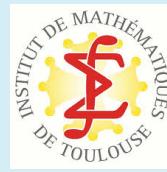


Modélisations mathématique et numérique de fluides géophysiques. Exemples en glaciologie, hydrologie et océanographie



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Institut de Mathématiques de Toulouse*



**** Synthèse de différentes études menées en étroite collaboration avec de nombreux collègues ****

Introduction

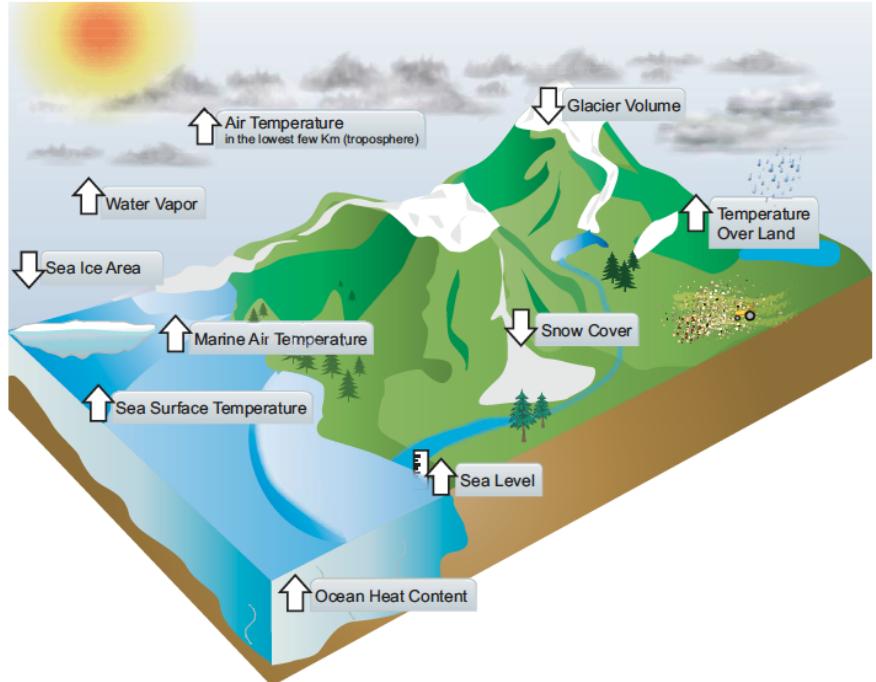
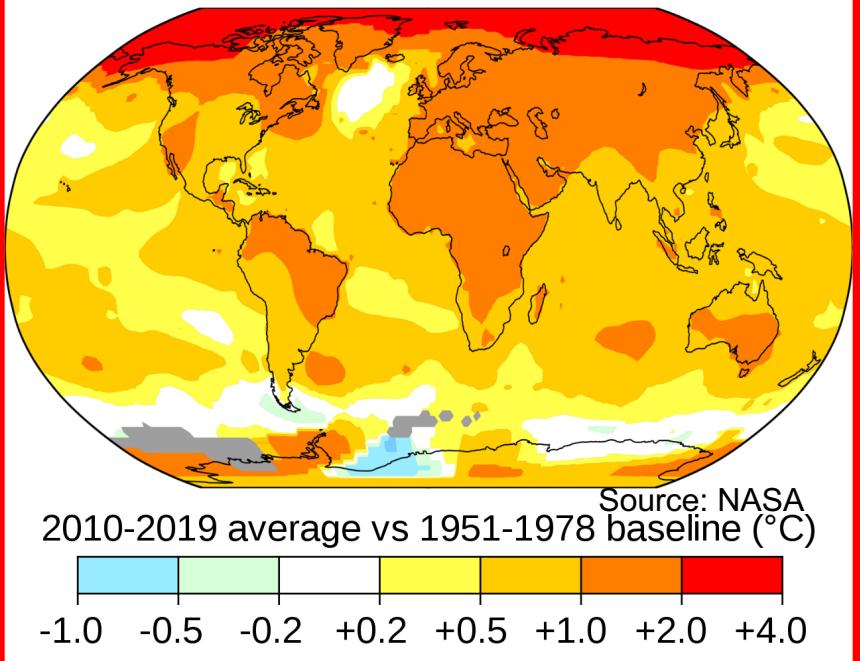
Exemple I en Glaciologie: Estimation de l'épaisseur des glaces en Antarctique.

J.M., J. Zhu (postdoc IMT-INSA-CNES), stages-projets INSA.
Collaboration UCI – NASA/JPL (Ca, USA).

Exemple II en Hydrologie: Estimation du débit des rivières à partir de mesures altimétriques.

J.M., K. Larnier (CS group/INSA) et al., stages-projets INSA.
Et aussi: T. Malou (thèse INSA/CLS group), P.-A. Garambois (INRAE) et al.
Collaborations CS group, INRAE, U. Strasbourg, CNES-NASA, CLS.

Temperature change in the last 50 years



**Réchauffement climatique → Nombreux changements observés et surtout à venir.
Dérèglements...**

Parmi les multiples conséquences,

- **Le niveau des mers** s'élève. Lien avec les écoulements des calottes polaires:
Etude #1 en glaciologie.
- Les **côtes** sont en tension... **Etude #2 en océanographie.**
- Les **ressources en eaux** (rivières, lacs) ont besoin d'être d'autant mieux gérées:
Etude #3 en hydrologie.

Ecoulements géophysiques: des données, beaucoup, mais hétérogènes, peu denses,
des phénomènes multi-physiques, multi-échelles, incertains,
des écoulements non reproductibles ...



→ **Modèles mathématiques** (équations différentielles, EDP) **multi-échelles, non linéaires.**
Des modèles forcément imparfaits, à paramètres incertains....
Base = Physique / lois de conservations / mécanique des milieux continus. Incertitudes.

- **Données in-situ:** riches mais très hétérogènes, locales / régionales, rarement globales,
- **Données satellitaires:** immenses et précieuses bases de données globales

mais pas forcément évidentes à « assimiler » ... cf slides suivantes.

Remarque. En vue d'analyses statistiques préliminaires de ces données, « nettoyages » des BD: travail amont généralement très conséquent. Idem bien entendu en vue d'estimations avec des algorithmes de ML - IA eg. réseaux de neurones profonds.

Si objectif = modélisation - simulation représentatives puis prédictives...

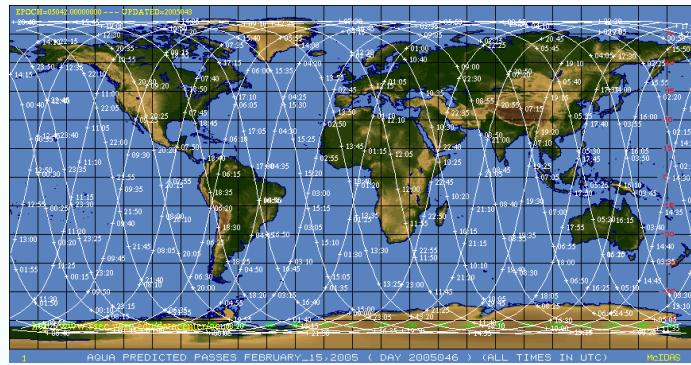
→ **Fusion de toute l'information disponible**

- Bases de données (satellitaires, in-situ, images-vidéos etc),
- Des modèles mathématiques de physique – mécanique,
- Des lois a-priori probabilistes-statistiques.

Fusion via des méthodes mathématiques.

Mots-Clefs: Assimilation de données, optimisation, modèles réduits,
Incertitudes, calculs hautes performances etc et aussi l'«I.A. Hybride » naissante.

L'observation spatiale: une révolution pour les Sciences de la Terre



- Base de données titaniques (To ++, « Big Earth data ») issues de « radars » e.g. Altimétrie, InSAR.
- Néanmoins,
 - des données *délicates à fusionner quantitativement* avec les modèles physiques ad-hoc,
 - des données à des échelles spatiales parfois supérieures à celles des phénomènes modélisés (longueurs d'ondes minimales observées grandes),
 - des données souvent moins fréquentes que la dynamique des phénomènes modélisés (basses fréquences observées).

A propos du niveau des mers

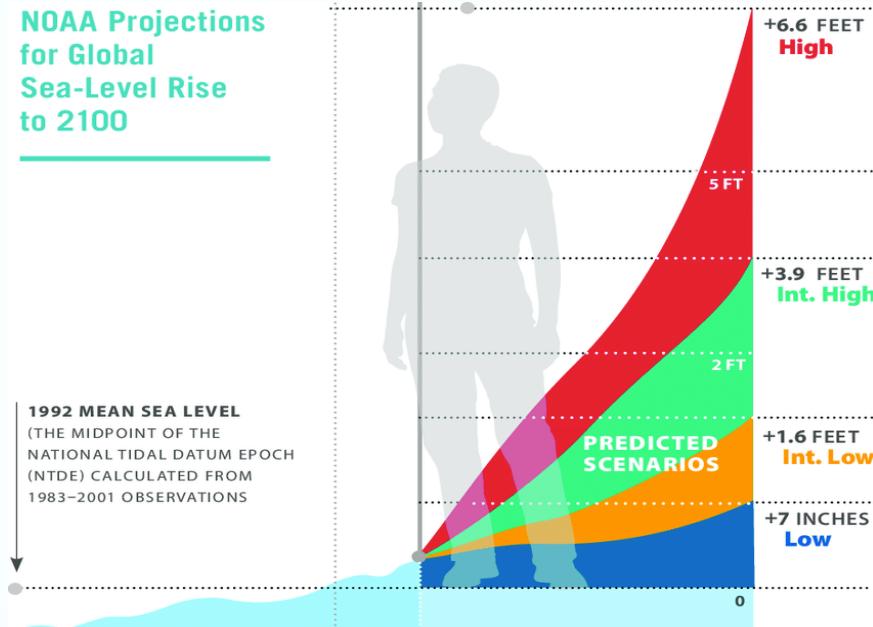
Le niveau moyen des mers s'élève
et va s'élèver plus fortement
dans les décennies à venir.

Implications:

- migration de populations,
- érosions des côtes,
- inondations côtières (tempêtes),
- salinisation des sols etc

NOAA Projections
for Global
Sea-Level Rise
to 2100

1992 MEAN SEA LEVEL
(THE MIDPOINT OF THE
NATIONAL TIDAL DATUM EPOCH
(NTDE) CALCULATED FROM
1983–2001 OBSERVATIONS



• Sources de la montée *actuelle* des mers [GIEC] :

- 38% dilatation thermique des océans,
- 45% fonte glaciers « montagnes » (alors qu'ils ne représentent que 1% de la cryosphère),
- 17% calottes (Antarctique, Groenland).

La contribution à venir des calottes va s'accroître [GIEC].

• Volumes de la cryosphère en équivalent montée des mers:

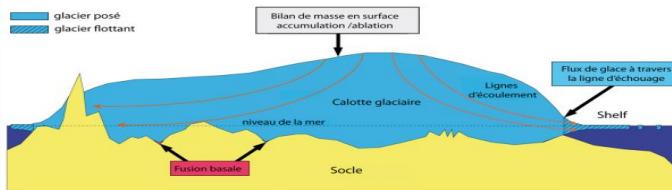
Glaciers « montagnes »: 0.2 m ; calotte Groenland: 7 m.

