Mathematical Models and Methods in Applied Sciences

FREE CONVECTION WITH RADIATIVE THERMAL TRANSFER OF GREY BODIES. ANALYSIS AND APPROXIMATION BY FINITE ELEMENT METHODS

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We study a steady-state free (or mixed) convection model in 2 or 3 dimensions of space, taking into account radiative thermal transfer of grey bodies separated by a non participating media. The existence of a weak solution is proved and the uniqueness is obtained when the viscosity and the thermal conductivity of the fluid are large enough. Then, we discretize the model using classical finite element schemes and we prove in detail the existence, uniqueness and the convergence of the discrete solution (when the viscosity and the thermal conductivity are large enough and the step size is small enough).
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1. Introduction

We study a steady-state free (or mixed) convection model in a bounded domain of $\mathbb{R}^n$ ($n = 2$ or 3). The equations are the Navier-Stokes equations coupled with the equations of heat transfer (the fluid is incompressible). The temperature appears in the right-hand side of the Navier-Stokes equations through the Boussinesq term, whereas the fluid velocity appears in the heat equation through the convection term. The thermal model takes into account the three different heat exchanges: convection, diffusion and radiation. The model of radiation takes into account the emission, the reflection and the absorption of the radiant energy. The emitted and reflected radiation are diffusely distributed. The surfaces are assumed to be opaque and to behave like grey bodies (the radiative exchanges do not depend on the wave length). Furthermore, they are separated by a non participating media. The radiative heat transfer is described by the radiosity (the radiosity is the radiant energy which flows away from a surface); and it is solution of the radiosity equation which is an integral equation on the boundary of the domain. The right-hand side of this integral equation depends on T. Therefore, the unknowns of the full model are the fluid velocity $\vec{u}$, its pressure p, its temperature T and the radiosity w. The quadruplet $(\vec{u}, p, T, w)$ is solution of the (classical) steady-state free convection model: the Navier-Stokes equations fully coupled with the advection-diffusion equation. But, the thermal boundary conditions are non linear, non monotone, they are of non local type and fully coupled with the integral equation.

Many papers treats of the free (or mixed) convection model *without* the radiative thermal transfer of grey bodies (see e.g. [1], [17] for a mathematical analysis and an approximation by finite element methods). In others respects, the present thermal model *with the fluid velocity $\vec{u}$ given* has already been studied: see [16], [4], [21], [22] and [14] for a mathematical analysis and [14] for a numerical analysis. To our knowledge, the free (or mixed) convection model with radiative thermal transfer of grey bodies has not been studied yet.

An outline of the paper is as follows. We write in Section 2 the physical model in a dimensionless form.
In Section 3, we prove the existence of a weak solution and we prove the uniqueness of the solution which satisfies the maximum principle, when the viscosity and the thermal conductivity of the fluid are large enough (Corollary 1). To this end, we proceed as follows. First, we use a result of [16] which gives an explicit expression of the radiosity w as a function of the temperature T (Proposition 1). Then, the model is reduced to three partial differential equations with $\vec{u}$, p and T as the only unknowns (equations (3.16)-(3.21)). The thermal boundary condition is of non local type, non linear and non monotone. Following [16], [14], we use a technique of truncation which consists to truncate the temperature on the boundary and in the right-hand side of the Navier-Stokes equations to "natural" values (i.e. to the inf-sup values of the temperature given on the boundary). Then, we can study the truncated problem in the standard Sobolev spaces. We write the variational formulations using the Sobolev spaces with divergence free velocity field and the unknown is reduced to $(\vec{u}, T)$. Thanks to this technique of truncation, we obtain estimates a priori of T and $\vec{u}$ (Lemma 4 and Lemma 5). Using the Leray-Schauder fixed point theorem, we obtain the existence of a weak solution (Theorem 1). We state a proof of uniqueness of the solution of the truncated problem, when the viscosity and the thermal conductivity of the fluid are large enough, using estimates in $H^1(\Omega)$ (Theorem 2). Finally, using the weak maximum principle (3.38), we obtain the existence and uniqueness (under the same conditions) of the solution to the initial model which satisfies the maximum principle, i.e. the existence and uniqueness of the *physical* solution (Corollary 1).
Let us point out that we use this technique of truncation because it presents the following advantages: a. it permits to obtain estimates a-priori; b. it permits to write the mathematical analysis in the standard Sobolev spaces; c. it permits to write the numerical analysis (see below), and finally we obtain the mathematical analysis and the numerical analysis in the same framework.
In Section 4, we consider a new truncated model (for a technical reason the truncation is smoother than previously), we discretize it using finite element methods and we present the analysis of the numerical schemes. We do not solve numerically the integral equation like we do in the continuous case and we do not consider discrete velocity field of divergence free. Hence, the unknowns are $\vec{u}_h$, p_h, T_h and w_h, and the velocity and pressure finite element spaces satisfy the Babuska-Brezzi inf-sup condition.
The main result of the section is the proof of the existence and uniqueness of the discrete solution $(\vec{u}_h, p_h, T_h, w_h)$ and its convergence towards the unique physical solution of the model, when the viscosity and the thermal conductivity of the fluid are large enough and when the step size h is small enough (Theorem 4). This result is obtained by proving that the equations fit into the framework of [3]. This previous paper deals with finite dimensional approximation of nonlinear problems of the form $F(\lambda; x) = 0$ where $F : \Lambda \times X \to X$, Λ is an interval of $I\!R$ and X is a Banach space. We consider non singular solutions (the linearized problem has to be well posed) and we apply the discrete implicit function theorem stated in [3]. The error estimates are established in detail (and the solution belongs a-priori to fractional order Sobolev spaces). Let us remark that the framework of [3] does not require the discrete maximum principle.
Finally, we detail four different finite element schemes in the bidimensional case. Two of them lead to first order methods and the two others lead to second order methods.

2. Physical Model

We denote by Ω a lipschitz bounded open set in $\mathbb{R}^n$ (n=2 or 3), by $\partial\Omega$ its boundary and we consider free (or mixed) convection in the "cavity" Ω. The particularity of the present model is to assume that the boundaries of the cavity emit, absorb and reflect radiative energy. The emitted and reflected radiation are diffusely distributed. The radiant surfaces behave like opaque and grey bodies (the radiative exchanges do not depend on the wave length). The fluid inside the cavity Ω is assumed to be a radiatively non participating media. Under these assumptions, the radiant heat transfer appear on the boundary and are described by the radiosity (the radiosity is the radiative energy which flows away from a surface), see e.g. [19]. The fluid velocity is denoted by $\vec{u}$, its pressure by p, its temperature by T and the radiosity by w. We denote by Ra and Pr the Rayleigh and the Prandtl numbers defined by: $Ra = \frac{|\vec{g}|\beta T^* L^{*3}}{\nu\lambda}$, $Pr = \frac{\nu}{\lambda}$, where $\vec{g}$ is the gravitational vector force, β is the thermal expansion coefficient, ν is the kinematic viscosity, λ is the thermal conductivity, T^* is a characteristic temperature of the flow and L^* is a characteristic length. We denote by Gr the Grashoff number, $Gr = \frac{Ra}{Pr} = \frac{|\vec{g}|\beta T^* L^{*3}}{\nu^2}$. The equations of the model (in a dimensionless form) are the following (see e.g. [11]):

$$-\Delta\vec{u} + \sqrt{Gr}\,(\vec{u}\cdot\vec{\nabla})\vec{u} + \vec{\nabla}p = \sqrt{Gr}\,\vec{g}\,T \quad \text{in } \Omega \tag{2.1}$$

$$div(\vec{u}) = 0 \quad \text{in } \Omega \tag{2.2}$$

$$-\Delta T + \sqrt{Gr}\,\vec{u}\cdot\vec{\nabla}T = 0 \quad \text{in } \Omega \tag{2.3}$$

where $(\vec{u}\cdot\vec{\nabla}) = \sum_{i=1}^{n} u_i\partial_i$.

The boundary conditions are of the following types:

i) Velocity: no slip condition ($\vec{u} = 0$) on $\partial\Omega$.

ii) Temperature: T given on Γ_d; convective and radiative heat transfer on Γ_f ($meas(\Gamma_d) > 0$ and $\Gamma_d \cup \Gamma_f = \partial\Omega$). (Of course, thermal data on the boundary are assumed to be such that a free convection flow is induced.)

Let us denote by h the thermal transfer coefficient. For a sake of simplicity, it is supposed to be independent of $\vec{u}$, it is a constant. It follows from the Fourier law that:

$$-\lambda\vec{\nabla}T.\vec{n} = h(T - T_0) + \varphi_{rad}$$

where T_0 is a temperature of boundary given, φ_{rad} is the radiative flux and $\vec{n}$ is the external normal to $\partial\Omega$. The radiative flux φ_{rad} can be expressed as follows, see e.g. [19]:

$$\varphi_{rad}(x) = \varepsilon(x)\sigma T^4(x) - \varepsilon(x)\int_{\partial\Omega}\phi(x,y)\,w(y)\,ds(y)$$

where the function $\varepsilon(x)$ is the surface emittance. It satisfies:

$$0 < \varepsilon_0 \le \varepsilon(x) \le \varepsilon_1 \le 1 \text{ on } \partial\Omega \tag{2.4}$$

(The case $\varepsilon = 1$ corresponds to a black body). The coefficient σ is the Stephan-Boltzmann's constant and the kernel $\phi \in L^1(\partial\Omega \times \partial\Omega)$ is the angle factor, it is positive, symmetric and satisfies:

$$\int_{\partial\Omega}\phi(x,y)\,ds(x) = 1 \tag{2.5}$$

In others respects, the radiosity w is solution of the following integral equation:

$$w(x) \;=\; Aw(x) \;+\; \varepsilon(x)\sigma T^4(x) \qquad \text{on } \partial\Omega \tag{2.6}$$

where

$$A\; w(x) \;=\; (1-\varepsilon(x)) \int_{\partial\Omega} \phi(x,y)\; w(y)\; ds(y), \tag{2.7}$$

We refer for example to [19] for a more detailed derivation of this radiative model.
Let us write the boundary conditions above in a dimensionless form. To this end, we denote by Bi the Biot number, $Bi = \frac{hL^*}{\lambda}$, by δ_1 and δ_2 the two following dimensionless numbers: $\delta_1 = \frac{L^*\sigma(T^*)^3}{\lambda}$, $\delta_2 = \frac{L^* w^*}{\lambda T^*}$; where w^* is a characteristic radiative energy. If we denote again by T and w the dimensionless variables, the boundary conditions are:

$$\vec{u} \;=\; 0 \;\; \text{on } \partial\Omega \tag{2.8}$$

$$T \;=\; T_d \;\; \text{on } \Gamma_d \tag{2.9}$$

$$-\frac{\partial T}{\partial n} \;=\; \Phi(T,w) \;\; \text{on } \Gamma_f \tag{2.10}$$

$$(I-A)w \;=\; \varepsilon\frac{\delta_1}{\delta_2}T^4 \;\; \text{on } \partial\Omega \tag{2.11}$$

where I denotes the mapping identity,

$$\Phi(T,w)(x) \;=\; Bi\;(T-T_0)(x) + \varepsilon(x)\;[\delta_1 T^4(x) \;-\; \delta_2 \int_{\partial\Omega} \phi(x,y)\; w(y)\; ds(y)] \tag{2.12}$$

and the temperatures T_d and T_0 are given.
In the remainder of the paper, $\vec{u}$, p, T and w denote the dimensionless variables, solution of (2.1)-(2.3)(2.8)-(2.11).

3. Existence and Uniqueness of the Solution

In this section, we prove the existence of a weak solution and we prove the uniqueness of the solution which satisfies the maximum principle, when the viscosity and the thermal conductivity of the fluid are large enough (Corollary 1). To this end, we recall an expression of the radiosity w as a function of T (Proposition 1). Then, we obtain partial differential equations with u, p and T as only unknowns (equations (3.16)-(3.21)). The thermal boundary condition is of non local type, non linear and non monotone. We study in Lemma 1 the integral operator B of this boundary condition. Then, following [16], [14], we truncate the temperature on the boundary and in the right-hand side of the Navier-Stokes equations to "natural" values, i.e. to the inf-sup values of the temperature given on the boundary (Problem $(\bar{P})$). We write the variational formulations in the standard Sobolev spaces with divergence free velocity field and the unknown is reduced to $(\vec{u}, T)$. Thanks to the truncation properties, we state some estimates a priori of T and $\vec{u}$ (Lemma 4 and Lemma 5) and we obtain the existence of a weak solution using the Leray-Schauder fixed point theorem (Theorem 1). Then, we write estimates in $H^1(\Omega)$ and we obtain the uniqueness of the solution of the truncated problem when the viscosity and the thermal conductivity of the fluid are large enough (Theorem 2). Finally, it follows straightforwardly from the maximum principle (3.38) that under these conditions, there exists an unique solution to the initial model which satisfies the maximum principle i.e. an unique *physical* solution (Corollary 1).

