalence $f\colon A_1 \longrightarrow A_2$ induces a homotopy equivalence $f_*\colon \mathrm{Map}_{\mathcal{P}}(A_0, A_1) \longrightarrow \mathrm{Map}_{\mathcal{P}}(A_0, A_2)$.*

Proof. One can identify $\mathrm{Map}_{\mathcal{P}}(A_0, A_1)$ with the simplicial set of strict maps of $\mathcal{P}$-algebras $\Omega_\phi \mathrm{Bar}_\phi(A_0) \longrightarrow A_1 \hat{\otimes} \Omega[\Delta^\bullet]$. The result then follows formally from $\Omega_\phi \mathrm{Bar}_\phi(A_0)$ being cofibrant (Proposition 2.41) and $A_1 \hat{\otimes} \Omega[\Delta^\bullet]$ being a fibrant simplicial resolution of A_1 [Hin97]. □

Proposition 2.45. *Let $\phi\colon \mathcal{C} \longrightarrow \mathcal{P}$ be a Koszul twisting morphism and consider the functor from the model category of $\mathcal{P}$-algebras to the simplicial category of $\mathcal{P}$-algebras*

$$j\colon \mathrm{Alg}^{\mathrm{cpl}}_{\mathcal{P}} \longrightarrow \mathbf{Alg}^{\mathrm{cpl}}_{\mathcal{P}}.$$

This exhibits $\mathbf{Alg}^{\mathrm{cpl}}_{\mathcal{P}}$ as the ∞-categorical localization of the category $\mathrm{Alg}^{\mathrm{cpl}}_{\mathcal{P}}$ at the filtered quasi-isomorphisms.

Proof. By Lemma 2.44, j sends filtered quasi-isomorphisms to homotopy equivalences, so it induces an essentially surjective functor $j\colon \mathrm{Alg}^{\mathrm{cpl}}_{\Omega\mathcal{C}}[\mathrm{w.e.}^{-1}] \longrightarrow \mathbf{Alg}^{\mathrm{cpl}}_{\Omega\mathcal{C}}$. This functor is fully faithful because the mapping spaces in $\mathbf{Alg}^{\mathrm{cpl}}_{\Omega\mathcal{C}}$ compute the derived mapping spaces in the model category of complete $\Omega\mathcal{C}$-algebras (by the proof of Lemma 2.44). □

2.5 Main example: curved L_∞-algebras

Recall [HM12, Section 3.2.2] that a *curved Lie algebra* is a complete (non-differential) graded vector space $\mathfrak{g}$, together with a Lie bracket $[-,-]$, a Lie algebra derivation ∇ of cohomological degree 1 and a degree 2 element $\omega \in F^1(\mathfrak{g})$ such that

$$\nabla(\omega) = 0 \qquad \text{and} \qquad \nabla^2(x) + [\omega, x] = 0.$$

(This notion is also frequently appears with the different convention $\nabla^2(x) = [\omega, x]$, e.g. in [Mau17].) This is a particular example of a curved L_∞-algebra [Fuk03, KS06, Get18], where all operations in arity ≥ 3 vanish:

Definition 2.46. A *(classical) curved L_∞-algebra* (over the field k) is a complete graded vector space $\mathfrak{g}$, endowed with operations

$$\ell_i\colon \mathrm{Sym}^i_k(\mathfrak{g}[1]) \longrightarrow \mathfrak{g}[1]$$

such that $\ell_0 \in F^1\mathfrak{g}$ and the following equations hold:

$$\sum_{\substack{p+q=n+1\\ q\geq 0,\ p\geq 1}} \sum_{\sigma\in \mathrm{Sh}^{-1}_{p-1,q}} \mathrm{sgn}(\sigma)(-1)^{(p-1)q}(\ell_p \circ_1 \ell_q)^\sigma = 0.$$

Equivalently, this is the data of a codifferential on the cofree complete coalgebra $\mathrm{Sym}^c_k(\mathfrak{g}[1])$, sending the element 1 to an element of filtration weight 1.

The purpose of this section is to describe the homotopy theory of curved L_∞-algebras from the perspective of the above operadic framework.

2.5.1 The Koszul morphism $\mathsf{ucoCom}\{1\} \to \mathsf{cLie}$

Let ucoCom denote the linear dual of the unital commutative operad, $\mathsf{ucoCom}(n) = k\mu_n$, for $n \geq 0$. This comes equipped with partial cocomposition maps (dual to the partial composition of the unital commutative operad). Note that the partial composition maps do *not* determine a total cocomposition map $\mathsf{ucoCom} \longrightarrow \mathsf{ucoCom} \circ \mathsf{ucoCom}$. We can endow ucoCom with two filtrations:

(a) the ('classical') filtration $\mathsf{ucoCom}^{\mathrm{cl}}$, where $F^0\mathsf{ucoCom}^{\mathrm{cl}} = \mathsf{ucoCom}$, $F^1\mathsf{ucoCom}^{\mathrm{cl}} = k\mu_0$ and $F^2\mathsf{ucoCom}^{\mathrm{cl}} = 0$.

(b) the ('mixed') filtration $\mathsf{ucoCom}^{\mathrm{mix}}$ where $F^0\mathsf{ucoCom}^{\mathrm{mix}} = \mathsf{ucoCom}$, $F^1\mathsf{ucoCom}^{\mathrm{mix}} = k\mu_0 \oplus k\mu_1$ and $F^2\mathsf{ucoCom}^{\mathrm{mix}} = 0$.

With these filtrations, $\mathsf{ucoCom}^{\mathrm{cl}}$ becomes a complete cooperad and $\mathsf{ucoCom}^{\mathrm{mix}}$ becomes a conilpotent cooperad in the complete sense, as in Definition 2.12.

Remark 2.47. Note that $\mathsf{ucoCom}^{\mathrm{cl}}$ is a counital cooperad, but (despite the name) $\mathsf{ucoCom}^{\mathrm{mix}}$ is not since the "counit" $\mathsf{ucoCom}(1) \to k$ is not compatible with the filtration.

Definition 2.48. The *mixed-curved L_∞-operad* is the complete operad

$$\mathsf{cLie}_\infty \coloneqq \Omega(\mathsf{ucoCom}^{\mathrm{mix}}\{1\}).$$

We will refer to algebras over cLie_∞ as *mixed-curved L_∞-algebras* (in contrast to the (classical) curved L_∞-algebras from Definition 2.46) and write $\mathrm{cLie}^{\mathrm{mix}}$ for the category of mixed-curved L_∞-algebras.

Unraveling the definition, as a graded operad, cLie_∞ is freely generated by an infinite collection of operations $(\ell_n)_{n\geq 0}$, where ℓ_n has arity n and degree $2-n$. The filtration is given by

$$F^p\mathsf{cLie}_\infty = \mathrm{span}\big\{\text{words containing at least } p \text{ times } l_0 \text{ or } l_1\big\}$$

and the differential reads

$$-\partial(\ell_n) = \sum_{\substack{p+q=n+1\\ q\geq 0,\ p\geq 1}} \sum_{\sigma\in \mathrm{Sh}^{-1}_{p-1,q}} \mathrm{sgn}(\sigma)(-1)^{(p-1)q}(\ell_p \circ_1 \ell_q)^\sigma \tag{2.49}$$

where we use the convention $\partial(f) = d\circ f - (-1)^{|f|} f \circ d$. Let us make the differential more explicit in low arity:

$(n = 0)$ $\quad -\partial(\ell_0) = \ell_1 \circ_1 \ell_0$

$(n = 1)$ $\quad -\partial(\ell_1) = \ell_1 \circ_1 \ell_1 + \ell_2 \circ_1 \ell_0$

$(n = 2)$ $\quad -\partial(\ell_2) = \ell_1 \circ_1 \ell_2 - \ell_2 \circ_1 \ell_1 + (\ell_2 \circ_1 \ell_1)^{(12)} + \ell_3 \circ_1 \ell_0.$

Proposition/definition 2.50. The filtered operad of *mixed-curved Lie algebras* is the operad cLie obtained from cLie_∞ by quotienting out the operadic dg-ideal generated by $\ell_3, \ell_4, \ldots$. The quotient map $\mathsf{cLie}_\infty \to \mathsf{cLie}$ is a weak equivalence.

Proof. On the associated graded of cLie_∞ let us consider a filtration by the number of times an element ℓ_0 appears. Taking a spectral sequence with respect to this filtration, on the 0-th page we observe that the differential acts essentially like the differential in the ordinary operad Lie_∞. Indeed, one can declare two ℓ_n-labelled trees in cLie_∞ to have the same skeleton if they have the same number of ℓ_0's and ℓ_1's, appearing in the same position, see Figure 1.

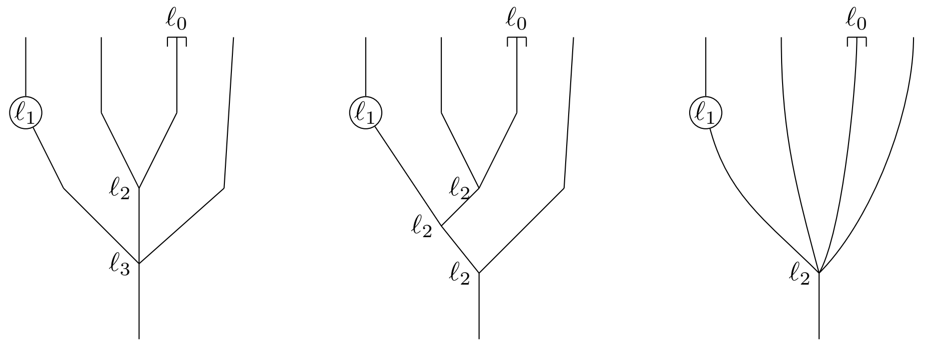

Figure 1: Three trees in cLie_∞ with the same skeleton.

It follows that map $\mathsf{cLie}_\infty \to \mathsf{cLie}$ on the 0-th page decomposes as a sum of maps

$$\bigoplus_{\text{skeleton types}} \mathsf{Lie}_\infty \to \bigoplus_{\text{skeleton types}} \mathsf{Lie},$$

which is a quasi-isomorphism. Since the (second) filtration is bounded above and exhaustive, it follows that the map $\mathrm{Gr}\,\mathsf{cLie}_\infty \to \mathrm{Gr}\,\mathsf{cLie}$ is also a quasi-isomorphism, i.e. $\mathsf{cLie}_\infty \to \mathsf{cLie}$ is a weak equivalence. ☐

Corollary 2.51. *The map* $\mathsf{ucoCom}^{\mathrm{mix}}\{1\} \to \mathsf{cLie}$ *mapping* μ_n *to* ℓ_n *for* $n = 0, 1, 2$ *is a Koszul twisting morphism.*

Corollary 2.52. *Every mixed-curved* L_∞*-algebra is filtered quasi-isomorphic to a mixed-curved Lie algebra.*

Notice that unlike Lie, the operad cLie has a differential and the complete complex underlying cLie is not minimal, since on the associated graded $\partial(\ell_1) = -\ell_2 \circ_1 \ell_0 \neq 0$. A cLie algebra just a complete cochain complex (V, d) equipped with operations ℓ_0, ℓ_1 increasing the filtration by 1 and ℓ_2 such that $(V, \omega = \ell_0, \nabla = \ell_1 + d, \ell_2)$ is a curved Lie algebra in the usual sense, as described above Definition 2.46.

Remark 2.53. One can apply a similar analysis starting instead from the cooperad $\mathsf{ucoCom}^{\mathrm{cl}}$. In this case one obtains an operad $\Omega(\mathsf{ucoCom}^{\mathrm{cl}})$ with the same generating operations and differential (2.49), but where only ℓ_0 is of filtration weight 1 and ℓ_1 is of weight 0. In particular, a curved L_∞-algebra in the classical sense of Definition 2.46 is simply a $\Omega(\mathsf{ucoCom}^{\mathrm{cl}})$-algebra structure on a complete graded vector space (with no differential). Equivalently, this is the data of a codifferential on the cosymmetric coalgebra $\mathsf{ucoCom}^{\mathrm{cl}} \circ (\mathfrak{g}[1])$ (where we consider $\mathsf{ucoCom}^{\mathrm{cl}}$ as a counital cooperad).

The complete operad $\Omega(\mathsf{ucoCom}^{\mathrm{cl}})$ is filtered acyclic, since $\mathsf{ucoCom}^{\mathrm{cl}}$ is counital. Likewise, the underlying unfiltered versions of cLie and cLie_∞ (either before or after completing with respect to the filtrations above) can be shown to be quasi-isomorphic to the unit operad. This would make the naïve homotopy theory of their algebras quite trivial.

2.5.2 Morphisms of mixed-curved L_∞-algebras.

Let us fix $\mathsf{ucoCom}^{\mathrm{mix}}\{1\} \longrightarrow \mathsf{cLie}_\infty$, $\mu_n \mapsto \ell_n$ the universal twisting morphism. From Definition 2.31 we have that an ∞-morphism $\phi\colon \mathfrak{g} \rightsquigarrow \mathfrak{h}$ between cLie_∞-algebras is determined by a map $\mathsf{ucoCom}^{\mathrm{mix}}\{1\}_+ \circ \mathfrak{g} \to \mathfrak{h}$. Notice that at the level of the underlying vector spaces we have $\mathsf{ucoCom}^{\mathrm{mix}}\{1\}_+ \circ \mathfrak{g} = \mathrm{Sym}^c(\mathfrak{g}[1])[-1] \oplus \mathfrak{g}$. Given this, ϕ is determined by maps

$$\phi_{\mathrm{lin}}\colon \mathfrak{g} \longrightarrow \mathfrak{h} \qquad\qquad \phi_n\colon \mathrm{Sym}^n(\mathfrak{g}[1])[-1] \longrightarrow \mathfrak{h} \qquad n \geq 0$$

with ϕ_n maps of cohomological degree $1-n$ and where ϕ_0 and ϕ_1 increase the filtration weight by 1. The map ϕ_{lin} is required to be a chain map and

$$\begin{aligned}\partial(\phi'_n) = &\sum_{\substack{p+q=n+1\\ q\geq 0,\ p\geq 1}} \sum_{\sigma\in \mathrm{Sh}^{-1}_{p-1,q}} \mathrm{sgn}(\sigma)(-1)^{(p-1)q}(\phi'_p \circ_1 \ell^{\mathfrak{g}}_q)^\sigma - \\ &- \sum_{\substack{k\geq 0\\ i_1+\cdots+i_k=n}} \sum_{\sigma\in \mathrm{Sh}^{-1}_{(i_1,\dots,i_k)}} \mathrm{sgn}(\sigma)\frac{(-1)^\epsilon}{k!}\ell^{\mathfrak{h}}_k(\phi'_{i_1},\dots,\phi'_{i_k})^\sigma, \end{aligned} \tag{2.54}$$

where $\phi'_n = \phi_n$ if $n \neq 1$, $\phi'_1 = \phi_1 + \phi_{\mathrm{lin}}$ and $\epsilon = \prod_{j=1}^{k-1}(k-j)(i_k-1)$. Notice that the first sum is finite (like the infinitesimal cocomposition of ucoCom) whereas the second sum is infinite (since it corresponds to the total cocomposition of ucoCom). For example, in the first term of an ∞-morphism $\phi\colon \mathfrak{g} \rightsquigarrow \mathfrak{h}$ we have

$$d_{\mathfrak{h}}(\phi_0) = \phi_{\mathrm{lin}}(\ell^{\mathfrak{g}}_0) + \phi_1(\ell^{\mathfrak{g}}_0) - \left(\ell^{\mathfrak{h}}_0 + \ell^{\mathfrak{h}}_1(\phi_0) + \frac{1}{2!}\ell^{\mathfrak{h}}_2(\phi_0,\phi_0) + \frac{1}{3!}\ell^{\mathfrak{h}}_3(\phi_0,\phi_0,\phi_0)\dots\right).$$

In particular, an ∞-morphism $0 \rightsquigarrow \mathfrak{h}$ into an uncurved Lie_∞-algebra is equivalent to a choice of a Maurer–Cartan element in $F^1\mathfrak{h}$.

Remark 2.55. Likewise, applying Definition 2.31 in the setting of $\Omega(\mathsf{ucoCom}^{\mathrm{cl}})$, one sees that an ∞-morphism between $\Omega(\mathsf{ucoCom}^{\mathrm{cl}})$-algebras is given by maps $\phi_{\mathrm{lin}}, \phi_i$ satisfying Equation 5.16, except that ϕ_1 is of filtration weight 0 instead of 1.

Now let $\mathfrak{g}$ and $\mathfrak{h}$ be two (classical) curved L_∞-algebras in the sense of Definition 2.46, corresponding to $\Omega(\mathsf{ucoCom}^{\mathrm{cl}})$-algebras with zero differential. Then an *∞-morphism of curved L_∞-algebras* $\mathfrak{g} \rightsquigarrow \mathfrak{h}$ is defined to be an ∞-morphism between the corresponding $\Omega(\mathsf{ucoCom}^{\mathrm{cl}})$-algebras such that $\phi_{\mathrm{lin}} = 0$. In other words, it is given by a collection of maps $\phi'_n\colon \mathrm{Sym}^n(\mathfrak{g}[1])[-1] \longrightarrow \mathfrak{h}$ satisfying Equation (2.54), where the left hand side is zero, because there is no differential. It is not hard to verify that this is equivalent to the data of a map of dg-coalgebras $\mathrm{Sym}^c(\mathfrak{g}[1]) \longrightarrow \mathrm{Sym}^c(\mathfrak{h}[1])$ (see Definition 2.46).

Example 2.56. A *curved* map between two cLie-algebras is an ∞-morphism $\phi\colon (\mathfrak{g},\omega_{\mathfrak{g}}) \rightsquigarrow (\mathfrak{h},\omega_{\mathfrak{h}})$ (ω denotes the curvature) such that $\phi_{\geq 2} = 0$. In this case, Equation 5.16 reduces to the more familiar notion of a map of curved Lie algebras that we find for example in [Mau17, Def 4.3] (up to signs): concretely, taking the linear term $\phi'_1 = \phi_{\mathrm{lin}} + \phi_1$ as before and the curved differential $d' = d + \ell_1$, one finds

$$\begin{aligned}\phi'_1([X,Y]) &= [\phi'_1(X), \phi'_1(Y)],\\ \phi'_1({}'X) &= d'\phi'_1(X) + [\phi_0, \phi'_1(X)],\\ \omega_{\mathfrak{h}} &= \phi'_1(\omega_{\mathfrak{g}}) + d\phi_0 + \frac{1}{2}[\phi_0,\phi_0].\end{aligned}$$

Remark 2.57 (General remark concerning signs)**.** Despite the large quantity of signs, most of them come from the degree shift in the space of generators, $\mathsf{ucoCom}\{1\}$. For instance, for the shifted operad $\mathsf{cLie}_\infty\{-1\} = \Omega(\mathsf{coCom})$, the equation for ∞-morphisms (2.54) takes the form

$$\partial(\phi_n) = \sum_{\substack{p+q=n+1 \\ q\geq 0,\ p\geq 1}} (\phi'_p \circ_1 \ell^{\mathfrak{g}}_q)^\sigma - \sum_{\substack{k\geq 0 \\ i_1+\cdots+i_k=n}} \sum_{\sigma\in \mathrm{Sh}^{-1}_{(i_1,\dots,i_k)}} \frac{1}{k!} \ell^{\mathfrak{h}}_k(\phi'_{i_1},\dots,\phi'_{i_k})^\sigma.$$

In practice, this allows us to abuse the notation later on, by replacing signs with $\pm$, since most of the signs are encompassed by this degree shift.

2.5.3 From mixed-curved L_∞-algebras to curved L_∞-algebras

One can think of mixed-curved L_∞-algebras as overdetermined versions of curved L_∞-algebras in the sense of Definition 2.46: indeed, they come with two derivations d, ℓ_1 (the second of filtration weight 1) instead of a single ℓ_1 of weight 0. Likewise, ∞-morphisms between cLie_∞-algebras come equipped with two linear components, one being of filtration weight 1. The classical homotopy theory of curved L_∞-algebras is usually formulated in terms of ∞-morphisms with a single linear part (cf. Remark 2.55).

Definition 2.58. The *∞-category* **cLie** *of curved L_∞-algebras* is defined to be following simplicially enriched category:

(0) objects are curved L_∞-algebras in the sense of Definition 2.46; equivalently, complete algebras over $\Omega(\mathsf{ucoCom}^{\mathrm{cl}})$ with zero differential (Remark 2.53).

(1) the simplicial sets of maps $\mathrm{Map}(\mathfrak{g},\mathfrak{h})$ are given in degree n by the set of ∞-morphisms $\mathfrak{g} \rightsquigarrow \mathfrak{h} \,\hat{\otimes}\, \Omega[\Delta^n]$ in the sense of Remark 2.55.

To study the properties of the ∞-category of curved L_∞-algebras, we will relate it to the ∞-category $\mathbf{cLie}^{\mathrm{mix}} := \mathbf{Alg}^{\mathrm{cpl}}_{\mathsf{cLie}_\infty}$ of mixed-curved L_∞-algebras, i.e. algebras over the operad cLie_∞ (Definition 2.48). Note that this latter ∞-category has very good abstract properties, since it arises from a combinatorial model category: for example, it has all limits and colimits. More precisely, observe that there is a functor of simplicially enriched categories

$$\mathrm{blend}\colon \mathbf{cLie}^{\mathrm{mix}} = \mathbf{Alg}^{\mathrm{cpl}}_{\mathsf{cLie}_\infty} \longrightarrow \mathbf{cLie} \tag{2.59}$$

sending a mixed-curved L_∞-algebra $(\mathfrak{g}, d, \ell_i)$ to the (classical) curved L_∞-algebra $(\mathfrak{g}, \ell'_i)$ where $\ell'_i = \ell_i$ for $i \neq 1$ and $\ell'_1 = d + \ell_1$. On ∞-morphisms, the functor sends $(\phi_{\mathrm{lin}}, \phi_i)\colon \mathfrak{g} \rightsquigarrow \mathfrak{h}$ to $(\phi'_n)\colon \mathrm{blend}(\mathfrak{g}) \longrightarrow \mathrm{blend}(\mathfrak{h})$ where $\phi'_n = \phi_n$ for $n \neq 1$ and $\phi'_1 = \phi_{\mathrm{lin}} + \phi_1$, as in Equation 5.16.

This forgets the redundancies is the definition of a curved L_∞-algebra by combining the differential d and its perturbation ℓ_1, resp. the linear map ϕ_{lin} and its perturbation ϕ_1. Alternatively, one can also get rid of redundancies by imposing further restrictions on d:

Definition 2.60. A *graded mixed-curved L_∞-algebra* is a graded complex $\mathfrak{g}$, together with the structure of a mixed-curved L_∞-algebra on $\mathrm{Tot}(\mathfrak{g})$.

There is a weight-graded operad $\mathsf{cLie}^{\mathrm{tot}}_\infty$ whose algebras are precisely the graded mixed-curved L_∞-algebras. Indeed, let $\mathsf{ucoCom}^{\mathrm{tot}}$ be the weight-graded cooperad spanned by operations n-ary operations μ^r_n of weight r, with $r \geq 0$ for $n \geq 2$ and $r \geq 1$ for $n = 0, 1$. The cocomposition is that of the cocommutative operad

$$\Delta(\mu^r_n) = \sum \mu^p_m \circ \left(\mu^{q_1}_{n_1}, \dots, \mu^{q_m}_{n_m}\right)$$

where the sum runs over all indices such that $n_1 + \cdots + n_m = n$ and $p + q_1 + \cdots + q_m = r$. Note that this sum is finite because $n_i \in \{0,1\}$ implies that $q_i \geq 1$, and that this defines a conilpotent cooperad. The desired operad is then given by

$$\mathsf{cLie}_\infty^{\mathrm{tot}} = \Omega(\mathsf{ucoCom}^{\mathrm{tot}}).$$

Here we use that the bar-cobar formalism (as in Section 2.3) works equally well in the weight-graded setting. Unraveling the definitions, an algebra over this operad comes equipped with operations

$$\ell_i^p \colon \mathrm{Sym}^i\big(\mathfrak{g}[1]\big) \longrightarrow \mathfrak{g}\langle p\rangle[1]$$

with $i \geq 2$ and $p \geq 0$ or $i = 0, 1$ and $p \geq 1$, such that $\sum_p \ell_i^p$ makes $\mathrm{Tot}(\mathfrak{g})$ a cLie_∞-algebra.

It follows that the category of graded mixed-curved L_∞-algebras carries a model structure and there is a notion of ∞-morphism between graded mixed-curved L_∞-algebras. Explicitly, an ∞-morphism $\mathfrak{g} \rightsquigarrow \mathfrak{h}$ is given by maps $\phi_{\mathrm{lin}} \colon \mathfrak{g} \longrightarrow \mathfrak{h}$ and

$$\phi_0^p \colon k \longrightarrow \mathfrak{h}\langle p\rangle \qquad \phi_1^p \colon \mathfrak{g} \longrightarrow \mathfrak{h}\langle p\rangle \qquad \phi_n^q \colon \mathrm{Sym}^n\big(\mathfrak{g}[1]\big)[-1] \longrightarrow \mathfrak{h}\langle q\rangle$$

for $p \geq 1$, $n \geq 2$ and $q \geq 0$. These maps have the property that ϕ_{lin} is a chain map and that the maps between total complexes $\phi_n = \sum_p \phi_n^p$ satisfy Equation 2.54. In other words, an ∞-morphism between graded mixed-curved L_∞-algebras $\mathfrak{g} \rightsquigarrow \mathfrak{h}$ is simply an ∞-morphism of mixed-curved L_∞-algebras $\mathrm{Tot}(\mathfrak{g}) \rightsquigarrow \mathrm{Tot}(\mathfrak{h})$ whose linear part respects the grading.

Definition 2.61. Let us denote by $\mathbf{cLie}^{\mathrm{mix}}$ and $\mathbf{cLie}^{\mathrm{gr-mix}}$ (the ∞-categories associated to) the simplicially enriched categories of (graded-)mixed-curved L_∞-algebras and ∞-morphisms, as in Definition 2.43.

The above discussion shows that there is a sequence of ∞-categories

$$\mathbf{cLie}^{\mathrm{gr-mix}} \xrightarrow{\mathrm{Tot}} \mathbf{cLie}^{\mathrm{mix}} \xrightarrow{\mathrm{blend}} \mathbf{cLie}.$$

Proposition 2.62. *The composite functor of ∞-categories is an equivalence.*

Proof. To see that $\mathrm{blend} \circ \mathrm{Tot}$ is essentially surjective, pick a curved L_∞-algebra $\big(\mathfrak{g}', \ell_i'\big)$. We can split the filtration on the complete graded vector space underlying $\mathfrak{g}$, i.e. we can write $\mathfrak{g}' = \mathrm{Tot}(\mathfrak{g})$ for some weight-graded (non-differential) graded vector space $\mathfrak{g}$. Using this splitting, the operations ℓ_i' can be decomposed by pure weight: we can write

$$\ell_0' = \ell_0^1 + \ell_0^2 + \ldots \qquad \ell_1' = d + \ell_1^1 + \ldots \qquad \ell_n' = \ell_n^0 + \ell_n^1 + \ldots \qquad (n \geq 2)$$

where each $\ell_n^r \colon \mathfrak{g}^{\otimes n} \longrightarrow \mathfrak{g}\langle r\rangle$ increases the weight by (exactly) r. We let d denote the part of ℓ_1 of pure weight 0, and since ℓ_0 was required to be of filtration weight 1, there is no ℓ_0^0.