Calotte Antarctique (2/3 de l'eau douce sur Terre): 58 m.

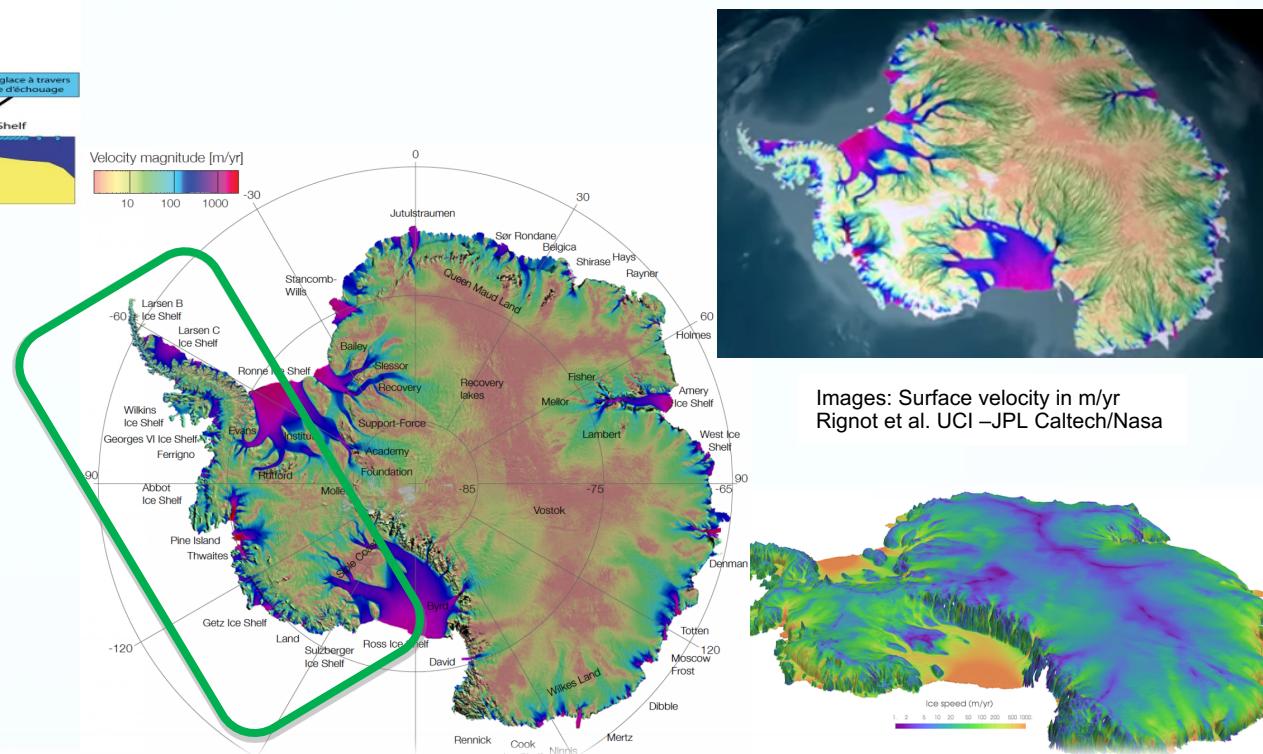


Partie I: Glaciologie.

Dynamique des glaces en Antarctique & Estimation du socle rocheux sous la calotte



Les glaces s'écoulent sous l'effet de la gravité.
10% du trait de côte évacue la quasi totalité de
la masse (fleuves de glace ~km/a).



Images: Surface velocity in m/yr
Rignot et al. UCI –JPL Caltech/Nasa

Concernant les décharges de glaces dans les océans,

• **l'Antarctique de l'Ouest** (de masse équiv. à 4.6 m mers) est actuellement le contributeur principal.

(80% des ~14 mm sur la période 1979-2017, [Rignot et al. 2019]).

Cette région se révèle être plus instable que toutes les prévisions-simulations numériques passées.

• **l'Antarctique de l'Est** (de masse équiv. à 53.3 m mers) contribue beaucoup moins. (20% d'après [Rignot et al. 2019]).

Quid de son (in-)stabilité à venir suite au réchauffement des océans alentours ?

Une contribution majeure à l'élévation des mers ? [Mengel et al. 2014, Ritz et al. 2015 etc].

Cependant encore de nombreuses d'inconnues sur cette région... Milieu très hostile, difficile à mesurer in-situ.

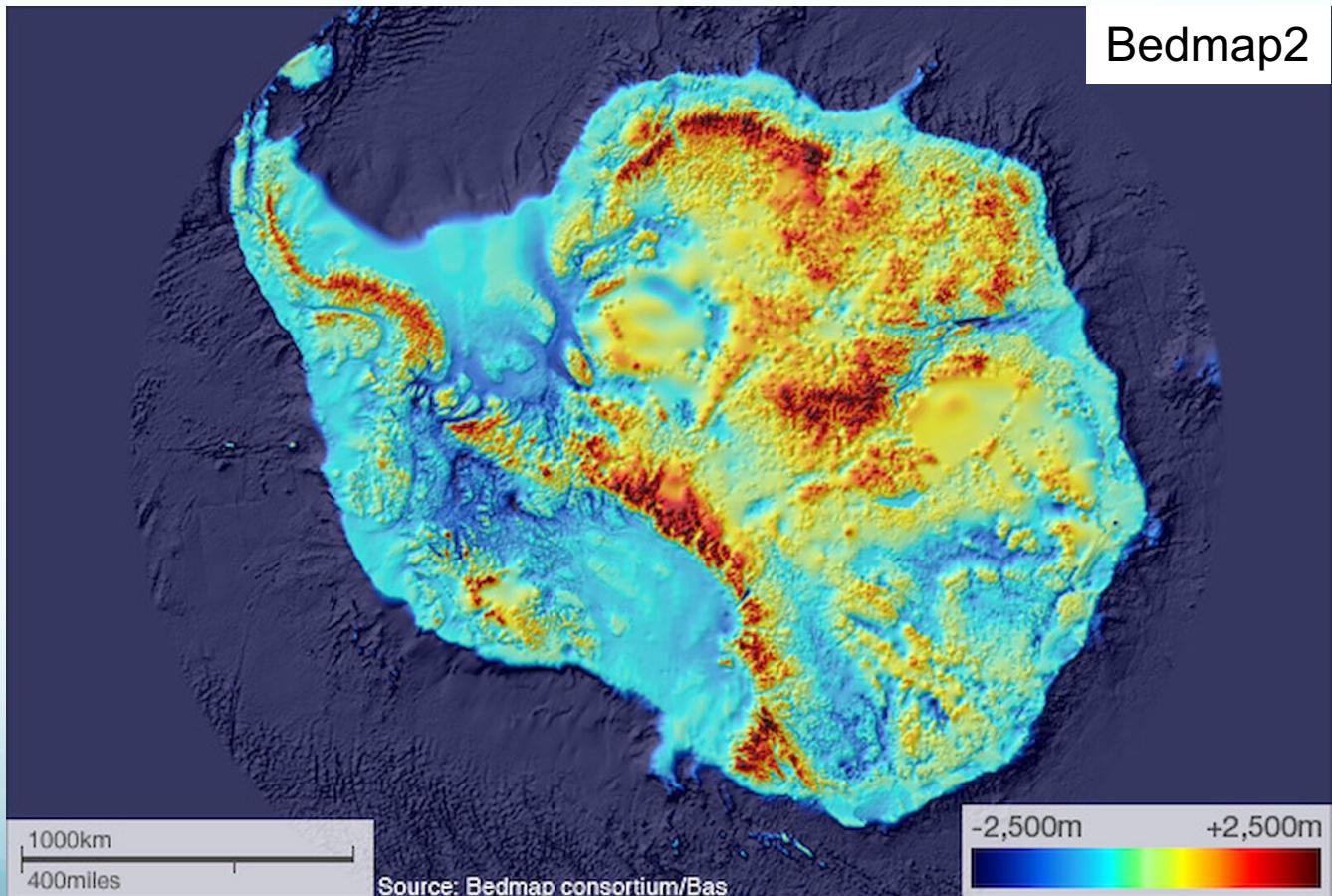
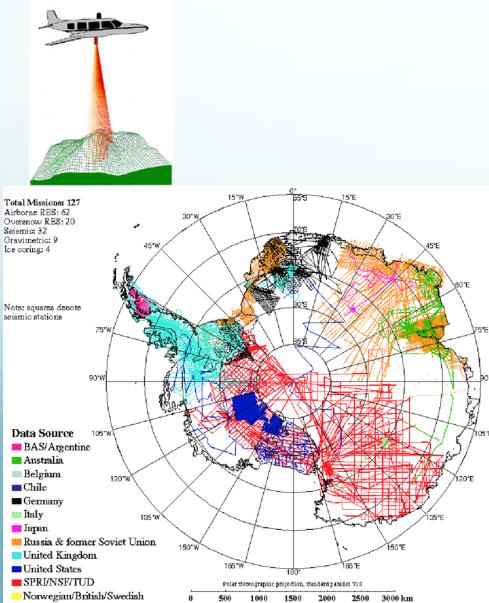
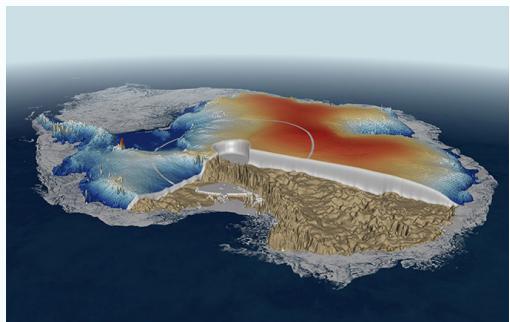
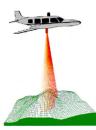
Objectif: Cartographie du socle rocheux sous la calotte Antarctique ↔ Estimer l'épaisseur de glace.

Bedmap2 [Fretwell-Pritchard, British Antarctic Survey, et al. 2013] = the reference bedrock map.

Built from: 25 millions survey measurements from airborne campaigns (IceBridge NASA, CReSIS)

+ Kriging method (basic statistical interpolation) in vicinities of the measurements.