3.1. *Resolution of the integral equation*

Let us consider the integral equation (2.11) with a right-hand side given. More precisely, we consider the following problem:

$$\left\{ \begin{array}{l} \text{Given } e \in L^q(\partial\Omega),\ 1 \leq q \leq \infty, \text{ find } w \in L^q(\partial\Omega) \text{ such that:} \\ (I \ - \ A)\ w \ = \ e \end{array} \right. \tag{3.13}$$

We have the
Proposition 1 *(Perret and Witomski, [16]) The operator A is a contracting operator in $L^q(\partial\Omega)$ for all $q \in [1, \infty]$ and Problem (3.13) has a unique solution $w \in L^q(\partial\Omega)$. Furthermore w has the following expression:*

$$w(x) \ = \ \int_{\partial\Omega} K(x,y)\ e(y)\ ds(y) \ + \ e(x) \tag{3.14}$$

where the kernel $K(x,y)$ is positive, belongs to $L^1(\partial\Omega \times \partial\Omega)$ and satisfies:

$$\int_{\partial\Omega} K(x,y)\ \varepsilon(y)\ ds(y) \ = \ 1 - \varepsilon(x) \tag{3.15}$$

We refer to [16] for the expression of $K(x,y)$ and for the proof of this proposition (see also [21] for the existence and uniqueness of w). □

It follows from (2.10), (2.11) and (3.14) that we can formulate the model with $\vec{u}$, p and T as only unknowns:

$$-\Delta\vec{u} \ + \ \sqrt{Gr}\ (\vec{u}\cdot\vec{\nabla})\vec{u} \ + \ \vec{\nabla}p = \sqrt{Gr}\ \vec{g}\ T \quad \text{in } \Omega \tag{3.16}$$

$$div(\vec{u}) = 0 \quad \text{in } \Omega \tag{3.17}$$

$$\vec{u} = 0 \quad \text{on } \partial\Omega \tag{3.18}$$

$$-\Delta T \ + \ \sqrt{Gr}\ \vec{u}\cdot\vec{\nabla}T = 0 \quad \text{in } \Omega \tag{3.19}$$

$$T = T_d \quad \text{on } \Gamma_d \tag{3.20}$$

$$-\frac{\partial T}{\partial n} = Q(T) \quad \text{on } \Gamma_f \tag{3.21}$$

where

$$Q(T)(x) \ = \ Bi\ (T - T_0)(x) \ + \ \varepsilon(x)\ \delta_1(I - B)(T^4)(x) \quad x \in \partial\Omega \tag{3.22}$$

with

$$B(g)(x) \ = \ \int_{\partial\Omega} \phi(x,y)\ [\ \int_{\partial\Omega} K(y,z)\ \varepsilon(z)\ g(z)\ ds(z) \ + \ \varepsilon(y)\ g(y)\ ]\ ds(y) \tag{3.23}$$

The triplet $(\vec{u}, p, T)$ is solution of (3.16)-(3.21) if and only if $(\vec{u}, p, T, w)$, with w defined by (3.14), is solution of (2.1)-(2.3)(2.8)-(2.11).

3.2. *The nonlocal boundary condition*

We study the integral operator B defined by (3.23).

Lemma 1 *The operator B is linear from $L^q(\partial\Omega)$ into $L^q(\partial\Omega)$ for all $q \in [1,\infty]$. Furthermore, it satisfies:*
i) the estimate

$$||B||_{L^q(\partial\Omega)} \;\leq\; (\frac{\varepsilon_1}{\varepsilon_0})^{\frac{1}{q}} \qquad 1 \leq q \leq \infty \tag{3.24}$$

ii) if g is a function constant on $\partial\Omega$ then $B(g) = g$.

Proof. i) Let us prove the result for $q = \infty$. B is a linear operator and:
$\forall g \in L^\infty(\partial\Omega)$,

$$\begin{aligned} |B(g)(x)| &\leq \int_{\partial\Omega}\int_{\partial\Omega} \phi(x,y)\; K(y,z)\; \varepsilon(z)\; |g(z)|\; ds(z)\; ds(y) \\ &\quad + \int_{\partial\Omega} \phi(x,y)\; \varepsilon(y)\; |g(y)|\; ds(y) \\ &\leq ||g||_{L^\infty(\partial\Omega)}\; [\; \int_{\partial\Omega} \phi(x,y) \int_{\partial\Omega} K(y,z)\; \varepsilon(z)\; ds(z)\; ds(y) \\ &\quad + \int_{\partial\Omega} \phi(x,y)\; \varepsilon(y)\; ds(y)\;] \end{aligned}$$

Using (3.15), we obtain:

$$|B(g)(x)| \;\leq\; ||g||_{L^\infty(\partial\Omega)} \text{ a.e.}$$

Hence: $||B||_{L^\infty(\partial\Omega)} \leq 1$.

Let us prove the result for $q = 1$. For all g in $L^1(\partial\Omega)$,

$$\begin{aligned} \int_{\partial\Omega} |B(g)(x)|\; ds(x) &\leq \int_{\partial\Omega}\int_{\partial\Omega}\int_{\partial\Omega} \phi(x,y)\; K(y,z)\; \varepsilon(z)\; |g(z)|\; ds(z)\; ds(y)\; ds(x) \\ &\quad + \int_{\partial\Omega}\int_{\partial\Omega} \phi(x,y)\; \varepsilon(y)\; |g(y)|\; ds(y)\; ds(x) \\ &\leq \int_{\partial\Omega}\int_{\partial\Omega} K(y,z)\; \varepsilon(z)\; |g(z)| \int_{\partial\Omega} \phi(x,y)\; ds(x)\; ds(z)\; ds(y) \\ &\quad + \int_{\partial\Omega} \varepsilon(y)\; |g(y)| \int_{\partial\Omega} \phi(x,y)\; ds(x)\; ds(y) \end{aligned}$$

Using (2.5), we obtain:

$$\int_{\partial\Omega} |B(g)(x)|\; ds(x) \;\leq\; \int_{\partial\Omega} \varepsilon(z)\; |g(z)|\; [\; \int_{\partial\Omega} K(y,z)\; ds(y) \;+\; 1\;]\; ds(z)$$

Using (3.15), we have:

$$\int_{\partial\Omega} K(x,y)\; ds(x) \;\leq\; \frac{1-\varepsilon_0}{\varepsilon_0}$$

It follows that:

$$||B(g)||_{L^1(\partial\Omega)} \;\leq\; \frac{\varepsilon_1}{\varepsilon_0}\; ||g||_{L^1(\partial\Omega)}$$

Hence: $||B||_{L^1(\partial\Omega)} \leq \dfrac{\varepsilon_1}{\varepsilon_0}$.

It follows from the Riesz-Thorin theorem (see e.g. [2]), that:
$B : L^p(\partial\Omega) \to L^p(\partial\Omega),\ 1 < p < \infty$. Furthermore, using an inequality of interpolation (see e.g. [2]), we have:

$$||B||_{L^r(\partial\Omega)} \ \leq \ ||B||^{\alpha}_{L^1(\partial\Omega)} \ ||B||^{1-\alpha}_{L^\infty(\partial\Omega)} \ \ , \frac{1}{r} = \alpha, \ 0 \leq \alpha \leq 1$$

and (3.24) holds.

ii) Let g be a function constant on $\partial\Omega$: $\forall x \in \partial\Omega,\ g(x) = c,\ c \in \mathbb{R}$. We have:

$$B(g)(x) \ = \ g \int_{\partial\Omega} \phi(x,y) \ [\ \int_{\partial\Omega} K(y,z) \ \varepsilon(z) \ ds(z) \ + \ \varepsilon(y) \ ] \ ds(y)$$

Using (2.5) and (3.15), we obtain the result. □

Remark 1 *It follows from Lemma 1 i) that B is an operator contractant in $L^\infty(\partial\Omega)$ and it is not in $L^q(\partial\Omega)$, $1 \leq q < \infty$. Nevertheless, if $\varepsilon_0 = \varepsilon_1$ (i.e. if ε is constant), then it is contractant in $L^q(\partial\Omega)$, $1 \leq q \leq \infty$.*

3.3. *The truncated problem*

It follows from the Sobolev embedding theorem and Lemma 1 i) that in the three dimensional case, if $T \in H^1(\Omega)$ then $T^4 \in L^1(\partial\Omega)$ and $Q(H^1(\Omega)) \subset L^1(\partial\Omega)$. Then, the natural space to study the thermal model (3.19)-(3.21) is the Banach space: $H^1(\Omega) \cap L^5(\partial\Omega)$. A mathematical analysis of this thermal model (*with u given*) is written in $H^1(\Omega) \cap L^5(\partial\Omega)$ in [22]. In the present paper, we do not work in this space. As a matter of fact, following [16] and [14], we use a technique of truncation which consists to truncate the temperature on the boundary Γ_f and in the right-hand side of the Navier-Stokes equations to the inf-sup values of the temperature given on the boundary. Then, we can study the truncated problem in the standard Sobolev spaces. Thanks to this technique of truncation, we obtain estimates a priori of T and u (Lemma 4 and Lemma 5). Using the Leray-Schauder fixed point theorem, we obtain the existence of a weak solution (Theorem 1). We state a proof of uniqueness of the solution of the truncated problem, when the viscosity and the thermal conductivity of the fluid are large enough (Theorem 2). Finally, it follows from the weak maximum principle (3.38) that the unique solution of the truncated problem is also solution of the initial problem. We obtain the existence and uniqueness (under the same conditions), of the *physical* solution to the initial model (Corollary 1).

Let us define the operator of truncation. The temperatures T_d and T_0 are given, positive and belong respectively to $H^{\frac{3}{2}}(\Gamma_d)$ and $H^{\frac{1}{2}}(\Gamma_f) \cap L^\infty(\Gamma_f)$. Following [16], we define:

$$T_{inf} = Min(\inf_{\Gamma_d} T_d,\ \inf_{\Gamma_f} T_0);\ T_{sup} = Max(\sup_{\Gamma_d} T_d,\ \sup_{\Gamma_f} T_0) \tag{3.25}$$

And the operator of truncation φ is defined as follows:

$$\varphi(T)(x) \ = \begin{cases} T_{inf} & if \quad T(x) \ \leq \ T_{inf} \\ T(x) & if \quad T_{inf} \ \leq \ T(x) \leq T_{sup} \\ T_{sup} & if \quad T(x) \ \geq \ T_{sup} \end{cases} \tag{3.26}$$

We have $\varphi(L^1(\partial\Omega)) \subset L^\infty(\partial\Omega)$ and $\varphi(H^1(\Omega)) \subset (H^1(\Omega) \cap L^\infty(\Omega))$. Then, we define the truncated operator $\bar{Q}$ as follows:

$$\bar{Q}(T) \; = \; Q \circ \varphi(T)$$

hence

$$\bar{Q}(T)(x) \; = \; Bi \; (\varphi(T) - T_0)(x) \; + \; \varepsilon(x) \; \delta_1 \; (I - B)(\varphi(T)^4)(x), \quad x \in \partial\Omega \qquad (3.27)$$

And, we define the truncated problem:

$$(\bar{P}) \left\{ \begin{array}{rcll} \multicolumn{4}{l}{\text{Find } (\vec{u}, p, T) \text{ satisfying:}} \\ -\Delta\vec{u} \; + \; \sqrt{Gr} \; (\vec{u} \cdot \vec{\nabla})\vec{u} \; + \; \vec{\nabla} p & = & \sqrt{Gr} \; \vec{g} \; \varphi(T) & \text{in } \Omega \\ div(\vec{u}) & = & 0 & \text{in } \Omega \\ \vec{u} & = & 0 & \text{on } \partial\Omega \\ -\Delta T \; + \; \sqrt{Gr} \; \vec{u} \cdot \vec{\nabla} T & = & 0 & \text{in } \Omega \\ T & = & T_d & \text{on } \Gamma_d \\ -\dfrac{\partial T}{\partial n} & = & \bar{Q}(T) & \text{on } \Gamma_f \end{array} \right.$$

It follows from Lemma 1 that: $\forall T \in H^1(\Omega)$, $\bar{Q}(T) \in L^\infty(\partial\Omega)$, and the boundary condition

$$-\frac{\partial T}{\partial n} = \bar{Q}(T) \qquad \text{on } \Gamma_f$$

has an integral representation with test function in $H^1(\Omega)$. Then, we can write a weak formulation of Problem $(\bar{P})$ in the standard Sobolev spaces.

Let us present some extra properties of this truncated operator.