Note that $\ell_1' \circ \ell_1' + \ell_2' \circ_1 \ell_0$ implies that $\ell_1' \circ \ell_1'$ is of filtration degree 1; this implies that $d^2 = 0$. This means that $\big(\mathfrak{g}', d, \ell_0', \ell_1' - d, \ell_2', \ldots\big)$ is a mixed-curved L_∞-algebra. By definition, this implies that $\big(\mathfrak{g}, d, \ell_n^i\big)$ is a mixed-curved L_∞-algebra, which maps to $\mathfrak{g}'$ under the functor $\mathrm{blend} \circ \mathrm{Tot}$.

Concerning full faithfulness, pick two graded mixed-curved L_∞-algebras $\mathfrak{g}, \mathfrak{h}$. The set of *graded mixed* ∞-morphisms $\mathfrak{g} \rightsquigarrow \mathfrak{h}$ coincides with the set of those *mixed* ∞-morphisms $\phi \colon \mathrm{Tot}(\mathfrak{g}) \rightsquigarrow \mathrm{Tot}(\mathfrak{h})$ such that ϕ_{lin} is homogeneous of weight 0. The functor blend sends such an ∞-morphism to $\mathrm{blend}(\phi)$ with $\mathrm{blend}(\phi)_n = \phi_n$ for $n \neq 1$ and $\mathrm{blend}(\phi)_1 = \phi_{\mathrm{lin}} + \phi_1$. This is a bijection: indeed, for any ∞-morphism ϕ' between curved L_∞-algebras, we can always decompose ϕ_1' *uniquely* into a part ϕ_{lin} which is *homogeneous* of weight 0 and a part ϕ_1 of filtration weight ≥ 1. The resulting map ϕ_{lin} is then a chain map for weight reasons.

Replacing $\mathfrak{h}$ by $\mathfrak{h} \mathbin{\hat{\otimes}} \Omega[\Delta^n]$ shows that the functor $\mathrm{blend} \circ \mathrm{Tot}$ is fully faithful. In fact, it is even (strictly) fully faithful as a functor of simplicial categories. $\square$

Proposition 2.63. *The functor* $\mathrm{Tot}\colon \mathbf{cLie}^{\mathrm{gr-mix}} \longrightarrow \mathbf{cLie}^{\mathrm{mix}}$ *is a right adjoint functor between presentable ∞-categories. Furthermore, it fits into a homotopy pullback square of ∞-categories*

$$\begin{tikzcd} \mathbf{cLie}^{\mathrm{gr-mix}} \arrow[r, "\mathrm{Tot}"] \arrow[d, "\mathrm{forget}"'] & \mathbf{cLie}^{\mathrm{mix}} \arrow[d, "\mathrm{forget}"] \\ \mathbf{Mod}_k^{\mathrm{gr}} \arrow[r, "\mathrm{Tot}"'] & \mathbf{Mod}_k^{\mathrm{cpl}}. \end{tikzcd}$$

Proof. Note that taking total complexes defines a right Quillen functor from graded mixed-curved L_∞-algebras and mixed-curved L_∞-algebras. In light of Proposition 2.45, the functor $\mathrm{Tot}\colon \mathbf{cLie}^{\mathrm{gr-mix}} \longrightarrow \mathbf{cLie}^{\mathrm{mix}}$ is the associated derived functor between ∞-categorical localizations, hence a right adjoint between presentable ∞-categories [Hin16].

For the second statement, note that by Remark 2.18, the ∞-category of (derived) complete complexes can be modeled by the model category on complete complexes. In particular, we can describe the bottom row of the square in terms of simplicial categories as well: we simply take the simplicially enriched categories of complete (resp. graded) complexes with maps given in simplicial degree n by chain maps $V \longrightarrow W \hat{\otimes} \Omega[\Delta^n]$.

Using this and our description of ∞-morphisms between graded mixed L_∞-algebras, one then sees that the above square arises from a (strict) pullback square of categories enriched in Kan complexes. To conclude, it suffices to verify that the right vertical functor is a *fibration* of simplicially enriched categories.

To see this, we have to verify two conditions: first, given $\mathfrak{g}, \mathfrak{h} \in \mathrm{cLie}^{\mathrm{mix}}$, we have to verify that sending an ∞-morphism $\phi\colon \mathfrak{g} \rightsquigarrow \mathfrak{h} \otimes \Omega[\Delta^\bullet]$ to its base map $\phi_{\mathrm{lin}}\colon \mathfrak{g} \longrightarrow \mathfrak{h} \otimes \Omega[\Delta^n]$ gives a Kan fibration of simplicial sets. Unraveling the definitions, the Kan condition translates into the lifting condition

$$\begin{tikzcd} \mathrm{Free}(\mathfrak{g}) \arrow[r] \arrow[d] & \mathfrak{h} \otimes \Omega[\Delta^n] \arrow[d] \\ \Omega_\phi \mathrm{Bar}_\phi(\mathfrak{g}) \arrow[r] \arrow[ur, dotted] & \mathfrak{h} \otimes \Omega[\Lambda^n_i] \end{tikzcd}$$

where $\Omega_\phi \mathrm{Bar}_\phi(\mathfrak{g})$ is the bar-cobar resolution for the canonical twisting morphism. The left vertical map is a cofibration by Proposition 2.41 and the right vertical map is an acyclic fibration, so the desired lift exists.

Second, we have to verify that the induced map on homotopy categories is an isofibration: if $i\colon V \longrightarrow \mathfrak{g}$ is a homotopy equivalence between complete complexes and $\mathfrak{g}$ carries a curved Lie_∞-structure, then there exists an ∞-morphism of curved Lie_∞-algebras i_∞ whose base map is (homotopic to) i. To do this, we can extend i to a homotopy retract by Proposition 2.34 and then apply the Homotopy Transfer Theorem 2.35. $\square$

Corollary 2.64. *The ∞-category* $\mathbf{cLie}$ *of curved L_∞-algebras is a presentable ∞-category, which arises as the pullback of ∞-categories*

$$\mathbf{cLie} \simeq \mathbf{cLie}^{\mathrm{mix}} \times_{\mathbf{Mod}_k^{\mathrm{cpl}}} \mathbf{Mod}_k^{\mathrm{gr}}.$$

2.5.4 Other types of curved algebras

The above results generalize from curved L_∞-algebras to other types of curved algebras. More precisely, let us fix the following data (in unfiltered complexes). First, let $\mathcal{C}$ be a graded cooperad concentrated in arity ≥ 2 and let us think of the symbol $\mathrm{u}\mathcal{C}$ be a "counital extension", in the sense that for $n \geq 2$ we have

$$\mathrm{u}\mathcal{C}(n) = \mathcal{C}(n), \qquad \mathrm{u}\mathcal{C}(0) = k \cdot \mu_0, \qquad \mathrm{u}\mathcal{C}(1) = k \cdot \mu_1 \text{ in degree } 0.$$

Furthermore, $\mathrm{u}\mathcal{C}$ comes equipped with partial cocomposition maps $\mathrm{u}\mathcal{C}(r) \longrightarrow \mathrm{u}\mathcal{C}(n+1) \circ \mathrm{u}\mathcal{C}(r-n)$ which are coassociative and extend the partial cocomposition of $\mathcal{C}$. We can endow this with two filtrations:

Definition 2.65. Let $\mathrm{u}\mathcal{C}^{\mathrm{cl}}$ be the complete cooperad where μ_0 has weight 1 and all other operations have weight 0. Then $\mathrm{u}\mathcal{C}^{\mathrm{cl}}$ is a counital, non-conilpotent complete cooperad, i.e. a counital coalgebra for $\circ$ on the category of symmetric complete complexes.

Similarly, let us call the "mixed variant" $\mathrm{u}\mathcal{C}^{\mathrm{mix}}$ the complete cooperad where μ_0 and μ_1 have weight 1 and all other operations have weight 0. Then $\mathrm{u}\mathcal{C}^{\mathrm{mix}}$ is a conilpotent non-counital cooperad.

Furthermore, let $\mathcal{P}$ be a graded operad and $\phi\colon \mathcal{C} \longrightarrow \mathcal{P}$ be a Koszul twisting morphism. In particular the map $\Omega\mathcal{C} \twoheadrightarrow \mathcal{P}$ is surjective. We then have the following three versions of 'curved $\mathcal{P}$-algebras':

Definition 2.66. A *(classical) curved $\mathcal{P}$-algebra* is a graded vector space $\mathfrak{g}$, together with a codifferential on the cofree $\mathrm{u}\mathcal{C}^{\mathrm{cl}}$-coalgebra $\mathrm{u}\mathcal{C}^{\mathrm{cl}}(\mathfrak{g})$, such that the corresponding map onto cogenerators

$$\delta\colon \mathrm{u}\mathcal{C}^{\mathrm{cl}}(\mathfrak{g}) \longrightarrow \mathfrak{g}[1]$$

induces a map $\Omega\mathcal{C}^{\mathrm{cl}} \circ \mathfrak{g} \longrightarrow \mathfrak{g}$ vanishing on the kernel of $\Omega\mathcal{C}^{\mathrm{cl}} \longrightarrow \mathcal{P}$. An ∞-*morphism* of curved $\mathcal{P}$-algebras is a map of differential-graded $\mathrm{u}\mathcal{C}^{\mathrm{cl}}$-coalgebras.

Lemma 2.67. *Let us denote by $\mathrm{c}\mathcal{P}$ the quotient of the complete operad $\Omega(\mathcal{C}^{\mathrm{mix}})$ by the operadic ideal generated by $\ker(\phi) \subseteq \mathcal{C} \subseteq \mathrm{u}\mathcal{C}^{\mathrm{mix}}$. This inherits a differential and the quotient map $\Omega(\mathcal{C}^{\mathrm{mix}}) \longrightarrow \mathrm{u}\mathcal{P}$ is a filtered quasi-isomorphism.*

Proof. Exactly the same as Definition/proposition 2.50. □

Definition 2.68. A *mixed-curved $\mathcal{P}$-algebra* is an (filtered complete) algebra over the complete operad $\mathrm{c}\mathcal{P}$. A *graded mixed-curved $\mathcal{P}$-algebra* is a weight-graded complex $\mathfrak{g}$ equipped with a mixed-curved $\mathcal{P}$-algebra structure on $\mathrm{Tot}(\mathfrak{g})$.

Remark 2.69. The exact same discussion as below Definition 2.60 shows that there is a weight-graded operad $\mathrm{c}\mathcal{P}^{\mathrm{tot}}$ whose algebras are graded mixed-curved $\mathcal{P}$-algebras. Indeed, let $\mathrm{u}\mathcal{C}^{\mathrm{tot}}$ denote the weight-graded (conilpotent, noncounital) cooperad given by $\mathcal{C} = \mathrm{u}\mathcal{C}/\langle \mu_0, \mu_1\rangle$ in weight 0 and $\mathrm{u}\mathcal{C}$ in weight ≥ 1. The comultiplication is inherited from $\mathrm{u}\mathcal{C}$. Then $\mathrm{c}\mathcal{P}^{\mathrm{tot}}$ is the quotient of $\Omega(\mathcal{C}^{\mathrm{tot}})$ by the operadic ideal generated by $\ker(\phi) \subseteq \mathcal{C}$ (in all weights ≥ 0). An argument analogous to Proposition/definition 2.50 shows that $\mathrm{u}\mathcal{C}^{\mathrm{tot}} \longrightarrow \mathrm{c}\mathcal{P}^{\mathrm{tot}}$ is a Koszul twisting morphism.

In particular, there is a natural notion of ∞-morphism between mixed (resp. graded mixed) curved L_∞-algebras, given by maps between their bar constructions (which are conilpotent coalgebras over $\mathrm{u}\mathcal{C}^{\mathrm{mix}}$, resp. $\mathrm{u}\mathcal{C}^{\mathrm{tot}}$). We define the ∞-category of (graded mixed, mixed) curved $\mathcal{P}$-algebras as in Definition 2.43, using ∞-morphisms into $\mathfrak{h} \otimes \Omega[\Delta^\bullet]$ as morphisms.

Proposition 2.70. *There are functors of ∞-categories*

$$\mathbf{cAlg}_{\mathcal{P}}^{\mathrm{gr-mix}} := \mathbf{Alg}_{\mathrm{c}\mathcal{P}^{\mathrm{tot}}}^{\mathrm{gr}} \xrightarrow{\mathrm{Tot}} \mathbf{cAlg}^{\mathrm{mix}} := \mathbf{Alg}_{\mathrm{c}\mathcal{P}}^{\mathrm{cpl}} \xrightarrow{\mathrm{blend}} \mathbf{cAlg}_{\mathcal{P}}.$$

The total composite is an equivalence and the left functor Tot *fits into a homotopy pullback square of the form*

$$\begin{tikzcd} \mathbf{cAlg}_{\mathcal{P}}^{\mathrm{gr-mix}} \arrow[r, "\mathrm{Tot}"] \arrow[d, "\mathrm{forget}"'] & \mathbf{cAlg}_{\mathcal{P}}^{\mathrm{mix}} \arrow[d, "\mathrm{forget}"] \\ \mathbf{Mod}_k^{\mathrm{gr}} \arrow[r, "\mathrm{Tot}"'] & \mathbf{Mod}_k^{\mathrm{cpl}}. \end{tikzcd}$$

In particular, all the above ∞-categories are presentable.

Proof. The same proofs as for Proposition 2.62 and Proposition 2.63. $\square$

Example 2.71. We can endow the unit cooperad $\mathbf{1}$ with two filtrations: let $\mathbf{1}^{\mathrm{cl}}$ simply be $\mathbf{1}$ in filtration weight 0 and let $\mathbf{1}^{\mathrm{mix}}$ be $\mathbf{1}$ in filtration weight 1. This corresponds to the case where $\mathcal{C} = 0$ is the zero cooperad, except that we omit the nullary operation μ_0 (everything above holds in this setting as well). The cobar construction $\Omega(\mathbf{1}^{\mathrm{cl}})$ is the graded-commutative algebra $k[\delta]$ with δ of degree 1 and filtration weight 0, while $\Omega(\mathbf{1}^{\mathrm{mix}})$ has δ of filtration weight 1. In particular, a 'classical $\Omega(\mathbf{1}^{\mathrm{cl}})$-algebra' is nothing but a complete filtered complex and a graded mixed $\Omega(\mathbf{1}^{\mathrm{mix}})$-algebra is precisely a graded mixed complex (Definition 2.23). Proposition 2.70 then reproduces the equivalence of ∞-categories

$$\mathrm{Tot}\colon \mathbf{Mod}_k^{\mathrm{gr-mix}} = \mathbf{Alg}_{\Omega(1)}^{\mathrm{gr-mix}} \longrightarrow \mathbf{Alg}_{\Omega(1)} = \mathbf{Mod}_k^{\mathrm{cpl}}$$

announced in Proposition 2.27.

Example 2.72. A way to encounter operads that fit the framework above is to consider Koszul operads $\mathcal{P}$ such that their Koszul dual $\mathcal{P}^!$ is extendable in the sense of [DSV18, Section 4.6]: there is a *unital extension* $\mathrm{u}\mathcal{P}^!$ operad with a monomorphism $\mathcal{P} \to \mathrm{u}\mathcal{P}^!$.

It follows that taking $\mathcal{P}$ to be the operad of associative, permutative, gravity or Pois_n, there is a meaningful notion of curved $\mathcal{P}$-algebras [DSV18, Proposition 4.6.1]. Let us treat the associative case in a more detail:

The associative operad As is Koszul self dual and extendable to the operad governing unital associative algebras uAs. The cooperad $\mathsf{ucoAs}\{1\}$ has dimension $n!$ in every arity $n \geq 0$.

Following Definition 2.66, a curved As algebra is therefore a graded vector space $(A, \cdot, \mu_1, \theta)$, equipped with a product $\cdot\colon A \otimes A \to A$ satisfying the associativity relation, a degree 1 endomorphism $d\colon A \to A$ which is an algebra derivation and a "curvature" element $\theta \in A_2$ such that $d^2x = \theta x - x\theta$ for all $x \in A$. This corresponds precisely to the classical notion of a curved associative algebra, as in [Pos11, Section 3.1].

A mixed-curved associative algebra on the other hand, is an algebra over the complete operad cAs, i.e., a filtered dg associative algebra $(A, d, \cdot)$, complete with respect to the filtration and equipped furthermore with a degree 1 and filtration increasing algebra derivation $\mu_1\colon F^\bullet A^\bullet \to F^{\bullet+1}A^{\bullet+1}$ and a curvature element $\theta \in F^1A^2$ satisfying

$$(d + \mu_1)^2 x = \theta x - x\theta, \qquad \forall x \in A.$$

Example 2.73 (Example of example 2.72)**.** A classical example of a curved associative algebra is as follows: Given a manifold M and a vector bundle $E \to M$, one can consider the set of $\mathrm{End}(E)$-valued differential forms $\Omega(M, \mathrm{End}(E)) = \bigoplus_{p=0}^{\dim M} \Omega^p(M, \mathrm{End}(E))$ which is a graded associative algebra. The choice of a connection ∇ on E gives rise to the covariant exterior derivative $d_\nabla\colon \Omega^\bullet(M, \mathrm{End}(E)) \to \Omega^{\bullet+1}(M, \mathrm{End}(E))$. Interpreting the curvature of ∇ as an element $F_\nabla \in \Omega^2(M, \mathrm{End}(E))$, the tuple $(\Omega(M, \mathrm{End}(E)), \wedge, d_\nabla, F_\nabla)$ is a curved associative algebra, see for instance [Pos11, Appendix B.1].

3 Curved Lie algebras over filtered algebras

In Section 2 we have studied the homotopy theory of curved Lie algebras (or curved L_∞-algebras) over a field k from the perspective of filtered operadic homological algebra. In particular, we have seen that there is a simple operadic way to describe 'overdetermined variants' of curved Lie algebras, i.e. mixed-curved Lie algebras; it then remains to remove some redundancies, for example by choosing a splitting of the filtration.

The purpose of this section is to give a similar analysis of the homotopy theory of curved Lie algebras over a *filtered* commutative dg-algebra B:

Definition 3.1. Let B denote a complete filtered commutative dg-algebra. A *(classical) curved L_∞-algebra* over B is a complete B-module $\mathfrak{g}$, endowed with operations

$$\ell_i\colon \mathrm{Sym}^i_B(\mathfrak{g}[1]) \longrightarrow \mathfrak{g}[1]$$

which are B-linear for $i \neq 1$ and a B-module derivation for $i = 1$, such that $\ell_0 \in F^1\mathfrak{g}$ and the following equations hold:

$$\sum_{\substack{p+q=n+1\\ q\geq 0,\ p\geq}} \sum_{\sigma\in \mathrm{Sh}^{-1}_{p-1,q}} \mathrm{sgn}(\sigma)(-1)^{(p-1)q}(\ell_p \circ_1 \ell_q)^\sigma = 0.$$

This situation is more complicated because the operation ℓ_1 on $\mathfrak{g}$ (which does not square to zero and hence should be considered as algebraic data) is required to interact with the differential on B. To deal with this, we will assume throughout that the filtration on B splits multiplicatively, so that B arises as the totalization of the weight-graded algebra $\mathrm{Gr}(B)$. In this case, the differential on $B = \mathrm{Tot}(\mathrm{Gr}(B))$ decomposes as $d+\delta$, where d is homogeneous of weight 0 and δ is of filtration degree 1, endowing the weight-graded $\mathrm{Gr}(B)$ with the structure of a 'graded mixed cdga' in the following sense:

Definition 3.2. We define a *graded mixed cdga* B to be the datum of a graded cdga denoted B_{gr}, together with a derivation $\delta\colon B := \mathrm{Tot}(B_{\mathrm{gr}}) \longrightarrow \mathrm{Tot}(B_{\mathrm{gr}})[1] = B[1]$ of filtration weight 1 such that $\delta^2 + [d, \delta] = 0$.

By a *complete module* over a graded mixed cdga B we will mean a complete module of the associated complete filtered cdga (B, d), while a *graded module* is a weight-graded module over B_{gr}.

Remark 3.3. We will typically denote a graded mixed cdga B and its associated complete cdga (with the differential d, not the total differential $d+\delta$) by the same symbol B. One can then identify $B_{\mathrm{gr}} = \mathrm{Gr}(B)$ with the associated graded of B. The data of a graded mixed cdga can also be described explicitly at the graded level: the graded cdga B_{gr} comes equipped with derivations $\delta_p\colon B \longrightarrow B\langle p\rangle[1]$ for all $p \geq 1$, such that

$$[d, \delta_r] + \sum_{p+q=r} \delta_p \circ \delta_q = 0.$$

This corresponds to a shifted version of a multicomplex [LV12, 10.3.7].

Example 3.4 (De Rham algebra)**.** Given a cofibrant cdga A, we denote by $\mathrm{dR}(A)$ the graded mixed cdga given by its completed algebra of differential forms, with respect to the grading given by the form degree $\mathrm{dR}(A)\langle p\rangle = \Omega^p_{\mathrm{dR}}(A)$ and derivation δ given by the de Rham differential $\delta = d_{\mathrm{dR}}$.

In particular, if A is of the form $A = (\mathrm{Sym}(V), d)$, we have $\mathrm{dR}(A) = A \otimes \widehat{\mathrm{Sym}}(V[-1])$.

We will give an operadic description of mixed-curved L_∞-algebras over such graded mixed algebras B. Contrary to the case where $B = k$ is a field, the operad controlling such algebras is not augmented, and hence does not quite fit into the framework of Section 2. Nonetheless, there are analogues of ∞-morphisms and the homotopy transfer theorem over B as well.

3.1 Mixed-curved Lie algebras

Let (B, δ) be a graded mixed cdga. Throughout, we will consider B with the (internal) differential d and view δ as some additional algebraic structure, so that e.g. a B-module is simply a dg-module over (B, d). In this case, there are obvious mixed and graded mixed variants of curved L_∞-algebras over B:

Definition 3.5. A *mixed-curved L_∞-algebra* over a graded mixed cdga (B, δ) is a complete module $\mathfrak{g}$ over the complete filtered cdga B, equipped with the structure of a k-linear mixed-curved L_∞-algebra (Definition 2.48) such that:

- for each $n \geq 2$, the map $\ell_n : \mathrm{Sym}_k(\mathfrak{g}[1]) \longrightarrow \mathfrak{g}[2]$ is B-multilinear.
- the map $\ell_1 : \mathfrak{g} \longrightarrow \mathfrak{g}[-1]$ is a derivation over δ, in the sense that

$$\ell_1(b \cdot x) = \delta(b)x + (-1)^{|b|} b \cdot \ell_1(x), \qquad \forall x \in \mathfrak{g}, b \in B.$$

A mixed-curved Lie algebra over B is a curved L_∞-algebra over B for which the ℓ_n vanish for $n \geq 3$.

Definition 3.6. A *graded mixed-curved L_∞-algebra* over a graded mixed cdga (B, δ) is a weight-graded module $\mathfrak{g}_{\mathrm{gr}}$ over B_{gr}, together with the structure of a mixed-curved L_∞-algebra over B on $\mathrm{Tot}(\mathfrak{g}_{\mathrm{gr}})$.

It is not difficult to see that there exists a complete k-linear operad $\mathsf{cLie}_{\infty,B}$ whose algebras are mixed-curved L_∞-algebras over B. We can describe this operad more concretely by expressing it as a distributive law [LV12, Section 8.6]:

Lemma 3.7. *Let $\mathsf{cLie}_\infty = \Omega(\mathsf{ucoCom}^{\mathrm{mix}}\{1\})$ be the complete curved L_∞-operad and consider*

$$\phi\colon \mathsf{cLie}_\infty \circ B \longrightarrow B \circ \mathsf{cLie}_\infty$$

sending $\ell_n \circ (b_1, \dots, b_n) \mapsto (-1)^{(n-2)(|b_1|+\cdots+|b_n|)} b_1 \dots b_n \circ \ell_n$ for $n \geq 2$ and $\ell_1 \circ b \mapsto (-1)^{|b|} b \circ \ell_1 + \delta(b) \circ 1$ (and extended in the obvious way to compositions). This defines a distributive law, with associated k-linear operad $\mathsf{cLie}_{\infty,B} \cong B \circ \mathsf{cLie}_\infty$.

Similarly, one obtains a graded operad $\mathsf{cLie}^{\mathrm{gr}}_{\infty,B}$ whose algebras are graded mixed-curved L_∞-algebras.

Proof. We have to verify that ϕ is compatible with composition in cLie_∞ and B, i.e. that the two squares

$$\begin{tikzcd} \mathsf{cLie}_\infty \circ \mathsf{cLie}_\infty \circ B \arrow[r, "\phi"] \arrow[d, "\text{compose}"'] & \mathsf{cLie}_\infty \circ B \circ \mathsf{cLie}_\infty \arrow[d, "\phi"] \\ \mathsf{cLie}_\infty \circ B \arrow[r, "\phi"'] & B \circ \mathsf{cLie}_\infty \end{tikzcd} \qquad \begin{tikzcd} \mathsf{cLie}_\infty \circ B \circ B \arrow[r, "\phi"] \arrow[d, "\text{compose}"'] & B \circ \mathsf{cLie}_\infty \circ B \arrow[d, "\phi"] \\ \mathsf{cLie}_\infty \circ B \arrow[r, "\phi"'] & B \circ \mathsf{cLie}_\infty \end{tikzcd}$$

commute. The left square commutes since ϕ is defined on generators and extended to be composition-preserving in cLie_∞. For the right square, the only nontrivial thing to observe is that the bottom-left composite sends $\ell_1 \circ b_1 \circ b_2$ to $(-1)^{|b_1|+|b_2|}(b_1 b_2) \circ \ell_1 + \delta(b_1 b_2) \circ 1$, while the top-right composite sends it to $(-1)^{|b_1|+|b_2|}(b_1 b_2) \circ \ell_1 + (-1)^{|b_1|} b_1 \delta(b_2) \circ 1 + \delta(b_1) b_2 \circ 1$. These terms coincide since δ is a derivation.