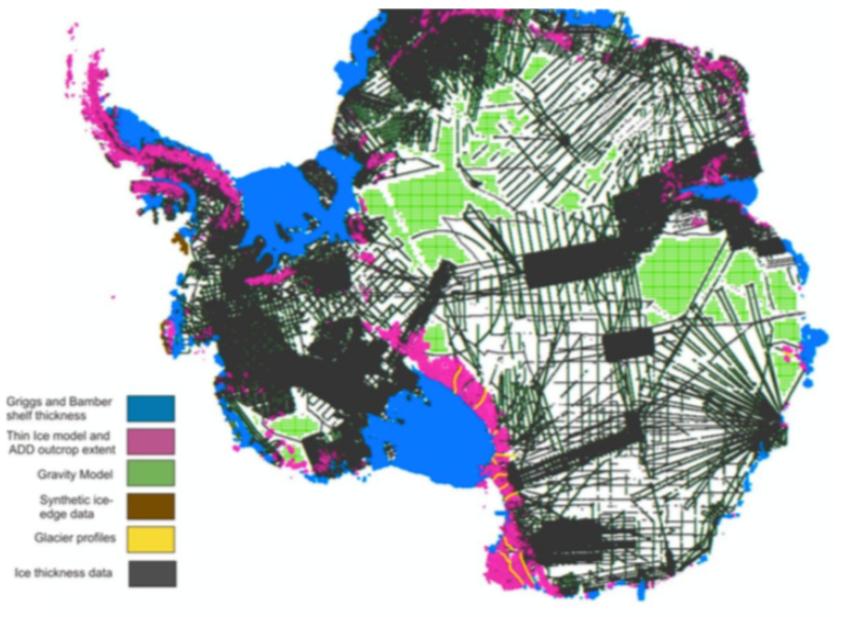
Beyond 50 km of any measurement: ice thickness estimation is based on the gravity-field inversion.



Objectif: Cartographie du socle rocheux sous la calotte Antarctique ↔ Estimer l'épaisseur de glace.

For interior sectors: gravimetry inversion → large uncertainties.
« Poles of ignorance » are ~200Km from nearest data.

Datasets used for Bedmap2



[Fretwell-Pritchard, British Antarctic Survey, et al. 2013]
Empirical uncertainties

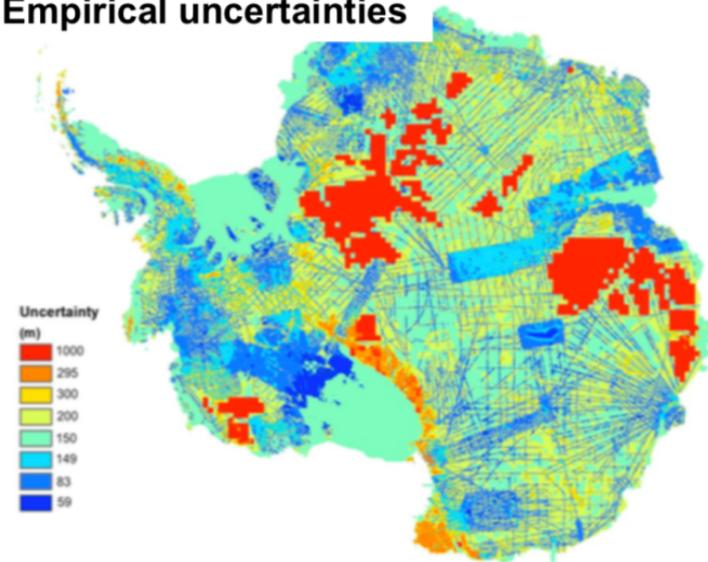
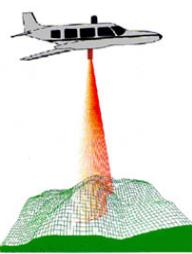


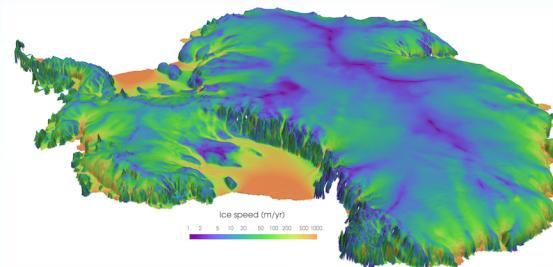
Fig. 11. Estimated uncertainty in ice thickness grid.



Lines: airborne measurements of ice thickness.
25 millions of survey points. IceBridge (NASA), CReSIS campaigns.

- Accurate close to the flights tracks only.
- Error up to +/- 1000 m in interior sectors (gravimetry uncertainties) ...

Notre objectif: estimer sur une base physique cette épaisseur de glace.

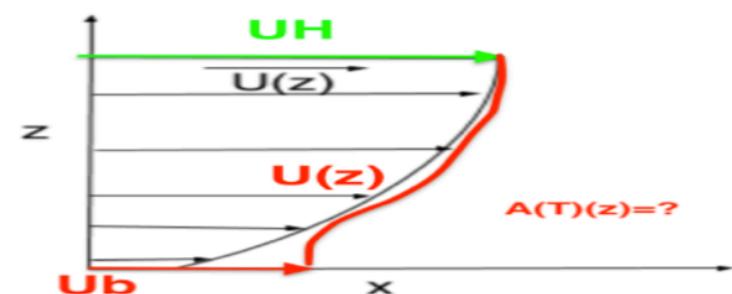
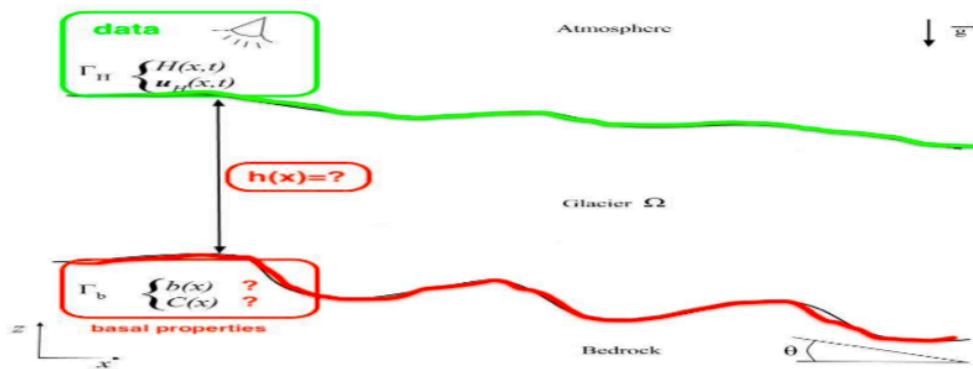


Etant données : - l'altitude de la surface $H(x)$, la vitesse en surface $U_H(x)$,
- des mesures très peu denses d'épaisseur de glace $h^{tracks}(s)$,

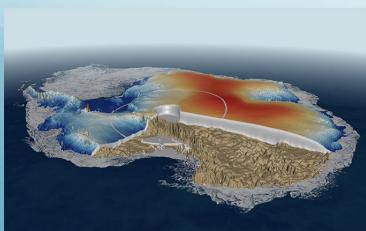
Estimer: l'épaisseur de glace $h(x)$ partout.

i.e. éditer une cartographie du fond rocheux $b(x)$ sous la calotte.

$$H(x) = b(x) + h(x)$$



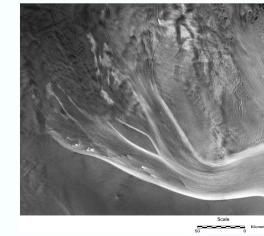
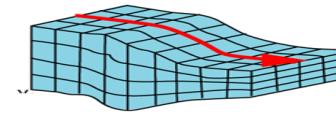
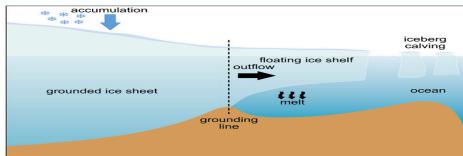
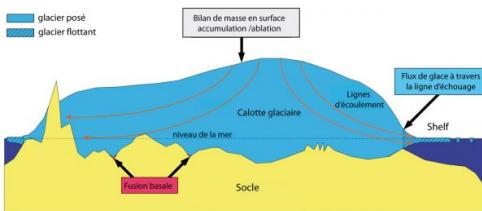
→ Identifier également la déformation interne $\bar{A}(x)$ & la friction au fond $C(x)$!...



Problème inverse a-priori « mal posé »...

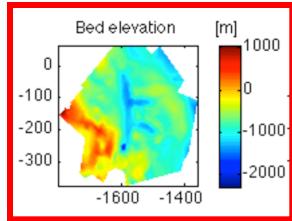
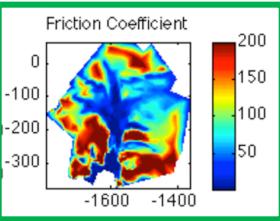
Modèle(s) d'écoulements = Problème direct

Les glaciers s'écoulent par gravitation (calotte en déséquilibre)



Input parameters

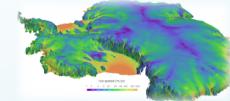
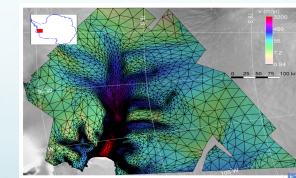
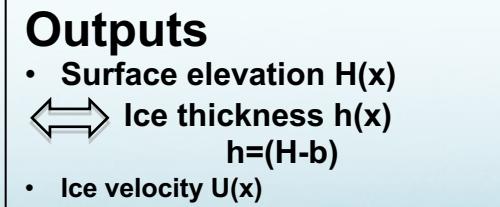
- Boundary + initial conditions
- Bedrock elevation $b(x)$
- Friction field at the base $C(x)$
- Fluid rheology parameters $A(x)$



Flow models (PDE, conservation laws)

Large scale \rightarrow 2D reduced & flow regime dependent models

$$\begin{cases} -\operatorname{div}(A(\|D(\mathbf{u})\|).\mathbf{u}) + \beta^2 \mathbf{u} = S \text{ dans } \Omega \\ \mathbf{u} = \mathbf{u}_\Gamma \text{ sur } \Gamma \\ \partial_t h + \mathbf{u} \cdot \nabla h = a \end{cases}$$



Various computational ice dynamics softwares for direct modeling are available for a decade now
eg. ISSM (JPL Nasa-Caltech/UCI); Elmer Ice (.eu) etc
also DassFlow (local INSA Toulouse) with original numerical schemes/algorithms.

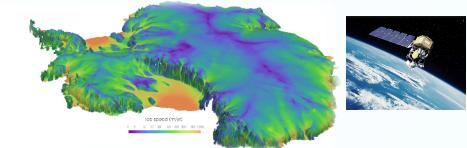
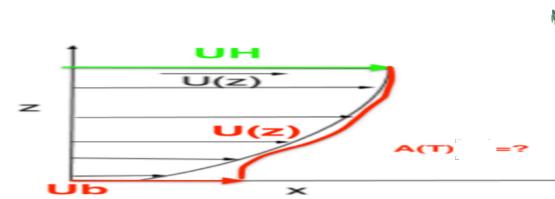
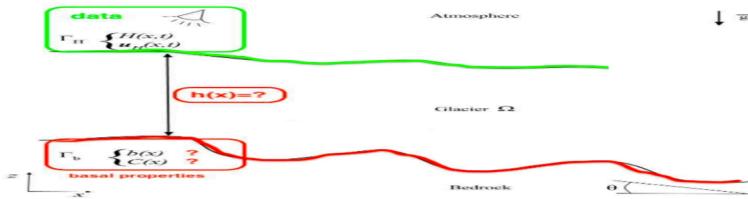
The inverse problem

Bedrock elevation $b(x) = ?$
Basal friction $C(x) = ?$
Internal deformation $A(x) = ?$

Model

$(H(x), U_H(x))$

Surface signature measured by satellites



The challenge.