Lemma 2 *The operator $\bar{Q}$ satisfies the following properties:*

i) *For all $t^{(1)}$, $t^{(2)} \in L^2(\partial\Omega)$,*

$$\|\bar{Q}(t^{(1)}) - \bar{Q}(t^{(2)})\|_{0,\partial\Omega} \; \leq \; \mathcal{C}_Q \; \|t^{(1)} - t^{(2)}\|_{0,\partial\Omega} \qquad (3.28)$$

with

$$\mathcal{C}_Q \; = \; [Bi \; + \; 4 \; \varepsilon_1 \; \delta_1 \; T_{sup}^3 \; (1 + \|B\|_{0,\partial\Omega})] \qquad (3.29)$$

ii) *The range of $L^2(\partial\Omega)$ by $\bar{Q}$ is bounded into $L^\infty(\partial\Omega)$. More precisely,*

$$\forall t \in L^2(\partial\Omega), \;\; |\bar{Q}(t)(x)| \; \leq \; B_i(T_{sup} - T_{inf}) \; + \; 2\delta_1\varepsilon_1 T_{sup}^4 \; a.e. \; on \; \partial\Omega \quad (3.30)$$

iii) *If $t(x) \geq T_{sup}$ a.e. (resp. $t(x) \leq T_{inf}$ a.e.) then $\bar{Q}(t)(x) \geq 0$ a.e. (resp. $\bar{Q}(t)(x) \leq 0$ a.e.).*

Proof. Following ([16], Proposition 5) and using Lemma 1, we obtain the result. □

3.4. *Weak formulations*

From now, we voluntarily omit the arrows above vectorial entities. We denote in bold face characters the vector spaces: $\mathbf{H^1}(\Omega) = (H^1(\Omega))^n$, $\mathbf{H_0^1}(\Omega) = (H_0^1(\Omega))^n$,

$\mathbf{H^{-1}}(\Omega) = (H^{-1}(\Omega))^n$ and $\mathbf{L^p}(\Omega) = (L^p(\Omega))^n$, n being the space dimension. We define the following spaces:

$$\left\{ \begin{array}{l} \mathbf{V}(\Omega) = \{v \in \mathbf{H_0^1}(\Omega);\ div(v) = 0 \text{ in } \Omega\} \\ L_0^2(\Omega) = \{q \in L^2(\Omega),\ \int_\Omega q\ dx = 0\} \\ W_0(\Omega) = \{t \in H^1(\Omega);\ t = 0 \text{ on } \Gamma_d\} \end{array} \right. \tag{3.31}$$

and the affine sub-space:

$$W_t(\Omega) = \{t \in H^1(\Omega);\ t = T_d \text{ on } \Gamma_d\} \tag{3.32}$$

We define the trilinear mapping $b : \mathbf{V}(\Omega) \times \mathbf{V}(\Omega) \times \mathbf{V}(\Omega) \to I\!R$ by

$$b(u, v, w) \ = \ \sqrt{Gr} \sum_{i,j=1}^{n} \int_\Omega u_i\ \partial_i v_j\ w_j\ dx \tag{3.33}$$

the trilinear mapping $a : \mathbf{V}(\Omega) \times W_0(\Omega) \times W_0(\Omega) \to I\!R$ by

$$a(u; \theta, t) \ = \ \sqrt{Gr} \int_\Omega u\ \nabla\theta\ t\ dx \tag{3.34}$$

and the mapping $c : \mathbf{V}(\Omega) \times W_0(\Omega) \times W_0(\Omega) \to I\!R$ by

$$c(u; \theta, t) \ = \ (\nabla\theta, \nabla t) \ + \ a(u; \theta, t) \tag{3.35}$$

where $(.,.)$ denotes the inner product of $L^2(\Omega)$.
Let $\tilde{T}_d$ be a function such that $\tilde{T}_d \in H^1(\Omega)$, $\tilde{T}_d = T_d$ on Γ_d. We denote by $\mathcal{R}_\Omega$ the constant (depending only on Ω), such that

$$||\tilde{T}_d||_{1,\Omega} \le \mathcal{R}_\Omega ||T_d||_{\frac{1}{2},\Gamma_d} \tag{3.36}$$

We set $\theta = T - \tilde{T}_d$ and we consider the weak formulation of the truncated problem:

$$(\bar{\mathcal{P}}) \left\{ \begin{array}{l} \text{Find } (u,\theta) \in \mathbf{V}(\Omega) \times W_0(\Omega) \text{ such that:} \\ \forall w \in \mathbf{V}(\Omega), \\ \sum_{i=1}^{n} (\nabla u_i, \nabla w_i) \ + \ b(u, u, w) \ = \sqrt{Gr}\ (g\ \varphi(\theta + \tilde{T}_d), w) \quad (3.37.a) \\ \forall t \in W_0(\Omega), \\ c(u; \theta, t) \ + \ \int_{\Gamma_f} \bar{Q}(\theta + \tilde{T}_d)\ t\ ds \ = \ - c(u; \tilde{T}_d, t) \qquad (3.37.b) \end{array} \right. \tag{3.37}$$

Let us notice that the pressure p does not appear anymore in the weak formulation $(\bar{\mathcal{P}})$. But one knows (see e.g. [20]) that using the De Rham theorem, if $(u, \theta) \in \mathbf{V}(\Omega) \times W_0(\Omega)$ is solution of $(\bar{\mathcal{P}})$, then there exists a unique $p \in L_0^2(\Omega)$ such that $(u, p, T = \theta + \tilde{T}_d) \in \mathbf{H^1}(\Omega) \times L_0^2(\Omega) \times H^1(\Omega)$ satisfies the problem $(\bar{P})$.

3.5. *Estimates a-priori*

Let us introduce the following constants

$$\left\{\begin{array}{ll} \mathcal{C}_\Omega = \sup\limits_{v\in \mathbf{H}_0^1(\Omega)} \dfrac{\|v\|_{1,\Omega}^2}{\sum_{i=1}^n \|\nabla v_i\|_{0,\Omega}^2}; & \mathcal{C}_\Omega^* = \sup\limits_{t\in W_0(\Omega)} \dfrac{\|t\|_{1,\Omega}^2}{\|\nabla t\|_{0,\Omega}^2} \\ & \text{(Poincaré-Friedrich's inequality)} \\ \mathcal{S}_\Omega = \sup\limits_{v\in \mathbf{H}^1(\Omega)} \dfrac{\|v\|_{L^4(\Omega)}}{\|v\|_{1,\Omega}}; & \mathcal{S}_\Omega^* = \sup\limits_{t\in H^1(\Omega)} \dfrac{\|t\|_{L^4(\Omega)}}{\|t\|_{1,\Omega}} \quad \text{(Sobolev's inequalities)} \\ \mathcal{T}_\Omega^* = \sup\limits_{t\in H^1(\Omega)} \dfrac{\|t\|_{\frac{1}{2},\partial\Omega}}{\|t\|_{1,\Omega}} & \text{(Continuity of the trace mapping)} \end{array}\right.$$

These constants depend only on Ω.

Let us recall the following weak maximum principle:

Lemma 3 *([14], Proposition 2) Let u be given in $\mathbf{V}(\Omega)$ and let $\theta = (T-\tilde{T}_d) \in W_0(\Omega)$ be a solution of (3.37.b). Then,*

$$T_{inf} \ \le\ T \ \le\ T_{sup} \quad a.e. \ in\ \bar{\Omega} \tag{3.38}$$

□

We state an estimate a priori of θ in norm H^1:

Lemma 4 *Let u be given in $\mathbf{V}(\Omega)$ and let $\theta = (T - \tilde{T}_d) \in W_0(\Omega)$ be a solution of (3.37.b). Then,*

$$\begin{aligned} \|\theta\|_{1,\Omega} \ \le\ & \mathcal{C}_\Omega^* \{ \sqrt{|\Gamma_f|}\ \mathcal{T}_\Omega^*\ [B_i(T_{sup} - T_{inf}) \ +\ 2\delta_1\varepsilon_1 T_{sup}^4] \\ & +\ \mathcal{R}_\Omega\ \|T_d\|_{\frac{1}{2},\Gamma_d}[1 \ +\ \mathcal{S}_\Omega^* \sqrt{Gr}\ \|u\|_{L^4(\Omega)}]\ \} \end{aligned} \tag{3.39}$$

Proof. We choose $t = \theta$ as test function in (3.37.b). We obtain:

$$\|\nabla\theta\|_{0,\Omega}^2 \ +\ a(u;\theta,\theta) \ +\ \int_{\Gamma_f} \bar{Q}(\theta + \tilde{T}_d)\theta\ ds \ =\ -\ c(u;\tilde{T}_d,\theta) \tag{3.40}$$

We have:

$$\begin{aligned} |\int_{\Gamma_f} \bar{Q}(\theta+\tilde{T}_d)\ \theta\ ds| \ &=\ |\int_{\Gamma_f} \bar{Q}(T)\ \theta\ ds| \\ &\le\ \|\bar{Q}(T)\|_{0,\Gamma_f}\|\theta\|_{0,\Gamma_f} \\ &\le\ \mathcal{T}_\Omega^*\ |\Gamma_f|^{\frac{1}{2}}\ \|\bar{Q}(T)\|_{L^\infty(\Gamma_f)}\|\theta\|_{1,\Omega} \end{aligned}$$

and using (3.30),

$$|\int_{\Gamma_f} \bar{Q}(\theta+\tilde{T}_d)\ \theta\ ds| \ \le\ \mathcal{T}_\Omega^*\ |\Gamma_f|^{\frac{1}{2}}\ [B_i(T_{sup} - T_{inf}) \ +\ 2\delta_1\varepsilon_1 T_{sup}^4]\ \|\theta\|_{1,\Omega} \tag{3.41}$$

In addition,

$$\begin{aligned}|c(u;\tilde{T}_d,\theta)| &\leq ||\nabla \tilde{T}_d||_{0,\Omega}||\nabla\theta||_{0,\Omega} + \sqrt{Gr}\ ||u||_{L^p(\Omega)}||\nabla \tilde{T}_d||_{0,\Omega}||\theta||_{L^q}(\Omega)\\ &\leq \mathcal{R}_\Omega\ ||T_d||_{\frac{1}{2},\Gamma_d}(\ ||\nabla\theta||_{0,\Omega} + \sqrt{Gr}\ ||u||_{L^p(\Omega)}||\theta||_{L^q(\Omega)}\ )\end{aligned}$$

with $\frac{1}{2}+\frac{1}{p}+\frac{1}{q}=1$, $p>n$ and $q\leq 6$ if $n=3$, $q<\infty$ if $n=2$. We set $p=q=4$ and we obtain:

$$|c(u;\tilde{T}_d,\theta)| \leq \mathcal{R}_\Omega\ ||T_d||_{\frac{1}{2},\Gamma_d}(\ 1 + \mathcal{S}^*_\Omega\sqrt{Gr}\ ||u||_{L^4(\Omega)}\ )||\theta||_{1,\Omega} \tag{3.42}$$

In others respects, $\forall u\in \mathbf{V}(\Omega)$, $\forall\theta\in W_0(\Omega)$,

$$\begin{aligned}a(u;\theta,\theta) &= \frac{\sqrt{Gr}}{2}\int_\Omega u\nabla(\theta^2)\ dx\\ &= \frac{\sqrt{Gr}}{2}\int_\Omega div(\theta^2 u)\ dx\\ &= \frac{\sqrt{Gr}}{2}\int_{\partial\Omega}\theta^2\ u\cdot n\ ds\ =\ 0\end{aligned} \tag{3.43}$$

Using (3.40)-(3.43), we obtain:

$$\begin{aligned}||\nabla\theta||^2_{0,\Omega} \leq\ & [\ \mathcal{T}^*_\Omega\ |\Gamma_f|^{\frac{1}{2}}\ (B_i(T_{sup}-T_{inf}) + 2\delta_1\varepsilon_1 T^4_{sup})\\ &+ \mathcal{R}_\Omega||T_d||_{\frac{1}{2},\Gamma_d}(\ 1 + \mathcal{S}^*_\Omega\sqrt{Gr}\ ||u||_{L^4(\Omega)}\ )]\ ||\theta||_{1,\Omega}\end{aligned}$$

and using the Friedrichs-Poincaré inequality, the estimate (3.39) holds. □

We give an estimate a priori of u in norm H^1:

Lemma 5 *Let T be given in $W_t(\Omega)$, $T=\theta+\tilde{T}_d$ and let $u\in\mathbf{V}(\Omega)$ be a solution of (3.37.a). Then,*

$$||u||_{1,\Omega} \leq \mathcal{C}_\Omega\ \sqrt{Gr}\ |g|\ \sqrt{|\Omega|}\ T_{sup} \tag{3.44}$$

Proof. We take $w=u$ in (3.37.a) and we obtain:

$$\sum_{i=1}^n ||\nabla u_i||^2_{0,\Omega} + b(u,u,u) = \sqrt{Gr}\ (g\ \varphi(\theta+\tilde{T}_d),u)$$

and (see e.g. [20], Chap. II, Lemma 1.3)

$$\forall u\in\mathbf{V}(\Omega),\ \forall v\in\mathbf{H}^1_0(\Omega),\quad b(u,v,v)=0 \tag{3.45}$$

Hence,

$$\sum_{i=1}^n ||\nabla u_i||^2_{0,\Omega} \leq \sqrt{Gr}\ |g|\ ||\varphi(T)||_{0,\Omega}\ ||u||_{0,\Omega}$$

The Friedrichs-Poincaré inequality gives

$$||u||_{1,\Omega} \leq \mathcal{C}_\Omega\ \sqrt{Gr}\ |g|\ ||\varphi(T)||_{0,\Omega} \tag{3.46}$$

and (3.44) follows from the definition of the operator of truncation $\varphi(T)$ (see (3.26)). □

It follows from (3.39) and (3.44), that there exists a strictly positive constant $\mathcal{D}$ such that*:

$$||\theta||_{1,\Omega} \;\leq\; \mathcal{D} \tag{3.47}$$

3.6. *The thermal model with u given*

A mathematical analysis of the present thermal model *with u given*, has been written in previous papers: [16], [4],[22] and [14]. Let us recall some results useful for the remainder of the analysis.