Finally, one has to check that ϕ preserves the differentials. To this end, note that for $n \neq 1$, we can write $-[d, \ell_n] = [\ell_1, \ell_n] + T_n$, where T_n is a composition of ℓ_i with $i \neq 1$ (see

Equation 2.49). Ignoring the Koszul signs appearing in the definition of ϕ, one can verify that

$$\phi\Big([\ell_1,\ell_n]\circ(b_1,\dots,b_n)\Big) = (b_1\dots b_n)\circ[\ell_1,\ell_n] \qquad \phi\Big(T_n\circ(b_1,\dots,b_n)\Big) = (b_1\dots b_n)\circ T_n\Big).$$

Using this, one sees that ϕ sends $d(\ell_n\circ(b_1,\dots,b_n))$ to $d(b_1\cdot\dots\cdot b_n\circ\ell_n)$ for $n\neq 2$. For $n=1$, we have (in the case b is even, the odd case is similar)

$$\begin{aligned}\phi\big(d(\ell_1\circ b)\big) &= \phi\Big(-\ell_1^2\circ b-(\ell_2\circ_1\ell_0)\circ b-\ell_1\circ d(b)\Big)\\ &= -\Big(b\circ\ell_1^2+\delta^2(b)\circ 1\Big)-\Big(b\circ(\ell_2\circ_1\ell_0)\Big)-\Big(-d(b)\circ\ell_1+\delta(d(b))\circ 1\Big)\\ &= b\circ d(\ell_1)+d(b)\circ\ell_1+d(\delta(b))\circ 1 = d\big(\phi(\ell_1\circ b)\big)\end{aligned}$$

We thus obtain an operad $B\circ\mathsf{cLie}_\infty$ from the distributive law ϕ. By construction, an algebra over this operad is a B-module with the structure of an algebra over cLie_∞, such that ℓ_n is B-multilinear and ℓ_1 is a derivation over ℓ_1. This is precisely a mixed-curved L_∞-algebra. □

Being the category of algebras over a complete operad, the category of mixed-curved L_∞-algebras over B admits a model structure whose weak equivalences are the filtered quasi-isomorphisms and whose fibrations are surjections in every filtration degree (Theorem 2.15). Furthermore, the natural map of operads $B\longrightarrow\mathsf{cLie}_{\infty,B}$ induces a Quillen adjunction between B-modules and mixed-curved L_∞-algebras

$$\mathrm{Free}\colon \mathrm{Mod}_B \rightleftarrows \mathrm{cLie}_B\colon \mathrm{forget}.$$

To relate mixed-curved L_∞-algebras over B to curved L_∞-algebras in the sense of Definition 3.1, we will need a few more details on the homotopy theory of algebras over operads like $\mathsf{cLie}_{\infty,B} = B\circ\mathsf{cLie}_\infty$.

3.2 Some operadic results

Let $\mathcal{C}$ be a nonunital complete (or graded) cooperad over k, let $\mathcal{P}=\Omega(\mathcal{C})$ and let (B,δ) be a graded mixed cdga. Suppose that

$$\phi\colon \mathcal{P}\circ B\longrightarrow B\circ\mathcal{P}$$

is a well-defined distributive law, given on generating elements $c\in\mathcal{C}[-1]$ by

$$\begin{aligned} c\circ(b_1,\dots,b_n) &\mapsto (-1)^{(|b_1|+\dots|b_n|)|c|}(b_1\dots b_n)\circ c &\qquad \text{arity }\neq 1\\ c\circ b &\mapsto (-1)^{|b|}b\circ c+\lambda_c\cdot\delta(b)\circ 1 &\qquad \text{arity }=1\end{aligned}\tag{3.8}$$

for certain $\lambda_c\in k$. This endows $\mathcal{P}_B\coloneqq B\circ\mathcal{P}$ with the structure of a (unital) operad, which need not be augmented (it is augmented iff all $\lambda_c=0$). There are natural maps of operads $B\longrightarrow\mathcal{P}_B\longleftarrow\mathcal{P}$, so that every $\mathcal{P}_B$-algebra has an underlying B-module and an underlying (k-linear) $\mathcal{P}$-algebra. Explicitly, a $\mathcal{P}_B$-algebra is precisely a B-module M, equipped with the structure of an (k-linear) algebra over $\mathcal{P}=\Omega(\mathcal{C})$, such that the generating operations $c\in\mathcal{C}(p)$ interact with the B-module structure via

$$\begin{aligned} c(b_1\cdot m_1,\dots,b\cdot m_p) &= (-1)^{(|b_1|+\dots|b_n|)|c|}(b_1\cdot\dots\cdot b_p)\cdot c(m_1,\dots,m_p) \qquad \text{for } p\neq 2,\\ c(b\cdot m) &= (-1)^{|b|}b\cdot c(m)+\lambda_c\cdot\delta(b)\cdot m.\end{aligned}$$

By Theorem 2.15, the category of $\mathcal{P}_B$-modules carries a model structure. The purpose of this section is to gather some general results on this homotopy theory of $\mathcal{P}_B$-algebras.

Remark 3.9. Recall from Remark 2.69 that for any complete cooperad $\mathcal{C}$, there is a cooperad $\mathcal{C}^{\mathrm{tot}}$ given in weight i by $\mathcal{C}^{\mathrm{tot}}\langle i\rangle = F^i\mathcal{C}$. This weight-graded cooperad has the property that a weight-graded $\Omega(\mathcal{C}^{\mathrm{tot}})$-algebra is precisely a weight-graded complex V together with an $\Omega(\mathcal{C})$-algebra structure on the corresponding filtered complex $\prod_i V\langle i\rangle$. The distributive law (3.8) induces a distributive law in the weight-graded setting

$$\Omega(\mathcal{C}^{\mathrm{tot}})\circ B_{\mathrm{gr}} \longrightarrow B_{\mathrm{gr}}\circ\Omega(\mathcal{C}^{\mathrm{tot}}).$$

Let $\mathcal{P}^{\mathrm{gr}}_B = B_{\mathrm{gr}}\circ\Omega(\mathcal{C}^{\mathrm{tot}})$ denote the corresponding weight-graded operad, whose algebras are weight-graded B_{gr}-modules $\mathfrak{g}_{\mathrm{gr}}$ such that the complete B-module $\mathrm{Tot}(\mathfrak{g}_{\mathrm{gr}})$ carries a compatible $\mathcal{P}_B$-structure. All of the results from this section apply to this weight-graded operad as well (with easier proofs).

3.2.1 Homotopy transfer theorem

The notion of an ∞-morphism for algebras over $\mathcal{P}=\Omega(\mathcal{C})$ has an analogue for $\mathcal{P}_B$-algebras:

Definition 3.10. An *∞-morphism* of $\mathcal{P}_B$-algebras $\phi\colon \mathfrak{g}\rightsquigarrow\mathfrak{h}$ is an ∞-morphism between the underlying k-linear $\mathcal{P}$-algebras, i.e. a map of $\mathcal{C}$-coalgebras $\mathcal{C}_+\circ\mathfrak{g}\longrightarrow\mathcal{C}_+\circ\mathfrak{h}$, satisfying the following condition: forgetting differentials, each element $c\in\mathcal{C}_+(p)$ induces a B-multilinear map

$$\phi_c\colon \mathfrak{g}^{\otimes p}\longrightarrow\mathfrak{h}.$$

Remark 3.11. Given two ∞-morphisms $\phi\colon\mathfrak{g}\longrightarrow\mathfrak{h}$ and $\psi\colon\mathfrak{h}\longrightarrow\mathfrak{k}$, recall that the composition of their underlying ∞-morphisms of $\mathcal{P}$-algebras is given by

$$\big(\psi\phi\big)_c = \sum \psi_{c^{(1)}}\circ\big(\phi_{c^{(i_1)}},\dots,\phi_{c^{(i_p)}}\big)$$

where $\Delta(c)=\sum c^{(1)}\circ\big(c^{(i_1)},\dots,c^{(i_p)}\big)$ is the (total) cocomposition. If each ϕ_c,ψ_c is B-multilinear, certainly $(\psi\phi)_c$ is B-multilinear as well. It follows that ∞-morphisms of $\mathcal{P}_B$-algebras can be composed.

Using this notion of ∞-morphism, we have the following version of the homotopy transfer theorem:

Theorem 3.12 (Homotopy Transfer Theorem)**.** *Let (B,δ) be a graded mixed cdga, $\mathcal{C}$ a complete nonunital cooperad over k and suppose that $\mathcal{P}_B = B\circ\Omega(\mathcal{C})$ arises from the distributive law (3.8). Let W be a $\mathcal{P}_B$ algebra and consider a deformation retract of B-modules*

$$\begin{tikzcd} V \arrow[r, shift left, "i"'] & W \arrow[l, shift left, "p"'] \arrow[loop right, "h"] \end{tikzcd}$$

satisfying the side conditions $ph=0$, $hi=0$ and $h^2=0$. Then the B-module structure on V extends to a transferred *$\mathcal{P}_B$-algebra structure, and i extends to an ∞-morphism i_∞ of $\mathcal{P}_B$-algebras.*

Proof. Apply the k-linear Homotopy Transfer Theorem 2.35 to endow V with a transferred $\Omega\mathcal{C}$-algebra structure, together with a k-linear ∞-morphism $i_\infty\colon V\rightsquigarrow W$. We claim that the resulting $\mathcal{P}$-algebra structure on V is compatible with its B-module structure and that i_∞ is B-multilinear.

To see this, recall that for any generating operation $c\in\Omega(\mathcal{C})$, the resulting operation on V is given by a sum of trees, with vertices labeled by elements from $\mathcal{C}$, leaves labeled by i, internal edges labeled by h and the root labeled by p [LV12, §10.3.3]. Almost all operations appearing in this tree are B-multilinear, the only exception coming from vertices $c_1 \in \mathcal{C}(1)$, which can appear locally in the tree as

Even though c_1 does not define a B-linear map on W, the first 3 composites are all B-linear by the side conditions: for example,

$$(hc_1h)(b \cdot m) = b \cdot (hc_1h)(m) + \lambda_{c_1} \cdot \delta(b) \cdot h^2(m) = b \cdot (hc_1h)(m)$$

by the side condition $h^2 = 0$. The fourth operation appears exactly once, in the formula for the transferred operation $c_1^V : V \longrightarrow V$. Consequently, all transferred operations of arity $\neq 1$ are B-multilinear, while $c_1^V = p \circ c_1 \circ i + f$ with f B-linear. In particular, c_1^V satisfies

$$c_1^V(b \cdot m) = \big(pc_1i\big)(b \cdot m) + f(b \cdot m) = b \cdot c_1^V(m) + \lambda_{c_1} \cdot \delta(b) \cdot m$$

since $pi = 1$.

A very similar argument shows that all components of the map i_∞ are B-linear, since they are given by trees with vertices labeled by $\mathcal{C}$, leaves labeled by i and the root and internal edges labeled by h [LV12, §10.3.10]. The operations from $\mathcal{C}(1)$ now only appear in the form of the left two pictures, and hence contribute B-linear terms to the formula for i_∞. □

3.2.2 Cofibrant resolutions

Cofibrant $\mathcal{P}_B$-algebras can be studied efficiently using bar-cobar methods.

Lemma 3.13. *Every cofibrant $\mathcal{P}_B$-algebra is cofibrant as a B-module.*

Proof. Every cofibrant $\mathcal{P}_B$-algebra is a retract of a quasi-free $\mathcal{P}_B$-algebra with an increasing filtration on its generators. Since $\mathcal{P}_B = B \circ \mathcal{P}$ is cofibrant as a left B-module, such quasi-free algebras are cofibrant as B-modules as well (see also [BM03, Corollary 5.5]). □

Conversely, a $\mathcal{P}_B$-algebra which is cofibrant as a B-module admits an explicit cofibrant replacement by means of a B-linear extension of the 'bar-cobar construction' for the operad $\mathcal{P} = \Omega(\mathcal{C})$. To explain this, it will be convenient to phrase things in terms of symmetric B-bimodules, i.e. symmetric sequences X with actions

$$B \curvearrowright X(p) \curvearrowleft \Sigma_p \ltimes B^{\otimes p}.$$

The category of such symmetric B-bimodules has a monoidal structure given by the relative composition product $\circ_B$, such that unital associative algebras in symmetric B-bimodules are simply (k-linear, complete) operads equipped with an operad map from B. In particular, $\mathcal{P}_B$ is an algebra in this monoidal category.

Now recall that the differential on the cobar construction $\mathcal{P} = \Omega\mathcal{C}$ decomposes as a sum of two differentials $d_{\Omega\mathcal{C}} = d + \delta$, where d arises from the differential on $\mathcal{C}$ and δ is the cobar differential. For a distributive law as in (3.8), the differential on $\mathcal{P}_B = B \circ \Omega\mathcal{C}$ can be decomposed similarly as $d + \delta$, where both terms are (square zero) derivations for the operadic composition.

Let us now consider $\mathcal{C}_{B+} := B \circ \mathcal{C}_+$, equipped with the structure of a symmetric B-bimodule where $c \circ (b_1, \ldots, b_n) = (-1)^{|c|(|b_1|+\ldots|b_n|)}(b_1 \ldots b_n) \circ c$. Without differentials, let

$$\mathcal{M}_B := \mathcal{P}_B \circ_B \mathcal{C}_{B+} \circ_B \mathcal{P}_B$$

be the free $\mathcal{P}_B$-bimodule generated by $\mathcal{C}_{B+}$ (relative to B). This inherits an (internal) differential d from $\mathcal{P}_B$ and $\mathcal{C}_B$. Furthermore, it comes with an additional derivation δ (which is a derivation over the cobar differentials δ on $\mathcal{P}_B$) given on generators $b \circ c \in \mathcal{C}_{B+} = B \circ \mathcal{C}$ by

$$\delta(b \circ c) = \big(b \circ c^{(1)}\big) \circ \big(c^{(2)}\big) \circ \big(1\big) - \big(b\big) \circ \big(c^{(1)}\big) \circ \big(c^{(2)}\big) \qquad \text{in } \mathcal{P}_B \circ_B \mathcal{C}_{B+} \circ_B \mathcal{P}_B.$$

Here $\Delta^{(1)}(c) = c^{(1)} \circ c^{(2)}$ is the infinitesimal cocomposition. This derivation δ (graded-) commutes with the internal differential d and using coassociativity of the infinitesimal cocomposition, one sees that $\delta^2 = 0$. Taking the sum, this makes $\mathcal{M}_B$ a bimodule over $\mathcal{P}_B$ (with its total differential).

Remark 3.14. Taking $B = k$, the resulting $\mathcal{P}$-bimodule $\mathcal{M}_k$ is the usual bimodule such that $\mathcal{M}_k \circ_{\mathcal{P}} \mathfrak{g}$ takes the bar-cobar resolution of a $\mathcal{P}$-algebra. The symmetric sequence $\mathcal{M}_B$ is isomorphic to $B \circ \mathcal{M}_k$, but the right B-action is nontrivial and involves the distributive law.

Definition 3.15. The *bar-cobar construction* of a $\mathcal{P}_B$-algebra $\mathfrak{g}$ is the $\mathcal{P}_B$-algebra $\mathcal{M}_B \circ_{\mathcal{P}_B} \mathfrak{g}$.

Proposition 3.16. *Let $\mathfrak{g}$ and $\mathfrak{h}$ be $\mathcal{P}_B$-algebras. Then there is a natural bijection between the set of ∞-morphisms $\mathfrak{g} \rightsquigarrow \mathfrak{h}$ and the set of strict morphisms $\mathcal{M}_B \circ_{\mathcal{P}_B} \mathfrak{g} \longrightarrow \mathfrak{h}$.*

Proof. Note that the unit $k \to B$ induces a natural map of k-operads $\mathcal{P} \longrightarrow \mathcal{P}_B$ and a map of bimodules over it $\mathcal{M}_k \longrightarrow \mathcal{M}_B$. These induce a map of $\mathcal{P}$-algebras $\mathcal{M}_k \circ_{\mathcal{P}} \mathfrak{g} \longrightarrow \mathcal{M}_B \circ_{\mathcal{P}_B} \mathfrak{g}$. Without the differential, this map can be identified with the quotient map

$$\mathcal{P} \circ \mathcal{C}_+ \circ \mathfrak{g} \longrightarrow \mathcal{P}_B \circ_B \mathcal{C}_{B+} \circ_B \mathfrak{g}$$

from a free $\mathcal{P}$-algebra to the quotient of a free $\mathcal{P}_B$-algebra. This map sends maps the space of $\mathcal{P}$-algebra generators $\mathcal{C}_+ \circ \mathfrak{g}$ onto the module of $\mathcal{P}_B$-algebra generators $\mathcal{C}_{B+} \circ_B \mathfrak{g}$. Consequently, the set of $\mathcal{P}_B$-algebra maps $\mathcal{M}_B \circ_{\mathcal{P}} \mathfrak{g} \longrightarrow \mathfrak{h}$ is a subset of the set of $\mathcal{P}$-algebra maps $\mathcal{M}_k \circ_{\mathcal{P}} \mathfrak{g} \longrightarrow \mathfrak{h}$; the latter is naturally isomorphic to the set of k-linear ∞-morphisms of $\mathcal{P}$-algebras from $\mathfrak{g}$ to $\mathfrak{h}$.

Furthermore, a map of $\mathcal{P}$-algebras $\mathcal{M}_k \circ_{\mathcal{P}} \mathfrak{g} \longrightarrow \mathfrak{h}$ descends to a map of $\mathcal{P}_B$-algebras $\mathcal{M}_B \circ_{\mathcal{P}_B} \mathfrak{g} \longrightarrow \mathfrak{h}$ if and only at the level of generators, it descends from a k-linear map $\mathcal{C}_+ \circ \mathfrak{g} \longrightarrow \mathfrak{h}$ to a B-linear map $\mathcal{C}_{B+} \circ_B \mathfrak{g} \longrightarrow \mathfrak{h}$. This means precisely that the components of the $\mathcal{P}$-algebraic ∞-morphism are B-multilinear. $\square$

Proposition 3.17. *Let $\mathfrak{g}$ be a $\mathcal{P}_B$-algebra whose underlying B-module is cofibrant. Then the natural map of complete $\mathcal{P}_B$-algebras*

$$\mathcal{M}_B \circ_{\mathcal{P}_B} \mathfrak{g} \longrightarrow \mathfrak{g}$$

induces a quasi-isomorphism on the associated graded.

Proof. Since taking the associated graded commutes with taking relative composition products and the cobar construction, it suffices to prove the analogous statement in the setting where $\mathcal{C}$, B and $\mathfrak{g}$ are graded objects. In particular, $\mathcal{C}$ is a graded conilpotent cooperad and has an exhaustive coradical filtration. One can identify

$$\mathcal{M}_B \circ_{\mathcal{P}_B} \mathfrak{g} \cong \big(\mathcal{P}_B \circ_B \mathcal{C}_{B+}\big) \circ_B \mathfrak{g}.$$

The cobar differential applies the cocomposition to a vertex in $\mathcal{C}_{B+}$ and then either moves the bottom vertex to $\mathcal{P} = \Omega(\mathcal{C})$ or acts by the top vertex on $\mathfrak{g}$. In particular, this differential preserves the exhaustive filtration on $\mathcal{M}_B \circ_{\mathcal{P}_B} \mathfrak{g}$ induced by the coradical filtrations on $\mathcal{C}_+$, resp. on $\mathcal{C}[-1] \subseteq \Omega(\mathcal{C})$.

The associated graded can be associated with the composition product $X \circ_B \mathfrak{g}$, where $X = B \circ \Omega(\mathcal{C}) \circ \mathcal{C}_+$, together with the differential taking an element c in $\mathcal{C}_+$, replacing it by $c \circ 1$ and "moving" c to $\Omega(\mathcal{C})$ while increasing the degree by 1. As in Proposition 2.31, X admits a contracting homotopy and the natural map $X \longrightarrow B$ is a (graded) quasi-isomorphism. Since $\mathfrak{g}$ is cofibrant as a left B-module, the natural map $X \circ_B \mathfrak{g} \longrightarrow \mathfrak{g}$ is then a quasi-isomorphism as well. □

As indicated by the terminology, under good conditions the bar-cobar construction provides a cofibrant replacement $\mathcal{M}_B \circ_{\mathcal{P}_B} \mathfrak{g} \to \mathfrak{g}$. Let us proof a slightly stronger version of this fact.

Proposition 3.18. *Let $\mathfrak{g}$ be a $\mathcal{P}_B$-algebra whose underlying B-module is cofibrant. Then the natural map $\mathcal{P}_B \circ_B \mathfrak{g} \longrightarrow \mathcal{M}_B \circ_{\mathcal{P}_B} \mathfrak{g}$ is a cofibration of $\mathcal{P}_B$-algebras. In particular, $\mathcal{M}_B \circ_{\mathcal{P}_B} \mathfrak{g}$ is a cofibrant $\mathcal{P}_B$-algebra.*

We start by proving an auxiliary lemma.

Lemma 3.19. *Let $\mathfrak{g}$ be a $\mathcal{P}_B$-algebra whose underlying B-module is cofibrant. Then the natural map $v\colon \mathfrak{g} \longrightarrow \mathcal{M}_B \circ_{\mathcal{P}_B} \mathfrak{g}$ induced by the unit map $B \longrightarrow \mathcal{M}_B$ is a cofibration of B-modules. In particular, the bar-cobar construction $\mathcal{M}_B \circ_{\mathcal{P}_B} \mathfrak{g}$ is cofibrant as a B-module.*

Proof. By Proposition 2.22, it suffices to verify that v is a summand inclusion of quasiprojective B-modules and that the associated graded of $\mathcal{M}_B \circ_{\mathcal{P}_B} \mathfrak{g}$ is a cofibrant weight-graded module over $\mathrm{Gr}(B)$. For the first, recall that without differential we simply have that

$$\mathcal{M}_B \circ_{\mathcal{P}_B} \mathfrak{g} \cong \mathcal{P}_B \circ_B \mathcal{C}_{B+} \circ_B \mathfrak{g}.$$

Since all symmetric sequences in the above expression are quasiprojective as left B-modules, their composition product is quasiprojective as well. Note that $v\colon \mathfrak{g} \longrightarrow \mathcal{M}_B \circ_{\mathcal{P}_B} \mathfrak{g}$ is simply the inclusion of the summand corresponding to the summand $B \cdot 1 \subseteq \mathcal{P}_B \circ_B \mathcal{C}_{B+}$.

To see that the associated graded of $\mathcal{M}_B \circ_{\mathcal{P}_B} \mathfrak{g}$ is cofibrant over $\mathrm{Gr}(B)$, it suffices to work entirely at the graded level, since taking the associated graded preserves bar/cobar constructions, relative composition products and the cofibrancy of modules. Let us therefore work in the setting where $\mathcal{C}, B$ and $\mathfrak{g}$ are all weight-graded. In this graded setting, we can filter $\mathcal{M}_B \circ_{\mathcal{P}_B} \mathfrak{g}$ using the coradical filtration on the cooperad $\mathcal{C}$ (as in the proof of Proposition 3.17). This is an exhaustive increasing filtration whose graded is given in degree i by

$$\Big(B \otimes \mathrm{Gr}^i\left(\Omega(\mathcal{C}) \circ \mathcal{C}_+\right)\Big) \circ_B \mathfrak{g}.$$

Since $\mathfrak{g}$ is cofibrant as a weight-graded B-module by assumption and each $B \otimes \mathrm{Gr}^i\left(\Omega(\mathcal{C}) \circ \mathcal{C}\right)$ is manifestly cofibrant as a weight-graded B-module, the associated graded of our increasing filtration consists of cofibrant weight-graded B-modules. This implies that $\mathcal{M}_B \circ_{\mathcal{P}_B} \mathfrak{g}$ is a cofibrant weight-graded B-module itself as well. □

Proof (of Proposition 3.18). The proof is essentially the same as that of Proposition 2.41. Using Proposition 3.16, it suffices to show the following assertion: let $p\colon \mathfrak{h} \twoheadrightarrow \mathcal{M}_B \circ_{\mathcal{P}_B} \mathfrak{g}$ be an acyclic fibration, equipped with a B-linear map $s\colon \mathfrak{g} \longrightarrow \mathfrak{h}$ such that $ps\colon \mathfrak{g} \longrightarrow \mathcal{M}_B \circ_{\mathcal{P}_B} \mathfrak{g}$ is the linear map underlying the universal ∞-morphism $v_\infty\colon \mathfrak{g} \rightsquigarrow \mathcal{M}_B \circ_{\mathcal{P}_B} \mathfrak{g}$. We have to provide a refinement of s to an ∞-morphism s_∞ such that $ps_\infty = v_\infty$.

First, note that we can always replace $\mathfrak{h}$ by a cofibrant resolution $q\colon \mathfrak{h}' \overset{\sim}{\twoheadrightarrow} \mathfrak{h}$ and lift the map s to a map with values in $\mathfrak{h}'$. Using this, we may therefore assume that $\mathfrak{h}$ is cofibrant as a B-module (by Lemma 3.13). Now, the underlying linear map $v\colon \mathfrak{g} \longrightarrow \mathcal{M}_B \circ_{\mathcal{P}_B} \mathfrak{g}$ is a cofibration of B-modules by Lemma 3.19, so that we can find a B-linear section $i\colon \mathcal{M}_B \circ_{\mathcal{P}_B} \mathfrak{g} \longrightarrow \mathfrak{h}$ extending v. By the same argument as in Lemma 2.33, i and p are part of a deformation retract satisfying the side conditions. Lemma 2.42, using the Homotopy Transfer Theorem 3.12, then provides an extension of i to an ∞-morphism i_∞. The desired map is now given by $s_\infty = v_\infty i_\infty$.

□

3.2.3 The ∞-category of algebras

For a graded mixed cdga (B, δ) and $\mathcal{C}$ and $\mathcal{P}$ as above, we can consider the following simplicially enriched category of $\mathcal{P}_B$-algebras and ∞-morphisms between them:

Definition 3.20. Let $\mathbf{Alg}^{\mathrm{cpl}}_{\mathcal{P}_B}$ denote the simplicially enriched category defined as follows:

(0) objects are complete $\mathcal{P}_B$-algebras whose underlying complete B-module is cofibrant (in the model structure of Theorem 2.15).

(1) For any two objects $\mathfrak{g}$ and $\mathfrak{h}$, the simplicial set of morphisms between them is given in simplicial degree n by the set of ∞-morphisms

$$\mathrm{Map}_{\mathcal{P}_B}(\mathfrak{g}, \mathfrak{h})_n = \big\{\mathfrak{g} \rightsquigarrow \mathfrak{h} \mathbin{\hat{\otimes}} \Omega[\Delta^n]\big\}.$$

Remark 3.21. The same proof as for Lemma 2.44 shows that $\mathbf{Alg}^{\mathrm{cpl}}_{\mathcal{P}_B}$ is enriched in Kan complexes.

Proposition 3.22. *Let $\mathcal{C}$ be a nonunital complete cooperad, (B, δ) a graded mixed cdga and let $\mathcal{P}_B$ be the operad arising from a distributative of the form (3.8). Then the functor $\mathrm{Alg}^{\mathrm{cpl}}_{\mathcal{P}_B} \longrightarrow \mathbf{Alg}^{\mathrm{cpl}}_{\mathcal{P}_B}$ exhibits $\mathbf{Alg}^{\mathrm{cpl}}_{\mathcal{P}_B}$ as the ∞-categorical localization of the model category of complete $\mathcal{P}_B$-algebras at the filtered quasi-isomorphisms.*

Proof. The proof of Proposition 2.45 carries over to this situation. □

3.3 Homotopy theory of curved Lie algebras

Let us now return to our situation of interest, the homotopy theory of curved L_∞-algebras. If (B, δ) is a graded mixed cdga, we have now have three different ∞-categories of (types of) curved L_∞-algebras over B:

Definition 3.23. Let (B, δ) be a graded mixed cdga. We consider the following three simplicial categories:

(1) The ∞-category $\mathbf{cLie}_B$ of *curved L_∞-algebras over B*. Its objects are (classical) curved L_∞-algebras over B (Definition 3.1) whose underlying B-module is quasiprojective, such that the associated graded is a cofibrant $\mathrm{Gr}(B)$-module. The simplicial set of maps $\mathrm{Map}_{\mathbf{cLie}_B}(\mathfrak{g}, \mathfrak{h})$ consists of ∞-morphisms $\phi\colon \mathfrak{g} \rightsquigarrow \mathfrak{h} \otimes \Omega[\Delta^n]$ between the underlying k-linear curved L_∞-algebras, such that each map

$$\phi_n\colon \mathfrak{g}^{\otimes n} \longrightarrow \mathfrak{h} \otimes \Omega[\Delta^n]$$

is B-multilinear (forgetting the differentials on both sides).