Given the surface signature everywhere + sparse airborne measurements b_{tr} ,
separate the effects of the bed topography $b(x)$
from the basal friction $C(x)$ ones & the internal deformation $A(x)$ ones.

- Inverse problems are generally more complex than the direct problem.
- Inverse problems are very often “ill-posed”: few solutions are possible...
- Advanced mathematical methods are required.

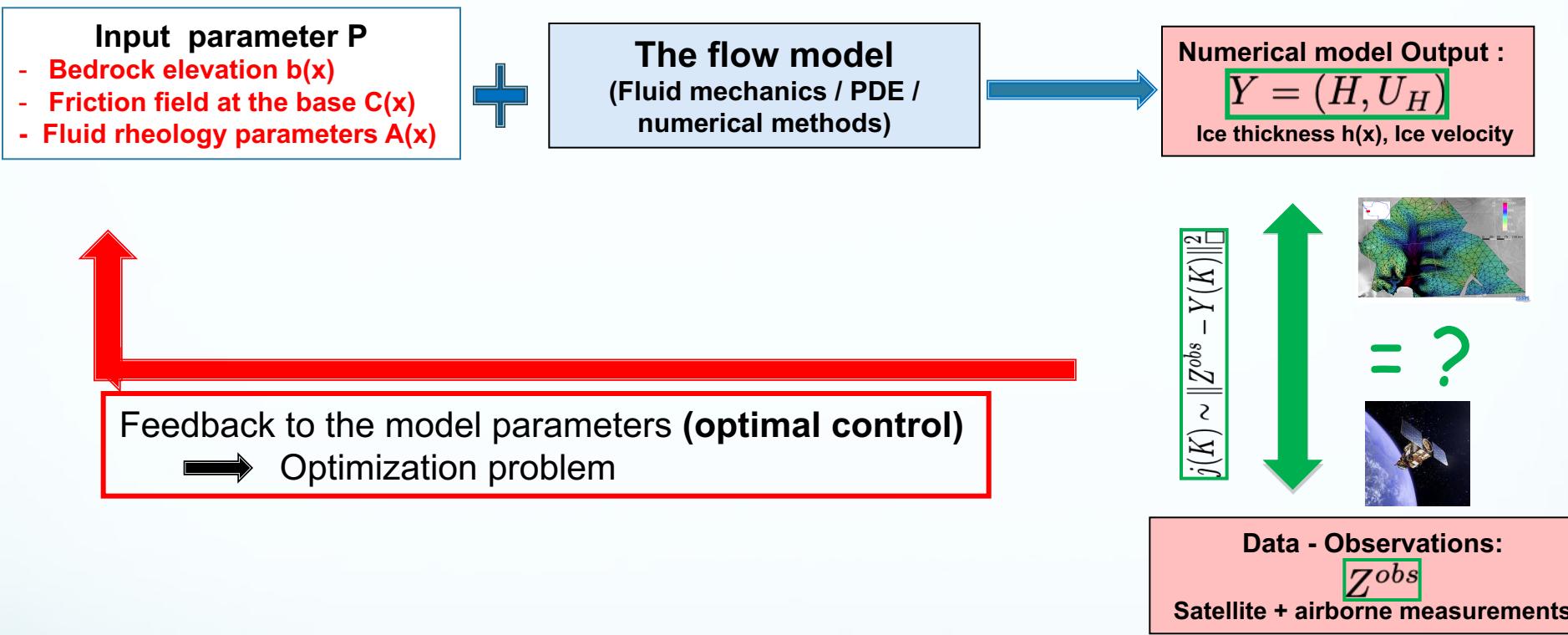
Keywords: mathematical transforms, optimization, Data Assimilation, Machine Learning, Computational Sciences etc

We are going here to develop two approaches to solve this inverse problem:

- **Data Assimilation.** Physically-informed estimations.
- **ML - Artificial Neural Networks.** Purely data driven estimations.

Variational Data Assimilation: fusing at best all the information to solve inverse problems

Present goal: to infer the bedrock $b(x)$ beneath the glaciers



Cost function: $j(P) \sim \|Z^{obs} - Y(P)\|_{\square}^2$

$$P = (b(x), C(x), A(x))$$

Optimization problem:

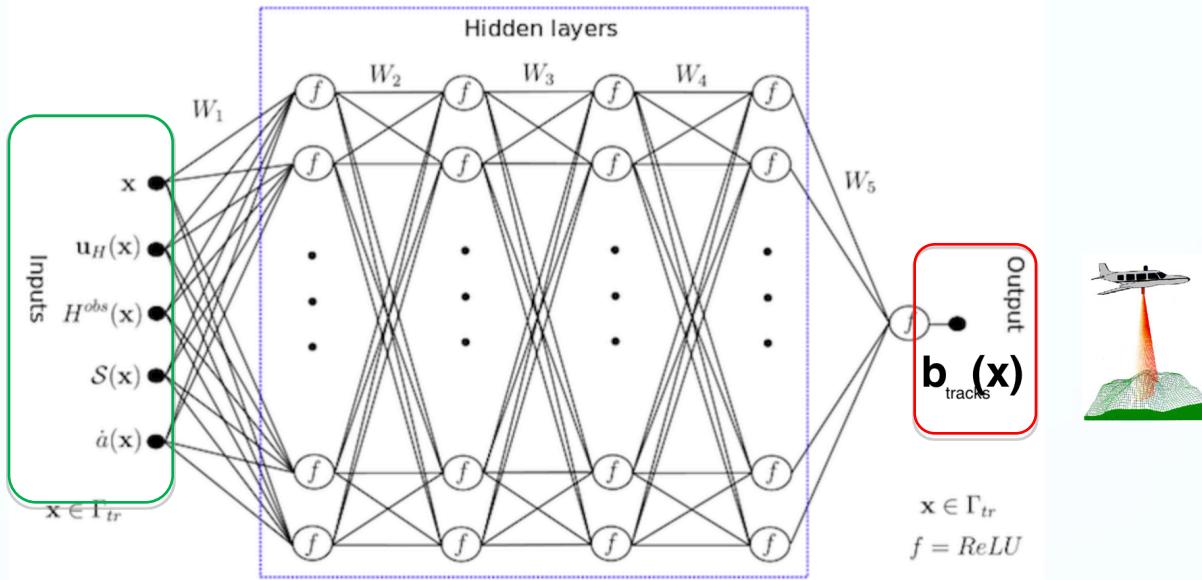
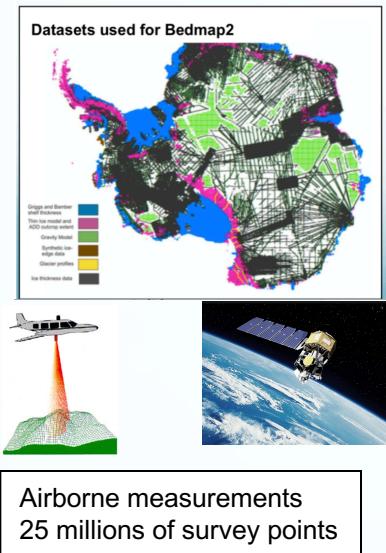
$$\min_P j(P)$$

under the constraint

« the physical model is satisfied: $Y = \mathcal{M}(P)$ »

(Deep) Artificial Neural Networks: a purely data-driven approach to solve inverse problems

Present goal: to infer the bedrock $b(x)$ beneath the glaciers



The model = the ANN architecture !

$$Y_i = b_i$$

Cost (loss) function: $j(\mathcal{D}_I; W) = \frac{1}{N_{ex}} \sum_{i=1}^{N_{ex}} (Y_i^{estim} - Y(H_i, S_i, |U_H|_i, \dot{a}_i))^2 \quad \forall i$

Optimization problem: $\min_W j(\mathcal{D}_I; W)$

DNN based estimations: greatly empirical, surprisingly highly efficient for numerous multi-scale, highly non-linear phenomena. However no intrinsic certification criteria.

Here, we did not manage to set up DNNs to infer $b(x)$ (equivalently the ice thickness $h(x)$). [INSA students projects]

Solving the inverse problem by a Data Assimilation method.

Question: what physical model should be considered ?

Bedrock elevation $b(x) = ?$
 Basal friction $C(x) = ?$
 Internal deformation $A(x) = ?$



Flow model
employed in the
reverse way



$(H(x), U_H(x))$
 Surface signature measured by
satellites
 + some sparse measurements of $b(x)$



- Issue.** If considering the (natural) original direct model as the model, the inverse problem is ill-posed !...
i.e. numerous solutions $(b(x), C(x), A(x))$ are possible !...
- An adequate inverse model has to respect a balance between
the model complexity - accuracy & inversion feasibility - stability ...
- Strategy followed here:** Derivation of a new flow model **intrinsically integrating the available data** ...

Keywords: law conservations in fluid mechanics, reduced flow models, uncertainties.

Resulting reduced uncertainty inverse model

$$-\operatorname{div} \left(\frac{|\mathbf{u}_H|}{S} \gamma h \nabla H \right) = \dot{a}$$

$$\bar{\mathbf{u}}(\mathbf{x}) = -\frac{|\mathbf{u}_H|}{S} \gamma \nabla H$$

$\frac{|\mathbf{u}_H|}{S}$: surface data ; $\eta = (\gamma h)$: unknown parameter

$$b = (H - h)$$

with γ an original dimensionless multi-physics number.

$$\gamma(x) = 1 - C^{\text{Rheology}} C^{\text{Regime}}(x)$$

Ref. [J.M. – DesBoscs (ex-INSA)'17]
 [J.M. – Zhu'19]

The resulting inversion strategy

Step 1) To infer the product $\gamma h \equiv \eta$ by **Variational Data Assimilation** (well-posed problem ☺)

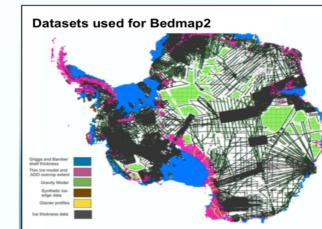
$$\min_{\eta} j(\eta)$$

under the model constraint $-\operatorname{div}\left(\frac{|\mathbf{u}_H|}{S} \eta \nabla H\right) = \dot{\mathbf{a}}$

$$j(\eta) = \|Z^{obs} - Y(\eta)\|_2^2$$

Step 2) Deduce the dimensionless number values along the flights tracks:

$$\gamma_{tr}(x) = \frac{\eta}{h_{tr}}(x) \text{ along flights tracks } (x \in \Gamma_{tr})$$



Step 3) How to separate h from γ everywhere i.e. outside the flights tracks ?

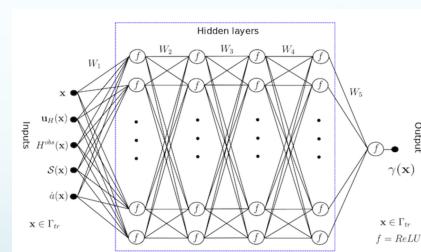
It turns out that a **DNN** enables to infer γ everywhere from its values along the tracks γ_{tr} !

$$\gamma(x) = 1 - C^{Rheology} C^{Regime}(x)$$

How is that possible (even in terms of fluid mechanics) ?