Proposition 2 *Let u be given in $\mathbf{L}^p(\Omega)$, $p > n$, such that $div(u) = 0$ in Ω and $u = 0$ on $\partial\Omega$. The equation (3.37.b) has a solution $\theta \in W_0(\Omega)$. Furthermore, this solution satisfies the weak maximum principle i.e. $T = (\theta + \tilde{T}_d) \in W_t(\Omega)$ satisfies (3.38).*

Proof. The proof is done in [14] with an operator $\bar{Q}$ slightly different. Using Lemma 2, the result can be extended straightforwardly to the present model. □

Using an idea of [22], we prove that when the linearized model is well posed, then the solution of (3.37.b) is unique. Hence, we study below the linearized thermal model with u given, namely the equation (3.37.b) linearized. Let us point out that this study is also useful for the numerical analysis of the full model (Section 4).

The linearized thermal model with u given Let u be given in $\mathbf{V}(\Omega)$ and let $\theta = (T - \tilde{T}_d) \in W_0(\Omega)$ be a solution of (3.37.b). This solution satisfies (3.38), hence $\bar{Q}(T) = Q(T)$. Then, we study the following linearized problem:

$$(\mathcal{P}_\theta^l) \quad \begin{cases} \text{Find } \theta^* = (T^* - \tilde{T}_d) \in W_0(\Omega) \text{ such that:} \\ \forall t \in W_0(\Omega), \quad c(u; \theta^* + \tilde{T}_d, t) \;+\; \displaystyle\int_{\Gamma_f} DQ(T).(\theta^* + \tilde{T}_d)\; t \; ds \;=\; 0 \end{cases}$$

where

$$DQ(T).T^* \;=\; Bi\; T^* \;+\; 4\varepsilon\; \delta_1 (I - B)(T^3 T^*)$$

and B is defined by (3.23).

Assumption 1 *Either the thermal transfer coefficient h is large enough or the thermal conductivity λ is large enough.*
Namely, either

$$\frac{Bi}{\delta_1} \;=\; \frac{h}{\sigma(T^*)^3} \;\geq\; 4\;\varepsilon_0\; T_{inf}^3 \;[\; (\frac{\varepsilon_1}{\varepsilon_0})^{\frac{3}{2}} \;\; \frac{T_{sup}^3}{T_{inf}^3} \;-\; 1\;] \tag{3.48}$$

---

*More precisely, we have:

$$\begin{aligned} \mathcal{D} \;&=\; \mathcal{C}_\Omega^*\{\; \sqrt{|\Gamma_f|}\; \mathcal{T}_\Omega^*[B_i(T_{sup} - T_{inf}) \;+\; 2\delta_1\varepsilon_1 T_{sup}^4] \\ &+\; \mathcal{R}_\Omega\; ||T_d||_{\frac{1}{2},\Gamma_d}[1 \;+\; \mathcal{C}_\Omega\; \mathcal{S}_\Omega\; \mathcal{S}_\Omega^*\; Gr\; |g|\; \sqrt{|\Omega|}\; T_{sup}]\;\} \end{aligned}$$

or

$$\left\{\begin{array}{lll} \dfrac{Bi}{\delta_1} & < & 4\,\varepsilon_0\,T^3_{inf}\,[\,(\dfrac{\varepsilon_1}{\varepsilon_0})^{\frac{3}{2}}\quad \dfrac{T^3_{sup}}{T^3_{inf}}\,-\,1\,] \\ and & & \\ \lambda & > & L^*\,C^*_\Omega\,[\,4\sigma(T^*)^3\varepsilon_0\,T^3_{inf}[\,(\dfrac{\varepsilon_1}{\varepsilon_0})^{\frac{3}{2}}\dfrac{T^3_{sup}}{T^3_{inf}}-1\,]\,-\,h\,] \end{array}\right. \tag{3.49}$$

Lemma 6 *Let u be given in $\mathbf{L}^{\mathbf{p}}(\Omega)$, $p > n$, such that $div(u) = 0$ in Ω and $u = 0$ on $\partial\Omega$. Under Assumption 1, the linearized thermal model $(\mathcal{P}^l_\theta)$ is well posed.*
Proof. We prove that when Assumption 1 holds, the problem is coercitive in $H^1(\Omega)$, and the result follows from the Lax-Milgram theorem.
We have: $\forall\theta^* \in W_0(\Omega)$,

$$c(u;\theta^*,\theta^*) \;=\; ||\nabla\theta^*||_{0,\Omega} \;+\; a(u;\theta^*,\theta^*)$$

Using (3.43) and the Friedrich-Poincaré's inequality, we have:

$$c(u;\theta^*,\theta^*) \;\geq\; \frac{1}{\mathcal{C}^*_\Omega}||\theta^*||^2_{1,\Omega} \tag{3.50}$$

In others respects,

$$\begin{aligned}\int_{\Gamma_f} DQ(T).(\theta^*)\;\theta^*\;ds \;&=\; \int_{\Gamma_f}[Bi\;\theta^* \;+\; 4\varepsilon\;\delta_1(I-B)(T^3\theta^*)]\;\theta^*\;ds \\ &=\; Bi||\theta^*||^2_{0,\partial\Omega} \;+\; 4\delta_1[\int_{\Gamma_f}\varepsilon\;T^3(\theta^*)^2\;ds \;-\; \int_{\Gamma_f}\varepsilon B(T^3\theta^*)\;\theta^*\;ds]\end{aligned} \tag{3.51}$$

Using Lemma 1, we have:

$$\int_{\Gamma_f}\varepsilon B(T^3\theta^*)\;\theta^*\;ds \;\leq\; \varepsilon_1\;(\frac{\varepsilon_1}{\varepsilon_0})^{\frac{1}{2}}\;T^3_{sup}\;||\theta^*||^2_{0,\partial\Omega} \tag{3.52}$$

Combining (3.50)-(3.52), we obtain: $\forall\theta^* \in W_0(\Omega)$,

$$\begin{aligned} c(u;\theta^*,\theta^*) \;+\; &\int_{\Gamma_f} DQ(T).(\theta^*)\;\theta^*\;ds \\ &\geq\; \frac{1}{\mathcal{C}^*_\Omega}||\theta^*||^2_{1,\Omega} \;+\; [Bi \;+\; 4\delta_1\varepsilon_0\;T^3_{inf}(1 \;-\; (\frac{\varepsilon_1}{\varepsilon_0})^{\frac{3}{2}}\frac{T^3_{sup}}{T^3_{inf}})]\;||\theta^*||^2_{0,\partial\Omega}\end{aligned}$$

Therefore if (3.48) holds then the problem is coercitive in $H^1(\Omega)$ and the result follows. If (3.48) does not hold then the term $I = [Bi + 4\delta_1\varepsilon_0\,T^3_{inf}(1 - (\frac{\varepsilon_1}{\varepsilon_0})^{\frac{3}{2}}\frac{T^3_{sup}}{T^3_{inf}})]$ is negative. Then, we consider the norm of $H^{\frac{1}{2}}(\partial\Omega)$ defined by:

$$||\beta||_{\frac{1}{2},\partial\Omega} \;=\; \inf_{t\in H^1(\Omega)/\;trace(t)=\beta} ||t||_{1,\Omega}$$

and we have:

$$\forall t \in W_0(\Omega),\;\; ||t||^2_{0,\partial\Omega} \;\leq\; ||t||^2_{\frac{1}{2},\partial\Omega} \;\leq\; ||t||^2_{1,\Omega}$$

Hence, the problem is coercitive in $H^1(\Omega)$ if (3.49) holds and the result follows. □

Using an idea of [22], we obtain that when the linearized model is well posed, the solution of (3.37.b) is unique (see [14], Proposition 5 for the proof):

Proposition 3 *Let u be given in $\mathbf{L^p}(\Omega)$, $p > n$, such that $div(u) = 0$ in Ω and $u = 0$ on $\partial\Omega$. Under Assumption 1, the solution of (3.37.b) is unique.* □

3.7. *Existence of a solution to the truncated problem*

We prove the existence of a solution to the truncated problem $(\bar{\mathcal{P}})$ using the Leray-Schauder theorem. To this end, we consider the problem:

$$\left\{\begin{array}{l} \text{Let } (v,\eta) \text{ be given in } \mathbf{V}(\Omega)\times W_0(\Omega), \text{find } (u,\theta)\in \mathbf{V}(\Omega)\times W_0(\Omega) \text{ such that:}\\ \forall w\in \mathbf{V}(\Omega),\\ \displaystyle\sum_{i=1}^{n}(\nabla u_i,\nabla w_i) \;=\; -\,b(v,v,w) \;+\; \sqrt{Gr}\,[(g\;\varphi(\eta),w)] \qquad (3.53.a)\\ \forall t\in W_0(\Omega),\\ \displaystyle c(v;\theta,t) \;+\; \int_{\Gamma_f}\bar{Q}(\theta+\tilde{T}_d)\;t\;ds \;=\; -\;c(v;\tilde{T}_d,t) \qquad (3.53.b) \end{array}\right. \tag{3.53}$$

and we define the operator E as follows:

$$E : \mathbf{V}(\Omega)\times W_0(\Omega)\to \mathbf{V}(\Omega)\times W_0(\Omega) \;:\; (v,\eta)\mapsto (u,\theta)$$

where (u,θ) is solution of (3.53). Any fixed point of E is solution of $(\bar{\mathcal{P}})$.

Let us make an assumption on the domain Ω in order to obtain below some extra regularity on the solution. This extra regularity is useful to obtain some compactness.

Assumption 2 *The domain Ω is polyhedric (polygonal if $n = 2$) and there is no re-entrant dihedral (angles if $n = 2$) with the mixed boundary conditions (Dirichlet-Neumann).*

Theorem 1 *Under the assumptions 1 and 2, the truncated problem $(\bar{\mathcal{P}})$ has at least one solution.*

Proof. The proof is done in four steps and use the Leray-Schauder fixed point theorem.

1st step: For all real σ, $0\le\sigma\le 1$, the operator σE is well defined from $\mathbf{V}(\Omega)\times W_0(\Omega)$ into $\mathbf{V}(\Omega)\times W_0(\Omega)$.

Let $(u,\theta) = \sigma E(v,\eta)$. The function u satisfies $u\in\mathbf{V}(\Omega)$ and:

$$\sum_{i=1}^{n}(\nabla u_i,\nabla w_i) \;=\; -\,b(\sigma v,\sigma v,w) \;+\; \sqrt{Gr}\;(g\;\varphi(\sigma\eta),w) \;\; \forall w\in\mathbf{V}(\Omega)$$

We have $v\in\mathbf{V}(\Omega)$, hence $(v.\nabla)v\in\mathbf{L^q}(\Omega)$ for all $q<\infty$ if $n=2$ and $(v.\nabla)v\in\mathbf{L}^{\frac{3}{2}}(\Omega)$ if $n=3$. Therefore the mapping $b(v,v,w)$ is well defined from $\mathbf{V}(\Omega)\times\mathbf{V}(\Omega)\times\mathbf{V}(\Omega)$ into $\mathbb{R}$ and it is linear continuous with respect to w. It follows from the classical results of the Stokes equations (see e.g. [20]) that for all $\sigma\in[0,1]$, u exists and is unique.