(2) The ∞-category $\mathbf{cLie}_B^{\mathrm{mix}}$ of *mixed-curved L_∞-algebras* over (B, δ) (Definition 3.5), defined as the ∞-category associated to the model category of complete algebras over the operad $\mathsf{cLie}_{\infty,B}$ from Lemma 3.7. By Proposition 3.22, this can also be described as the simplicial category whose objects are mixed-curved L_∞-algebras whose underlying complete B-module is cofibrant, with morphisms given by simplicial sets of ∞-morphisms (see Definition 3.20).

(3) The ∞-category $\mathbf{cLie}_{B_{\mathrm{gr}}}^{\mathrm{gr-mix}}$ of graded mixed-curved L_∞-algebras over B (Definition 3.6), defined as the ∞-category associated to the model category of graded algebras over the operad $\mathsf{cLie}^{\mathrm{gr}}_{\infty,B}$ (Remark 3.9). An analogue of Proposition 3.22 shows that this ∞-category can also be described as the simplicial category whose objects are graded mixed-curved L_∞-algebras whose underlying graded B_{gr}-module is cofibrant, with morphisms given by simplicial sets of ∞-morphisms.

We now have the following analogues of Proposition 2.62 and 2.63:

Theorem 3.24. *Let (B, δ) be a graded mixed cdga. Then there is a sequence of functors of presentable ∞-categories whose composite is an equivalence*

$$\mathbf{cLie}_B^{\mathrm{gr-mix}} \xrightarrow{\mathrm{Tot}} \mathbf{cLie}_B^{\mathrm{mix}} \xrightarrow{\mathrm{blend}} \mathbf{cLie}_B.$$

Furthermore, the first functor fits into a pullback square of ∞-categories

$$\begin{tikzcd} \mathbf{cLie}_B^{\mathrm{gr-mix}} \arrow[r, "\mathrm{Tot}"] \arrow[d, "\mathrm{forget}"'] & \mathbf{cLie}_B^{\mathrm{mix}} \arrow[d, "\mathrm{forget}"] \\ \mathbf{Mod}_{B_{\mathrm{gr}}}^{\mathrm{gr}} \arrow[r, "\mathrm{Tot}"'] & \mathbf{Mod}_B^{\mathrm{cpl}}. \end{tikzcd}$$

Proof. The functor Tot sends a graded mixed-curved L_∞-algebra to its total complex and the functor blend is defined in exactly the same way like (2.59): it sends a mixed-curved L_∞-algebra $(\mathfrak{g}, d, \ell_i)$ to the curved L_∞-algebra $\big(\mathfrak{g}, \ell_1' = d+\ell_1, \ell_{i\neq 1}\big)$, and similarly on ∞-morphisms one sums up the component ϕ_{lin} and ϕ_1. The proofs of Proposition 2.62 and Proposition 2.63 now carry over verbatim: the functor $\mathbf{cLie}_B^{\mathrm{gr-mix}} \longrightarrow \mathbf{cLie}_B$ is an equivalence because every curved Lie algebra over B whose underlying B-module is quasiprojective admits a splitting, i.e. arises as the totalization of a graded B-module (which can then be endowed with a canonical graded mixed-curved L_∞-structure).

Furthermore, the above pullback square is a strict pullback square of simplicially enriched categories. To verify that it is a homotopy pullback, it suffices to verify that the right vertical functor is a fibration, which follows from the Homotopy Transfer Theorem 3.12. □

4 Curved Lie algebroids

In this section we recall the notion of a Lie algebroid (or L_∞-algebroid) over A, and introduce an obvious curved analogue as well. The main result (Theorem 4.23) provides an equivalence between the ∞-category of such curved Lie algebroids and a certain ∞-category of weight-graded non-curved Lie algebroids. Restricting to subcategories of Lie algebras, this gives yet another description of the homotopy theory of curved Lie algebras, purely in terms of Lie algebras without curvature.

4.1 Categories of (curved) Lie algebroids

From now on, A will denote a cdga in nonnegative cohomological degrees. Everything below will typically only be homotopically sound when A is furthermore cofibrant or smooth. We will denote by T_A the A-module of derivations of A, which comes with a k-linear Lie algebra structure given by the commutator bracket satisfying the Leibniz rule

$$[v, a \cdot w] = a \cdot [v, w] + \mathcal{L}_v(a) \cdot w, \qquad a \in A, v, w \in T_A. \tag{4.1}$$

Definition 4.2. An L_∞*-algebroid* over A (relative to k) is given by an A-module $\mathfrak{L}$ whose underlying complex carries a k-linear L_∞-algebra structure, together with an *anchor map*

$$\rho\colon \mathfrak{L} \longrightarrow T_A.$$

This data has to satisfy the following conditions:

- The anchor map preserves both the A-module and L_∞-algebra structure.
- All brackets ℓ_n of arity $n \geq 3$ are A-multilinear and the binary bracket satisfies the Leibniz rule (4.1).

A *Lie algebroid* is an L_∞-algebroid whose brackets in arity ≥ 3 vanish.

There are evident filtered and weight-graded analogues of the above definition, where one treats T_A (and A) as being concentrated in filtration degree (weight-grading) 0. In particular, all elements of filtration degree ≥ 1 (resp. weight $\neq 0$) are contained in the kernel of the anchor map. In addition, one can add curvature to the above definition, where the curvature (being of filtration degree 1) is contained in the kernel of the anchor map:

Definition 4.3. A *curved* L_∞*-algebroid* over A is given by a k-linear curved L_∞-algebra $\mathfrak{L}$ (Definition 2.46) equipped with the structure of a complete graded A-module and an anchor map

$$\rho\colon \mathfrak{L} \longrightarrow T_A\langle 0\rangle$$

to T_A, concentrated in filtration degree 0. This data is required to satisfy two conditions:

- The anchor map is a map of complete graded A-modules and preserves the curved L_∞-algebra structure (strictly).
- The brackets ℓ_n of arity $n \geq 3$ are A-multilinear, the binary bracket satisfies the Leibniz rule (4.1) and ℓ_1 is an A-module derivation.

We define mixed-curved and graded mixed-curved L_∞-algebroids analogously. Note that none of these versions of L_∞-algebroids can be seen as algebras over an operad.

Example 4.4. The terminal (curved) L_∞-algebroid is T_A itself (in filtration degree 0), with the usual commutator bracket. Orthogonally, (curved) L_∞-algebroids with zero anchor map are precisely (curved) L_∞-algebras over A.

Example 4.5. Let $\mathfrak{g} \longrightarrow T_A$ be a (strict) map of k-linear curved L_∞-algebras. Then $A \otimes \mathfrak{g}$ has the structure of a curved L_∞-algebroid over A, where the anchor map and the ℓ_n with $n \geq 3$ are extended A-multilinearly, while ℓ_2 is extended according to the Leibniz rule (4.1). This construction defines a left adjoint to the forgetful functor from (curved) L_∞-algebroids to k-linear (curved) L_∞-algebras over T_A.

Example 4.6. If $\mathfrak{L}$ is an L_∞-algebroid, then the tensor products with differential forms on simplices

$$\mathfrak{L} \boxtimes \Omega[\Delta^n] \coloneqq \mathfrak{L} \otimes \Omega[\Delta^n] \times_{T_A \otimes \Omega[\Delta^n]} T_A$$

again have the structure of L_∞-algebroids (cf. [Nui19a, Construction 5.23] and [Vez15]). This provides a simplicial enrichment of the category of L_∞-algebroids, with simplicial sets of maps consisting of (strict) maps $\mathfrak{L} \longrightarrow \mathfrak{H} \boxtimes \Omega[\Delta^n]$. The same thing applies to curved, mixed-curved and graded mixed-curved L_∞-algebroids.

Example 4.7. Let $k \longrightarrow k'$ be a map of rings and let $A' = k' \otimes_k A$. Then there is an adjoint pair $\mathrm{cLie}(A/k) \leftrightarrows \mathrm{cLie}(A'/k')$ between (graded mixed, mixed) curved L_∞-algebroids over A relative to k and curved L_∞-algebroids over A' relative to k'. The left adjoint sends $\mathfrak{L} \longrightarrow T_{A/k}$ to $k' \otimes_k \mathfrak{L} \longrightarrow k' \otimes_k T_{A/k} \longrightarrow T_{A'/k'}$ while the right adjoint sends $\mathfrak{H} \longrightarrow T_{A'/k'}$ to the pullback $\mathfrak{H} \times_{T_{A'/k'}} T_{A/k}$.

Example 4.8. Let $\mathfrak{L}$ be an L_∞-algebroid. There are three canonical (complete) filtrations that one can put on $\mathfrak{L}$:

(0) *trivial filtration* $\mathfrak{L}\langle 0\rangle$*:* the filtered L_∞-algebroid with $F^1\mathfrak{L}\langle 0\rangle = 0$, $F^0\mathfrak{L}\langle 0\rangle = \mathfrak{L}$.

(1) *Hodge filtration* $\mathfrak{L}\langle 1\rangle$*:* given by $F^0\mathfrak{L}\langle 1\rangle = 0$, $F^{-1}\mathfrak{L}\langle 1\rangle = \mathfrak{L}$.

(2) *anchor filtration* $\mathfrak{L}^{\mathrm{anc}}$*:* the fiber product $\mathfrak{L}\langle 0\rangle \times_{T_A\langle 0\rangle} T_A\langle 1\rangle$. Explicitly, this is a filtered Lie algebroid over $T_A\langle 1\rangle$ given by

$$\begin{array}{ccccccccccc} \text{Filtration:} & & 1 & & 0 & & -1 & & -2 & & \\ \dots & \longrightarrow & 0 & \hookrightarrow & \mathfrak{n} & \longrightarrow & \mathfrak{L} & \xrightarrow{=} & \mathfrak{L} & \xrightarrow{=} & \dots \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\rho} & & \downarrow & & \\ \dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & T_A & \xrightarrow{=} & T_A & \xrightarrow{=} & \dots \end{array}$$

where $\mathfrak{n}$ is the kernel of the anchor map.

Definition 4.9. Let $\mathfrak{L}$ be a curved L_∞-algebroid. Its *Chevalley–Eilenberg complex* is given by the complete graded vector space

$$C^*(\mathfrak{L}) \coloneqq \mathrm{Hom}_A\big(\mathrm{Sym}_A(\mathfrak{L}[1]), A\big)$$

equipped with the Chevalley–Eilenberg differential, given (modulo Koszul signs) by

$$\begin{aligned}(d_{\mathrm{CE}}\alpha)(x_1, \dots, x_n) = d_A\big(\alpha(x_1, \dots, x_n)\big) &+ \sum_i \mathcal{L}_{\rho(x_i)}\alpha(x_1, \dots, \hat{x_i}, \dots, x_n) \\ &- \sum_{k\geq 0} \sum_{\sigma \in \mathrm{Sh}^{-1}(k,n-k)} \alpha\big(\ell_k(x_{\sigma(1)}, \dots, x_{\sigma(k)}), x_{\sigma(k+1)}, \dots, x_{\sigma(n)}\big).\end{aligned}$$

The usual exterior product endows $C^*(\mathfrak{L})$ with the structure of a complete k-linear cdga.

Remark 4.10. Abstractly, the Chevalley–Eilenberg complex can be identified as follows. The Lie algebra action of T_A on A (by derivations) endows $\mathrm{Sym}^c_k(T_A[1]) \otimes_k A$ with a differential δ making it a dg-comodule over the "bar construction" $\mathrm{Sym}^c_k(T_A[1])$ (cf. [Hal92]). On the other hand, the anchor map ρ induces a map of (complete) dg-coalgebras $\mathrm{Sym}^c_k(\mathfrak{L}[1]) \longrightarrow \mathrm{Sym}^c_k(T_A[1])$. Corestricting along this map, we obtain a dg-comodule structure on $\mathrm{Sym}^c_k(\mathfrak{L}[1]) \otimes_k A$.

The complex of comodule maps $\mathrm{Sym}^c_k(\mathfrak{L}[1]) \longrightarrow \mathrm{Sym}^c_k(\mathfrak{L}[1]) \otimes_k A$ can then be identified with the complete graded vector space $\mathrm{Hom}_k(\mathrm{Sym}^c_k(\mathfrak{L}[1]), A)$, endowed with the Chevalley–Eilenberg differential d_{CE} described above. The subspace $\mathrm{Hom}_A(\mathrm{Sym}^c_A(\mathfrak{L}[1]), A)$ of A-multilinear maps is closed under this differential.

Example 4.11. Let $\mathfrak{L}$ be an ordinary L_∞-algebroid over A. Then $C^*(\mathfrak{L}) = C^*(\mathfrak{L}\langle 0\rangle)$ is the usual Chevalley–Eilenberg complex, concentrated in filtration weight 0. On the other hand, $C^*(\mathfrak{L}\langle 1\rangle)$ is the Chevalley–Eilenberg complex of $\mathfrak{L}$ equipped with the *Hodge filtration*, in which p-forms are of filtration weight p. In particular, $C^*(\mathfrak{L}\langle 1\rangle)$ has a canonical graded mixed structure, with the grading given by form degree.

Finally, $C^*(\mathfrak{L}^{\mathrm{anc}})$ is the Chevalley–Eilenberg complex of $\mathfrak{L}$, endowed with the *anchor filtration* where a form has filtration weight p if it is zero when applied to $\geq p$ elements coming from the kernel of $\rho\colon \mathfrak{L} \longrightarrow T_A$.

4.2 Homotopy theory of curved Lie algebroids

The categories of mixed-curved and graded mixed-curved L_∞-algebroids almost carry a model structure:

Theorem 4.12. *The category of mixed-curved L_∞-algebroids over A carries a (left) semi-model structure (cf. [Fre09, Section 12.1]) such that:*

(1) weak equivalences are weak equivalences between the underlying complete A-modules.

(2) fibrations are maps that induce surjections in each filtration weight.

Furthermore, every cofibrant object is (in particular) cofibrant as a complete A-module.

Similarly, there are (left) semi-model structures on the categories of graded mixed-curved L_∞-algebroids and (weight-graded/complete) L_∞-algebroids over A.

One does not obtain model structures in the strict sense because there are (mixed-curved) L_∞-algebroids which do not admit a fibrant replacement $\mathfrak{L} \xrightarrow{\sim} \mathfrak{L}^{\mathrm{fib}}$ (i.e. one for which the anchor map is surjective) [Nui19a, Example 3.2]. However, this does not pose a problem from the point of view of ∞-categories: the associated ∞-categories still have all expected properties, e.g. limits and colimits that are computed as homotopy limits and colimits.

Proof. For ordinary L_∞-algebroids, without filtrations or curvature, this is proven in [Nui19a]. The proofs of loc. cit. carry over verbatim to this case; we will briefly outline the argument in the mixed-curved case, the other cases are easier. The desired semi-model structure is obtained by transfer along the free-forgetful adjunction $\mathrm{Free}\colon \mathrm{Mod}_k^{\mathrm{cpl}}/T_A \leftrightarrows \mathrm{cLie}(A/k)^{\mathrm{mix}}\colon U$. To establish the existence of the semi-model structure, it suffices to verify that for every cofibrant $\mathfrak{L}$ and every contractible complete complex Z over T_A, the map $\mathfrak{L} \longrightarrow \mathfrak{L}\amalg\mathrm{Free}(Z)$ is a trivial cofibration of complete A-modules [Fre09, Theorem 12.1.4]. We will prove something stronger: let us say that an object $\mathfrak{L}$ is *good* if it satisfies the following two conditions:

(a) Without differential, it is the retract of a free mixed-curved L_∞-algebroid $\mathrm{Free}(V_0)$ on a complete graded vector space over T_A.

(b) The functor $\mathfrak{L} \amalg \mathrm{Free}(-)$ sends (trivial) cofibrations of complete complexes over T_A to (trivial) cofibrations of complete A-modules.

Furthermore, we will say that a map $\mathfrak{L} \longrightarrow \mathfrak{H}$ is good if both $\mathfrak{L}$ and $\mathfrak{H}$ are good, and for every complete complex X, the map $\mathfrak{L} \amalg \mathrm{Free}(X) \longrightarrow \mathfrak{H} \amalg \mathrm{Free}(X)$ is a cofibration of complete A-modules. We now claim that every cofibration with cofibrant domain is a good map, which implies the existence of the semi-model structure, as well as the fact that all cofibrant objects are (in particular) cofibrant as complete A-modules.

To verify the claim, note that good morphisms are closed under transfinite compositions and retracts. Next, suppose that $\mathfrak{L}$ is good and let $V \longrightarrow W$ be a cofibration of complete complexes. Then any pushout $\mathfrak{L} \longrightarrow \mathfrak{L} \amalg_{\mathrm{Free}(V)} \mathrm{Free}(W)$ is good. To see this, let X be any other complex over T_A an consider the map

$$\mathfrak{L} \amalg \mathrm{Free}(X) \longrightarrow \mathfrak{L} \amalg_{\mathrm{Free}(V)} \mathrm{Free}(W \oplus X). \tag{4.13}$$

Note that without differential, the inclusion $V \longrightarrow W$ is the inclusion of a summand. Using Example 4.5 to compute free mixed-curved Lie algebroids in terms of free mixed-curved Lie algebras, the above map then takes the form

$$A \otimes \mathsf{cLie}_\infty(V_0 \oplus X) \longrightarrow A \otimes \mathsf{cLie}_\infty(V_0 \oplus W/V \oplus X)$$

where the target has some differential. As a map of complete A-modules, we can filter this map by word length in W/V. The associated graded can then be identified with $\mathfrak{L} \amalg \mathrm{Free}(W/V \oplus X)$. Since $\mathfrak{L}$ was good by assumption, this gives that the map (4.13) is a cofibration of complete A-modules, and that its codomain is a good object as well.

This implies that all cofibrations with a good domain are themselves good. It now remains to verify that the initial mixed-curved L_∞-algebroid is good, i.e. that $\mathrm{Free}(-) = A \otimes \mathsf{cLie}_\infty(-)$ sends (trivial) cofibrations of complete complexes to (trivial) cofibrations of complete A-modules. This is immediate. $\square$

Our next goal will be to give a more explicit description of the ∞-categories associated to the model categories from Theorem 2.15, in terms of ∞-morphisms:

Definition 4.14. Let $\mathfrak{L}$ and $\mathfrak{H}$ be curved L_∞-algebroids. An ∞-morphism $\phi\colon \mathfrak{L} \rightsquigarrow \mathfrak{H}$ is an ∞-morphism between the underlying k-linear curved L_∞-algebras satisfying the following two conditions:

(1) The composite ∞-morphism $\mathfrak{L} \rightsquigarrow \mathfrak{H} \xrightarrow{\rho_{\mathfrak{H}}} T_A$ agrees with the strict morphism $\rho_{\mathfrak{L}}$.

(2) Each component ϕ_n defines an A-multilinear map $\phi_n\colon \mathrm{Sym}^n_A(\mathfrak{L}[1])[-1] \longrightarrow \mathfrak{H}$.

The same definition applies to (graded) mixed-curved L_∞-algebroids and (complete, weight-graded) L_∞-algebroids.

We have the following version of the Homotopy Transfer Theorem (see also [PS20, Cam19]):

Theorem 4.15 (Homotopy Transfer Theorem)**.** *Let $\mathfrak{L}$ be a mixed-curved L_∞-algebroid over A and consider a deformation retract of complete A-modules* $V \underset{i}{\overset{p}{\leftrightarrows}} \mathfrak{L} \circlearrowleft h$ *relative to T_A (i.e. both i and p commute with the projection to T_A), satisfying the side conditions $ph = 0$, $hi = 0$ and $h^2 = 0$. Then the A-module structure on V extends to a* transferred *mixed-curved L_∞-algebroid structure, and i extends to an ∞-morphism i_∞ of mixed-curved L_∞-algebroids.*

Proof. The proof is similar to Theorem 3.12: we apply the Homotopy Transfer Theorem for k-linear mixed-curved L_∞-algebras to obtain a transferred mixed-curved L_∞-structure on V and a k-linear ∞-morphism $i_\infty\colon V \rightsquigarrow \mathfrak{L}$. Note that the homotopy h takes values in the kernel of the anchor map $\rho_{\mathfrak{L}}$, since $\rho_{\mathfrak{L}} h = \rho_V(ph)$. The formula for i_∞ then implies that all nonlinear components of $\rho_{\mathfrak{L}} \circ i_\infty$ vanish, i.e. the underlying (A-linear) map $\rho_{\mathfrak{L}} \circ i$ is a strict map of mixed-curved L_∞-algebras.

Next, recall that all operations ℓ_n for $n \neq 2$ on $\mathfrak{L}$ are A-linear. On the other hand, using that h takes values in the kernel of the anchor map and satisfies the side conditions, one sees that $h \circ (\ell_2 \circ_1 h)$, $p \circ (\ell_2 \circ_1 h)$ and $h \circ \ell_2 \circ (i, i)$ are all A-linear as well. This implies that i_∞ is A-linear and that there is only one term in the transferred structure that is not A-multilinear, namely the term $p \circ \ell_2 \circ (i, i)$ in the formula for the transferred operation ℓ_2'.

Since $p \circ \ell_2 \circ (i, i)$ precisely satisfies the Leibniz rule (and all other terms contributing to ℓ_2 are A-bilinear), the transferred k-linear curved-mixed L_∞-structure makes V a curved-mixed L_∞-algebroid. $\square$

Proposition 4.16. *Let $\mathfrak{L}$ be a mixed-curved L_∞-algebroid. Then there exists a unique mixed-curved L_∞-algebroid $Q(\mathfrak{L})$ together with a natural bijection*

$$\Big\{\textit{structure-preserving maps } Q(\mathfrak{L}) \longrightarrow \mathfrak{H}\Big\} \cong \Big\{\infty\textit{-morphisms } \mathfrak{L} \rightsquigarrow \mathfrak{H}\Big\}.$$

If $\mathfrak{L}$ is cofibrant as a complete A-module, then $Q(\mathfrak{L})$ is a cofibrant mixed-curved L_∞-algebroid and the natural map $Q(\mathfrak{L}) \longrightarrow \mathfrak{L}$ is a weak equivalence. Similarly for graded mixed-curved L_∞-algebroids and (complete, weight-graded) L_∞-algebroids.

Proof. The proof is similar to Proposition 3.18 and is slightly different from the non-curved, unfiltered case treated in [Nui19a, Section 5]. First, the existence and uniqueness of the object $Q(\mathfrak{L})$ follows from category theoretic reasons: the functor sending $\mathfrak{H}$ to the set of ∞-morphisms $\mathfrak{L} \rightsquigarrow \mathfrak{H}$ preserves limits and filtered colimits, and is hence corepresentable. In particular, there is a canonical map $\pi\colon Q(\mathfrak{L}) \longrightarrow \mathfrak{L}$ corresponding to the identity $\mathfrak{L} \rightsquigarrow \mathfrak{L}$ and a canonical ∞-morphism $v_\infty\colon \mathfrak{L} \rightsquigarrow Q(\mathfrak{L})$ corresponding to the identity on $Q(\mathfrak{L})$.

Underlying graded A-module. Next, let us identify the linear map $v_{\text{lin}}\colon \mathfrak{L} \longrightarrow Q(\mathfrak{L})$ as a map of complete graded A-modules, ignoring the differentials. We start by noting that the universal property of $Q(\mathfrak{L})$ realizes it as a certain quotient of the mixed-curved L_∞-algebroid $A \otimes \Omega\,\mathrm{Bar}(\mathfrak{L})$, where $\mathfrak{L}$ is viewed as a k-linear mixed-curved L_∞-algebra (cf. Example 4.5). Without differential, $Q(\mathfrak{L})$ can therefore be identified with the quotient of $A \otimes \big(\mathsf{cLie}_\infty \circ \mathsf{ucoCom}_+^{\mathrm{mix}} \circ \mathfrak{L}\big)$ by the following relation: viewing elements in this complex as certain height 2 trees with root labeled by A and leaves labeled by $\mathfrak{g}$, rescaling a leaf by $a \in A$ is equivalent to rescaling the root by a. Using that $\mathfrak{L}$ is (the retract of) a free A-module, this implies that $Q(\mathfrak{L})$ is a quasiprojective complete A-module.

Underlying A-module. We will now show that $Q(\mathfrak{L})$ is cofibrant as a complete A-module and that the map $\pi\colon Q(\mathfrak{L}) \longrightarrow \mathfrak{L}$ is a weak equivalence.

To this end, let us endow $\mathfrak{L}$ with an *additional, increasing* filtration such that $F_0(\mathfrak{L}) = 0 \subseteq \mathfrak{L} = F_1(\mathfrak{L})$. This induces a nonnegative increasing filtration on $Q(\mathfrak{L})$ and the map π respects these filtrations. In now suffices to show that the associated graded of $Q(\mathfrak{L})$ is cofibrant as a complete A-module and that π induces an equivalence on the associated graded.

This is most easily seen using the Rees construction, sending a complete k-module V with an increasing filtration to the $\hbar$-torsion free $k[\hbar]$-module $\bigoplus_n \hbar^n F_n(V)$. The associated graded is then the fiber at $\hbar = 0$. The Rees construction of $\mathfrak{L}$ can then be identified with the mixed-curved L_∞-algebroid $\mathfrak{L}_\hbar$ over $A[\hbar]$ given by $A[\hbar] \otimes_A \mathfrak{L}$, with brackets given by $\hbar \cdot \ell_n$ and anchor map given by

$$\hbar \cdot \rho\colon A[\hbar] \otimes_A \mathfrak{L} \longrightarrow A[\hbar] \otimes_A T_A \subseteq T_{A[\hbar]}.$$

Likewise, the Rees construction of the map $\pi\colon Q(\mathfrak{L}) \longrightarrow \mathfrak{L}$ coincides with the natural map $\pi_\hbar\colon Q(\mathfrak{L}_\hbar) \longrightarrow \mathfrak{L}_\hbar$. We therefore have to show that $\pi_\hbar$ induces a weak equivalence between cofibrant complete A-modules after setting $\hbar = 0$.

Using the adjunction from Example 4.7, ones sees that after setting $\hbar = 0$, the map $\pi_\hbar$ coincides with the map $Q(\mathfrak{L}_0) \longrightarrow \mathfrak{L}_0$ for the *trivial* mixed-curved L_∞-algebroid $\mathfrak{L}_0$, i.e. $\mathfrak{L}$ with zero brackets and zero anchor map. For these, $Q(\mathfrak{L}_0)$ coincides with the bar-cobar construction for mixed-curved L_∞-algebras in complete A-modules; this is indeed cofibrant as an A-module and equivalent to $\mathfrak{L}_0$ (cf. Proposition-Definition 2.31 and Section 3.2.2).

Cofibrancy. We have shown that $Q(\mathfrak{L})$ is cofibrant as a complete A-module and it remains to verify that it is also cofibrant as a mixed-curved L_∞-algebroid. We now argue as in Proposition 2.41 and Proposition 3.18: it suffices to verify that for any acyclic fibration $p\colon \mathfrak{H} \longrightarrow Q(\mathfrak{L})$, there exists an ∞-morphism $s_\infty\colon \mathfrak{L} \longrightarrow \mathfrak{H}$ such that $ps_\infty = v_\infty$ is the universal ∞-morphism. In fact, we can replace $\mathfrak{H}$ by a cofibrant resolution and hence assume that it is cofibrant as an A-module.