This is not clear yet... However the estimations are robust...

Recall. DNNs were not able to directly infer the ice depth h from h_{tracks} ...

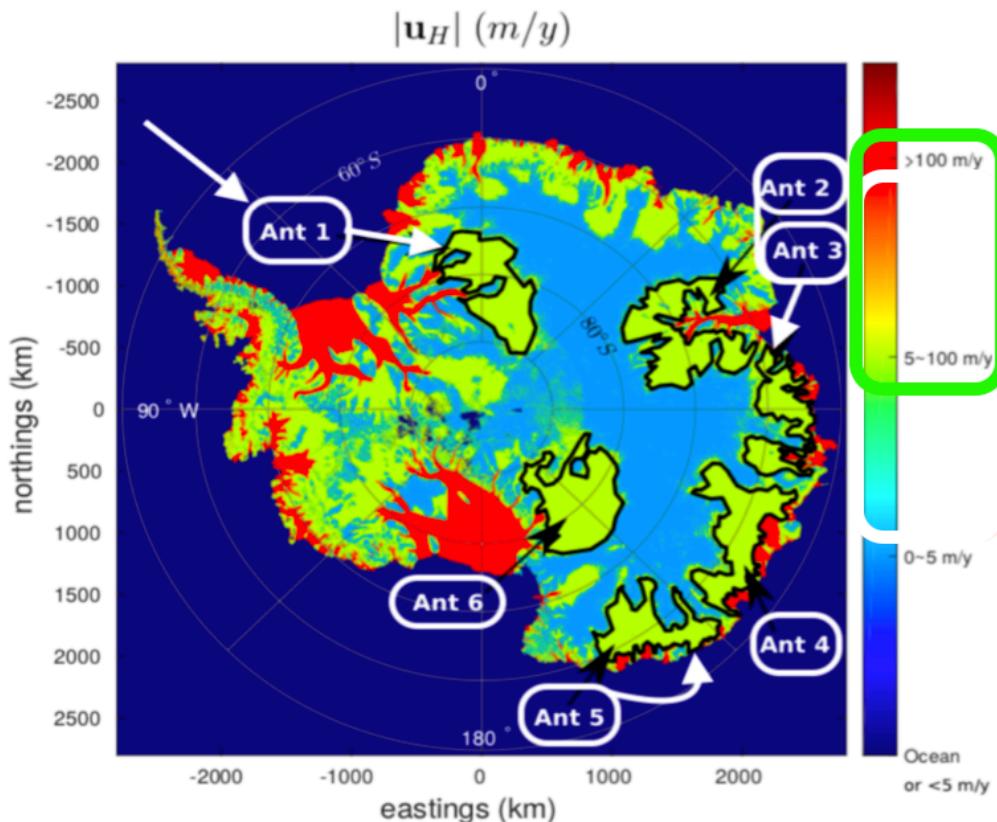


Final result: the ice thickness $h(x)$ everywhere therefore the bedrock elevation $b=(H-h)$.

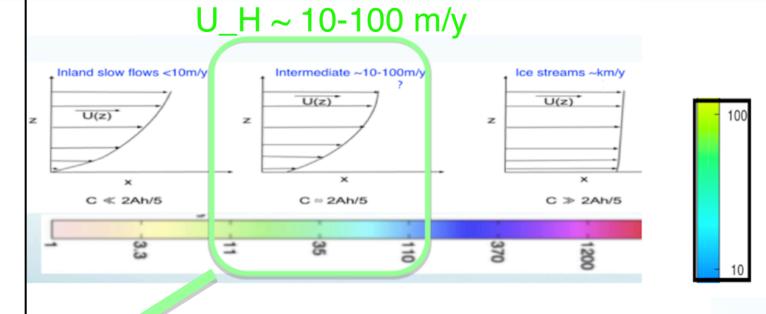
Example of numerical results

The inverse flow model is mechanically consistent for the **challenging intermediate velocity regions** in large East Antarctica regions where:

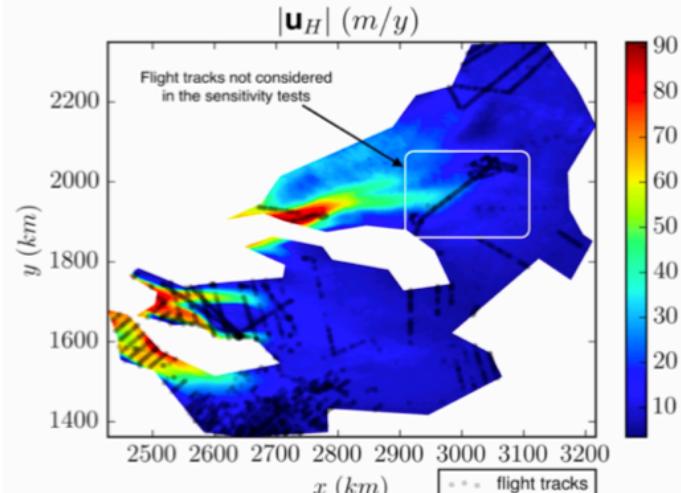
- uncertainties on $b(x)$ are large ,
- measured surface velocity values $U_H(x)$ are large enough to be reliable.



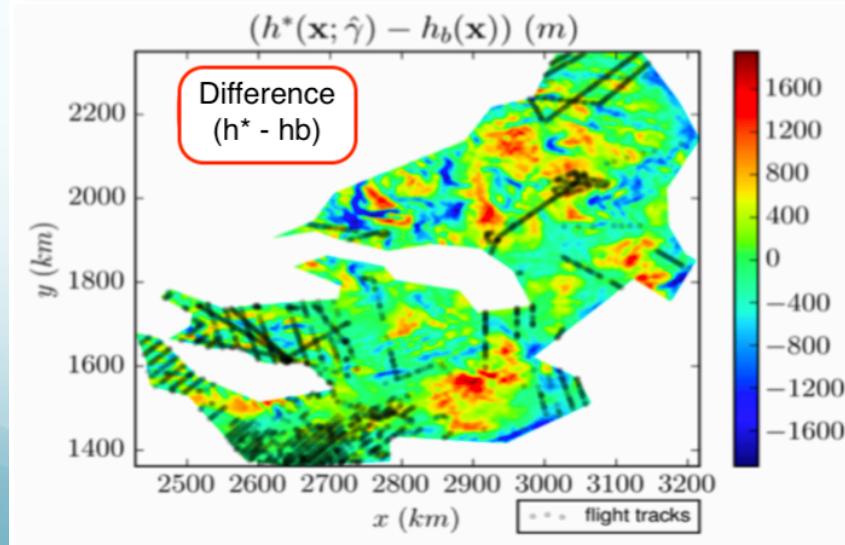
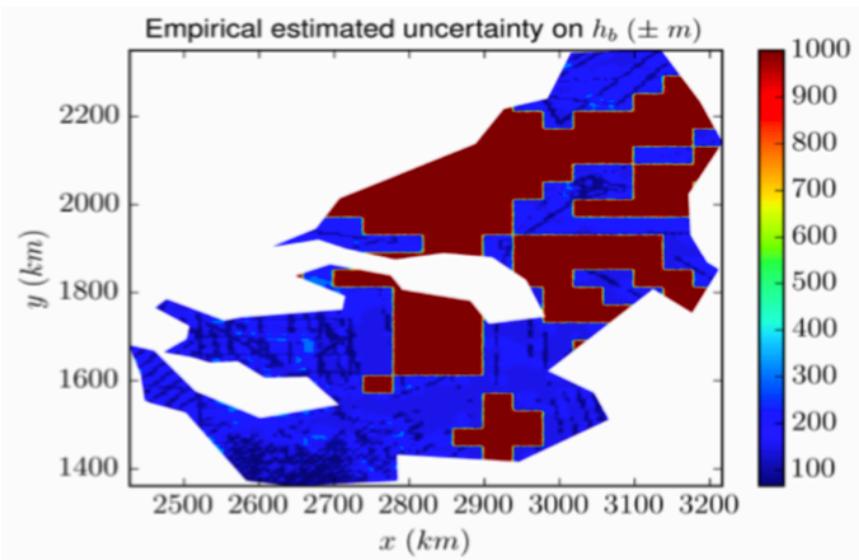
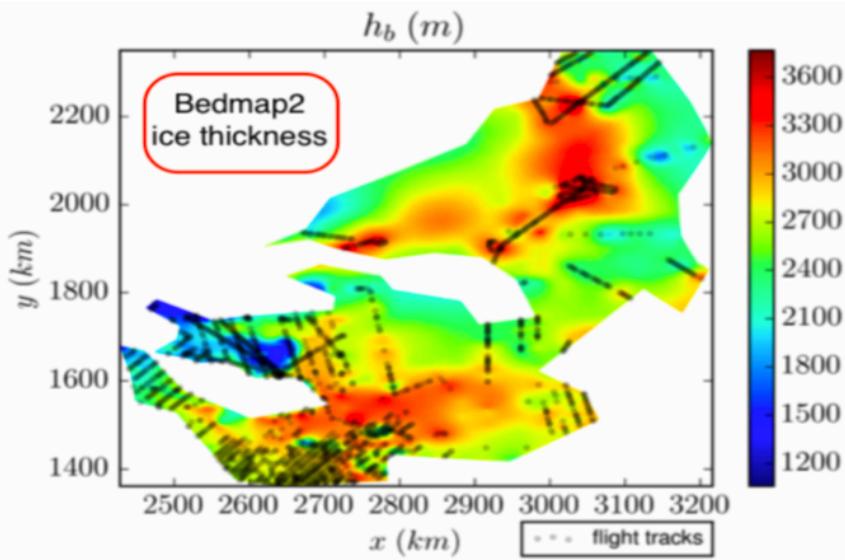
Flow regimes velocity vertical profiles.
From very slow (L) to very fast (R) flows.



Refs. [J.M. – J. Zhu '19]
& [J.M., Zhu] (in revision / science ok).



Example of numerical results



► First guesses : depth $h^{(0)} = h_b$ = Bedmap2; SMB $\dot{a}^{(0)}$ = Racmo2 [Noel et al.]'18.

Bedmap2 [Fretwell et al.'13] : ArcGIS Topogrid routine based on the ANUDEM algorithm [Hutchinson et al.]'89, with statistically-based uncertainty estimations.