It is proved in [7] that under Assumption 2 and with a right-hand side in $H^{-\frac{1}{2}}(\Omega)$, one has $u\in\mathbf{V}(\Omega)\cap\mathbf{H}^{1+\alpha}(\Omega)$ with $\alpha<\frac{1}{2}$. Hence this result holds with a right hand

side in $L^{\frac{3}{2}}(\Omega)$ since $L^{\frac{3}{2}}(\Omega)$ is continuously embedded into $H^{-\frac{1}{2}}(\Omega)$.
In others respects, we have $\theta \in W_0(\Omega)$ and:

$$c(\sigma v;\theta,t) \;+\; \int_{\Gamma_f} \bar{Q}(\theta+\tilde{T}_d)\; t\; ds \;=\; -\,c(\sigma v;\tilde{T}_d,t) \tag{3.54}$$

It follows from Proposition 2 and 3 that for all $\sigma \in [0,1]$, there exists a unique $\theta \in W_0(\Omega)$ which satisfies (3.54), and this solution is such that (3.38) holds. Furthermore, we have some extra regularity on this solution θ (or $T = \theta + \tilde{T}_d$). As a matter of fact, let us write the classical formulation of (3.53.b).

$$\left\{ \begin{array}{rcll} -\Delta T \;+\; \sqrt{Gr}\; u\nabla T & = & 0 & \text{in } \Omega \\ T & = & T_d & \text{on } \Gamma_d \\ -\dfrac{\partial T}{\partial n} & = & \bar{Q}(T) & \text{on } \Gamma_f \end{array} \right. \tag{3.55}$$

The velocity field u is given and belongs to $\mathbf{V}(\Omega)\cap\mathbf{H}^{1+\alpha}(\Omega)$, $\alpha = \frac{1}{2}-\varepsilon$, for all $\varepsilon > 0$; and there exists a unique solution T in $H^1(\Omega)$ to the problem (3.55). Then, let us consider the non principal part of the operator of (3.55) (the term $\sqrt{Gr}\; u\nabla T$) and $\bar{Q}(T)$ as data of the problem. It follows from Lemma 2 ii) that $\bar{Q}(T) = g \in L^\infty(\Gamma_f)$ and using the Sobolev embedding theorem, we have $f = -\sqrt{Gr}\; u\nabla T \in H^{-\frac{1}{2}}(\Omega)$. Therefore, the solution T satisfies:

$$\left\{ \begin{array}{rcll} -\Delta T & = & f & \text{in } \Omega \\ T & = & T_d & \text{on } \Gamma_d \\ \dfrac{\partial T}{\partial n} & = & g & \text{on } \Gamma_f \end{array} \right. \tag{3.56}$$

Since Assumption 2 holds, it follows from the results of regularity stated in [10] that there exists a real β, $0 < \beta < \frac{1}{2}$, such that $T \in H^{1+\beta}(\Omega)$.

2nd step: The operator E is compact from $\mathbf{V}(\Omega)\times W_0(\Omega)$ into itself.
This property of compactness follows straightforwardly from the regularity of (u,θ) and the compactness of the injection from $H^{1+\gamma}(\Omega)$, $\gamma > 0$, into $H^1(\Omega)$.

3rd step: Estimate a-priori.
Let $\sigma \in [0,1]$ given and (u,θ) be a fixed point of σE: $(u,\theta) = \sigma E(u,\theta)$. Following the proof of Lemma 5, we have:

$$\sum_{i=1}^{n} ||\nabla u_i||^2_{0,\Omega} \;+\; \sigma^2 b(u,u,u) \;=\; \sqrt{Gr}\;(g\;\varphi(\sigma T), u)$$

Using (3.45) and the Friedrichs-Poncaré's inequality,

$$||u||_{1,\Omega} \;\le \mathcal{C}_\Omega\; \sqrt{Gr}\; |g|\; \sqrt{|\Omega|}\; T_{sup} \tag{3.57}$$

In other respects, using Lemma 4, we have:

$$\begin{array}{rcl} ||\theta||_{1,\Omega} & \le & \mathcal{C}^*_\Omega\{\; \sqrt{|\Gamma_f|}\; \mathcal{T}^*_\Omega[B_i(T_{sup}-T_{inf}) \;+\; 2\delta_1\varepsilon_1 T^4_{sup}] \\ & + & \mathcal{R}_\Omega\; ||T_d||_{\frac{1}{2},\Gamma_d}(1 \;+\; \sigma\; \mathcal{S}^*_\Omega\sqrt{Gr}\; ||u||_{L^4(\Omega)})\;\} \end{array} \tag{3.58}$$

Combining (3.57) and (3.58), we obtain that:

$$||\theta||_{1,\Omega} \;\leq\; \mathcal{D} \tag{3.59}$$

where $\mathcal{D}$ is the constant introduced in (3.47).

4th step: Application of the Leray-Schauder fixed point theorem.
We define: $X = \mathbf{V}(\Omega) \times W_0(\Omega)$, and the operator E is compact from X into itself. Using (3.57) and (3.59), we have: for all $\sigma \in [0,1]$, for all $x = (u,\theta) \in X$ such that $x = \sigma E x$, $||x||_X = ||u||_{1,\Omega} + ||\theta||_{1,\Omega} \leq C$ where C is a constant independent of σ, u and θ. Then, it follows from the Leray-Schauder fixed point theorem (see e.g. [8], Chapter 11, Theorem 11.3) that E has at least one fixed point (u,θ) and the result follows □.

3.8. *Uniqueness of the solution to the truncated problem*

We prove the uniqueness of the solution by stating estimates in the standard Sobolev spaces. We obtain the result under the condition that the thermal conductivity λ and the viscosity ν are large enough.

Assumption 3 *The thermal conductivity λ and the viscosity ν are large enough.* †

Theorem 2 *Under the assumptions 1, 2 and 3, the truncated problem $(\bar{\mathcal{P}})$ has one and only one solution.*

Proof. Let $(u^{(1)},\theta^{(1)})$ and $(u^{(2)},\theta^{(2)})$ be two solutions of $(\bar{\mathcal{P}})$. We set $u = u^{(1)} - u^{(2)}$, $T^{(i)} = \theta^{(i)} + \tilde{T}_d$, $i = 1,2$, and $\theta = \theta^{(1)} - \theta^{(2)} = T$.
First, we consider the thermal equation. We have: $\forall t \in W_0(\Omega)$,

$$c(u^{(i)};\theta^{(i)},t) \;+\; \int_{\Gamma_f} \bar{Q}(T^{(i)})\; t\; ds \;=\; -\,c(u^{(i)};\tilde{T}_d,t) \quad i = 1,2$$

hence,

$$\begin{aligned} c(u^{(1)};\theta^{(1)},t) \;-\; c(u^{(2)};\theta^{(1)},t) \;+\; c(u^{(2)};\theta^{(1)},t) \;-\; c(u^{(2)};\theta^{(2)},t) \\ +\; \int_{\Gamma_f} (\bar{Q}(T^{(1)}) - \bar{Q}(T^{(2)}))\; t\; ds \;=\; -\,a(u;\tilde{T}_d,t) \end{aligned}$$

---

† More precisely, if we define:

$$\mathcal{L}_1 \;=\; [\;\mathbf{Bi} \;+\; 4\;\varepsilon_1\;\delta_1\;T^3_{sup}\;(1 + \sqrt{\frac{\varepsilon_1}{\varepsilon_0}})\;]\;\mathcal{C}^*_\Omega\;\mathcal{T}^{*2}_\Omega$$

and

$$\mathcal{L}_2 \;=\; \mathbf{Gr}\;T_{sup}\;\mathcal{C}^2_\Omega\;\mathcal{S}^2_\Omega\;|g|\;\sqrt{|\Omega|}$$

then, the following three inequalities hold:
i)

$$\mathcal{L}_1 \;<\; 1 \tag{3.60}$$

ii)

$$\mathcal{L}_2 \;<\; 1 \tag{3.61}$$

iii)

$$\begin{aligned} &[1 \;-\; \mathcal{L}_1]\,[1 \;-\; \mathcal{L}_2] \\ &-\;4\;\mathbf{Gr}\;T^3_{sup}\;|g|\mathcal{C}^*_\Omega\mathcal{C}_\Omega\;\mathcal{S}^*_\Omega\;\mathcal{S}_\Omega\;\{\;\mathcal{C}^*_\Omega\;\mathcal{T}^*_\Omega|\Gamma_f|^{\frac{1}{2}}\;[B_i(T_{sup} - T_{inf}) + 2\delta_1\varepsilon_1 T^4_{sup}] \\ &+\mathcal{R}_\Omega||T_d||_{\frac{1}{2},\Gamma_d}[\mathcal{C}^*_\Omega + \mathcal{C}^*_\Omega\;\mathcal{C}_\Omega\;\mathcal{S}^*_\Omega\;\mathcal{S}_\Omega\;\mathbf{Gr}\;|g|\;\sqrt{|\Omega|}\;T_{sup} + 1]\;\}\;>\;0 \end{aligned} \tag{3.62}$$

$$a(u;\theta^{(1)},t) + (\nabla\theta,\nabla t) + a(u^{(2)};\theta,t) + \int_{\Gamma_f} (\bar{Q}(T^{(1)}) - \bar{Q}(T^{(2)}))\ t\ ds \ = \ -a(u;\tilde{T}_d,t)$$

We take $t = \theta = T$ and it follows from (3.43):

$$||\nabla\theta||^2_{0,\Omega} \ = \ -\ [a(u;T^{(1)},\theta) \ + \ \int_{\Gamma_f} (\bar{Q}(T^{(1)}) - \bar{Q}(T^{(2)}))\ \theta\ ds] \qquad (3.63)$$

It follows from Lemma 2 i) that

$$|\int_{\Gamma_f} (\bar{Q}(T^{(1)}) - \bar{Q}(T^{(2)}))\ \theta\ ds| \ \leq \ \mathcal{C}_Q\ \mathcal{T}^{*2}_\Omega\ ||\theta||^2_{1,\Omega}$$

the constant $\mathcal{C}_Q$ being defined by (3.29). Hence, the Friedrichs-Poincaré inequality and the continuity of the bilinear form a(.,.) give:

$$||\theta||_{1,\Omega} \ \leq \ \mathcal{C}^*_\Omega\ [\ \sqrt{Gr}\ \mathcal{S}^*_\Omega\ ||u||_{L^4(\Omega)}\ ||\nabla T^{(1)}||_{0,\Omega} \ + \ \mathcal{C}_Q\ \mathcal{T}^{*2}_\Omega\ ||\theta||_{1,\Omega}\ ] \qquad (3.64)$$

and,

$$(1 \ - \ \mathcal{C}_Q\ \mathcal{C}^*_\Omega\ \mathcal{T}^{*2}_\Omega)\ ||\theta||_{1,\Omega} \ \leq \ \sqrt{Gr}\ \mathcal{C}^*_\Omega\ \mathcal{S}^*_\Omega\ ||u||_{L^4(\Omega)}\ ||\nabla T^{(1)}||_{0,\Omega} \qquad (3.65)$$

In other respects, (3.39) and (3.36) give

$$\begin{aligned} ||\nabla T^{(1)}||_{0,\Omega} \ &\leq \ ||\theta^{(1)}||_{1,\Omega} \ + \ \mathcal{R}_\Omega\ ||T_d||_{\frac{1}{2},\Gamma_d} \\ &\leq \ \mathcal{C}^*_\Omega\{\ \mathcal{T}^*_\Omega\ |\Gamma_f|^{\frac{1}{2}}\ [B_i(T_{sup} - T_{inf}) \ + \ 2\delta_1\varepsilon_1 T^4_{sup}] \\ &\quad + \mathcal{R}_\Omega\ ||T_d||_{\frac{1}{2},\Gamma_d}[1 + \mathcal{S}^*_\Omega\sqrt{Gr}||u^{(1)}||_{L^4(\Omega)}]\} + \mathcal{R}_\Omega\ ||T_d||_{\frac{1}{2},\Gamma_d} \end{aligned} \qquad (3.66)$$

and (3.44) gives

$$||u^{(1)}||_{L^4(\Omega)} \ \leq \mathcal{S}_\Omega\ \mathcal{C}_\Omega\ \sqrt{Gr}\ |g|\ \sqrt{|\Omega|}\ T_{sup} \qquad (3.67)$$

Combining (3.65)-(3.67), we obtain:

$$\mathcal{C}^\theta_1\ ||\theta||_{1,\Omega} \ \ \leq \ \ \mathcal{C}^\theta_2\ ||u||_{1,\Omega} \qquad (3.68)$$

with

$$\mathcal{C}^\theta_1 \ = \ (1 \ - \ \mathcal{C}_Q\ \mathcal{C}^*_\Omega\ \mathcal{T}^{*2}_\Omega) \qquad (3.69)$$

and

$$\begin{aligned} \mathcal{C}^\theta_2 \ &= \ \sqrt{Gr}\ \mathcal{C}^*_\Omega\ \mathcal{S}^*_\Omega\ \mathcal{S}_\Omega\ \{\ \mathcal{C}^*_\Omega\ \mathcal{T}^*_\Omega\ |\Gamma_f|^{\frac{1}{2}}\ [B_i(T_{sup} - T_{inf}) \ + \ 2\delta_1\varepsilon_1 T^4_{sup}] \\ &\quad + \ \mathcal{R}_\Omega\ ||T_d||_{\frac{1}{2},\Gamma_d}[\mathcal{C}^*_\Omega \ + \ \mathcal{C}^*_\Omega\ \mathcal{C}_\Omega\ \mathcal{S}^*_\Omega\ \mathcal{S}_\Omega\ Gr\ |g|\ \sqrt{|\Omega|}\ T_{sup} \ + \ 1]\ \} \end{aligned} \qquad (3.70)$$