Since $Q(\mathfrak{L})$ is cofibrant as an A-module, there exists an A-linear section i of p and an A-linear homotopy that realizes $Q(\mathfrak{L})$ as a deformation retract of $\mathfrak{H}$ relative to T_A. We can now apply the argument from Lemma 2.42, using the Homotopy Transfer Theorem 4.15. $\square$

Definition 4.17. We will denote by $\mathbf{cLie}(A/k)^{\mathrm{mix}}$ the ∞-category corresponding to the following simplicially enriched category:

(0) objects are *fibrant* mixed-curved L_∞-algebroids over A whose underlying complete A-module is cofibrant.

(1) simplicial sets of morphisms consist of ∞-morphisms $\mathfrak{L} \rightsquigarrow \mathfrak{H} \boxtimes \Omega[\Delta^n]$, using the tensoring from Example 4.6.

Likewise, we will write $\mathbf{Lie}(A/k), \mathbf{Lie}(A/k)^{\mathrm{gr}}$ and $\mathbf{cLie}(A/k)^{\mathrm{gr-mix}}$ for the ∞-categories of L_∞-algebroids, weight-graded L_∞-algebroids and graded mixed-curved L_∞-algebroids, respectively. In each case, objects are required to have a cofibrant underlying (weight-graded) A-module and a surjective anchor map, and morphisms are ∞-morphisms.

Corollary 4.18. *The simplicially enriched category* $\mathbf{cLie}(A/k)^{\mathrm{mix}}$ *presents the ∞-categorical localization of the category of mixed-curved L_∞-algebroids at the weak equivalences. In particular,* $\mathbf{cLie}(A/k)^{\mathrm{mix}}$ *is a presentable ∞-category. The same assertions holds for graded mixed-curved L_∞-algebroids and (weight-graded) L_∞-algebroids.*

Proof. Exactly as Proposition 2.45. Note that the ∞-categorical localization of the semi-model category $\mathrm{cLie}(A/k)^{\mathrm{mix}}$ is presentable because it is equivalent to that of the (Quillen equivalent) combinatorial model category $\tilde{0}/\mathrm{cLie}(A/k)^{\mathrm{mix}}$ of mixed-curved L_∞-algebroids under a fibrant-cofibrant replacement of the initial object. $\square$

Definition 4.19. The ∞-category $\mathbf{cLie}(A/k)$ of curved L_∞-algebroids over A is the ∞-category associated to the simplicially enriched category whose:

(0) objects are curved L_∞-algebroids such that the anchor $\mathfrak{L} \longrightarrow T_A\langle 0\rangle$ is surjective, the underlying complete A-module is quasiprojective and $\mathrm{Gr}(A)$ is a cofibrant graded A-module.

(1) simplicial sets of morphisms consist of ∞-morphisms $\mathfrak{L} \rightsquigarrow \mathfrak{H} \boxtimes \Omega[\Delta^n]$.

Proposition 4.20. *There is a sequence of functors between presentable ∞-categories*

$$\mathbf{cLie}(A/k)^{\mathrm{gr-mix}} \xrightarrow{\mathrm{Tot}} \mathbf{cLie}(A/k)^{\mathrm{mix}} \xrightarrow{\mathrm{blend}} \mathbf{cLie}(A/k)$$

whose composite is an equivalence. In particular, the ∞-category of curved L_∞-algebroids is presentable.

Proof. As Proposition 2.62 and Theorem 3.24; the assumption that all objects are quasiprojective as complete A-modules implies that we can split their filtration A-linearly. $\square$

Let us conclude with the following observation about the three canonical filtrations on an L_∞-algebroid from Example 4.8:

Proposition 4.21. *The three filtrations from Example 4.8 determine fully faithful right adjoint functors of ∞-categories*

$$\begin{aligned}
\mathbf{Lie}(A/k) &\xrightarrow{\mathfrak{L}\mapsto\mathfrak{L}\langle 0\rangle} \mathbf{cLie}(A/k)\\
\mathbf{Lie}(A/k) &\xrightarrow{\mathfrak{L}\mapsto\mathfrak{L}\langle 1\rangle} \mathbf{cLie}(A/k)_{/T_A\langle 1\rangle}\\
\mathbf{Lie}(A/k) &\xrightarrow{\mathfrak{L}\mapsto\mathfrak{L}^{\mathrm{anc}}} \mathbf{cLie}(A/k)_{/T_A\langle 1\rangle}.
\end{aligned}$$

Proof. We will only treat the 'anchor filtration', the others are similar but easier. We will present the ∞-functor $\mathfrak{L}\mapsto\mathfrak{L}^{\mathrm{anc}}$ by a right Quillen functor; to do this, let $\mathfrak{t}_A \xrightarrow{\sim} T_A$ be an equivalent L_∞-algebroid whose underlying A-module is cofibrant, and let $\tilde{0}_A$ denote the free L_∞-algebroid generated by the map of A-modules $\mathfrak{t}_A[0,-1]\longrightarrow\mathfrak{t}_A\longrightarrow T_A$ from the path space of $\mathfrak{t}_A$. Then $\tilde{0}_A\simeq 0$ is weakly equivalent to the initial L_∞-algebroid.

Consider the category $\tilde{0}_A/\mathrm{Lie}(A/k)/\mathfrak{t}_A$ of L_∞-algebroids that fit into a diagram $\tilde{0}_A\longrightarrow\mathfrak{L}\longrightarrow\mathfrak{t}_A$. Equivalently, this is the category of $\mathfrak{L}\longrightarrow\mathfrak{t}_A$ which come equipped with an A-linear section (which need not preserve differentials), inducing a decomposition $\mathfrak{L}=\mathfrak{t}_A\oplus\mathfrak{n}$. This carries a model structure induced from the semi-model structure on all L_∞-algebroids, and forgetting the maps from $\tilde{0}_A$ and to $\mathfrak{t}_A$ relate the two by a zig-zag of Quillen equivalences.

Likewise, let $\mathrm{cLie}(A/k)^{\mathrm{gr-mix}}/\mathfrak{t}_A\langle 1\rangle$ denote the category of graded mixed-curved L_∞-algebroids with a map to $\mathfrak{t}_A\langle 1\rangle$. The forgetful functor to all graded mixed-curved L_∞-algebroids is a right Quillen equivalence. The functor $\mathfrak{L}\mapsto\mathfrak{L}^{\mathrm{anc}}$ can then be presented by the right Quillen functor

$$\tilde{0}_A/\mathrm{Lie}(A/k)/\mathfrak{t}_A \longrightarrow \mathrm{cLie}(A/k)^{\mathrm{gr-mix}}/\mathfrak{t}_A\langle 1\rangle; \qquad \big(\mathfrak{L}=\mathfrak{t}_A\oplus\mathfrak{n}\to\mathfrak{t}_A\big)\longmapsto\mathfrak{L}^{\mathrm{anc}}$$

where the graded mixed-curved L_∞-algebroid $\mathfrak{L}^{\mathrm{anc}}$ is given in weight 0 by $\mathfrak{n}$ and in weight -1 by $\mathfrak{t}_A$ (and the mixed-curved L_∞-structure on the total complex $\mathfrak{n}\oplus\mathfrak{t}_A$ is that of $\mathfrak{L}$). The left adjoint Φ sends a graded mixed-curved L_∞-algebroid $\mathfrak{H}$ to (a) its quotient by the ideal generated by all $\mathfrak{H}\langle p\rangle$ with $p\neq 0,1$ and by $\ker\big(\mathfrak{H}\langle 1\rangle\longrightarrow\mathfrak{t}_A\langle 1\rangle\big)$ (in particular, the result has no curvature), and (b) then takes the associated total L_∞-algebroid (forgetting the filtration).

Now notice that the set of (curved) ∞-morphisms $\mathfrak{L}^{\mathrm{anc}}\longrightarrow\mathfrak{H}^{\mathrm{anc}}$ is isomorphic to the set of ∞-morphisms $\mathfrak{L}\longrightarrow\mathfrak{H}$. In particular, there is no ϕ_0 because $\mathfrak{H}^{\mathrm{anc}}$ is zero is positive weights. By adjunction, this means that the functor Φ preserves the 'bar-cobar resolution' of Proposition 4.16: $\Phi(Q(\mathfrak{L}^{\mathrm{anc}}))\cong Q(\mathfrak{L})$. Since $Q(\mathfrak{L}^{\mathrm{anc}})\longrightarrow\mathfrak{L}^{\mathrm{anc}}$ is a cofibrant resolution whenever $\mathfrak{L}$ is cofibrant as an A-module, this implies that the derived counit $\mathbb{L}\Phi(\mathfrak{L}^{\mathrm{anc}})\longrightarrow\mathfrak{L}$ is an equivalence, hence $(-)^{\mathrm{anc}}$ induces a fully faithful functor of ∞-categories. □

4.3 Curved Lie algebroids as non-curved Lie algebroids

The goal of this section is to give a more intrinsic description of the ∞-category of curved L_∞-algebroids in terms on *non-curved* L_∞-algebroids, using (the Koszul dual of) the *Rees construction.* Note that taking $A=k$ the base field, this also gives a description of the ∞-category of curved L_∞-algebras studied in Section 2.5.

Definition 4.22. Let us denote by $\mathcal{R}(T_A)$ the weight-graded Lie algebroid over A

$$\mathcal{R}(T_A) \coloneqq T_A\ltimes A\langle -1\rangle[-1]$$

given by the direct sum of T_A (in weight 0) and the free A-module on a generator θ of weight 1 and degree 1, such that $[\theta,\theta]=0$ and $[v,a\cdot\theta]=\mathcal{L}_v(a)\cdot\theta$ for $v\in T_A$.

Theorem 4.23. *There are equivalences of presentable ∞-categories*

$$\mathbf{cLie}(A/k) \xleftarrow{\sim} \mathbf{cLie}(A/k)^{\mathrm{gr-mix}} \xrightarrow{\sim} \mathbf{Lie}(A/k)^{\mathrm{gr}}/_{\mathcal{R}(T_A)}$$

between the ∞-categories of curved L_∞-algebroids, graded mixed-curved L_∞-algebroids and graded L_∞-algebroids over $\mathcal{R}(T_A) = T_A \ltimes A\langle -1\rangle[-1]$.

Remark 4.24. From a geometric perspective, one can informally think of weight-graded Lie algebroids over A as maps of formal stacks

$$\mathrm{Spec}(A) \times B\mathbb{G}_m \longrightarrow Y \longrightarrow \mathrm{Spec}(A)_{\mathrm{dR}} \times B\mathbb{G}_m$$

where the first (equivalently second) map is a nil-isomorphism. The structure map to $B\mathbb{G}_m$ gives rise to the weight-grading. The weight-graded Lie algebroid $\mathcal{R}(T_A)$ then corresponds to the formal stack $\mathrm{Spec}(A)_{\mathrm{dR}} \times \widehat{\mathbb{A}}^1/\mathbb{G}_m$. In other words, curved L_∞-algebroids over A describe nil-isomorphisms of formal stacks

$$\mathrm{Spec}(A) \times B\mathbb{G}_m \longrightarrow Y \longrightarrow \mathrm{Spec}(A)_{\mathrm{dR}} \times \widehat{\mathbb{A}}^1/\mathbb{G}_m.$$

In particular, the structure morphism to $\widehat{\mathbb{A}}^1/\mathbb{G}_m$ endows Y with a complete filtration, and Y maps to $\mathrm{Spec}(A)_{\mathrm{dR}}$ equipped with the *trivial* filtration.

Let us start by constructing the functor from graded mixed-curved L_∞-algebroids to graded L_∞-algebroids over $\mathcal{R}(T_A)$.

Construction 4.25. Suppose that $\mathfrak{L}$ is a graded mixed-curved L_∞-algebroid over A and consider the weight-graded (non-differential) graded A-module

$$\mathcal{R}(\mathfrak{L}) = \mathfrak{L} \oplus A\langle -1\rangle[-1].$$

The anchor map of $\mathfrak{L}$ induces a map $\mathcal{R}(\mathfrak{L}) \longrightarrow T_A \ltimes A\langle -1\rangle[-1]$. Recall from Definition 2.60 that $\mathfrak{L}$ comes with brackets

$$\ell_n^p \colon \mathfrak{L}^{\otimes n} \longrightarrow \mathfrak{L}$$

of weight $p \geq 0$ (except for ℓ_0^p and ℓ_1^p, which are only defined for weights $p \geq 1$). Using these, we define n-ary operations ℓ_n of weight 0 on $\mathcal{R}(\mathfrak{L})$ by

$$\ell_n\big(\theta, \dots, \theta, x_{p+1}, \dots, x_n\big) := p! \cdot \ell_{n-p}^p\big(x_{p+1}, \dots, x_n\big)$$

for p copies of θ and $x_{p+1}, \dots, x_n \in \mathfrak{L}$. In particular, all maps ℓ_n take values in $\mathfrak{L} \subseteq \mathcal{R}(\mathfrak{L})$.

Proposition 4.26. *The operations $\{\ell_n\}$ make the map $\mathcal{R}(\mathfrak{L}) \longrightarrow \mathcal{R}(T_A)$ a map of weight-graded L_∞-algebroids if and only if the operations $\{\ell_n^p\}$ make $\mathfrak{L}$ a curved L_∞-algebroid.*

Proof. Note that in the weight-graded case, all ℓ_n^p are A-multilinear except ℓ_2^0, which satisfies the Leibniz rule. One easily sees that this is equivalent to the ℓ_n all being A-linear, except for ℓ_2 which satisfies the Leibniz rule.

It then suffices to verify that the $\{\ell_n\}$ define a k-linear L_∞-structure on $\mathcal{R}(\mathfrak{L})$ if and only if the ℓ_n^p define a k-linear graded mixed-curved L_∞-algebra structure on $\mathfrak{L}$. To see this, note that unshuffles σ of an n-element set $(\theta, \dots, \theta, x_{p+1}, \dots, x_n)$ are in 1-1 correspondence with pairs consisting of an unshuffle τ of the p-element set $(\theta, \dots, \theta)$ and an unshuffle σ' of the set $(x_{p+1},\dots,x_n)$. Denoting $m=n-p$, the L_∞-condition for the ℓ_n then translates into

$$\begin{aligned}
\big[d,\ell_n\big](\theta,\dots,\theta,x_{p+1},\dots,x_n) &\overset{!}{=} \sum_{i+j=n+1}\ \sum_{\sigma\in\mathrm{Sh}^{-1}_{i-1,j}} \pm\big(\ell_i\circ_1\ell_j\big)^{\sigma}(\theta,\dots,\theta,x_{p+1},\dots,x_n)\\
&= \sum_{\substack{i'+j'=m+1\\ q+r=p}}\ \sum_{\substack{\sigma'\in\mathrm{Sh}^{-1}_{i'-1,j'}\\ \tau\in\mathrm{Sh}^{-1}_{q,r}}} \pm q!r!\big(\ell^q_{i'}\circ_1\ell^r_{j'}\big)^{\sigma'}(x_{p+1},\dots,x_n)\\
&= \sum_{q+r=p}\ \sum_{i'+j'=m+1}\ \sum_{\sigma'\in\mathrm{Sh}^{-1}_{i'-1,j'}} \pm p!\cdot\big(\ell^q_{i'}\circ_1\ell^r_{j'}\big)^{\sigma'}(x_{p+1},\dots,x_n)
\end{aligned}$$

where the $\pm$ signs are determined by Remark 2.57. Here we used that the unshuffles τ (which only permutes copies of θ) leave the values invariant. Since $\big[d,\ell_n\big](\theta,\dots,x_n)=p!\cdot\big[d,\ell^p_m\big](x_{p+1},\dots,x_n)$, the above equation can be identified with the graded mixed-curved L_∞-equation for $\mathfrak{L}$. □

Example 4.27. Let $T_A\langle1\rangle$ be as in Example 4.8. Viewing $T_A\langle1\rangle$ as a graded mixed-curved Lie algebroid (with trivial curvature), the Rees construction of $T_A\langle1\rangle$ is given by $T_A\langle1\rangle\ltimes A\langle-1\rangle[1]$. The Chevalley–Eilenberg algebra of this graded Lie algebroid is given by the $\hbar$-de Rham complex (cf. Example 3.4)

$$\mathrm{dR}(A)[\![\hbar]\!],\qquad\qquad d\alpha=d_A(\alpha)+\hbar\cdot d_{\mathrm{dR}}\alpha,\qquad d\hbar=0,$$

where $\hbar$ has degree 0 (weight -1) and 1-forms have weight 1. Using the equivalence between graded $k[\![\hbar]\!]$-algebras and filtered algebras (via the classical Rees construction), this corresponds to the de Rham algebra of A, endowed with the Hodge filtration. From the perspective of Remark 4.24, the weight-graded L_∞-algebroid $\mathcal{R}\big(T_A(1)\big)$ corresponds to the *Hodge stack*

$$\mathrm{Spec}(A)\times B\mathbb{G}_m\longrightarrow\mathrm{Spec}(A)^{\mathrm{Hodge}}\longrightarrow\mathrm{Spec}(A)_{\mathrm{dR}}\times\widehat{\mathbb{A}}^1/\mathbb{G}_m.$$

This is the formal stack whose structure map to $\widehat{\mathbb{A}}^1/\mathbb{G}_m$ encodes the Hodge filtration on the de Rham complex.

Remark 4.28. Consider the zero curved L_∞-algebroid 0 and notice that 0 is *not* the initial curved L_∞-algebroid: instead, the category of curved L_∞-algebroids under 0 is simply the category of (non-curved) L_∞-algebroids. In fact, the space of ∞-morphisms $0\rightsquigarrow\mathfrak{L}$ can be identified with the space of Maurer–Cartan elements in $F^1\ker(\mathfrak{L}\longrightarrow T_A)$ (cf. Section 2.5.2). The Rees construction $\mathcal{R}(0)$ is the trivial graded Lie algebroid $A\langle-1\rangle[1]$, with zero bracket and anchor map. The corresponding Chevalley–Eilenberg algebra is simply $A[\![\hbar]\!]$ with $\hbar$ of degree 0 and weight -1. Geometrically, this corresponds to $\mathrm{Spec}(A)\times\widehat{\mathbb{A}}^1/\mathbb{G}_m$. Informally, we can therefore think of the curvature of a curved L_∞-algebroid $\mathfrak{L}$ in terms of the corresponding formal stack Y as the obstruction to finding a dotted lift

$$\begin{tikzcd}
\mathrm{Spec}(A)\times B\mathbb{G}_m \arrow[rr] \arrow[d] & & Y \arrow[d]\\
\mathrm{Spec}(A)\times\widehat{\mathbb{A}}^1/\mathbb{G}_m \arrow[r] \arrow[urr, dotted] & \mathrm{Spec}(A)_{\mathrm{dR}}\times\widehat{\mathbb{A}}^1/\mathbb{G}_m. &
\end{tikzcd}$$

Proof of Theorem 4.23. The equivalence $\mathbf{cLie}(A/k)^{\mathrm{gr-mix}}\longrightarrow\mathbf{cLie}(A/k)$ already appeared in Proposition 4.20. The equivalence $\mathbf{cLie}(A/k)^{\mathrm{gr-mix}}\longrightarrow\mathbf{Lie}(A/k)^{\mathrm{gr}}/\mathcal{R}(T_A)$ arises from a Quillen equivalence. Indeed, let $\tilde{0}$ denote the free weight-graded L_∞-algebroid generated by the contractible complex $k(-1)[-1,-2]$ in weight 1. There is a canonical map $\tilde{0} \longrightarrow \mathcal{R}(T_A)$ sending the degree 1 generator of $\tilde{0}$ to the element $\theta \in A(-1)[-1]/\mathcal{R}(T_A)$.

Now consider the category $\mathcal{C}$ of weight-graded L_∞-algebroids over A equipped with maps $\tilde{0} \longrightarrow \mathfrak{L} \longrightarrow \mathcal{R}(T_A)$. Equivalently, this is the category of weight-graded L_∞-algebroids $\mathfrak{L}$ over $\mathcal{R}(T_A)$, equipped with a compatible A-linear decomposition $\mathfrak{L} \cong \mathfrak{L}' \oplus A\langle -1\rangle[-1]$ (which need not respect differentials). This carries a semi-model structure induced by the semi-model structure on all weight-graded L_∞-algebroids from Theorem 4.12. Since $\tilde{0}$ is cofibrant and weakly contractible, forgetting the splitting induces a Quillen equivalence to the category of weight-graded L_∞-algebroids over $\mathcal{R}(T_A)$, whose associated ∞-category is exactly $\mathbf{Lie}(A/k)^{\mathrm{gr}}/_{\mathcal{R}(T_A)}$.

On the other hand, using Proposition 4.26 one sees that the functor $\mathfrak{L} \mapsto \mathcal{R}(\mathfrak{L})$ defines an equivalence of categories from the category of graded mixed-curved L_∞-algebroids to $\mathcal{C}$. This equivalence identifies the model structures on both sides, from which the result follows. $\square$

Remark 4.29. Although we do not need this, one can verify that the Rees construction (Construction 4.25) is compatible with ∞-morphisms as well.

5 The equivalence between curved Lie algebras and Lie algebroids

The goal of this section is to prove that for a nonpositively graded cdga A satisfying a certain finiteness condition, there is an equivalence between the ∞-category of Lie algebroids over A and the ∞-category of certain types of curved Lie algebras over its (completed) de Rham algebra $\mathrm{dR}(A)$, endowed with the Hodge filtration.

To this end, let us recall that a nonpositively graded cdga is said to be *locally of finite presentation* if it is a compact object in the category of nonpositively graded cdgas; equivalently, it is the retract of a cdga with finitely many generators and relations. Note that a nonpositively graded cdga A is cofibrant and locally of finite presentation if and only if it is the retract of a quasi-free cdga with finitely many generators, each of which is in nonpositive degree. Indeed, one can always realize such a cofibrant A as a retract of a nonpositively graded quasi-free cdga with infinitely many generators; if A is a compact object, it must already be a retract of a subalgebra on finitely many generators.

Theorem 5.1. *Let A be a nonpositively graded cdga. Suppose that A is smooth or cofibrant and locally of finite presentation, and let $\mathrm{dR}(A)$ denote the de Rham algebra of A, endowed with the Hodge filtration. Then there is a fully faithful functor of ∞-categories*

$$\mathrm{curv}\colon \mathbf{Lie}(A/k) \hookrightarrow \mathbf{cLie}_{\mathrm{dR}(A)}$$

whose essential image consists of curved L_∞-algebras $\mathfrak{g}$ over $\mathrm{dR}(A)$ such that the canonical map $\mathrm{gr}_0(\mathfrak{g}) \otimes_{\mathrm{gr}_0(\mathrm{dR}(A))} \mathrm{gr}(\mathrm{dR}(A)) \longrightarrow \mathrm{gr}(\mathfrak{g})$ is an equivalence.

In fact, we will deduce this from a more general result about curved L_∞-algebroids over complete filtered algebras of the form $C^*(\mathfrak{t})$, where $\mathfrak{t}$ is a complete L_∞-algebroid over A; for the filtration on $C^*(\mathfrak{t})$ to behave like the Hodge filtration on $\mathrm{dR}(A)$, we require $F^0(\mathfrak{t}) = 0$. We will then prove the following result:

Theorem 5.2. *Let A be a nonpositively graded cdga and let $\mathfrak{t}$ be a complete L_∞-algebroid over A such that $F^0(\mathfrak{t}) = 0$ and each $F^i(\mathfrak{t})$ is finitely generated quasiprojective as an A-module. Then there is an equivalence of ∞-categories*

$$\mathrm{curv}\colon \mathbf{cLie}(A/k)/_{\mathfrak{t}} \xrightarrow{\sim} \mathbf{cLie}_{C^*(\mathfrak{t})}.$$

Example 5.3. In the situation of Theorem 5.1, one can take $\mathfrak{t} = T_A\langle 1\rangle$ the tangent complex of A, put in filtration degree -1 (Example 4.8), whose Chevalley–Eilenberg algebra is precisely the de Rham algebra $\mathrm{dR}(A)$, equipped with the Hodge filtration (Example 4.11). Theorem 5.2 provides an equivalence between curved L_∞-algebras over $\mathrm{dR}(A)$ (with the Hodge filtration) and curved L_∞-algebroids over A with a map to $T_A\langle 1\rangle$. By Theorem 4.23 one can also identify this second ∞-category with the ∞-category of weight-graded L_∞-algebroids over the Rees construction of $T_A\langle 1\rangle$, which corresponds to the Hodge stack of $\mathrm{Spec}(A)$ (Example 4.27).

The majority of this section is devoted to the proof of Theorem 5.2. Throughout, we will assume that $\mathfrak{t}$ is a complete L_∞-algebroid over A as in the theorem, and we will write

$$B := C^*(\mathfrak{t}) \cong \mathrm{Hom}_A\big(\mathrm{Sym}_A(\mathfrak{t}[1]), A\big) \cong \mathrm{Sym}_A\big(\mathfrak{t}^\vee[-1]\big)$$

for its (filtered) Chevalley–Eilenberg algebra. The last isomorphism of complete graded algebras holds because $F^0\mathfrak{t} = 0$ and each $F^i\mathfrak{t}$ is a dualizable A-module, so that each $F^i\mathrm{Sym}_A(\mathfrak{t}[1])$ is a dualizable A-module as well.

5.1 Curved L_∞-algebras from curved L_∞-algebroids

In this section we will show how to associate a curved L_∞-algebra over B to a curved L_∞-algebroid over $\mathfrak{t}$, equipped with a linear section $\sigma\colon \mathfrak{t} \longrightarrow \mathfrak{L}$. Furthermore, we will show how ∞-morphisms between such curved L_∞-algebroids over $\mathfrak{t}$ give rise to ∞-morphisms between the associated curved L_∞-algebras over B. In Section 5.2, we will then show how these constructions give rise to a functor between simplicially enriched categories which presents the desired functor 'curv' in Theorem 5.2.

5.1.1 Definition on objects

Let us start by describing how to associate a curved L_∞-algebra over B to certain curved L_∞-algebroids with a map to $\mathfrak{t}$. More precisely, our goal will be to prove the following:

Proposition 5.4. *Let $\mathfrak{n}$ be a complete graded A-module and let $\pi\colon \mathfrak{t}\oplus\mathfrak{n} \longrightarrow \mathfrak{t}$ denote the canonical projection. Then there is a natural bijection (given by Construction 5.11) between:*

(1) curved L_∞-algebroid structures on $\mathfrak{t}\oplus\mathfrak{n}$ such that π is a (strict) map of curved L_∞-algebroids.

(2) curved L_∞-algebra structures on the complete graded B-module $B\otimes_A\mathfrak{n}$.

To prove this, let us start with some preliminary observations.

Remark 5.5. Because $F^0\mathfrak{t} = 0$ and each $F^i\mathfrak{t}$ is a dualizable complete A-module by assumption, we have that $\mathrm{Sym}_A(\mathfrak{t}[1])$ is a dualizable complete A-module as well, whose dual is $\mathrm{Sym}_A(\mathfrak{t}^\vee[-1])$. Consequently, for every complete graded A-module $\mathfrak{n}$, there are isomorphisms of complete graded vector spaces

$$B\otimes_A\mathfrak{n} \cong \mathrm{Sym}_A(\mathfrak{t}^\vee[-1])\otimes_A\mathfrak{n} \cong \mathrm{Hom}_A\big(\mathrm{Sym}_A(\mathfrak{t}[1]),\mathfrak{n}\big) = \prod_p \mathrm{Hom}_A\big(\mathrm{Sym}^p_A(\mathfrak{t}[1]),\mathfrak{n}\big).$$

In other words, we can identify $B\otimes_A\mathfrak{n}$ with $\mathfrak{n}$-valued forms on $\mathfrak{t}$.