Surface $ \Omega_1 $	370809 km ²		
Mean ice thickness of h_b (h_b = Bedmap2)	2696.2 m		
# mesh vertices : in Ω_1 / on flight tracks	57661 / 2152		
RU-SIA model output (with $\hat{\gamma}$)	Median	Mean	Max
$ H(h_b) - H^{obs} $ (before h -inversion)	7.0 m	10.3 m	53.0 m
$ H(h^*) - H^{obs} $ (after h -inversion)	2.6 m	3.7 m	44.6 m
Inferred RHS \dot{a}			
$ \dot{a}^* - \dot{a}_b $	0.7 cm/y	0.7 cm/y	3.6 cm/y
$ \dot{a}^* - \dot{a}_b / \dot{a}_b $	19.2 %	17.0 %	20.0 %
Inferred thickness h			
$ h^* - h_b $	275.2 m	356.8 m	1953.5 m
$ h^* - h_b / h_b $	10.4 %	13.3 %	65.7 %
Ice volume change in km ³ / in %	$1.6 \cdot 10^4 \text{ km}^3 / 1.6 \%$		

In summary

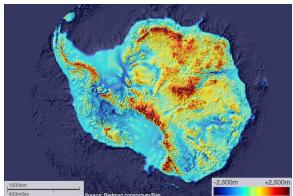
- This challenging a-priori ill-posed mathematical problem has been successfully solved here by « optimally » combining:

- + large heterogeneous datasets,

- + an original reduced multi-physics model intrinsically taking into account the data,

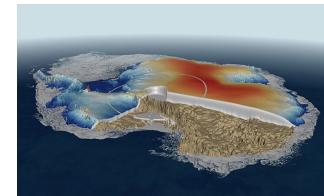
- + advanced Data Assimilation formulations combining deterministic and probabilistic laws,

- + (standard) Deep Neural Networks.



$$-\operatorname{div} \left(\frac{|\mathbf{u}_H|}{S} \gamma \mathbf{h} \nabla H \right) = \dot{\mathbf{a}}$$

$$\min_P j(P)$$

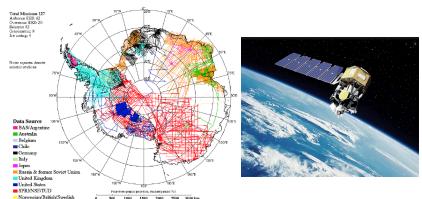


- In terms of final product: are these bedrock estimations more accurate than present ones ?

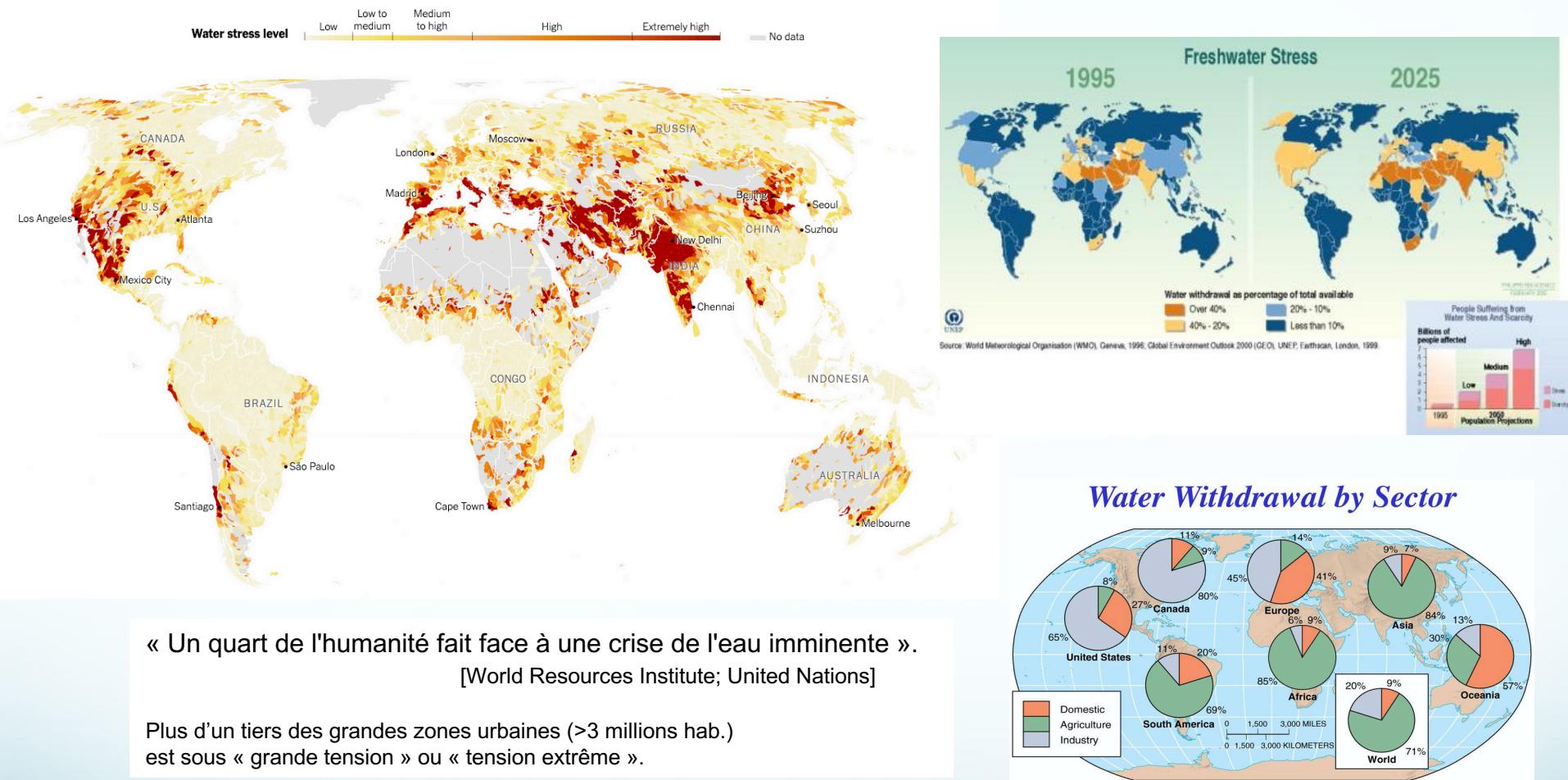
Multi-disciplinary studies need now to be pursued to analyse in detail the results.

One need very likely to introduce some geomorphology-consistent regularizations in the optimization formulation...

Indeed, reduced physically-informed models are necessarily incomplete, unperfect...



Exemple II) Hydrologie. Estimation du débit des rivières à partir de mesures altimétriques.

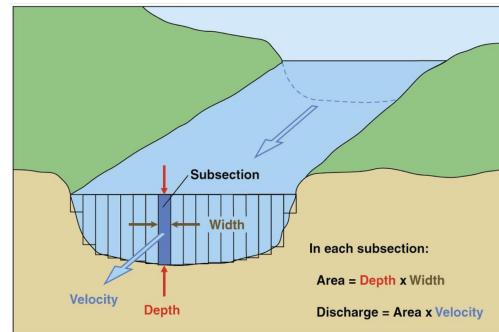


Une estimation fine du débit des rivières à l'échelle planétaire serait bénéfique pour mieux

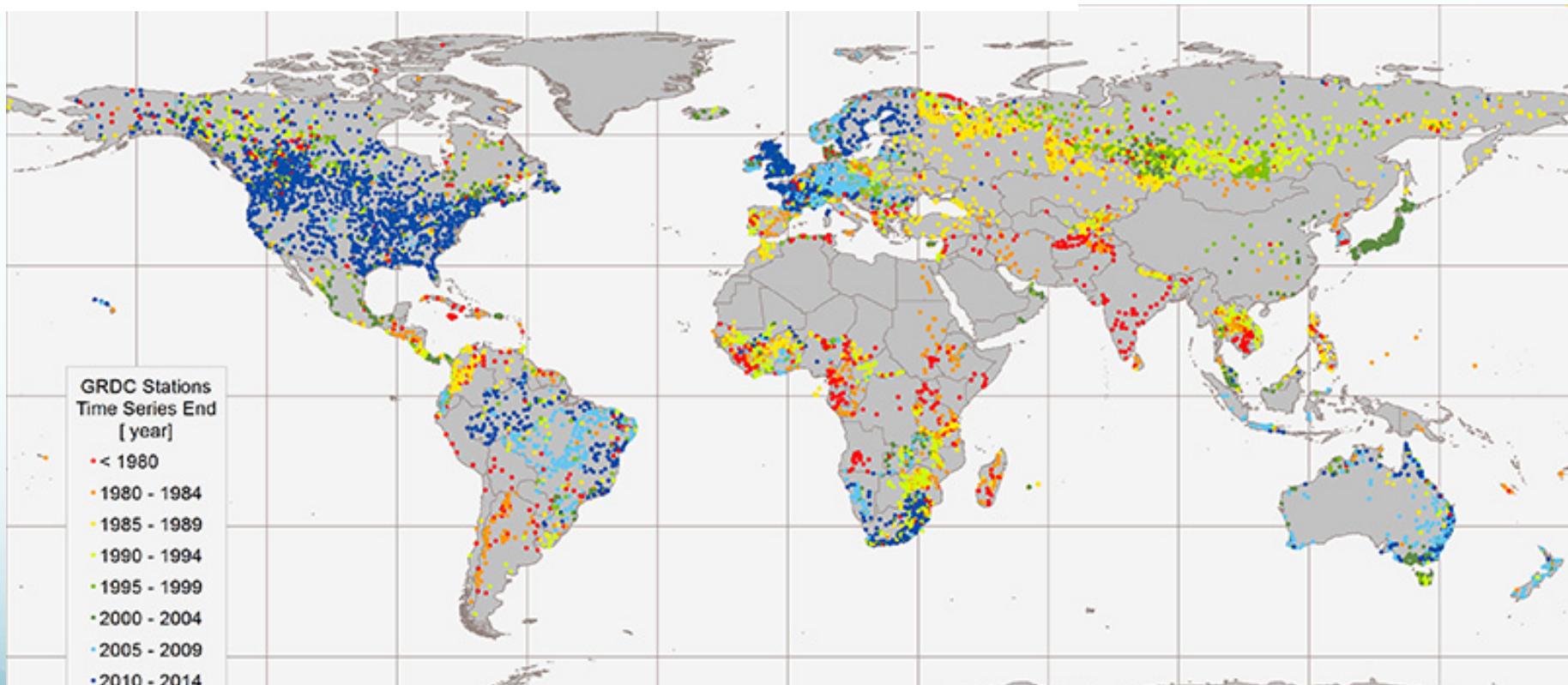
- gérer l'eau: consommations urbaines, agricoles, industrielles,
- modéliser les inondations et les risques environnementaux liés,
- contribuer aux diverses politiques publiques.