Now, let us consider the fluid equation. We have: $\forall w \in \mathbf{V}(\Omega)$,

$$\sum_{j=1}^n (\nabla u^{(i)}_j, \nabla w_j) \ + \ b(u^{(i)}, u^{(i)}, w) \ = \sqrt{Gr}\ (g\ \varphi(T^{(i)}), w) \quad i = 1,2$$

hence,

$$\sum_{j=1}^n (\nabla u_j, \nabla w_j) + b(u^{(1)}, u^{(1)}, w) - b(u^{(2)}, u^{(2)}, w) \ = \sqrt{Gr}\ (g\ [\varphi(T^{(1)}) - \varphi(T^{(2)})], w)$$

We take $w = u$ and we obtain:

$$\sum_{j=1}^{n} ||\nabla u_j||^2_{0,\Omega} \; + \; b(u^{(1)}, u^{(1)}, u) \; - \; b(u^{(2)}, u^{(1)}, u) \; + \; b(u^{(2)}, u^{(1)}, u)$$

$$- \; b(u^{(2)}, u^{(2)}, u) \; = \sqrt{Gr} \; (g \; [\varphi(T^{(1)}) - \varphi(T^{(2)})], u)$$

Using (3.45) and the Friedrich-Poincaré inequality,

$$||u||^2_{1,\Omega} \; \leq \; \mathcal{C}_\Omega \; [\; |b(u, u^{(1)}, u)| \; + \; \sqrt{Gr} \; |g| \; ||\varphi(T^{(1)}) - \varphi(T^{(2)})||_{0,\Omega} \; ||u||_{1,\Omega} \;] \quad (3.71)$$

Furthermore,

$$|\varphi(T^{(1)}) - \varphi(T^{(2)})| \; \leq \; 4 \; T^3_{sup} \; |T^{(1)} - T^{(2)}| \text{ a.e. on } \partial\Omega \quad (3.72)$$

We bound by above the term $|b(u, u^{(1)}, u)| : \; \forall u, v, w \in \mathbf{H^1_0}(\Omega)$,

$$|b(u, v, w)| \; \leq \; \sum_{j=1}^{n} \sqrt{Gr} \; ||u||_{L^4(\Omega)} \; ||\nabla v_j||_{0,\Omega} \; ||w||_{L^4(\Omega)}$$

hence,

$$|b(u, u^{(1)}, u)| \; \leq \; \sqrt{Gr} \; ||u||^2_{L^4(\Omega)} \; \sum_{j=1}^{n} ||\nabla u_j^{(1)}||_{0,\Omega} \quad (3.73)$$

Combining (3.71)-(3.73),

$$||u||_{1,\Omega} \; \leq \; \mathcal{C}_\Omega \; \sqrt{Gr} \; [\; \mathcal{S}^2_\Omega \; ||u||_{1,\Omega} \; \sum_{j=1}^{n} ||\nabla u_j^{(1)}||_{0,\Omega} \; + \; 4 \; T^3_{sup} \; |g| \; ||\theta||_{0,\Omega} \;] \quad (3.74)$$

In other respects, (3.44) gives:

$$\sum_{j=1}^{n} ||\nabla u_j^{(1)}||_{0,\Omega} \; \leq \mathcal{C}_\Omega \; \sqrt{Gr} \; |g| \; \sqrt{|\Omega|} \; T_{sup} \quad (3.75)$$

therefore,

$$\mathcal{C}^u_1 \; ||u||_{1,\Omega} \; \leq \; \mathcal{C}^u_2 \; ||\theta||_{0,\Omega} \quad (3.76)$$

with

$$\mathcal{C}^u_1 \; = \; 1 \; - \; Gr \; \mathcal{C}^2_\Omega \; \mathcal{S}^2_\Omega \; |g| \; \sqrt{|\Omega|} \; T_{sup} \quad (3.77)$$

and

$$\mathcal{C}^u_2 \; = \; 4 \; \sqrt{Gr} \; T^3_{sup} \; \mathcal{C}_\Omega \; |g| \quad (3.78)$$

Let us assume that $\mathcal{C}^\theta_1$ and $\mathcal{C}^u_1$ are positive i.e the inequalities (3.60) and (3.61) hold. Then, combining (3.68) and (3.76) we obtain:

$$(\mathcal{C}^\theta_1 \; \mathcal{C}^u_1 \; - \; \mathcal{C}^\theta_2 \; \mathcal{C}^u_2) \; ||\theta||_{1,\Omega} \; \leq \; 0 \quad (3.79)$$

Therefore if $(\mathcal{C}^\theta_1 \; \mathcal{C}^u_1 \; - \; \mathcal{C}^\theta_2 \; \mathcal{C}^u_2) > 0$ i.e. if (3.62) holds, then $\theta^{(1)} = \theta^{(2)}$ a.e. in $\bar{\Omega}$ and since (3.76), $u^{(1)} = u^{(2)}$ a.e. in $\bar{\Omega}$. □

3.9. *Existence and uniqueness to the initial problem*

Corollary 1 *Under the assumptions 1, 2 and 3, the problem (2.1)-(2.3)(2.8)-(2.11) has one and only one weak solution* (u,p,T,w) *in* $\mathbf{H^1}(\Omega)\times L^2_0(\Omega)\times H^1(\Omega)\times L^\infty(\partial\Omega)$ *which satisfies:*

$$T_{inf} \;\leq\; T \;\leq\; T_{sup} \quad a.e.\ in\ \bar{\Omega} \tag{3.80}$$

In others words, under these assumptions, the problem has one and only one physical solution.
Proof. Let the assumptions 1, 2 and 3 be satisfied. There exists a unique $(u,\theta)\in \mathbf{V}(\Omega)\times W_0(\Omega)$ solution to $(\bar{\mathcal{P}})$ (Theorem 1). And using the De Rham theorem, there exists a unique $p\in L^2_0(\Omega)$ such that $(u,p,T=\theta+\tilde{T}_d)\in \mathbf{H^1}(\Omega)\times L^2_0(\Omega)\times H^1(\Omega)$ is the unique (weak) solution to $(\bar{P})$. In others respects, it follows from Lemma 3 that T satisfies (3.80). Therefore $\varphi(T)=T$ a.e. and the solution of $(\bar{P})$ is also solution to (3.16)-(3.21). Consequently, there exists a unique solution (u,p,T) in $\mathbf{H^1}(\Omega)\times L^2_0(\Omega)\times H^1(\Omega)$ to (3.16)-(3.21) which satisfies the weak maximum principle (3.80). And the final result follows from Proposition 1. □

4. Approximation by finite element methods

In this section, we consider the formulation of the problem in the primitive variables, the unknowns are u, p, T and w. Indeed, we do not solve numerically the integral equation like we did in the continuous case and we do not consider discrete velocity field of divergence free: we consider the formulation (2.1)-(2.3)(2.8)-(2.11). Nevertheless, as previously, we truncate the temperature in the right-hand sides of the Navier-Stokes equations and the integral equation, also in the thermal boundary condition of non local type. We discretize the equations using classical finite element methods and we write the numerical analysis of the schemes. (Let us notice that for a technical reason, the truncation is in this section, smoother than previously; it is a C^2 truncation). The main result of this section is the proof of the existence and uniqueness of the discrete solution (u_h,p_h,T_h,w_h) and its convergence towards the unique physical solution of the model, when the viscosity and the thermal conductivity of the fluid (and the thermal transfer coefficient) are large enough and when the step size h is small enough (Theorem 4).

The numerical analysis below is based on the analysis written in [3]. The result of [3] we use, is recalled in Theorem 3. This previous paper deals with finite dimensional approximation of nonlinear problems of the form $F(\lambda;x)=0$ where $F:\Lambda\times X\to X$, Λ is an interval of $\mathbb{R}$ and X is a Banach space. Let us describe the different steps of the present numerical analysis. First, we write the equations of the (continuous) model in the form $F(x)=0$ (Section 4.1, Lemma 7). To obtain later on the convergence of the discrete solution, we assume some extra regularity on the solution (Assumption 4). In Section 4.2, we linearize the equations and using the Lax-Milgram theorem, we prove that the linearized problem is well posed when the viscosity ν and the thermal conductivity λ of the fluid (and the thermal transfer coefficient h) are large enough; by definition, we obtain a branch of non singular solutions (Proposition 4). In Section 4.3 and following [9], we consider general finite element spaces and we discretize the equations. The discrete velocity field is not divergence free and the finite element spaces considered satisfy the Babuska-Brezzi inf-sup condition. We write the discrete problem in the form $F_h(x_h)=0$. Finally, we prove in Section 4.4 that this general finite element scheme fits into the framework of Theorem 3. We obtain the existence, uniqueness and convergence of the discrete solution towards the unique physical solution, locally in h and when the

viscosity, the thermal conductivity (and the thermal transfer coefficient) are large enough (Theorem 4).
Let us notice that Theorem 4 deals with the analysis of the centered scheme, in which no stabilization procedure is introduced (such as artificial diffusion). In others respects, the error estimates are established in detail and the solution belongs -a priori- to fractional order Sobolev spaces.

In Section 4.5, we consider the bidimensional case and we present four different finite element schemes. Two of them lead to first order methods and the two others lead to second order methods.

4.1. *The continuous problem formulated as a fixed point problem*

Following [14], we consider the problem in primitive variables and we truncate the temperature in the right-hand sides and in the boundary conditions. This new truncated problem is:

$$
(P_\varphi)\quad \left\{
\begin{array}{rcll}
\multicolumn{4}{l}{\text{Find } (u,p,T,w) \text{ satisfying:}}\\
-\Delta u \;+\; \sqrt{Gr}\,(u\cdot\nabla)u \;+\; \nabla p &=& \sqrt{Gr}\; g\varphi(T) & \text{in } \Omega\\
div(u) &=& 0 & \text{in } \Omega\\
u &=& 0 & \text{on } \partial\Omega\\
-\Delta T \;+\; \sqrt{Gr}\; u\cdot\nabla T &=& 0 & \text{in } \Omega\\
T &=& T_d & \text{on } \Gamma_d\\
-\dfrac{\partial T}{\partial n} &=& \Phi_\varphi(T,w) & \text{on } \Gamma_f\\
(I-A)w &=& \varepsilon\frac{\delta_1}{\delta_2}\varphi(T)^4 & \text{on } \partial\Omega
\end{array}\right.
$$

where

$$
\Phi_\varphi(T,w)(x) \;=\; Bi\;(\varphi(T)-T_0)(x) \;+\; \varepsilon(x)\;[\delta_1\varphi(T)^4(x) \;-\; \delta_2\int_{\partial\Omega}\phi(x,y)\;w(y)\;ds(y)] \tag{4.81}
$$

the operator A is defined by (2.7) and from now, φ denotes the C^2-truncation defined as follows. Let ξ be a real strictly positive such that $T_{inf}-\xi>0$. We define:

$$
\varphi(T)(x) \;=\; \left\{
\begin{array}{ccc}
T_{inf} & if & T(x)\;\le\;T_{inf}-\xi\\
P_{inf}(T)(x) & if & T_{inf}-\xi\;\le\;T(x)\;\le\;T_{inf}\\
T(x) & if & T_{inf}\;\le\;T(x)\;\le\;T_{sup}\\
P_{sup}(T)(x) & if & T_{sup}\;\le\;T(x)\;\le\;T_{sup}+\xi\\
T_{sup} & if & T_{sup}+\xi\;\le\;T(x)
\end{array}\right. \tag{4.82}
$$

where $P_{inf}(T)$ and $P_{sup}(T)$ are polynomial such that $\varphi(T)$ defined from $I\!R$ into itself is of class C^2. Namely, P_{inf} is such that:

$$
\left\{
\begin{array}{c}
P_{inf}(T_{inf}-\xi) \;=\; P_{inf}(T_{inf}) \;=\; T_{inf}\\
P'_{inf}(T_{inf}-\xi) \;=\; 0;\;\; P'_{inf}(T_{inf}) \;=\; 1\\
P''_{inf}(T_{inf}-\xi) \;=\; P''_{inf}(T_{inf}) \;=\; 0
\end{array}\right. \tag{4.83}
$$

The condition (4.83) defines a unique polynomial P_{inf} of degree 5. Similarly, there exists a unique polynomial P_{sup} of degree 5 which fulfills the conditions such that $\varphi(T)$ is C^2 (see [14]).