To compare the structure maps of $\mathfrak{t}\oplus\mathfrak{n}$ and $B\otimes_A\mathfrak{n}$, let us consider the sets of maps

$$\begin{aligned}\mathcal{H}(p) &:= \mathrm{Hom}_B\Big(\mathrm{Sym}^p_B\big(B\otimes_A\mathfrak{n}[1]\big), B\otimes_A\mathfrak{n}[2]\Big)\\ \mathcal{H}(1) &:= \mathrm{Der}_B\Big(B\otimes_A\mathfrak{n}[1], B\otimes_A\mathfrak{n}[2]\Big)\end{aligned}\tag{5.6}$$

where in the second line we take (graded) derivations with respect to the Chevalley–Eilenberg differential on $B = C^*(\mathfrak{t})$. The structure maps of a curved L_∞-structure on $B \otimes_A \mathfrak{n}$ are exactly contained in these sets.

Lemma 5.7. *For $p \neq 1$, there are bijections*

$$\mathcal{H}(p) \cong \prod_i \operatorname{Hom}_A\Big(\operatorname{Sym}^i_A(\mathfrak{t}[1]) \otimes_A \operatorname{Sym}^p_A(\mathfrak{n}[1]), \mathfrak{n}[2]\Big).$$

Furthermore, there is an inclusion

$$\mathcal{H}(1) \longleftrightarrow \textstyle\prod_i \operatorname{Hom}_k\Big(\operatorname{Sym}^i_A(\mathfrak{t}[1]) \otimes \mathfrak{n}[1], \mathfrak{n}[2]\Big).$$

whose image consists precisely of tuples of maps $f_1^{(i)} \colon \operatorname{Sym}^i_A(\mathfrak{t}[1]) \otimes \mathfrak{n}[1] \longrightarrow \mathfrak{n}[2]$ that are A-multilinear for $i \neq 0, 1$, while for $x \in \mathfrak{n}[1]$, $t \in \mathfrak{t}[1]$:

$$\begin{aligned} f_1^{(0)}(a \cdot x) &= d_A(a) \cdot x + a \cdot f_1^{(1)}(x) \\ f_1^{(1)}(a_1 \cdot t, a_2 \cdot x) &= a_1 \mathcal{L}_{\rho(t)}(a_2) \cdot x + a_1 a_2 \cdot f_1^{(1)}(t, x). \end{aligned} \tag{5.8}$$

In the above lemma and throughout, we will use the following convention: maps with p inputs from $\mathfrak{n}[1]$ and i inputs from $\mathfrak{t}[1]$ will be denoted by $f_p^{(i)}$.

Proof. The domain of a map in $\mathcal{H}(p)$ is the free B-module on $\operatorname{Sym}^p_A(\mathfrak{n}[1])$. In particular, for $p \neq 1$ we can identify maps $f \in \mathcal{H}(p)$ with A-linear maps $f \colon \operatorname{Sym}^p_A(\mathfrak{n}[1]) \longrightarrow B \otimes_A \mathfrak{n}[2]$. Using Remark 5.5, such A-linear maps f correspond by adjunction to tuples of A-linear maps

$$f_p^{(i)} \colon \operatorname{Sym}^i_A(\mathfrak{t}[1]) \otimes_A \operatorname{Sym}^p_A(\mathfrak{n}[1]) \longrightarrow \mathfrak{n}[2].$$

For $p = 1$, note that a derivation $f_1 \colon B \otimes_A \mathfrak{n}[1] \longrightarrow B \otimes_A \mathfrak{n}[1]$ is uniquely determined by a map $f_1 \colon \mathfrak{n}[1] \longrightarrow B \otimes_A \mathfrak{n}[2]$ with the property that for all $x \in \mathfrak{n}[1]$, $a \in A$

$$f_1(a \cdot x) = d_B(a) \cdot x + a \cdot f_1(x).$$

Again, it follows from Remark 5.5 that f_1 is determined by adjunction by a family of maps $f_1^{(i)} \colon \operatorname{Sym}^i_A(\mathfrak{t}[1]) \otimes_k \mathfrak{n}[1] \longrightarrow \mathfrak{n}[2]$. Using that $d_B(a) = d_A(a) + \mathcal{L}_{\rho(-)}(a)$, the above derivation rule then translates into the equations (5.8). $\square$

Remark 5.9. Let $\{f_1^{(i)}\}_{i \geq 0}$ be a family of maps $f_1^{(i)} \colon \operatorname{Sym}^i_A(\mathfrak{t}[1]) \otimes \mathfrak{n}[1] \longrightarrow \mathfrak{n}[2]$ as in Lemma 5.7. Then the induced derivation f_1 on $B \otimes_A \mathfrak{n}[1]$ can be computed explicitly as follows: for $\alpha \colon \operatorname{Sym}_A(\mathfrak{t}[1]) \longrightarrow \mathfrak{n}$ an $\mathfrak{n}$-valued form and $t_i \in \mathfrak{t}[1]$, we have

$$\begin{aligned} f_1(\alpha)(t_1, \ldots, t_n) = \sum_{i \geq 1} \sum_{\sigma \in \mathrm{Sh}^{-1}_{i,n-i}} \Big(& f_1^{(i)}\big(t_{\sigma(1)}, \ldots, t_{\sigma(i)}, \alpha(t_{\sigma(i+1)}, \ldots, t_{\sigma(n)})\big) \\ & - \alpha\big(\ell_{\mathfrak{t}}^{(i)}(t_{\sigma(1)}, \ldots, t_{\sigma(i)}), t_{\sigma(i+1)}, \ldots, t_{\sigma(n)}\big)\Big). \end{aligned}$$

Here $\ell_{\mathfrak{t}}^{(i)} \colon \operatorname{Sym}^n(\mathfrak{t}[1]) \longrightarrow \mathfrak{t}[2]$ denotes the L_∞-algebroid structure on $\mathfrak{t}$.

Observation 5.10. The collection of $\mathcal{H}(p)$ has a structure similar to a convolution Lie algebra, in the sense that for any $f_p \in \mathcal{H}(p)$ and $f_q \in \mathcal{H}(q)$, there is a B-linear map

$$[f_p, f_q] \colon \operatorname{Sym}^{p+q-1}_B\big(B \otimes_A \mathfrak{n}[1]\big) \longrightarrow B \otimes_A \mathfrak{n}[3]$$

given by the graded commutator

$$[f_p, f_q] = \sum_{\sigma\in \mathrm{Sh}^{-1}_{p-1,q}} \big(f_p \circ_1 f_q\big)^{\sigma} + \sum_{\tau\in \mathrm{Sh}^{-1}_{q-1,p}} \big(f_q \circ_1 f_p\big)^{\tau}.$$

Let us identify this commutator in terms of the decomposition from Lemma 5.7, i.e. identifying $f_p = \sum_i f_p^{(i)}$ as a sum of maps with i inputs from $\mathfrak{t}$ and p inputs from $\mathfrak{n}$. We have for $p, q \neq 1$, that

$$[f_p, f_q] = \sum_{i,j} \sum_{\tau\in \mathrm{Sh}^{-1}_{i,j}} \Bigg(\sum_{\sigma\in \mathrm{Sh}^{-1}_{p-1,q}} \big(f_p^{(i)} \circ_{\mathfrak{n}} f_q^{(j)}\big)^{\sigma\times\tau} + \sum_{\sigma\in \mathrm{Sh}^{-1}_{q-1,p}} \big(f_q^{(j)} \circ_{\mathfrak{n}} f_p^{(i)}\big)^{\sigma\times\tau}\Bigg).$$

Here we symmetrize with respect to the inputs coming from $\mathfrak{n}$, as well as all inputs from $\mathfrak{t}$, and use $f_p^{(i)} \circ_{\mathfrak{n}} f_q^{(j)}$ to denote partial composition along the first $\mathfrak{n}$-variable.

The description of the commutator $[f_p, f_1]$ is slightly more involved, since f_1 is extended from its restriction to $\mathfrak{n}[1]$ as a derivation. By Remark 5.9, we have that

$$[f_p, f_1] = \sum_{i,j} \Bigg(\sum_{\tau\in \mathrm{Sh}^{-1}_{i,j}} \Bigg(\big(f_1^{(j)} \circ_{\mathfrak{n}} f_p^{(i)}\big)^{\tau} + \sum_{\sigma\in \mathrm{Sh}^{-1}_{p-1,1}} \big(f_p^{(i)} \circ_{\mathfrak{n}} f_1^{(j)}\big)^{\sigma\times\tau}\Bigg) + \sum_{\tau\in \mathrm{Sh}^{-1}_{i-1,j}} \big(f_p^{(i)} \circ_{\mathfrak{t}} \ell_{\mathfrak{t}}^{(j)}\big)^{\tau}\Bigg).$$

Here the first term is just the commutator (suitably symmetrized in the $\mathfrak{t}$-variables), and in the second term $- \circ_{\mathfrak{t}} \ell_{\mathfrak{t}}^{(j)}$ takes the partial composition in the first $\mathfrak{t}$-variable with a structure map of $\mathfrak{t}$.

Let us now turn to the main construction behind Proposition 5.4:

Construction 5.11. Let $\mathfrak{L} = \mathfrak{t} \oplus \mathfrak{n}$ be a complete graded A-module, let $\pi\colon \mathfrak{L} \longrightarrow \mathfrak{t}$ be the natural projection and let $\ell_{\mathfrak{t}}^{(n)}$ denote the n-ary bracket of the L_∞-algebroid structure on $\mathfrak{t}$. Then a curved L_∞-algebroid structure on $\mathfrak{L}$ such that π is a (strict) map of curved L_∞-algebroids has structure maps of the form

$$\big(\ell_{\mathfrak{t}}^{(s)} \circ \pi, \mathcal{L}_s\big)\colon \mathrm{Sym}^s_k\big(\mathfrak{t}[1] \oplus \mathfrak{n}[1]\big) \longrightarrow \mathfrak{t}[2] \oplus \mathfrak{n}[2].$$

In other words, the $\mathfrak{t}$-component of the bracket on $\mathfrak{t} \oplus \mathfrak{n}$ is given by the brackets of $\mathfrak{t}$. Expanding binomially, such a curved L_∞-algebroid structure is therefore determined by maps

$$\ell_p^{(i)}\colon \mathrm{Sym}^i_k\big(\mathfrak{t}[1]\big) \otimes \mathrm{Sym}^p_k\big(\mathfrak{n}[1]\big) \longrightarrow \mathfrak{n}[2],$$

such that

$$\mathcal{L}_s = \sum_i \sum_{\gamma\in \mathrm{Sh}^{-1}_{i,s-i}} \big(\ell_{s-i}^{(i)}\big)^{\gamma}. \tag{5.12}$$

(Here we view maps from a symmetric power as symmetric functions from a tensor power; the sums over unshuffles then guarantee that the above indeed gives a symmetric function.) Note that $\mathcal{L}_s$ is A-multilinear for $s \neq 1, 2$ and that $\mathcal{L}_1$ and $\mathcal{L}_2$ have the derivation properties

$$\mathcal{L}_1(a \cdot x) = d_A(a) \cdot x + a \cdot \mathcal{L}_1(x) \qquad\qquad \mathcal{L}_2(x, a \cdot y) = a \cdot \mathcal{L}_2(x, y) + \mathcal{L}_{\rho(x)}(a) \cdot y.$$

This is equivalent to all maps $\ell_p^{(i)}$ being graded A-linear, except for the maps

$$\ell_1^{(0)}\colon \mathfrak{n} \longrightarrow \mathfrak{n}[1] \qquad \text{and} \qquad \ell_1^{(1)} = [-,-]\colon \mathfrak{t} \otimes \mathfrak{n} \longrightarrow \mathfrak{n}$$

which satisfy equation (5.8). In other words, Lemma 5.7 implies that the maps $\mathcal{L}_s$ are A-multilinear (resp. derivations for $n = 1, 2$) if and only if each

$$\ell_p := \sum_i \ell_p^{(i)}$$

defines an element in $\mathcal{H}(p)$.

Proof (of Proposition 5.4). In light of Lemma 5.7 and Construction 5.11, it suffices to verify that the maps $\ell_p = \sum_i \ell_p^{(i)}$ define a curved L_∞-structure on $B \otimes_A \mathfrak{n}$ if and only if the maps $\mathcal{L}_s = \sum_i \sum_\gamma \big(\ell_{s-i}^{(i)}\big)^\gamma$ define a curved L_∞-structure on $\mathfrak{t} \oplus \mathfrak{n}$.

To this end, let us consider all ℓ_p together as a single element $\sum_p \ell_p \in \prod_p \mathcal{H}(p)$, with $\mathcal{H}(p)$ as in (5.6). The condition of being a curved L_∞-algebra translates into the equation (where we can suppress all additional signs by working with the shift $\mathfrak{n}[1]$, see Remark 2.57)

$$\frac{1}{2}\sum_{p,q}[\ell_p, \ell_q] = \sum_{p,q} \sum_{\sigma \in \mathrm{Sh}_{p-1,q}^{-1}} \big(\ell_p \circ_1 \ell_q\big)^\sigma = 0.$$

Let us unravel this equation in terms of the maps $\ell_p^{(i)} \colon \mathrm{Sym}_A^i(\mathfrak{t}[1]) \otimes \mathrm{Sym}_A^p(\mathfrak{n}[1]) \longrightarrow \mathfrak{n}[1]$, as in Construction 5.11. Using the formulas for partial composition and commutator from Observation 5.10, this yields

$$\sum_{p,q,i,j} \sum_{\substack{\sigma \in \mathrm{Sh}_{p-1,q}^{-1} \\ \tau \in \mathrm{Sh}_{i,j}^{-1}}} \big(\ell_p^{(i)} \circ_\mathfrak{n} \ell_q^{(j)}\big)^{\sigma \times \tau} + \sum_{p,i,j} \sum_{\tau \in \mathrm{Sh}_{i-1,j}^{-1}} \big(\ell_p^{(i)} \circ_\mathfrak{t} \ell_\mathfrak{t}^{(j)}\big)^\tau = 0.$$

The above equation is equivalent to a certain system of equations $E(r, k) = 0$, collecting all terms consisting of maps with r inputs from $\mathfrak{n}$ and k inputs from $\mathfrak{t}$. In turn, this is equivalent to a system of equations

$$\begin{aligned} 0 = \sum_{r,k} \sum_{\gamma \in \mathrm{Sh}_{r-k,r}^{-1}} E(r-k, k)^\gamma = &\sum_{p,q,i,j} \sum_{\gamma \in \mathrm{Sh}_{p+q-1,i+j}^{-1}} \sum_{\substack{\sigma \in \mathrm{Sh}_{p-1,q}^{-1} \\ \tau \in \mathrm{Sh}_{i,j}^{-1}}} \Big(\big(\ell_p^{(i)} \circ_\mathfrak{n} \ell_q^{(j)}\big)^{\sigma\times\tau}\Big)^\gamma \\ &+ \sum_{p,i,j} \sum_{\gamma \in \mathrm{Sh}_{p,i-1+j}^{-1}} \sum_{\tau \in \mathrm{Sh}_{i-1,j}^{-1}} \Big(\big(\ell_p^{(i)} \circ_\mathfrak{t} \ell_\mathfrak{t}^{(j)}\big)^\tau\Big)^\gamma. \end{aligned}$$

For fixed r, this consists of maps $\mathrm{Sym}_k^r(\mathfrak{t}[1] \oplus \mathfrak{n}[1]) \longrightarrow \mathfrak{n}[1]$. By our definition of the maps $\ell_{r-k}^{(k)}$ in terms of the curved Lie_∞-structure on $\mathfrak{t} \oplus \mathfrak{n}$ (5.12), the above equation is then equivalent to the equation

$$\sum_{t,s\geq 1} \sum_{\sigma \in \mathrm{Sh}_{s-1,t}^{-1}} (\mathcal{L}_s \circ_\mathfrak{n} \mathcal{L}_t)^\sigma + \sum_{\sigma \in \mathrm{Sh}_{s-1,t}^{-1}} \big(\mathcal{L}_s \circ_\mathfrak{t} \ell_\mathfrak{t}^{(t)}\big)^\sigma = 0.$$

Here the first term involves partial composition along $\mathfrak{n}$, while the second term involves partial composition along $\mathfrak{t}$. Unraveling the definitions, the above equation means precisely that the pair $\big(\ell_\mathfrak{t}^{(s)} \circ \pi, \mathcal{L}_s\big) \colon \mathrm{Sym}_k^s(\mathfrak{t}[1] \oplus \mathfrak{n}[1]) \longrightarrow \mathfrak{t}[2] \oplus \mathfrak{n}[2]$ defines a curved L_∞-structure on $\mathfrak{t} \oplus \mathfrak{n}$. $\square$

5.1.2 Behaviour on ∞-morphisms

Next, let us discuss how Proposition 5.4 interacts with ∞-morphisms. To this end, let $\mathfrak{t}$ and $B = C^*(\mathfrak{t})$ be as in Theorem 5.2 and let

$$\mathfrak{L} = \mathfrak{t} \oplus \mathfrak{n}, \qquad \mathfrak{H} = \mathfrak{t} \oplus \mathfrak{m}.$$

Assume that $\mathfrak{L}$ and $\mathfrak{H}$ come equipped with curved L_∞-algebroid structures such that the projection to $\mathfrak{t}$ is a (strict) map of curved L_∞-algebroids and consider an ∞-morphism of curved L_∞-algebroids which fits into a commuting diagram

$$\begin{tikzcd} \mathfrak{L} \arrow[rr, rightsquigarrow, "\Phi"] \arrow[dr, "\pi_{\mathfrak{g}}"'] & & \mathfrak{H} \arrow[dl, "\pi_{\mathfrak{h}}"] \\ & \mathfrak{t}. & \end{tikzcd} \tag{5.13}$$

In particular, this ∞-morphism is uniquely determined by A-linear maps of the form

$$\Phi_s = (0, \Phi'_s)\colon \mathrm{Sym}^s_A(\mathfrak{t}[1] \oplus \mathfrak{n}[1]) \longrightarrow \mathfrak{t}[1] \oplus \mathfrak{m}[1] \qquad p \neq 1$$

$$\Phi_1 = \big(\pi_{\mathfrak{t}}, \Phi'_1\big)\colon \mathfrak{t}[1] \oplus \mathfrak{n}[1] \longrightarrow \mathfrak{t}[1] \oplus \mathfrak{m}[1]$$

where $\pi_{\mathfrak{t}}$ projects onto $\mathfrak{t}$. As in Construction 5.11, we decompose $\Phi'_s = \sum_i \sum_{\gamma \in \mathrm{Sh}^{-1}_{i,s-i}} \big(\phi^{(i)}_{s-i}\big)^\gamma$ binomially into maps of complete graded A-modules

$$\phi^{(i)}_p\colon \mathrm{Sym}^i_A(\mathfrak{t}[1]) \otimes_A \mathrm{Sym}^p_A(\mathfrak{n}[1]) \longrightarrow \mathfrak{m}[1].$$

Since $\mathfrak{t}$ is a dualizable A-module, arguing as in Lemma 5.7 shows that such families of A-linear maps correspond bijectively to families of B-linear maps

$$\phi_p\colon \mathrm{Sym}^p_B\big(B \otimes_A \mathfrak{n}[1]\big) \longrightarrow B \otimes_A \mathfrak{m}[1].$$

Here ϕ_p denotes (using Remark 5.5) the B-linear extension of the map

$$\sum \phi^{(i)}_p\colon \mathrm{Sym}^p_A(\mathfrak{n}[1]) \longrightarrow B \otimes_A \mathfrak{m}[1] \cong \prod_i \mathrm{Hom}_A\big(\mathrm{Sym}^i_A(\mathfrak{t}[1]), \mathfrak{m}[1]\big).$$

Proposition 5.14. *Suppose that $\mathfrak{L}$ and $\mathfrak{H}$ are curved L_∞-algebroids over $\mathfrak{t}$ as above. Then the maps ϕ_p constructed above define an ∞-morphism $\phi\colon \mathrm{curv}(\mathfrak{L}) \rightsquigarrow \mathrm{curv}(\mathfrak{H})$ of curved L_∞-algebras over B if and only if the maps Φ_s define an ∞-morphism of curved L_∞-algebroids as in (5.13).*

Throughout, let us write ℓ_p and κ_q for the curved L_∞-structure maps on $B \otimes_A \mathfrak{n}$ and $B \otimes_A \mathfrak{m}$, respectively. Likewise, the structure maps of $\mathfrak{L} = \mathfrak{t} \oplus \mathfrak{n}$ and $\mathfrak{H} = \mathfrak{t} \oplus \mathfrak{m}$ have the form $(\ell^{(s)}_{\mathfrak{t}}, \mathcal{L}_s)$, respectively $(\ell^{(s)}_{\mathfrak{t}}, \mathcal{K}_s)$, where $\ell^{(s)}_{\mathfrak{t}}$ is the L_∞-structure on $\mathfrak{t}$.

Observation 5.15. Let $\phi_p\colon \mathrm{Sym}^p_B(B \otimes_A \mathfrak{n}[1]) \longrightarrow B \otimes_A \mathfrak{m}[1]$. For $q \neq 1$ and $k \neq 1$, the composite maps

$$\sum_{\sigma \in \mathrm{Sh}^{-1}_{p-1,q}} \big(\phi_p \circ_1 \ell_q\big)^\sigma \qquad \text{and} \qquad \sum_{\sigma \in \mathrm{Sh}^{-1}_{(p_1,\dots,p_k)}} \Big(\kappa_k\big(\phi_{p_1}, \dots, \phi_{p_k}\big)\Big)^\sigma$$

are B-multilinear. In particular, they are determined by the maps

$$\phi^{(i)}_p\colon \mathrm{Sym}^i_A(\mathfrak{t}[1]) \otimes_A \mathrm{Sym}^p_A(\mathfrak{n}[1]) \longrightarrow \mathfrak{m}[1]$$

$$\ell^{(j)}_q\colon \mathrm{Sym}^j(\mathfrak{t}[1]) \otimes \mathrm{Sym}^q_A\big(\mathfrak{n}[1]\big) \longrightarrow \mathfrak{n}[2]$$

$$\kappa^{(j)}_k\colon \mathrm{Sym}^j(\mathfrak{t}[1]) \otimes \mathrm{Sym}^k_A\big(\mathfrak{m}[1]\big) \longrightarrow \mathfrak{m}[2]$$

via

$$\sum_{\sigma\in\mathrm{Sh}^{-1}_{p-1,q}} \big(\phi_p \circ_1 \ell_q\big)^{\sigma} = \sum_{i,j} \sum_{\substack{\sigma\in\mathrm{Sh}^{-1}_{p-1,q}\\ \tau\in\mathrm{Sh}^{-1}_{i,j}}} \big(\phi_p^{(i)} \circ_{\mathfrak{n}} \ell_q^{(j)}\big)^{\sigma\times\tau}$$

$$\sum_{\sigma\in\mathrm{Sh}^{-1}_{(p_1,\dots,p_k)}} \Big(\kappa_k\big(\phi_{p_1},\dots,\phi_{p_k}\big)\Big)^{\sigma} = \sum_{i,j_1,\dots,j_k} \sum_{\substack{\sigma\in\mathrm{Sh}^{-1}_{(p_1,\dots,p_k)}\\ \tau\in\mathrm{Sh}^{-1}_{(i,j_1,\dots,j_k)}}} \Big(\kappa_k^{(i)} \circ_{\mathfrak{m}} \big(\phi_{p_1}^{(j_1)},\dots,\phi_{p_k}^{(j_k)}\big)\Big)^{\sigma\times\tau}$$

where $\circ_{\mathfrak{n}}$ denotes partial composition via the first $\mathfrak{n}$-variable and $\circ_{\mathfrak{m}}$ denotes total composition along the $\mathfrak{m}$-variable.

On the other hand, the maps $\sum_\sigma \big(\phi_p \circ_1 \ell_1\big)^{\sigma}$ and $\kappa_1 \circ \phi_p$ both define maps

$$\mathrm{Sym}^p(B\otimes_A \mathfrak{n}[1]) \longrightarrow B\otimes_A \mathfrak{m}[2]$$

that are derivations over the Chevalley–Eilenberg differential on B in each variable. Such derivations are again determined uniquely by their restriction $\mathrm{Sym}^p_A(\mathfrak{n}[1]) \longrightarrow B\otimes_A \mathfrak{m}[2]$. Using Remark 5.9, these can be expressed in terms of the maps $\phi_p^{(i)}, \ell_1^{(j)}$ and $\kappa_1^{(j)}$ as

$$\sum_{\sigma\in\mathrm{Sh}^{-1}_{p-1,1}} \big(\phi_p \circ_1 \ell_1\big)^{\sigma} = \sum_{i,j} \sum_{\substack{\sigma\in\mathrm{Sh}^{-1}_{p-1,1}\\ \tau\in\mathrm{Sh}^{-1}_{i,j}}} \big(\phi_p^{(i)} \circ_{\mathfrak{n}} \ell_1^{(j)}\big)^{\sigma\times\tau}$$

$$\kappa_1\circ\phi_p = \sum_{i,j} \sum_{\tau\in\mathrm{Sh}^{-1}_{j,i}} \big(\kappa_1^{(j)} \circ_{\mathfrak{m}} \phi_p^{(i)}\big)^{\tau} - \sum_{i,j} \sum_{\tau\in\mathrm{Sh}^{-1}_{i-1,j}} \big(\phi_p^{(i)} \circ_{\mathfrak{t}} \ell_{\mathfrak{t}}^{(j)}\big)^{\tau}.$$

Here $\ell_{\mathfrak{t}}^{(j)}$ is the L_∞-structure on $\mathfrak{t}$ and $\circ_{\mathfrak{t}}$ takes the partial composition in the $\mathfrak{t}$-variable.