River Discharge Monitoring Network

Discharge Q = wetted cross- section A x mean velocity U
 $Q = A \cdot U \text{ (m}^3/\text{s)}$



In-situ measurements of discharge Q (m^3/s): monthly values

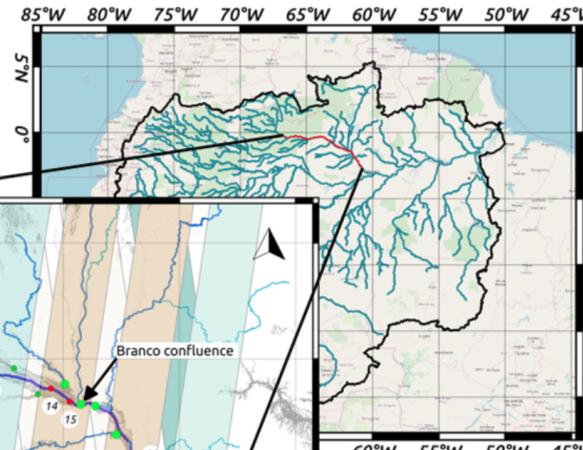
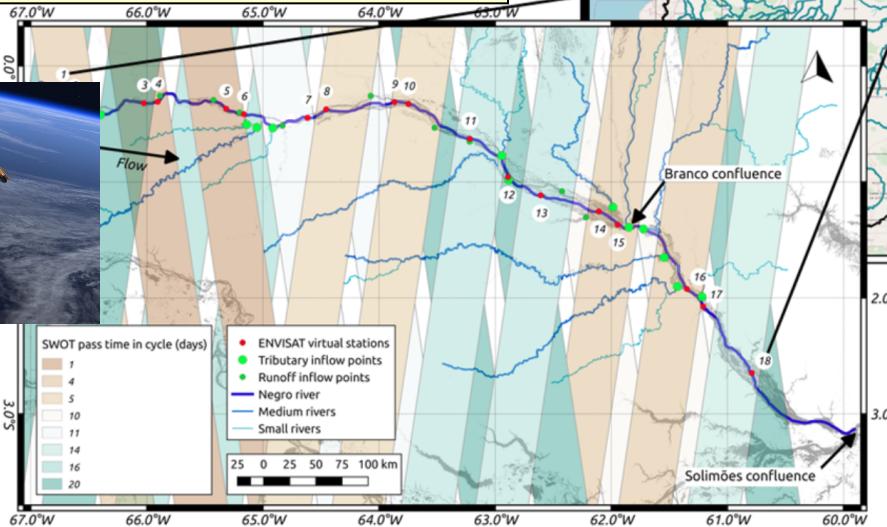


- Stations de mesures pas forcément bien réparties et surtout de moins en moins nombreuses...
- Aussi toutes les données in-situ dont nous disposons ne sont pas toujours partagées...

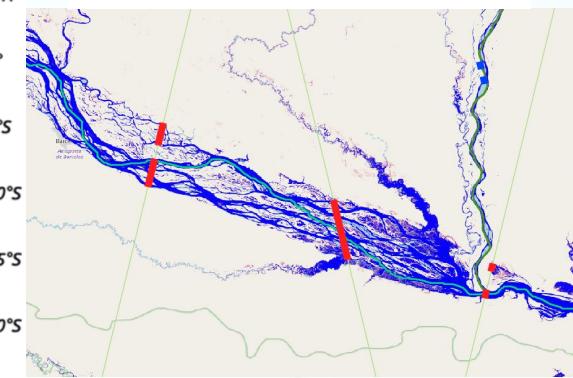
Spatial Hydrology

Numerous satellites datasets
from « pointwise » altimeters

eg. Sentinel-x, EnviSat (ESA), Jason (Nasa-Cnes) etc

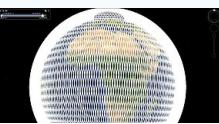
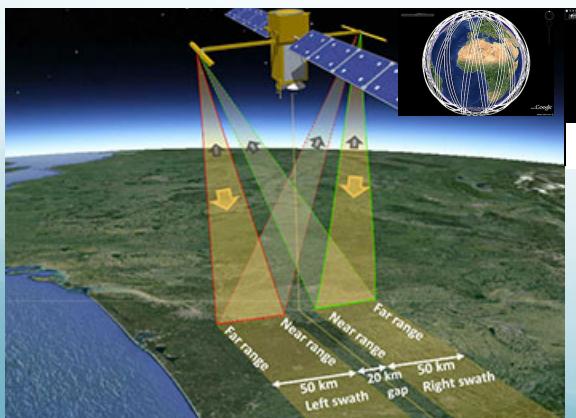


Sentinel-3b tracks & measurements
over the river

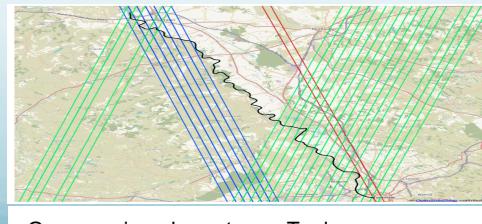


Rivers in the Amazonian basin
Test case considered in on-going PhDs
T. Malou (INSA-IMT/INRAE/CLS group)
L. Pujol (U. Strasbourg/INRAE/INSA-IMT/CNES)

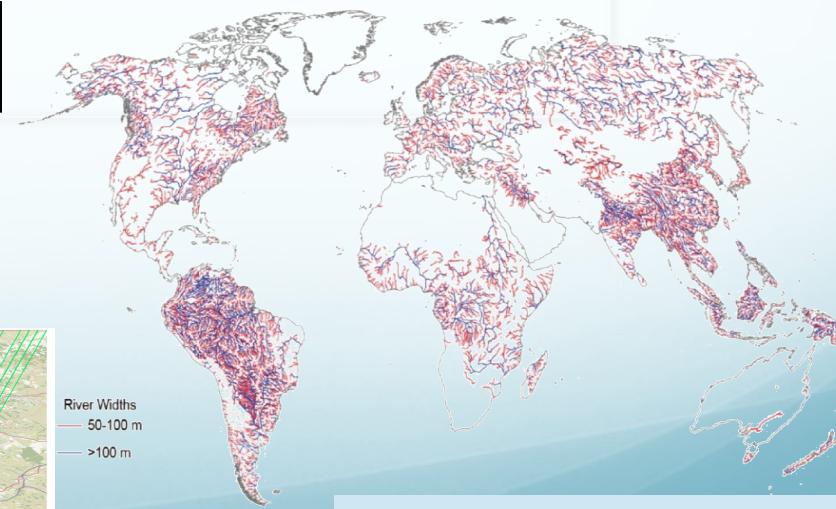
A promising forthcoming instrument :
SWOT (Surface Water & Ocean Topography). NASA-CNES 2022.



Repeat cycle 22 days

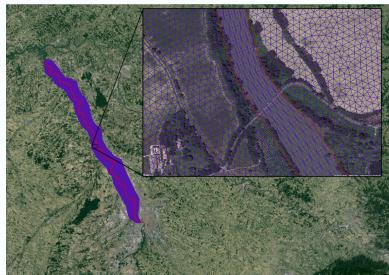


Garonne river downstream Toulouse.
SWOT swaths during the calibration phase

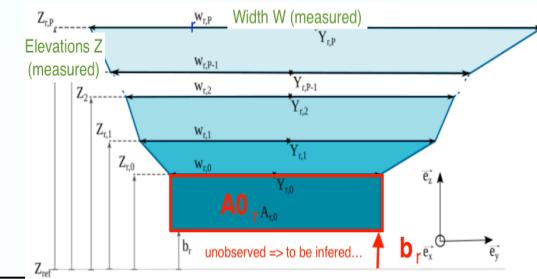
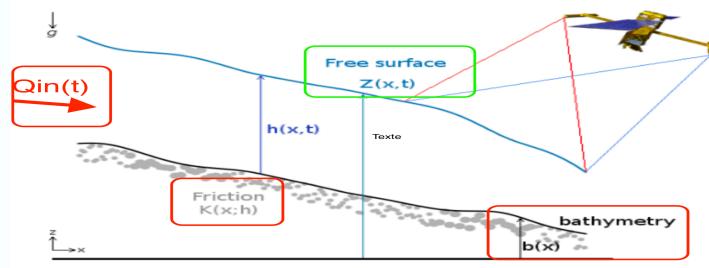


Database under construction:
~265 000 river portions to be observed.
[SWOT Science Team, Pavelski et al.]

Flow model set up (direct problem)



Garonne river downstream Toulouse.
Data: IMFT; Computations: INSA-IMT, CS group.



Wetted-cross section $A_0(x)$... equiv. bathymetry $b(x)$
Friction parameter $K(x, h(x,t))$

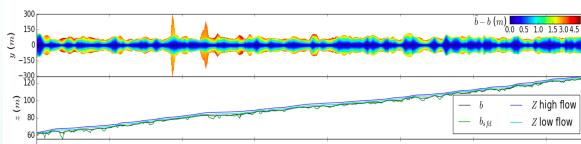


$$\begin{cases} \partial_t A + \partial_x Q \\ \partial_t Q + \partial_x \left(\frac{Q^2}{A} \right) + g A \partial_x Z \end{cases} = -g A S_f(A, Q; K)$$

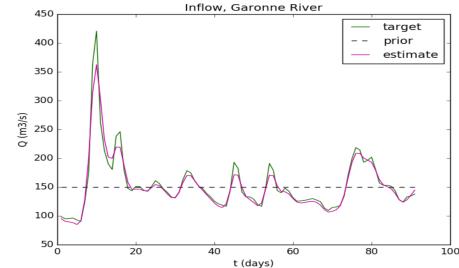
+ Boundary Conditions $Qin(t)$ + Initial Conditions



Output: $Y = (A, Q)(x, t)$



Discharge estimation $Q(.,t)$



The inverse problem

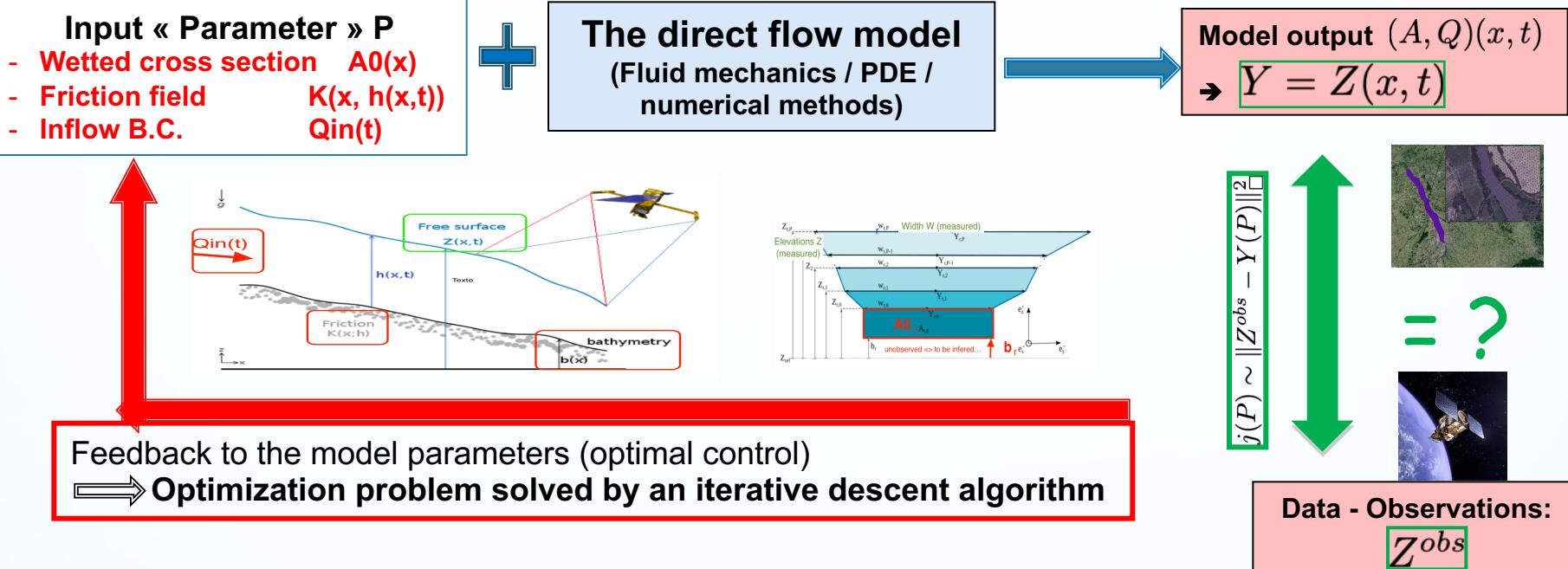
Given the surface measurements: the water elevation $Z(x,t)$ & width $W(x,t)$,

Measurements at large scale (~1 / 5 km) and low frequency (~10 / 20 days) only.

estimate the unobserved lowest cross-section $A_0(x)$ (bed topography $b(x)$),

the friction $K(x, h(x,t))$ and the inflow discharge value $Qin(t)$ (= the main target).