Let us make some remarks concerning the introduction of this new truncated problem. First and as previously, we introduce a truncated problem because of the lack of regularity of the boundary condition in the 3d case. As a matter of fact, in order to write the numerical analysis, we formulate the problem as a fixed point problem and this lack of regularity on Γ_f prevents us from doing it in the standard Sobolev space $H^1(\Omega)$. In others respects, we do not consider the same truncated problem as in the continuous analysis because the technique of the present numerical analysis requires a C^2 truncation. Second, let us point out that the truncated problem (P_φ) has a unique solution *and* the temperature satisfies the weak maximum principle.

The equivalent fixed point problem We formulate Problem (P_φ) as a fixed point problem, in the form: $I\!F(x) = 0$. To this end, we define the Stokes operator $I\!S : f^* \mapsto -(u^*, p^*) : \mathbf{H^{-1}}(\Omega) \to \mathbf{H^1}(\Omega) \times (L^2(\Omega)/I\!R)$, where (u^*, p^*) is the unique weak solution of the Stokes problem:

$$\left\{ \begin{array}{rcll} -\Delta u^* + \nabla p^* & = & f^* & \text{in } \Omega \\ div(u^*) & = & 0 & \text{in } \Omega \\ u^* & = & 0 & \text{on } \partial\Omega \end{array} \right. \tag{4.84}$$

We define the operator $I\!L$ corresponding to the thermal partial differential equation:

$$\begin{array}{rcl} I\!L : (h^*, \Phi^*, T_d) & \mapsto & -T^* \\ H^{-1}(\Omega) \times H^{-\frac{1}{2}}(\Gamma_f) \times H^{\frac{1}{2}}(\Gamma_d) & \to & H^1(\Omega) \end{array}$$

where T^* is the unique weak solution of the problem:

$$\left\{ \begin{array}{rcll} -\Delta T^* & = & h^* & \text{in } \Omega \\ T^* & = & T_d & \text{on } \Gamma_d \\ -\dfrac{\partial T^*}{\partial n} & = & \Phi^* & \text{on } \Gamma_f \end{array} \right. \tag{4.85}$$

We define the operator $I\!E : e^* \mapsto -w^* : L^2(\partial\Omega) \to L^2(\partial\Omega)$, where

$$(I - A) w^* = e^* \quad \text{on } \partial\Omega \tag{4.86}$$

The three operators $I\!S$, $I\!L$ and $I\!E$ are well defined. We write:

$$X = \mathbf{H^1}(\Omega) \times (L^2(\Omega)/I\!R) \times H^1(\Omega) \times L^2(\partial\Omega) \tag{4.87}$$

$$Y = \mathbf{H^{-1}}(\Omega) \times (H^{-1}(\Omega) \times H^{-\frac{1}{2}}(\Gamma_f) \times H^{\frac{1}{2}}(\Gamma_d)) \times L^2(\partial\Omega) \tag{4.88}$$

We define the operator $I\!T : Y \to X$,

$$I\!T = \begin{pmatrix} I\!S & 0 & 0 \\ 0 & I\!L & 0 \\ 0 & 0 & I\!E \end{pmatrix} \tag{4.89}$$

and the operator $I\!G : x = (v, q, t, r) \mapsto G(x)$ by

$$I\!G(x) = \begin{pmatrix} \sqrt{Gr}\,[\, g\varphi(t) - (v.\nabla)v \,] \\ (-\sqrt{Gr}\, v\nabla t,\ \Phi_\varphi(t, r),\ T_d) \\ \varepsilon\, \frac{\delta_1}{\delta_2}\, \varphi(t)^4 \end{pmatrix} \tag{4.90}$$

Let us notice that if $t \in H^1(\Omega)$ then $\varphi(t)^4 \in H^{\frac{1}{2}}(\partial\Omega) \cap L^\infty(\partial\Omega)$. In others respects, if $\varepsilon(x)$ is regular enough (for example a lipschitz function) then the term $\varepsilon(x)\frac{\delta_1}{\delta_2}\varphi(t)^4$ belongs to $H^{\frac{1}{2}}(\partial\Omega) \cap L^\infty(\partial\Omega)$. From now, we assume that $\varepsilon(x)$ is lipschitz function which satisfies (2.4).
Let $x = (v, q, t, r) \in X$, it follows from the Sobolev embedding theorem that: $(v.\nabla)v \in \mathbf{L}^{\frac{3}{2}}(\Omega)$, $v\nabla t \in L^{\frac{3}{2}}(\Omega)$ and $(\varepsilon\ \frac{\delta_1}{\delta_2}\ \varphi(t)^4) \in H^{\frac{1}{2}}(\partial\Omega) \cap L^\infty(\partial\Omega)$. We define

$$Z = \mathbf{L}^{\frac{3}{2}}(\Omega) \times (L^{\frac{3}{2}}(\Omega) \times L^2(\Gamma_f) \times H^{\frac{3}{2}}(\Gamma_d)) \times (H^{\frac{1}{2}}(\partial\Omega) \cap L^\infty(\partial\Omega)) \tag{4.91}$$

Let us recall that T_d, the temperature given on Γ_d, belongs to $H^{\frac{3}{2}}(\Gamma_d) \cap L^\infty(\Gamma_d)$. Therefore

the operator G maps X into Z

Furthermore, since Z is compactly embedded into Y, the operator T is linear compact from Z into X.
Finally, we define the operator $I\!F : X \to X$ as follows:

$$I\!F(x) \;=\; x \;+\; I\!T I\!G(x) \tag{4.92}$$

and we have
Lemma 7 *The quadruplet* $(u, p, T, w) \in \mathbf{H^1}(\Omega) \times (L^2(\Omega)/I\!R) \times H^1(\Omega) \times L^2(\partial\Omega)$ *is solution of* (P_φ) *if and only if*

$$I\!F(x) = 0 \tag{4.93}$$

with $x = (u, p, T, w)$ *and* $I\!F$ *defined by (4.92).*

□

Regularity of the solution In order to prove the convergence of the discrete solution, one need in the sequel some extra regularity on the solution of the problem $I\!F(x) = 0$ i.e. $x = -I\!T I\!G(x)$ (see Theorem 3). Under Assumption 2, one can assume some extra regularity on the solution of the integral equation, on the solution of the Stokes problem, [7] and on the solution of the Laplace equation, [10]. Then, one obtain some extra regularity on $x^* = I\!T z^*$, $\forall z^* \in Z$. A such result would imply extra regularity on $x = -I\!T I\!G(x)$, $I\!G(x)$ belonging to Z. Let us assume a such result of regularity.
Assumption 4 *(Regularity of* $x^* = I\!T z^*$, $z^* \in Z$*).*
Let $x^* = I\!T z^* = (u^*, p^*, T^*, w^*)$, $z^* \in Z$. *There exists two reals* s *and* β *strictly greater than* $(\frac{n}{2} - 1)$ *and a real* γ *strictly positive such that:*

$$(u^*, p^*) \in \mathbf{H^{1+s}}(\Omega) \times H^s(\Omega)$$

$$T^* \in H^{1+\beta}(\Omega)$$

and

$$w^* \in H^\gamma(\partial\Omega) \cap L^\infty(\partial\Omega)$$

4.2. *The linearized problem*

Let the assumptions 1, 2 and 3 be satisfied and let $x = (u, p, T, w) \in \mathbf{H^1}(\Omega) \times (L^2(\Omega)/I\!R) \times H^1(\Omega) \times L^\infty(\partial\Omega)$ be the unique solution of (4.93). We establish an extra sufficient condition on the physical data ν, λ and h (they have to be large enough), such that x is a non singular solution of (4.93) i.e. such that $D_x\ I\!F(x)$ is

an isomorphism of X.
The linear operator $D_x I\!F(x)$ is an isomorphism of X if and only if for all $y = (v, q, t, r) \in X$, there exists a unique $x^* = (u^*, p^*, T^*, w^*) \in X$ such that

$$x^* \; + \; I\!T(D_x I\!G(x).x^*) \; = \; y, \tag{4.94}$$

the mapping $y \in X \mapsto x^* \in X$ being continuous.
Let us recall that since T satisfies the weak maximum principle (3.80) then $\varphi(T) = T$ a.e.. Hence we have: $D_x I\!G(x).x^* =$

$$\left(\begin{array}{c} \sqrt{Gr}\,[\; gT^* - (u.\nabla)u^* \; - \; (u^*.\nabla)u \;] \\ (-\sqrt{Gr}\,[u\nabla T^* \; + \; u^*\nabla T], \;\; Bi\; T^* + \varepsilon[4\delta_1 T^3 T^* - \delta_2 \int_{\partial\Omega} \phi(x,y)\; w^*(y)\; ds(y)], \;\; 0) \\ 4\varepsilon\, \frac{\delta_1}{\delta_2}\; T^3 T^* \end{array} \right)$$

and (4.94) is equivalent to:

$$-\Delta u^* + \sqrt{Gr}\,[(u.\nabla)u^* + (u^*.\nabla)u] + \nabla p^* \; = \; -\Delta v + \nabla q + \sqrt{Gr}\, g\, T^* \quad \text{in } \Omega \tag{4.95}$$

$$div(u^*) \; = \; div(v) \qquad \text{in } \Omega \tag{4.96}$$

$$u^* \; = \; v \qquad \text{on } \partial\Omega \tag{4.97}$$

$$-\Delta T^* \; + \; \sqrt{Gr}\,[u\nabla T^* + u^*\nabla T] \; = \; - \Delta t \qquad \text{in } \Omega \tag{4.98}$$

$$T^* \; = \; t \qquad \text{on } \Gamma_d \tag{4.99}$$

$$-\frac{\partial T^*}{\partial n} \; = \; -\frac{\partial t}{\partial n} + Bi\; T^* \; + \; \varepsilon[\; 4\delta_1 T^3 T^* - \delta_2 \int_{\partial\Omega} \phi(x,y)\; w^*(y)\; ds(y)\;] \qquad \text{on } \Gamma_f \tag{4.100}$$

$$(I - A)\; w^* \; = \; (I - A)\; r \; + \; 4\varepsilon\, \frac{\delta_1}{\delta_2}\; T^3\; T^* \qquad \text{on } \partial\Omega \tag{4.101}$$

One knows (see for instance [9] p299) that there exists a unique $(u^*, p^*) \in \mathbf{H^1}(\Omega) \times (L^2(\Omega)/I\!R)$ solution of (4.95)-(4.97) if and only if for f given in $\mathbf{H^{-1}}(\Omega)$, there exists a unique $(u^*, p^*) \in \mathbf{H^1}(\Omega) \times (L^2(\Omega)/I\!R)$ solution of:

$$-\Delta u^* \; + \; \sqrt{Gr}\,[(u.\nabla)u^* + (u^*.\nabla)u] \; + \; \nabla p^* = \sqrt{Gr}\; g\; T^* \; + \; f \tag{4.102}$$

$$div(u^*) = 0 \tag{4.103}$$

$$u^* = 0 \tag{4.104}$$

In others respects, using Proposition 1 and setting $t = r = 0$, the equations (4.100)-(4.101) become:

$$-\frac{\partial T^*}{\partial n} \; = \; L(T^*) \qquad \text{on } \Gamma_f \tag{4.105}$$

where

$$L(T^*) \; = \; DQ(T).T^* \; = \; Bi\; T^* \; + \; 4\; \varepsilon\; \delta_1\; (I - B)(T^3 T^*) \qquad \text{a.e. on } \partial\Omega \tag{4.106}$$

and B is defined by (3.23).