Proof. By Section 2.5.2 and Definition 3.10, the maps ϕ_p define an ∞-morphism of curved L_∞-algebras over B if and only if for each $n\geq 0$

$$0 = \sum_{p+q=n+1} \sum_{\sigma\in\mathrm{Sh}^{-1}_{p-1,q}} \big(\phi_p \circ_1 \ell_q\big)^{\sigma} - \sum_{\substack{k\geq 0\\ p_1+\dots+p_k=n}} \sum_{\sigma\in\mathrm{Sh}^{-1}_{(p_1,\dots,p_k)}} \frac{1}{k!}\kappa_k(\phi_{p_1},\dots,\phi_{p_k})^{\sigma} \tag{5.16}$$

The signs are determined by Remark 2.57 and are exactly as written above if we work at the level of shifted objects. As in the proof of Proposition 5.4, the proof boils down to take the sum of these equations over n and then appropriately rewriting the involved sums "diagonally". More precisely, using Observation 5.15, the above equation is equivalent to a system of equations

$$\begin{aligned} 0 = &\sum_{i,j,p,q} \sum_{\substack{\sigma\in\mathrm{Sh}^{-1}_{p-1,q}\\ \tau\in\mathrm{Sh}^{-1}_{i,j}}} \big(\phi_p^{(i)} \circ_{\mathfrak{n}} \ell_q^{(j)}\big)^{\sigma\times\tau} \\ &- \sum_{\substack{i,k,j_1,\dots,j_k\\ p_1,\dots,p_k}} \frac{1}{k!} \sum_{\substack{\sigma\in\mathrm{Sh}^{-1}_{(p_1,\dots,p_k)}\\ \tau\in\mathrm{Sh}^{-1}_{(i,j_1,\dots,j_k)}}} \Big(\kappa_k^{(i)} \circ_{\mathfrak{m}} \big(\phi_{p_1}^{(j_1)},\dots,\phi_{p_k}^{(j_k)}\big)\Big)^{\sigma\times\tau} \\ &+ \sum_{i,j,p} \sum_{\tau\in\mathrm{Sh}^{-1}_{i-1,j}} \big(\phi_p^{(i)} \circ_{\mathfrak{t}} \ell_{\mathfrak{t}}^{(j)}\big)^{\tau}. \end{aligned}$$

As in the proof of Proposition 5.4, this is a system of equations $E(r, m) = 0$: for fixed r and m, this is an equation between maps $\mathrm{Sym}^r(\mathfrak{n}[1]) \otimes \mathrm{Sym}^m(\mathfrak{t}[1]) \longrightarrow \mathfrak{m}[2]$. This is equivalent to the system of equations

$$0 = \sum_{r,m} \sum_{\gamma \in \mathrm{Sh}^{-1}_{r-m,m}} E(r-m, m)^{\gamma} = \sum_{i,j,p,q} \sum_{\gamma,\sigma,\tau} \Big(\big(\phi^{(i)}_p \circ_{\mathfrak{n}} \ell^{(j)}_q\big)^{\sigma \times \tau} \Big)^{\gamma}$$
$$- \sum_{\substack{i,k,j_1,\dots,j_k \\ p_1,\dots,p_k}} \frac{1}{k!} \sum_{\gamma,\sigma,\tau} \Big(\Big(\kappa^{(i)}_k \circ_{\mathfrak{m}} \big(\phi^{(j_1)}_{p_1}, \dots, \phi^{(j_k)}_{p_k}\big) \Big)^{\sigma \times \tau} \Big)^{\gamma}$$
$$+ \sum_{i,j,p} \sum_{\gamma,\tau} \Big(\big(\phi^{(i)}_p \circ_{\mathfrak{t}} \ell^{(j)}_{\mathfrak{t}}\big)^{\tau} \Big)^{\gamma}.$$

Here we sum over unshuffles σ of the variables within $\mathfrak{n}$, unshuffles τ from the variables within $\mathfrak{t}$ and finally the unshuffles γ of the sets of variables from $\mathfrak{t}$ and $\mathfrak{n}$. For fixed r, this is an equation between maps $\mathrm{Sym}^r\big(\mathfrak{t}[1] \oplus \mathfrak{n}[1]\big) \longrightarrow \mathfrak{m}[1]$.

Now, the terms in the first and third line with $p+i = s$ and $q+j = t$ sum up to the maps

$$\sum_{\sigma \in \mathrm{Sh}^{-1}_{s-1,t}} \big(\Phi'_s \circ_1 \mathcal{L}_t\big)^{\sigma} \qquad \text{and} \qquad \sum_{\sigma \in \mathrm{Sh}^{-1}_{s-1,t}} \big(\Phi'_s \circ_1 \ell^{(t)}_{\mathfrak{t}}\big)^{\sigma} \tag{5.17}$$

of the form $\mathrm{Sym}^{s+t-1}(\mathfrak{t}[1] \oplus \mathfrak{n}[1]) \longrightarrow \mathfrak{m}[1]$. Likewise, the terms in the second line with fixed $k + i = s$ and $p_\alpha + j_\alpha = t_\alpha$ sum up to

$$\sum_{a} \sum_{\sigma \in \mathrm{Sh}^{-1}_{i,t_1,\dots,t_{s-i}}} \frac{1}{(s-i)!} \Big(\mathcal{K}_s \circ \big(\underbrace{\pi_{\mathfrak{t}}, \dots, \pi_{\mathfrak{t}}}_{i\times}, \Phi'_{t_1}, \dots, \Phi'_{t_{s-i}} \big) \Big)^{\sigma}. \tag{5.18}$$

Here $\pi_{\mathfrak{t}} \colon \mathfrak{t} \oplus \mathfrak{n} \longrightarrow \mathfrak{t}$ denotes the projection onto $\mathfrak{t}$. The sum of (5.18) over all $t_1, \dots, t_{s-a}$, can then be identified in terms of the structure maps Φ_p of (5.13) with

$$\frac{1}{s!} \sum_{t_1,\dots,t_s} \sum_{\sigma \in \mathrm{Sh}^{-1}_{t_1,\dots,t_s}} \Big(\mathcal{K}_s \circ \big(\Phi_{t_1}, \dots, \Phi_{t_s}\big) \Big)^{\sigma}. \tag{5.19}$$

Indeed, for each $i \geq 0$ there are $\binom{s}{i}$ terms in (5.19) where i different copies t_α are equal to 1 (so that $\Phi_{t_\alpha} = \Phi_1 = (\pi_{\mathfrak{t}}, \Phi'_1)$); in turn, each of these $\binom{s}{i}$ terms can be identified with $i!$ copies of the expression (5.18), since the size of $\mathrm{Sh}^{-1}(1, \dots, 1, t_1, \dots, t_{s-i})$ is $i!$ times the size of $\mathrm{Sh}^{-1}(i, t_1, \dots, t_{s-i})$.

The sums of (5.17) and (5.19) now precisely give the equation for $\Phi \colon \mathfrak{g} \rightsquigarrow \mathfrak{h} = \mathfrak{t} \oplus \mathfrak{m}$ being an ∞-morphism of curved L_∞-algebroids. □

5.2 The functor curv and proof of the main theorem

We will now use the explicit computations from Section 5.1 to construct the desired equivalence of ∞-categories of Theorem 5.2

$$\mathrm{curv} \colon \mathbf{cLie}(A/k)_{/\mathfrak{t}} \longrightarrow \mathbf{cLie}_B.$$

Recall that both ∞-categories $\mathbf{cLie}(A/k)$ and $\mathbf{cLie}_B$ are modeled by concrete simplicially enriched categories (Definition 4.17 and Definition 3.20). Likewise, the domain of the putative functor 'curv' can be modeled by an explicit simplicially enriched ∞-category, as follows:

Definition 5.20. Let $\mathbf{C}$ denote the simplicially enriched category whose:

(0) objects are given by curved L_∞-algebroids $\mathfrak{L}$ over A, equipped with a (strict) map of L_∞-algebroids $\pi\colon \mathfrak{L} \longrightarrow \mathfrak{t}$ and a section of complete graded A-modules $\sigma\colon \mathfrak{t} \longrightarrow \mathfrak{L}$. This section induces an equivalence $\mathfrak{L} \cong \mathfrak{t} \oplus \mathfrak{n}$ and we require $\mathfrak{n}$ to be projective as a complete graded A-module.

(1) simplicial sets of morphisms are given by the simplicial sets of ∞-morphisms

$$\begin{tikzcd} \mathfrak{L} \arrow[r, rightsquigarrow, "\Phi"] \arrow[dr] & \mathfrak{H} \boxtimes_{\mathfrak{t}} \Omega[\Delta^\bullet] := \mathfrak{H} \otimes \Omega[\Delta^\bullet] \times_{\mathfrak{t}\otimes\Omega[\Delta^\bullet]} \mathfrak{t} \arrow[dl] \\ & \mathfrak{t}. \end{tikzcd}$$

Lemma 5.21. *There is a natural equivalence of ∞-categories $\mathbf{C} \xrightarrow{\sim} \mathbf{cLie}(A/k)_{/\mathfrak{t}}$.*

Proof. Let us denote by $\mathbf{cLie}(A/k)_{\not{/}\mathfrak{t}}$ the simplicially enriched slice category of $\mathbf{cLie}(A/k)$ over $\mathfrak{t}$: objects are maps $\mathfrak{g} \longrightarrow \mathfrak{t}$ and simplicial maps of morphisms consist exactly of maps as in Definition 5.20. The simplicially enriched category $\mathbf{cLie}(A/k)_{\not{/}\mathfrak{t}}$ is *not* a model for the slice ∞-category $\mathbf{cLie}(A/k)_{/\mathfrak{t}}$, but there is a comparison map of ∞-categories $\mathbf{cLie}(A/k)_{\not{/}\mathfrak{t}} \longrightarrow \mathbf{cLie}(A/k)_{/\mathfrak{t}}$ [Lur18, Tag 01ZN].

Composing this map with the natural simplicially enriched functor $\mathbf{C} \longrightarrow \mathbf{cLie}(A/k)_{\not{/}\mathfrak{t}}$ produces the desired map $\mathbf{C} \longrightarrow \mathbf{cLie}(A/k)_{/\mathfrak{t}}$. It is essentially surjective because every curved L_∞-algebroid over $\mathfrak{t}$ is homotopy equivalent to one for which the projection $\mathfrak{L} \longrightarrow \mathfrak{t}$ is surjective and $\mathfrak{L}$ is a projective complete graded A-module (by Theorem 4.12). Since $\mathfrak{t}$ was assumed to be projective as a graded complete A-module, such curved L_∞-algebroids $\mathfrak{L}$ admit a splitting $\mathfrak{L} \cong \mathfrak{t} \oplus \mathfrak{n}$ where $\mathfrak{n}$ is projective as well.

Furthermore, for two objects $\pi_1\colon \mathfrak{L} \longrightarrow \mathfrak{t}$ and $\pi_2\colon \mathfrak{H} \longrightarrow \mathfrak{t}$ in $\mathbf{C}$, the simplicial set of maps between them fits into a pullback square of simplicial sets

$$\begin{tikzcd} \mathrm{Map}_{\mathbf{C}}(\mathfrak{L}, \mathfrak{H}) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathbf{cLie}(A/k)}(\mathfrak{L}, \mathfrak{H}) \arrow[d, "(\pi_2)_*"] \\ \{\pi_1\} \arrow[r] & \mathrm{Map}_{\mathbf{cLie}(A/k)}(\mathfrak{L}, \mathfrak{t}). \end{tikzcd}$$

Unraveling the definitions, one sees that the right vertical map is a Kan fibration, so that the above square is homotopy cartesian. This implies that the functor $\mathbf{C} \longrightarrow \mathbf{cLie}(A/k)_{/\mathfrak{t}}$ is fully faithful (cf. [Lur18, Tag 01ZT]). □

Construction 5.22. For every map of curved L_∞-algebroids $\mathfrak{L} \longrightarrow \mathfrak{t}$, together with a splitting $\mathfrak{L} \cong \mathfrak{t} \oplus \mathfrak{n}$, let us now define

$$\mathrm{curv}(\mathfrak{L}) := B \otimes_A \mathfrak{n}$$

equipped with the curved L_∞-structure over B from Proposition 5.4. Observe that this definition is compatible with tensoring with forms on the simplex, i.e.

$$\mathrm{curv}\left(\mathfrak{L} \boxtimes_{\mathfrak{t}} \Omega[\Delta^n]\right) \cong \mathrm{curv}(\mathfrak{L}) \otimes \Omega[\Delta^n]. \tag{5.23}$$

For any ∞-morphism Φ as in Definition 5.20, we then let

$$\mathrm{curv}(\Phi)\colon \mathrm{curv}(\mathfrak{L}) \rightsquigarrow \mathrm{curv}(\mathfrak{H}) \otimes \Omega[\Delta^\bullet]$$

be the associated ∞-morphism of curved L_∞-algebras over B from Proposition 5.14.

Lemma 5.24. *Construction 5.22 defines a simplicially enriched functor*

$$\mathrm{curv}\colon \mathbf{C} \longrightarrow \mathbf{cLie}_B$$

to the simplicial category of curved Lie algebras over B from Definition 3.20.

Proof. If $\Psi\colon \mathfrak{F} \rightsquigarrow \mathfrak{L}$ and $\Phi\colon \mathfrak{L} \rightsquigarrow \mathfrak{H}$ are ∞-morphisms of curved L_∞-algebroids (over $\mathfrak{t}$), their composite $\Phi \circ \Psi$ has components (cf. [LV12, Proposition 10.2.7])

$$(\Phi\circ\Psi)_s = \sum_{k, s_1+\dots+s_k=s} \sum_{\sigma\in \mathrm{Sh}^{-1}_{(s_1,\dots,s_k)}} \frac{1}{k!}\Phi_k(\Psi_{s_1},\dots,\Psi_{s_k})^\sigma.$$

On the other hand, ∞-morphisms of curved L_∞-algebras over B have the same composition formula. Going through the arguments of Proposition 5.14, one then sees that $\mathrm{curv}(\Phi\circ\Psi) = \mathrm{curv}(\Phi)\circ\mathrm{curv}(\Psi)$. Alternatively, this can be seen in terms of coalgebras as follows: an ∞-morphism $\mathfrak{L} = \mathfrak{t}\oplus\mathfrak{n} \rightsquigarrow \mathfrak{t}\oplus\mathfrak{m} = \mathfrak{H}$ over $\mathfrak{t}$ is given by a certain map of graded complete A-linear cocommutative coalgebras

$$\mathrm{Sym}^c_A(\mathfrak{g}[1]) \cong \mathrm{Sym}^c_A(\mathfrak{t}[1])\otimes \mathrm{Sym}^c_A(\mathfrak{n}[1]) \xrightarrow{\ \Phi\ } \mathrm{Sym}^c_A(\mathfrak{t}[1])\otimes\mathrm{Sym}^c_A(\mathfrak{m}[1]) \cong \mathrm{Sym}^c_A(\mathfrak{h}[1])$$

over $\mathrm{Sym}^c_A(\mathfrak{t}[1])$. Using that $\mathrm{Sym}^c_A(\mathfrak{t}[1])$ is a dualizable complete A-module, such a map of coalgebras over $\mathrm{Sym}^c(\mathfrak{t}[1])$ is equivalent to a map of B-linear coalgebras

$$B\otimes_A \mathrm{Sym}^c_A(\mathfrak{n}[1]) = \mathrm{Sym}_A(\mathfrak{t}^\vee[-1])\otimes_A \mathrm{Sym}^c_A(\mathfrak{n}[1]) \longrightarrow B\otimes_A\mathrm{Sym}^c_A(\mathfrak{n}[1])$$

This is precisely the B-linear coalgebra map corresponding to the ∞-morphism of Proposition 5.14. The above construction manifestly preserves composition of coalgebra maps, which implies that curv preserves composition of ∞-morphisms.

For functoriality on n-simplices in the mapping spaces, note that the composition of $\Psi\colon \mathfrak{F}\rightsquigarrow \mathfrak{L}\boxtimes_{\mathfrak{t}}\Omega[\Delta^n]$ and $\Phi\colon \mathfrak{L}\rightsquigarrow\mathfrak{H}\boxtimes_{\mathfrak{t}}\Omega[\Delta^n]$ is given by

$$\mathfrak{F} \overset{\Psi}{\rightsquigarrow} \mathfrak{L}\boxtimes_{\mathfrak{t}}\Omega[\Delta^n] \overset{\Phi}{\rightsquigarrow} \Big(\mathfrak{H}\boxtimes_{\mathfrak{t}}\Omega[\Delta^n]\Big)\boxtimes_{\mathfrak{t}}\Omega[\Delta^n] \cong \mathfrak{H}\boxtimes_{\mathfrak{t}}\Omega[\Delta^n\times\Delta^n] \xrightarrow{\ \Delta^*\ } \mathfrak{H}\boxtimes_{\mathfrak{t}}\Omega[\Delta^n]$$

where the last map is induced by the map of cdgas $\Delta^*\colon \Omega[\Delta^n\times\Delta^n]\longrightarrow\Omega[\Delta^n]$ restricting along the diagonal. By functoriality at the level of ∞-morphisms and Equation (5.23), one sees that curv also respects composition at the level of n-simplices. □

Finally, we turn to the proofs of Theorem 5.2 and Theorem 5.1.

Proof of Theorem 5.2. Lemma 5.21 and Lemma 5.24 furnish a zig-zag of functors of ∞-categories

$$\mathbf{cLie}(A/k)_{/\mathfrak{t}} \xleftarrow{\ \sim\ } \mathbf{C} \xrightarrow{\ \mathrm{curv}\ } \mathbf{cLie}_B.$$

Taking the inverse of the left functor and composing it with the right functor gives the desired functor $\mathrm{curv}\colon \mathbf{cLie}(A/k)_{/\mathfrak{t}} \longrightarrow \mathbf{cLie}_B$. To see that it is an equivalence, it suffices to verify that the right functor is an equivalence. Indeed, Proposition 5.4 implies that it is essentially surjective and Proposition 5.14 implies that it is (strictly) fully faithful as a map between simplicially enriched categories. □

Proof of Theorem 5.1. If A is a cofibrant cdga locally of finite presentation (or smooth), then T_A is a finitely generated projective graded A-module. Consequently, the filtered Lie algebroid $T_A\langle 1\rangle$ satisfies the conditions of Theorem 5.2 (see Example 5.3). Let us now consider the composite functor

$$\mathbf{Lie}(A/k) \xhookrightarrow{(-)^{\mathrm{anc}}} \mathbf{cLie}(A/k)_{/T_A\langle 1\rangle} \xrightarrow[\sim]{\mathrm{curv}} \mathbf{cLie}_{\mathrm{dR}(A)} \tag{5.25}$$

where $\mathrm{dR}(A) = C^*(T_A\langle 1\rangle)$ is the de Rham algebra of A with its natural filtration. The second functor is the equivalence of Theorem 5.2 and the first functor is the fully faithful inclusion of Proposition 4.21. Unraveling the constructions, one sees that the composite sends an L_∞-algebroid of the form $\mathfrak{L} \simeq T_A \oplus \mathfrak{n}$ to a curved L_∞-algebra of the form $\mathrm{dR}(A) \otimes_A \mathfrak{n}$, where $\mathfrak{n}$ is in filtration weight 0. Such curved L_∞-algebras have the property that the natural map

$$\mathrm{Gr}^0\left(\mathrm{dR}(A) \otimes_A \mathfrak{n}\right) \otimes_{\mathrm{Gr}^0(\mathrm{dR}(A))} \mathrm{Gr}(\mathrm{dR}(A)) \longrightarrow \mathrm{Gr}^0\left(\mathrm{dR}(A) \otimes_A \mathfrak{n}\right)$$

is an isomorphism. Conversely, suppose that $\mathfrak{g}$ is a curved L_∞-algebra over B such that

$$\mathrm{Gr}^0(\mathfrak{g}) \otimes_{\mathrm{Gr}^0(\mathrm{dR}(A))} \mathrm{Gr}(\mathrm{dR}(A)) \longrightarrow \mathrm{Gr}(\mathfrak{g}) \tag{5.26}$$

is an equivalence. We may assume that $\mathfrak{g}$ arises from a mixed-curved L_∞-algebra over B under the functor blend: $\mathrm{cLie}_B^{\mathrm{mix}} \longrightarrow \mathrm{cLie}_B$ from Theorem 3.24. There then exists a cofibrant dg-A-module $\mathfrak{n}$ (in filtration degree 0) together with a map

$$\phi\colon \mathrm{dR}(A) \otimes_A \mathfrak{n} \longrightarrow \mathfrak{g}$$

which induces a quasi-isomorphism on the associated graded. Using the Homotopy Transfer Theorem 3.12, one can endow $\mathrm{dR}(A) \otimes_A \mathfrak{n}$ with the structure of a mixed-curved L_∞-algebra over $\mathrm{dR}(A)$ such that ϕ becomes an ∞-equivalence of mixed-curved L_∞-algebras. Proposition 5.4 now implies that this equivalent curved L_∞-algebra $\mathrm{dR}(A) \otimes_A \mathfrak{n}$ is isomorphic to $\mathrm{curv}(T_A\langle 1\rangle \oplus \mathfrak{n})$, for a certain curved L_∞-algebroid structure on $T_A\langle 1\rangle \oplus \mathfrak{n}$. Such (curved) L_∞-algebroids are precisely images of ordinary L_∞-algebroids under the functor $(-)^{\mathrm{anc}}$. We conclude that the essential image of (5.25) indeed consists of those curved L_∞-algebras over $\mathrm{dR}(A)$ for which the map (5.26) is an equivalence. □

6 Examples and applications

In this section we will give some concrete computations and applications of our main results.

In Section 6.1 we will establish an explicit formula for the functor curv for Lie algebroids with surjective anchor and discuss the less explicit case of a non-surjective anchor.

In Section 6.2 we will see that the formalism developed for curved Lie algebras produces non-trivial results even when restricting to uncurved Lie algebras, by exhibiting the Maurer–Cartan space as a mapping space.

Finally, in Section 6.3 we show a $\mathcal{C}^\infty$ variant of our main result and also interpret that result in the case where L_∞-algebroids arise from vector bundles.

6.1 A more explicit functor curv

While the construction of the functor curv resorts to ∞-categorical methods, its description on objects can be made quite explicit when applied to a strict Lie algebroid $\rho\colon \mathfrak{L} \to T_A$. The construction depends (even if it is homotopically independent) of a choice of a section of ρ as a map of graded A-modules, so we first treat the case in which the anchor admits an A-linear section and then we show how to explicitly obtain such an algebroid when the anchor is not surjective.

If the anchor splits

Suppose $\mathfrak{L}$ is a Lie algebroid over a cofibrant cdga A with surjective anchor $\rho\colon \mathfrak{L} \twoheadrightarrow T_A$ and let us fix an A-linear section $s\colon T_A \to \mathfrak{L}$ of ρ, which is not necessarily compatible with the differentials nor the Lie brackets.

Denoting $\mathfrak{n} = \ker\rho$, using the section we can split $\mathfrak{L}$ as a direct sum of (non-differential) A-modules $\mathfrak{L} = T_A \oplus \mathfrak{n}$.

The bracket and the differential with respect to this decomposition now take the form

$$\begin{aligned} d_{\mathfrak{L}}(v,\xi) &= \big(dv, d\xi + \ell^{(1)}(v)\big) && \in T_A \oplus \mathfrak{n} \\ \big[(v,\xi),(w,\eta)\big] &= \big([v,w], [\xi,\eta] + \nabla_v(\eta) - \nabla_w(\xi) + \ell_0^{(2)}(v,w)\big) && \in T_A \oplus \mathfrak{n}, \end{aligned}$$

for some uniquely determined k-(bi)linear maps $\ell_0^{(1)}(-)$, $\ell_0^{(2)}(-,-)$ and $\nabla_-(-)$.

It follows from the Leibniz rule that $\nabla\colon T_A \otimes \mathfrak{n} \longrightarrow \mathfrak{n}$ is a connection and from the Jacobi identity on $\mathfrak{L}$ we observe that for a fixed $v \in T_A$ we have that $\nabla_v\colon \mathfrak{n} \longrightarrow \mathfrak{n}$ is a derivation with respect to the Lie bracket.

Let us consider the $\mathrm{dR}(A)$-module $\mathrm{dR}(A) \otimes_A \mathfrak{n}$ equipped with the differential $d = d_{\mathrm{dR}(A)} + d_{\mathfrak{n}}$. Consider the $\mathrm{dR}(A)$-linear endomorphism ℓ_1 obtained by extending linearly the connection $\nabla\colon \mathfrak{n} \longrightarrow \Omega^1_A \otimes_A \mathfrak{n}$ and let us interpret $\ell_0^{(1)} \in \mathrm{dR}^1(A) \otimes_A \mathfrak{n}$ and $\ell_0^{(2)} \in \mathrm{dR}^2(A) \otimes_A \mathfrak{n}$.

Proposition 6.1. *The* $\mathrm{dR}(A)$*-module* $\mathrm{dR}(A) \otimes_A \mathfrak{n}$*, equipped with:*

- *the* $\mathrm{dR}(A)$*-linear extension of the Lie bracket on* $\mathfrak{n}$*,*
- *the pre-differential* $d + \ell_1$*,*
- *and the curvature* $\ell_0 = \ell_0^{(1)} + \ell_0^{(2)}$

is the curved Lie algebra obtained as the image of the functor curv *from Theorem 5.1 applied to* $\mathfrak{L}$*.*

Proof. This is a particular case of Proposition 5.4. A direct verification that $\mathrm{dR}(A) \otimes_A \mathfrak{n}$ is a curved Lie algebra can be done using that:

$[d, \ell_0^{(1)}] = 0$ since the differential on $\mathfrak{L}$ squares to zero.

$[d, \nabla_v] - \nabla_{dv} = \mathrm{ad}_{\ell_0^{(1)}(v)}$ since the differential on $\mathfrak{L}$ derivation for the Lie bracket.

$[\nabla_v, \nabla_w] - \nabla_{[v,w]} = \mathrm{ad}_{\ell_0^{(2)}(v,w)}$ due to the Jacobi identity on $\mathfrak{L}$.

□

If the anchor does not split

Next, let us consider a Lie algebroid over A whose anchor $\rho\colon \mathfrak{L} \to T_A$ does not admit a splitting. Proposition 6.1 and Proposition 5.4 do not apply in this situation, but they do apply to a fibrant replacement of $\mathfrak{L}$. Such fibrant replacements always exist for formal reasons (Theorem 4.12), but are typically big and not very explicit. In the case where $\mathfrak{L}$ is a Lie algebroid (or L_∞-algebroid) whose underlying A-module is perfect, i.e. finitely generated and projective without differential, one can also construct a fibrant replacement by geometric means: this provides a fibrant replacement of $\mathfrak{L}$ which is again perfect as an A-module. Let us briefly describe this construction, which is inspired by [GG20].

Construction 6.2. Let A be a smooth algebra or a cofibrant cdga which is locally of finite presentation (so that Ω^1_A is a dualizable A-module). Let us write $A\hat{\otimes}A$ for the complete filtered cdga given by the formal completion at the diagonal, i.e. by the adic completion of $A\otimes A$ at the kernel I of the multiplication map. Note that since $I/I^2\cong\Omega^1_A$, there is a canonical $\widehat{\mathrm{Sym}}_A(\Omega^1_A)\cong\mathrm{Gr}(A\hat{\otimes}A)$ with Ω^1_A of weight 1. By a *formal exponential* we will mean an isomorphism of complete graded algebras that fits into a commuting diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i_1"'] & \widehat{\mathrm{Sym}}_A(\Omega^1_A) \arrow[d] \\ A\hat{\otimes}A \arrow[r] \arrow[ur, dotted, "\exp^*"] & A \end{tikzcd}$$

and induces the canonical isomorphism on the associated graded (equivalently, on the first graded piece it induces the canonical identification $I/I^2\cong\Omega^1_A$). Geometrically, such an isomorphism identifies the formal completion of the diagonal with the tangent bundle of A. In particular, there then exists some differential on $\widehat{\mathrm{Sym}}_A(\Omega^1_A)$ so that $A\hat{\otimes}A\cong\widehat{\mathrm{Sym}}_A(\Omega^1_A)$ as complete dg-A-algebras, where $A\hat{\otimes}A$ is considered an A-algebra *from the left*. Unraveling the definitions, the data of a formal exponential (or rather its inverse) comes down to providing a left A-linear section of $F^1(A\hat{\otimes}A)\twoheadrightarrow\mathrm{Gr}^1(A\hat{\otimes}A)\cong\Omega^1_A$; in particular, such a section exists by our assumptions on A.