Parameters estimation by Variational Data Assimilation



Cost function: $j(P) \sim \|Z^{obs} - Y(P)\|_□^2$

$P = (A_0(x), K(x, h), Q_{in}(t))$

Optimization problem:

$$\min_P j(P)$$

under the constraint

« the physical model is satisfied: $Y = \mathcal{M}(P)$ »

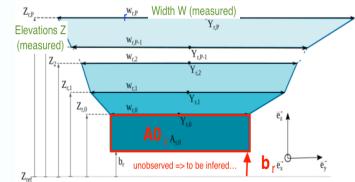
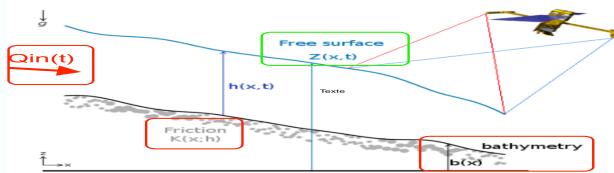
The considered inverse problem is ill-posed...

Lowest cross-section
Friction
Inflow discharge

$A_0(x)$
 $K(x; h(x,t))$
 $Q_{in}(t)$

The flow model
more or less complex...

$(Z, W)(x, t)$
Surface signature
measured by satellite(s)

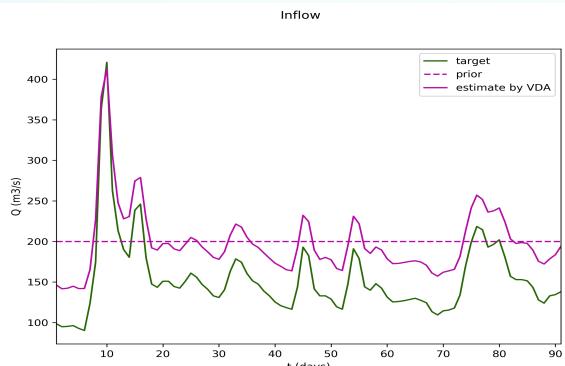


Issues

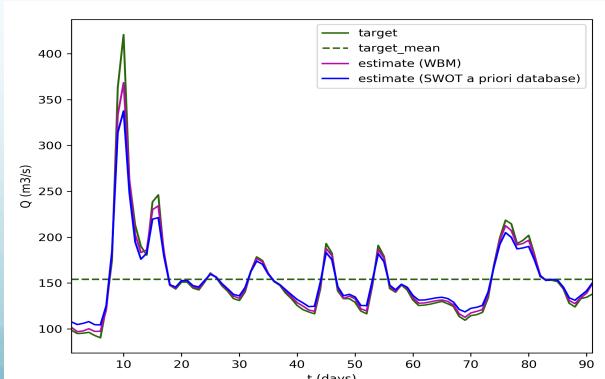
- If « Flow model » = the fundamental equilibrium (Manning-Strickler's law), one can infer the ratio $(Q/K)(x, t)$ only !
- If « Flow model » = the complete dynamics flow model $\begin{cases} \partial_t A + \partial_x Q \\ \partial_t Q + \partial_x \left(\frac{Q^2}{A} \right) + g A \partial_x Z \end{cases} = -g A S_f(A, Q; K)$
+ Boundary Conditions $Q_{in}(t)$ + Initial Conditions.

we show that the unknowns $A_0(x), Q_{in}(t)$ are identifiable but up to (any) multiplicative factor...

e.g. up to the prior mean values of $\bar{Q}^{(0)}$ of the iterative optimization process ...

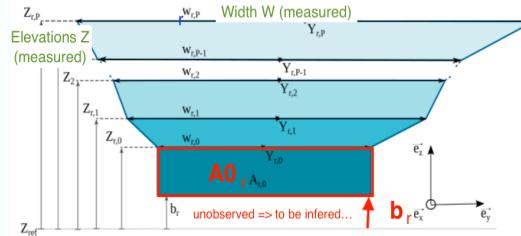
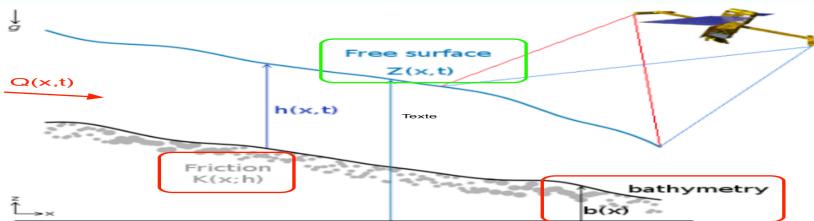


Garonne river downstream Toulouse.
Data: IMFT; Computations: INSA-IMT, CS group.



Refs
[Larnier – J.M. et al '19]
[Brisset (ex-INSA) - J.M. et al. '18]
[Garambois- J.M. '15]

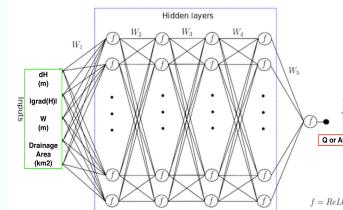
Enriched approach: mixing Machine Learning (DNN) with additional information & Physical models inversions (VDA)



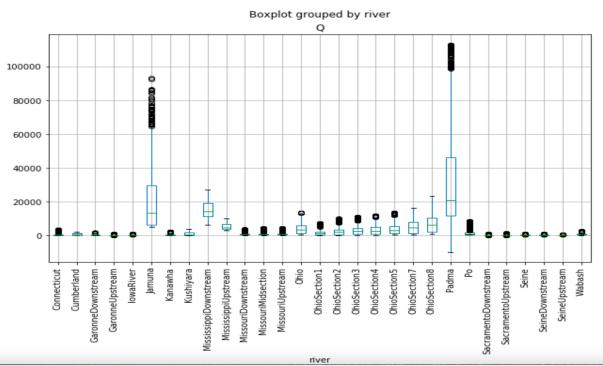
Step 1) Setting $\bar{Q}^{(0)}$ or $\bar{A}^{(0)}$ by ML from enriched datasets

From the following inputs variables $(dZ, |\nabla Z|, W)_i \oplus \text{DrainageArea}_i$ (km²)

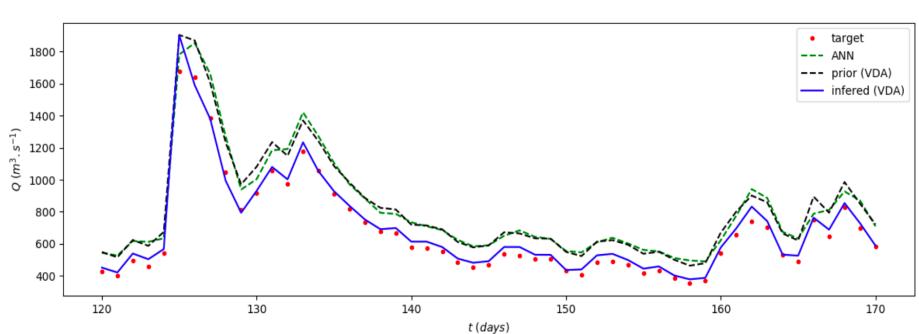
a Deep Neural Network (DNN) enables to estimate quite accurately either Q_i or $(A_0)_i$



Datasets built up by the SWOT Science Team
(multi-disciplinary US, Fr, Europe, Canada etc researchers)



Step 2) From $\bar{Q}^{(0)} \equiv Q^{ML}$, estimate the space-time variations of $Q(x,t)$ by VDA



Refs [Larnier – J.M.'20]
INSA projects - internships

Garonne river portion (Tonneins-La Reole portion).
The DNN estimation is already very good (<10% error).
Next, the VDA solution is excellent.

In summary

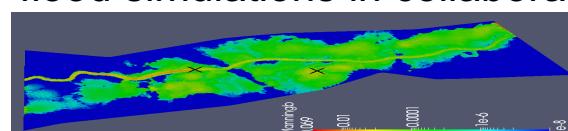
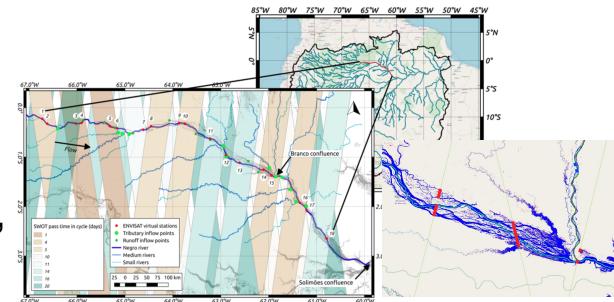
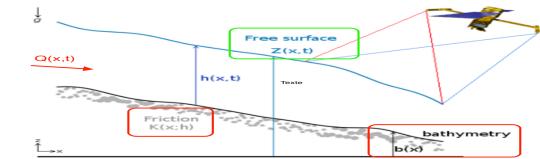
- A flow model analysis + Data Assimilation enables to solve such a challenging inverse geophysical problem.
- Here, ML (DNN) enables to estimate quite accurate mean values at the observations time and locations.
- The physical models enable to « physically filter » these ML values.

The resulting calibrated model is next employed as a space – time interpolator and as an « extrapolator » for prediction purposes.

- The presented HiVDI algorithm has been implemented onto the CNES server. [Larnier - J.M. et al.]

HiVDI algorithm constitutes the French-European Discharge algorithm in SWOT community, completing two US algorithms based on different scientific approaches.

- Numerous improvements are under investigations :
Multi-scale models (hydrological basin – river portions),
Model reductions for real-time computations etc.
- Similar methods are developed for flood simulations in collaboration with INRAE et al.



In short...

Earth Sciences is a multi-disciplinary fascinating scientific topic.

Applied mathematics, Computational Sciences, Hybrid I.A.

have a great role to play to improve our understandings of our little planet.