Likewise, if E is a perfect dg-A-module, then a *formal parallel transport* on E is an isomorphism of graded $(A\hat{\otimes}A)$-modules

$$\mathrm{PT}\colon E\otimes_A(A\hat{\otimes}A)\xrightarrow{\cong}(A\hat{\otimes}A)\otimes_A E$$

which is the identity when tensored up along $A\hat{\otimes}A\longrightarrow A$. To provide some intuition, notice that if E arises from a vector bundle, an isomorphism $E\otimes_A(A\otimes A)\cong(A\otimes A)\otimes_A E$ represents a continuous identification of the fibers $E_x\cong E_y$ for each point $(x,y)\in\mathrm{Spec}(A)\times\mathrm{Spec}(A)$. A formal parallel transport provides such an identification for infinitesimally close points.

A formal parallel transport is induced by a (left) A-linear section of the map $(A\hat{\otimes}A)\otimes_A E\twoheadrightarrow E$ (not necessarily preserving differentials), and hence exists since E is perfect.

These constructions can be recovered directly from connection data:

Lemma 6.3. *Suppose that A is a smooth algebra or a cofibrant cdga. Every connection ∇ on T_A (not required to preserve differentials) induces a natural formal exponential.*

Proof. Notice that the complete filtered algebra $A\hat{\otimes}A$ is the (filtered) left A-linear dual of the universal enveloping algebra $\mathcal{U}(T_A)$, equipped with its PBW filtration [Rin63]; the duality arises from the canonical (filtered) left A-linear pairing

$$\langle -,-\rangle\colon\mathcal{U}(T_A)\hat{\otimes}(A\hat{\otimes}A)\longrightarrow A;\quad\langle D,a\otimes b\rangle=a\cdot D(b).$$

On the associated graded, the PBW theorem [Rin63] identifies this with the canonical (nondegenerate) pairing $\mathrm{Sym}_A(T_A)\otimes\widehat{\mathrm{Sym}}_A(\Omega^1_A)$ with T_A and Ω^1_A in filtration degrees -1 and 1, respectively (see e.g. [CRvdB10, Sec. 4.4]).

Following [LS19], recall that the connection ∇ gives rise recursively to an A-linear map $\exp\colon\mathrm{Sym}_A(T_A)\to\mathcal{U}(T_A)$, which sends $X_i\in T_A$ to $X_i\in\mathcal{U}(T_A)$ and satisfies

$$\exp(X_1\dots X_n)=\frac{1}{n}\sum_{k=1}^{n}\pm\left(X_k\exp(X_1\dots\hat{X}_k\dots X_n)-\exp\left(\nabla_{X_k}(X_1\dots\hat{X}_k\dots X_n)\right)\right).$$

At the associated graded level, exp coincides with the PBW isomorphism and is therefore an isomorphism itself [LS19, Prop. 4.2]. Taking (filtered) A-linear duals, we obtain the left A-linear isomorphism $\exp^*\colon A\hat{\otimes}A \to \widehat{\mathrm{Sym}}_A(\Omega^1_A)$; this preserves the multiplication by [LS19, Thm. 4.3]. $\square$

Lemma 6.4. *Let A be a smooth algebra or a cofibrant cdga over k, equipped with a formal exponential. If E is a perfect A-module, then every connection ∇^E on E (not required to respect differentials) induces a formal parallel transport on E.*

Geometrically, the algebraic construction in the proof below can be viewed as follows: for every k-point $x \in \mathrm{Spec}(A)$, one can restrict the (graded) vector bundle with connection E to the formal neighbourhood $\mathrm{Spec}(A)^\wedge_x$ around x. The formal exponential map identifies $\mathrm{Spec}(A)^\wedge_x$ with the tangent space $T_x\mathrm{Spec}(A)$. We then produce a formal parallel transport by taking the usual parallel transport along straight lines in $T_x\mathrm{Spec}(A)$, starting at the origin.

Proof. The connection ∇^E induces a left A-linear connection on the base change $(A\otimes A)\otimes_A E$ (i.e. a connection with respect to the tangent bundle of $A\otimes A$ relative to $A \otimes k$). This connection induces a left A-linear connection on the complete $A\hat{\otimes}A$-module $(A\hat{\otimes}A)\otimes_A E$. Using the formal exponential, $(A\hat{\otimes}A) \cong \widehat{\mathrm{Sym}}_A(\Omega^1_A) =: B$, we obtain a left A-linear connection on the perfect complete B-module $F := B \otimes_A E$. Note that F comes with the natural adic filtration, and $\mathrm{gr}^0(F) \cong E$ as A-modules. We have to provide a natural left A-linear section of the map $F \longrightarrow \mathrm{gr}^0(F)$.

To this end, let us work in a setting where there is a further "*projective*" grading. Consider the algebra $\mathrm{Sym}_A(\Omega^1_A \oplus A\cdot t)$ where A has projective weight 0, Ω^1_A has weight 1 and t has weight -1, and let B_t be its t-adic completion ($B_t = \mathrm{Sym}_A(\Omega^1_A)[\![t]\!]$ but with different projective grading). Then B_t is a complete graded ring and there is a natural map of complete A-algebras

$$B \longrightarrow B_t; \quad \Omega^1_A \ni \alpha \longmapsto t\cdot\alpha.$$

This map is an isomorphism onto the part of B_t in projective weight 0; in fact, B_t is given in projective weight 0 by $\prod \mathrm{Sym}^n_A(\Omega^1_A)\otimes t^n$. Now consider the perfect (projective graded) B_t-module $F_t := B_t \otimes_B F$, which coincides with F in projective weight 0. Explicitly, it has elements $b(t)\otimes f$, with $b(t) \in B_t \cong \mathrm{Sym}_A(\Omega^1_A)[\![t]\!]$ and $f \in F$, subject to the relation

$$b(t)\otimes(\alpha\cdot f) \sim b(t)\cdot t\cdot\alpha\otimes f$$

for every $\alpha \in \Omega^1_A$ (and extended by continuity for the adic topology). We now claim that F_t has a canonical connection $\nabla_t\colon F_t \longrightarrow F_t\otimes dt$ in the t-direction. Indeed, let $\nabla\colon F \longrightarrow F\otimes_B\Omega^1_{B/A}$ be the A-linear connection we had on F, and denote $\nabla(f) = \nabla^{(1)}(f)\otimes\nabla^{(2)}(f)$. We then define

$$\nabla_t\big(b(t)\otimes f\big) = \Big(\frac{db(t)}{dt}\otimes f + \big(b(t)\cdot t\cdot\nabla^{(2)}(f)\big)\otimes\nabla^{(1)}(f)\Big)dt.$$

One can verify that this is well-defined and indeed defines a connection in the t-direction on F_t. But for t-connections on a perfect (projectively graded) module F_t over a power series algebra $B_t = \mathrm{Sym}_A(\Omega^1_A)[\![t]\!]$, we know that the composite

$$\ker(\nabla_t\colon F_t \longrightarrow F_t) \longrightarrow F_t \longrightarrow F_t/t\cdot F_t$$

is an isomorphism of projectively graded $\mathrm{Sym}_A(\Omega^1_A)$-modules. Finally, considering only the part in projective weight 0, one obtains a sequence of left A-linear maps

$$\ker(\nabla_t)(0) \longrightarrow F_t(0) \cong F \longrightarrow (F_t/tF_t)(0) \cong \mathrm{gr}^0(F)$$

where the composition is an isomorphism. This provides the desired left A-linear section of $F \longrightarrow \mathrm{gr}^0(F)$, which concludes the proof. $\square$

Proposition 6.5. *Let $\mathfrak{L}$ be a perfect L_∞-algebroid over A and fix a formal exponential and formal parallel transport on $\mathfrak{L}$. Then the mapping cylinder $\mathfrak{L}\oplus T_A[-1]\oplus T_A$ admits a natural a L_∞-algebroid structure over A, which fits into a commuting diagram*

$$\begin{tikzcd} \mathfrak{L} \arrow[rr, rightsquigarrow] \arrow[dr, "\rho"'] & & \mathfrak{L}\oplus T_A[-1]\oplus T_A \arrow[dl, "\pi"] \\ & T_A. & \end{tikzcd} \tag{6.6}$$

Here the ∞-morphism has linear part given by $(\mathrm{id}, 0, \rho)$ *(in particular, it is a quasi-isomorphism) and π is the projection onto the last factor.*

Proof. Recall that for a perfect A-module $\mathfrak{L}$ equipped with an A-linear map $\rho\colon \mathfrak{L} \longrightarrow T_A$, there is an equivalence between L_∞-algebroid structures on $\mathfrak{L}$ with anchor ρ and differentials on the complete filtered symmetric algebra $\widehat{\mathrm{Sym}}(\mathfrak{L}^\vee[-1]\langle 1\rangle)$ making

$$\mathrm{dR}(A) \xrightarrow{\rho^*} \widehat{\mathrm{Sym}}(\mathfrak{L}^\vee[-1]\langle -1\rangle) \longrightarrow A$$

a sequence of complete cdgas. In this case, the middle term agrees with the Chevalley–Eilenberg complex $C^*(\mathfrak{L}\langle 1\rangle)$.

Now suppose that $\mathfrak{L}$ is an L_∞-algebroid and consider the map of complete cdgas $\mathrm{dR}(A)\otimes C^*(\mathfrak{L}\langle 1\rangle) \longrightarrow A\otimes A \longrightarrow A$. Let us write $\hat{R}$ for the adic completion of $\mathrm{dR}(A)\otimes C^*(\mathfrak{L}\langle 1\rangle)$ at the kernel of this map; unraveling the definitions, $\hat{R}$ can be identified without differential with the complete symmetric algebra

$$\hat{R} = \widehat{\mathrm{Sym}}_{A\hat{\otimes}A}\Big(\pi_1^*\Omega^1_A[-1] \oplus \pi_2^*\mathfrak{L}^\vee[-1]\Big). \tag{6.7}$$

Here the filtration arises from the filtration on $A\hat{\otimes}A$ and

$$\pi_1^*\Omega^1_A = \Omega^1_A \otimes_A (A\hat{\otimes}A) \qquad\qquad \pi_2^*\mathfrak{L}^\vee = (A\hat{\otimes}A)\otimes_A \mathfrak{L}^\vee$$

where Ω^1_A and $\mathfrak{L}^\vee$ are both of filtration weight 1. There are natural maps of complete cdgas $i_1\colon \mathrm{dR}(A) \longrightarrow \hat{R}$, induced by the left inclusion of $\mathrm{dR}(A)$, and $\phi\colon \hat{R} \longrightarrow C^*(\mathfrak{L}\langle 1\rangle)$ induced by

$$(\rho^*, \mathrm{id})\colon \mathrm{dR}(A)\otimes C^*(\mathfrak{L}\langle 1\rangle) \longrightarrow C^*(\mathfrak{L}\langle 1\rangle).$$

In terms of (6.7), i_1 includes A into $A\hat{\otimes}A$ from the left and maps Ω^1_A into $\pi_1^*\Omega^1_A$. Furthermore, ϕ is given by the diagonal $A\hat{\otimes}A \longrightarrow A$, acts as ρ^* on $\pi_1^*\Omega^1_A$ and as the identity on $\pi_2^*\mathfrak{L}^\vee$.

Applying formal parallel transport, we can identify $\hat{R}$ with the complete graded algebra

$$\hat{R} \cong \widehat{\mathrm{Sym}}_{A\hat{\otimes}A}\Big(\pi_1^*\Omega^1_A[-1] \oplus \pi_1^*\mathfrak{L}^\vee[-1]\Big)$$

In this presentation, i_1 is still the obvious inclusion, but the map $\hat{R} \longrightarrow C^*(\mathfrak{L}\langle 1\rangle)$ only acts as the identity on $\pi_1^*\mathfrak{L}^\vee[-1]$ up to first order.

Next, using the formal exponential map we can identify $A\hat{\otimes}A \cong \widehat{\mathrm{Sym}}_A(\Omega^1_A)$ as complete A-algebras, where $A\hat{\otimes}A$ is seen as an A-algebra from the left. Using this, we find that the sequence of complete cdgas $\mathrm{dR}(A) \longrightarrow \hat{R} \longrightarrow C^*(\mathfrak{L}\langle 1\rangle)$ can be identified without differential with the sequence of A-algebra maps

$$\widehat{\mathrm{Sym}}_A\big(\Omega^1_A[-1]\big) \xrightarrow{i_1} \widehat{\mathrm{Sym}}_A\Big(\Omega^1_A \oplus \Omega^1_A[-1] \oplus \mathfrak{L}^\vee[-1]\Big) \xrightarrow{\phi} \widehat{\mathrm{Sym}}_A\big(\mathfrak{L}^\vee[-1]\big). \tag{6.8}$$

Here i_1 is simply induced by the summand inclusion $\Omega^1_A[-1] \longrightarrow \Omega^1_A \oplus \Omega^1_A[-1] \oplus \mathfrak{L}^\vee[-1]$. The map ϕ is given on the summand $\Omega^1_A[-1]$ by ρ^*, while the restriction to $\Omega^1_A \oplus \mathfrak{L}^\vee[-1]$ is given by the projection onto $\mathfrak{L}^\vee[-1]$ up to higher order terms.

The differential on $\hat{R}$ endows $\widehat{\mathrm{Sym}}_A\big(\Omega^1_A \oplus \Omega^1_A[-1] \oplus \mathfrak{L}^\vee[-1]\big)$ with a differential, such that the associated graded is precisely the symmetric algebra on the mapping cylinder. This makes (6.8) a diagram of cdgas, dual to the desired diagram of L_∞-algebroids (6.6). □

Even if A is a smooth discrete algebra and $\mathfrak{L}$ is a Lie algebroid over it concentrated in degree 0, the mapping cylinder $\mathfrak{L} \oplus T_A[-1] \oplus T_A$ will generally only come with an L_∞-algebroid structure. The higher brackets depend on higher order terms appearing in the formal exponential map $\widehat{\mathrm{Sym}}_A(\Omega^1_A) \cong A \hat{\otimes} A$ and the formal parallel transport map $\mathfrak{L} \otimes_A (A \hat{\otimes} A) \cong (A \hat{\otimes} A) \otimes_A \mathfrak{L}$.

Corollary 6.9 (cf. [GG20])**.** *Let A be a smooth algebra or a cofibrant cdga of finite type and let $\mathfrak{L} \to T_A$ be a Lie algebroid over A whose underlying A-module is perfect. Then the complete* $\mathrm{dR}(A)$*-module* $\mathrm{dR}(A) \otimes_A (\mathfrak{L} \oplus T_A[-1])$ *admits the structure of a curved L_∞-algebra which gives a model for* $\mathrm{curv}(\mathfrak{L})$.

6.2 The Maurer–Cartan space

In this section we show how to use our constructions at the level of curved Lie algebras to recover results concerning the classical homotopy theory of L_∞-algebras over a field of characteristic 0.

Let $\mathrm{Alg}^{c.p.f.}_{L_\infty}$ denote the category of complete positively filtered L_∞ algebras, with ∞-morphisms. Due to the assumption of positive filtration we can associate to such an L_∞-algebra $\mathfrak{g}$ its Maurer–Cartan set

$$\mathrm{MC}(\mathfrak{g}) = \{x \in \mathfrak{g} | dx + \frac{1}{2!}\ell_2(x,x) + \frac{1}{3!}\ell_3(x,x,x) + \cdots = 0\}.$$

This construction extends to the Maurer–Cartan space, which is the functor

$$\begin{aligned} \mathcal{MC}\colon \mathrm{Alg}^{c.p.f.}_{L_\infty} &\longrightarrow \text{Kan complexes} \\ \mathfrak{g} & \mathrm{MC}(\mathfrak{g} \otimes \Omega[\Delta^\bullet]) \end{aligned}$$

where $\Omega[\Delta^\bullet]$ denotes the polynomial differential forms on the simplex, see [DR15].

It is not entirely trivial to see that the Maurer–Cartan functor is well defined on ∞-morphisms, let alone that it satisfies the right homotopical properties. It turns out that under our framework this functor takes the form of a mapping space if we consider the larger category of curved L_∞-algebras, which establishes in particular that $\mathcal{MC}$ is a representable functor.

Proposition 6.10. *Let $\mathfrak{g}$ be a positively filtered L_∞ algebra, seen as an object in the simplicial category of curved L_∞-algebras* **cLie***. Then*

$$\mathcal{MC}(\mathfrak{g}) = \mathrm{Map}_{\mathbf{cLie}}(0, \mathfrak{g}).$$

Proof. Following the explicit form of an ∞-morphism as presented in section 2.5.2, since 0 is the trivial curved L_∞-algebra, an ∞-morphism $\phi\colon 0 \to \mathfrak{g}$ can only have one non-trivial piece, namely ϕ_0, which is represented by a filtration 1, degree 1 element satisfying precisely the Maurer–Cartan condition.

The result follows from $F^1\mathfrak{g} = \mathfrak{g}$. □

Since in an ∞-category mapping spaces preserve weak equivalences, as a corollary we obtain the main result of [DR15].

Theorem 6.11 (Theorem 1.1 of [DR15])**.** *Let $\mathfrak{g} \rightsquigarrow \mathfrak{h}$ be a weak equivalence of positively filtered complete L_∞-algebras. Then, $\mathcal{MC}(\mathfrak{g}) \to \mathcal{MC}(\mathfrak{h})$ is a homotopy equivalence of simplicial sets.*

Furthermore, by expressing the Maurer–Cartan functor as a mapping space, the corollary above we can show the ∞-categorical version of a result due to Rogers [Rog20].

Theorem 6.12 (See Theorem 3 of [Rog20])**.** *The Maurer–Cartan functor $\mathcal{MC}\colon \mathrm{Alg}_{L_\infty}^{c.p.f.} \to$* Kan complexes *induces a left exact functor at the ∞-categorical level.*

Proof. Following Proposition 6.10, at the ∞-categorical level $\mathcal{MC}$ is a (covariant) mapping space functor, which is therefore left exact. $\square$

6.3 A differential-geometric variant

Let us conclude with a brief discussion of Theorem 5.1 in the differential-geometric setting, where a similar construction has been studied in [GG20]. Let M be a (Hausdorff, second countable) smooth manifold and let $A = \mathcal{C}^\infty(M)$ be the ring of $\mathcal{C}^\infty$-functions on M. Let $\mathrm{Sh}_{\mathcal{O}_M}$ denote the category of sheaves of $\mathcal{O}_M$-modules on M, where $\mathcal{O}_M = \mathcal{C}^\infty(-)$ is the structure sheaf. Then there is an adjoint pair

$$(-)^\sim\colon \mathrm{Mod}_{\mathcal{C}^\infty(M)} \rightleftarrows \mathrm{Sh}_{\mathcal{O}_M}(M)\colon \Gamma$$

where the fully faithful right adjoint takes global sections and the left adjoint sends an $\mathcal{C}^\infty(M)$-module V to the associated sheaf of $V \otimes_{\mathcal{C}^\infty(M)} \mathcal{C}^\infty(-)$ [Joy19, Section 5.4]. In other words, $\mathcal{O}_M$-module sheaves are completely determined by their global sections.

Furthermore, the Serre–Swan theorem asserts that Γ restricts to a monoidal equivalence between locally free sheaves – i.e. vector bundles on M – and finitely generated projective $\mathcal{C}^\infty(M)$-modules. For example, the module $T_{\mathcal{C}^\infty(M)}$ of algebra derivations of $\mathcal{C}^\infty(M)$ agrees with the module $\Gamma(M, TM)$ of vector fields on M and its $\mathcal{C}^\infty(M)$-linear dual agrees with the module $\Gamma(M, T^*M)$ of 1-forms on M (however, its natural *pre*dual, consisting of Kähler differentials, is not finitely generated).

Definition 6.13. Let $\mathfrak{g}$ be an L_∞-algebroid over $\mathcal{C}^\infty(M)$ in the sense of Definition 4.2. We will say that $\mathfrak{g}$ is:

(1) a *sheaf of L_∞-algebroids* on M if the $\mathcal{C}^\infty(M)$-module underlying $\mathfrak{g}$ arises as the global sections of a complex of sheaves of $\mathcal{O}_M$-modules on M.

(2) a *differential-geometric* L_∞-algebroid if $\mathfrak{g}$ is bounded above and each $\mathfrak{g}_n$ arises from a vector bundle, i.e. it is a finitely generated projective $\mathcal{C}^\infty(M)$-module.

To justify this terminology, let us point out that under the Serre–Swan theorem, differential-geometric L_∞-algebroids over M indeed correspond to the usual definition of an L_∞-algebroid over a smooth manifold considered in the literature [LGLS18]; the only possible exception is that we allow *unbounded* complexes of vector bundles, while one typically only considers nonpositively graded complexes of vector bundles. In particular, $T_{\mathcal{C}^\infty(M)}$ itself corresponds to the usual tangent Lie algebroid T_M and its Chevalley–Eilenberg complex $C^*(T_{\mathcal{C}^\infty(M)}) \cong \Omega^*(M)$ is isomorphic to the usual de Rham complex of M.

On the other hand, suppose that $\mathfrak{g} \longrightarrow T_{\mathcal{C}^\infty(M)}$ is a sheaf of L_∞-algebroids in the above sense. For every open U, the map

$$\mathfrak{g} \otimes_{\mathcal{C}^\infty(M)} \mathcal{C}^\infty(U) \longrightarrow T_{\mathcal{C}^\infty(M)} \otimes_{\mathcal{C}^\infty(M)} \mathcal{C}^\infty(U) \cong T_{\mathcal{C}^\infty(U)}$$

is the anchor map of a natural L_∞-algebroid $\mathfrak{g}\otimes_{\mathcal{C}^\infty(M)}\mathcal{C}^\infty(U)$ over $\mathcal{C}^\infty(U)$. Taking associated sheaves, one sees that $\mathfrak{g}^\sim \longrightarrow T^\sim_{\mathcal{C}^\infty(M)} = \mathcal{X}(-)$ gives a sheaf of L_∞-algebroids, with anchor map taking values in the sheaf of vector fields on M. By our assumption on $\mathfrak{g}$, the global sections of this sheaf of L_∞-algebroids coincides with $\mathfrak{g}$ itself. In this way, sheaves of L_∞-algebroids on M embed fully faithfully in L_∞-algebroids over $\mathcal{C}^\infty(M)$, by taking global sections.

Theorem 6.14. *Let M be a differentiable manifold and $\mathcal{C}^\infty(M)$. Then there is a fully faithful inclusion of ∞-categories*

$$\mathrm{curv}\colon \mathbf{Lie}(\mathcal{C}^\infty(M)/\mathbb{R}) \hookrightarrow \mathbf{cLie}_{\Omega^*(M)}$$

from the ∞-category of L_∞-algebroids over $\mathcal{C}^\infty(M)$ to the ∞-category of curved L_∞-algebras over the de Rham complex $\Omega^*(M)$, filtered by form degree. Furthermore, curv *restricts to equivalence between the full subcategories of:*

(1) sheaves of L_∞-algebroids on M and curved L_∞-algebroids $\mathfrak{h}$ over $\Omega^*(M)$ such that $\mathrm{Gr}^i(\mathfrak{h}) \simeq \Omega^i(M)\otimes_{\mathcal{C}^\infty(M)} \mathrm{Gr}^0(\mathfrak{h})$ is an equivalence and $\mathrm{Gr}^0(\mathfrak{h})$ arises as the global sections of a complex of sheaves of $\mathcal{O}_M$-modules.

(2) differential-geometric L_∞-algebroids on M and curved L_∞-algebras over $\Omega^*(M)$ equivalent to a curved L_∞-algebra of the form $\Omega^*(M)\otimes_{\mathcal{C}^\infty(M)} E$, with E a bounded above graded vector bundle on M.

Proof. The Lie algebroid $T_{\mathcal{C}^\infty(M)}$ satisfies the conditions of Theorem 5.2. Furthermore, the ∞-category of (algebraic) L_∞-algebroids over $T_{\mathcal{C}^\infty(M)}$ embeds into the ∞-category of curved L_∞-algebroids over $T_{\mathcal{C}^\infty(M)}\langle 1\rangle$ via the functor $(-)^{\mathrm{anc}}$. The proof of Theorem 5.1 now carries over verbatim to show that the essential image of $\mathbf{Lie}(\mathcal{C}^\infty(M)/\mathbb{R})$ consists of curved L_∞-algebroids $\mathfrak{h}$ with $\mathrm{Gr}^i(\mathfrak{h}) \simeq \Omega^i(M)\otimes_{\mathcal{C}^\infty(M)} \mathrm{Gr}^0(\mathfrak{h})$.

Now note from the construction of the functor curv that $\mathrm{Gr}^0(\mathrm{curv}(\mathfrak{g}))$ is weakly equivalent to the mapping fiber of the anchor $\rho\colon \mathfrak{g}\longrightarrow T_A$. In particular, $\mathrm{Gr}^0(\mathrm{curv}(\mathfrak{g}))$ arises as the global sections of a complex of sheaves of $\mathcal{O}_M$-modules if and only if $\mathfrak{g}$ does.

Likewise, $\mathrm{Gr}^0(\mathrm{curv}(\mathfrak{g}))$ is weakly equivalent to a complex of vector bundles if and only if $\mathfrak{g}$ is weakly equivalent to a complex of vector bundles. But if $\mathfrak{h}$ is a curved L_∞-algebra with $\mathrm{Gr}^0(\mathfrak{h})$ weakly equivalent to a complex of vector bundles, then $\mathfrak{h}$ is itself weakly equivalent to a curved L_∞-algebra of the form $\Omega^*(M)\otimes_{\mathcal{C}^\infty(M)} E$ for a graded vector bundle E, by the Homotopy Transfer Theorem 3.12. Similarly, if $\mathfrak{g}$ is an L_∞-algebroid whose underlying $\mathcal{C}^\infty(M)$-module is weakly equivalent to a bounded above complex of vector bundles, then $\mathfrak{g}$ is weakly equivalence to a differential-geometric L_∞-algebroid by the Homotopy Transfer Theorem 4.15. □

Remark 6.15. Part (2) of Theorem 6.14 can be made more explicit as follows. Given a curved L_∞-algebra over the filtered cdga $\Omega^*(M)$ whose underlying graded module is of the form $\Omega^*(M)\otimes_{\mathcal{C}^\infty(M)} E$ for a bounded above graded vector bundle E on M. Proposition 5.4 shows that the graded vector bundle $TM\oplus E$ carries a natural L_∞-algebroid structure, with the anchor given by the projection to TM.

Conversely, let $\rho\colon \mathfrak{g}\longrightarrow TM$ be a differential-geometric L_∞-algebroid. If ρ is surjective, we can choose a splitting $\mathfrak{g} = TM\oplus\mathfrak{n}$ and Proposition 5.4 determines a curved L_∞-structure on $\Omega^*(M)\otimes_{\mathcal{C}^\infty(M)}\mathfrak{n}$.

Remark 6.16. The reader with derived inclinations may also take A to be a nonpositively graded cdga of the form $\big(\mathcal{C}^\infty(M)[x_1,\dots,x_n], d\big)$ where the variables $x_1,\dots,x_n$ are of strictly negative degree. Such cdgas arise as the algebras of functions on derived manifolds, in which case the module of derivations T_A indeed models the tangent sheaf of the derived manifold. Part (1) of Theorem 6.14 holds in this setting as well.

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