Then, we consider the following linearized problem:

$$(P^l)\begin{cases} \text{Given } f \in \mathbf{H^{-1}}(\Omega),\ h \in H^{-1}(\Omega),\ k \in H^{\frac{1}{2}}(\Gamma_d),\ \varphi \in H^{-\frac{1}{2}}(\Gamma_f) \text{ and let} \\ x \in X \text{ be the unique physical solution of (4.93).} \\ \text{Find } x^* = (u^*, p^*, T^*, w^*) \in X \text{ such that:} \\ -\Delta u^* + \sqrt{Gr}\ [(u.\nabla)u^* + (u^*.\nabla)u] + \nabla p^* = \sqrt{Gr}\ g\ T^* + f & \text{in } \Omega \\ div(u^*) = 0 & \text{in } \Omega \\ u^* = 0 & \text{on } \partial\Omega \\ -\Delta T^* + \sqrt{Gr}\ [\ u\nabla T^* + u^*\nabla T] = 0 & \text{in } \Omega \\ T^* = T_d & \text{on } \Gamma_d \\ -\dfrac{\partial T^*}{\partial n} = L(T^*) & \text{on } \Gamma_f \end{cases}$$

Assumption 5 *The viscosity ν, the thermal conductivity λ and the thermal transfer coefficient h are large enough.* ‡

Proposition 4 *Let $x \in X$ be the unique solution of (4.93) which satisfies (3.80). Under Assumption 5, x is a non singular solution (i.e. $D_x\ I\!F(x)$ is an isomorphism of X).*

Proof. We prove that the problem (P^l) is well posed using the Lax-Milgram theorem, then it follows that x is a non singular solution.
We define: $E(\Omega) = \mathbf{V}(\Omega) \times W_0(\Omega)$. $E(\Omega)$ equipped with the induced norm is a Hilbert space. We define: $\theta^* = (T^* - \tilde{T}_d) \in W_0(\Omega)$, $\eta^* = (u^*, \theta^*) \in E(\Omega)$, $\xi = (w, t) \in E(\Omega)$ and the bilinear form $d(.,.)$ from $E(\Omega) \times E(\Omega)$ into $I\!R$ by

$$d(\eta^*, \xi) = \sum_{i=1}^{n} (\nabla u_i^*, \nabla w_i) + b(u, u^*, w) + b(u^*, u, w) - \sqrt{Gr}\ (g\ \theta^*, w)$$
$$+ c(u; \theta^*, t) + a(u^*; T, t) + \int_{\Gamma_f} L(\theta^*) t\ ds$$

The mapping d is continuous from $E(\Omega) \times E(\Omega)$ into $I\!R$. We define:

$$l(\xi) = \sqrt{Gr}\ (g\ \tilde{T}_d, w) + (f, w) - \int_{\Gamma_f} L(\tilde{T}_d)\ t\ ds - c(u; \tilde{T}_d, t)$$

The mapping l is linear continuous from $E(\Omega)$ into $I\!R$. With the notations above, the weak formulation of (P^l) is:

$$\begin{cases} \text{Find } \eta^* = (u^*, \theta^*) \in E(\Omega) \text{ such that:} \\ \forall \xi \in E(\Omega),\quad d(\eta^*, \xi) = l(\xi) \end{cases} \tag{4.107}$$

Let us prove that the bilinear form d is E-elliptic. For all $\eta^* \in E(\Omega)$,

$$d(\eta^*, \eta^*) = \sum_{i=1}^{n} ||\nabla u_i^*||_{0,\Omega}^2 + b(u, u^*, u^*) + b(u^*, u, u^*) - \sqrt{Gr}\ (g\ \theta^*, u^*)$$

‡More precisely, the following inequality holds:

$$\frac{1}{2}\ min\{\mathcal{C}_\Omega^*, \mathcal{C}_\Omega\} - \frac{1}{4}\sqrt{Gr}\ [\ \mathcal{S}_\Omega^*\ \mathcal{S}_\Omega\ (\mathcal{D} + ||\nabla \tilde{T}_d||_{0,\Omega}) + |g|\ ]$$
$$-Gr\ \mathcal{S}_\Omega^2\ \mathcal{C}_\Omega\ |g|\ \sqrt{|\Omega|}\ T_{sup} + [Bi - 4\delta_1\varepsilon_0 T_{inf}^3((\frac{\varepsilon_1}{\varepsilon_0})^{\frac{3}{2}}\ \frac{T_{sup}^3}{T_{inf}^3} - 1)] > 0$$

where $\mathcal{D}$ is the constant introduced in (3.47)

$$+ \|\nabla\theta^*\|^2_{0,\Omega} \; + \; a(u;\theta^*,\theta^*) \; + \; a(u^*;T,\theta^*) \; + \; \int_{\Gamma_f} L(\theta^*)\theta^* \; ds$$

Using (3.43) and (3.45), we obtain:

$$\begin{aligned} d(\eta^*,\eta^*) \; = \; & b(u^*,u,u^*) \; + \; a(u^*;T,\theta^*) \; - \; \sqrt{Gr} \; (g \; \theta^*, u^*) \\ & + \; \int_{\Gamma_f} L(\theta^*)\theta^* \; ds \; + \; \sum_{i=1}^{n} \|\nabla u_i^*\|^2_{0,\Omega} \; + \; \|\nabla\theta^*\|^2_{0,\Omega} \end{aligned} \quad (4.108)$$

We bound above (or below) the terms of (4.108). The first term gives:

$$\begin{aligned} |b(u^*,u,u^*)| \; & \leq \; \sqrt{Gr} \; \sum_{i,j=1}^{n} | \int_{\Omega} u_j^* \; (\partial_j u_i) \; u_i^* \; dx| \\ & \leq \; \sqrt{Gr} \; \|u^*\|^2_{L^4(\Omega)} \; \sum_{i=1}^{n} \|\nabla u_i\|_{0,\Omega} \\ & \leq \; \sqrt{Gr} \; \mathcal{S}_\Omega^2 \; \|u^*\|^2_{1,\Omega} \; \sum_{i=1}^{n} \|\nabla u_i\|_{0,\Omega} \end{aligned}$$

and using (3.44),

$$|b(u^*,u,u^*)| \; \leq \; Gr \; \mathcal{S}_\Omega^2 \; \mathcal{C}_\Omega \; |g| \; \sqrt{|\Omega|} \; T_{sup} \; \|u^*\|^2_{1,\Omega} \quad (4.109)$$

The second term:

$$|a(u^*;T,\theta^*)| \; \leq \; \sqrt{Gr} \; \|u^*\|_{L^4(\Omega)} \; \|\nabla T\|_{0,\Omega} \; \|\theta^*\|_{L^4(\Omega)}$$

and using (3.47),

$$\begin{aligned} |a(u^*;T,\theta^*)| \; & \leq \; \sqrt{Gr} \; \mathcal{S}_\Omega^* \; \mathcal{S}_\Omega \; (\mathcal{D} + \|\nabla \tilde{T}_d\|_{0,\Omega}) \; \|u^*\|_{1,\Omega} \; \|\theta^*\|_{1,\Omega} \\ & \leq \; \frac{1}{2}\sqrt{Gr} \; \mathcal{S}_\Omega^* \; \mathcal{S}_\Omega \; (\mathcal{D} + \|\nabla \tilde{T}_d\|_{0,\Omega}) \; (\|u^*\|^2_{1,\Omega} \; + \; \|\theta^*\|^2_{1,\Omega}) \end{aligned} \quad (4.110)$$

The third term of (4.108) gives

$$\begin{aligned} |(g\theta^*,u^*)| \; & \leq \; |g| \; \|\theta^*\|_{0,\Omega} \; \|u^*\|_{0,\Omega} \\ & \leq \; \frac{1}{2}|g| \; (\|\theta^*\|^2_{1,\Omega} \; + \; \|u^*\|^2_{1,\Omega}) \end{aligned} \quad (4.111)$$

It follows from (3.51)-(3.52) that the fourth term satisfies:

$$\int_{\Gamma_f} L(\theta^*)\theta^* \; ds \; \geq \; [Bi \; + \; 4 \; \delta_1 \; \varepsilon_0 \; T^3_{inf}(1 \; - \; (\frac{\varepsilon_1}{\varepsilon_0})^{\frac{3}{2}} \; \frac{T^3_{sup}}{T^3_{inf}})] \; \|\theta^*\|^2_{0,\partial\Omega} \quad (4.112)$$

Combining (4.108)(4.109)(4.110)(4.111)(4.112), we obtain:

$$d(\eta^*,\eta^*) \; \geq \; \sum_{i=1}^{n} \|\nabla u_i^*\|^2_{0,\Omega} \; + \; \|\nabla\theta^*\|^2_{0,\Omega}$$

$$
\begin{aligned}
&- \quad Gr\ \mathcal{S}_\Omega^2\ \mathcal{C}_\Omega\ |g|\ \sqrt{|\Omega|}\ T_{sup}\ ||u^*||_{1,\Omega}^2 \\
&- \quad \frac{1}{2}\sqrt{Gr}\ \mathcal{S}_\Omega^*\ \mathcal{S}_\Omega\ (\mathcal{D} + ||\nabla \tilde{T}_d||_{0,\Omega})\ (||u^*||_{1,\Omega}^2\ +\ ||\theta^*||_{1,\Omega}^2) \\
&+ \quad [Bi\ -\ 4\ \delta_1\ \varepsilon_0\ T_{inf}^3((\frac{\varepsilon_1}{\varepsilon_0})^{\frac{3}{2}}\ \frac{T_{sup}^3}{T_{inf}^3}\ -\ 1)]\ ||\theta^*||_{0,\partial\Omega}^2 \\
&- \quad \frac{1}{2}\sqrt{Gr}\ |g|\ (||u^*||_{1,\Omega}^2\ +\ ||\theta^*||_{1,\Omega}^2)
\end{aligned}
\tag{4.113}
$$

We consider the norm of $H^{\frac{1}{2}}(\partial\Omega)$ defined by:

$$
||\beta||_{\frac{1}{2},\partial\Omega}\ =\ \inf_{t\in H^1(\Omega)/\ trace(t)=\beta}\ ||t||_{1,\Omega}
$$

Hence,

$$
\forall t \in W_0(\Omega),\ \ ||t||_{0,\partial\Omega}^2\ \leq\ ||t||_{\frac{1}{2},\partial\Omega}^2\ \leq\ ||t||_{1,\Omega}^2 \tag{4.114}
$$

In other respects,

$$
\forall \xi \in E(\Omega),\ \ (||w||_{1,\Omega}^2\ +\ ||t||_{1,\Omega}^2)\ \geq\ \frac{1}{2}\ ||\xi||_{1,\Omega}^2 \tag{4.115}
$$

Combining (4.113)-(4.115), we obtain:

$$
\begin{aligned}
d(\eta^*,\eta^*)\quad \geq &\quad \frac{1}{2}\ min(\mathcal{C}_\Omega^*, \mathcal{C}_\Omega)\ ||\eta^*||_{1,\Omega}^2 \\
- &\quad \frac{1}{4}\sqrt{Gr}\ [\ \mathcal{S}_\Omega^*\ \mathcal{S}_\Omega\ (\mathcal{D}\ +\ ||\nabla \tilde{T}_d||_{0,\Omega})\ +\ |g|\ ]\ ||\eta^*||_{1,\Omega}^2 \\
- &\quad Gr\ \mathcal{S}_\Omega^2\ \mathcal{C}_\Omega\ |g|\ \sqrt{|\Omega|}\ T_{sup}\ ||u^*||_{1,\Omega}^2 \\
- &\quad [\ Bi\ -\ 4\ \delta_1\ \varepsilon_0\ T_{inf}^3((\frac{\varepsilon_1}{\varepsilon_0})^{\frac{3}{2}}\ \frac{T_{sup}^3}{T_{inf}^3}\ -\ 1)\ ]\ ||\theta^*||_{0,\partial\Omega}^2
\end{aligned}
$$

Therefore, under Assumption 5, the bilinear form $d(.,.)$ is E-elliptic. Using the Lax-Milgram theorem, the problem (4.107) is well posed and the result is proved. □

4.3. *The discrete equations*

We discretize the problem (P_φ) using classical finite element methods. We denote by $(\mathcal{T}_h)$ a regular and quasi-uniform family of triangulation, such that $\bar{\Omega} = \cup_{T\in\mathcal{T}_h} T$.

The finite element spaces Following [9], Section IV.4., we consider the finite-dimensional spaces N_h, M_h, L_h and R_h such that:

$$
\mathbf{N_h} \subset \mathbf{H^1}(\Omega),\ M_h \subset L^2(\Omega),\ L_h \subset H^1(\Omega),\ R_h \subset L^2(\partial\Omega)
$$

We assume that M_h and R_h contain the constant functions. We define:

$$
\mathbf{U_h} = \mathbf{N_h} \cap \mathbf{H_0^1}(\Omega) = \{v_h \in \mathbf{N_h},\ v_h = 0 \text{ on } \partial\Omega\}
$$

$$
P_h = M_h \cap (L^2(\Omega)/I\!R) = \{q_h \in M_h,\ q_h(dof_0) = 0\}
$$

where dof_0 is the first degree of freedom of the pressure.

$$W_h = L_h \cap W_0(\Omega) = \{t_h \in L_h,\ t_h = 0 \text{ on } \Gamma_d\}$$

Let us recall that for all $s > \frac{n}{2} - 1$, $H^{1+s}(\Omega) \subset C^0(\bar{\Omega})$ with continuous embedding. We make the following assumptions.

Assumption 6 *(Approximation property of $\mathbf{U_h}$). There exists an operator $\Pi_h^u \in \mathcal{L}(\mathbf{H^{1+s}}(\Omega) \cap \mathbf{H_0^1}(\Omega); \mathbf{U_h})$ and an integer l such that:*

$$\|v - \Pi_h^u v\|_{1,\Omega} \ \leq \ c\ h^s\ \|v\|_{1+s,\Omega}, \quad \forall v \in \mathbf{H^{1+s}}(\Omega),\ s \textit{ real},\ \frac{n}{2} - 1 < s \leq l \quad (4.116)$$

with the constant c independent of h.

Assumption 7 *(Approximation property of M_h). There exists an operator $\Pi_h^p \in \mathcal{L}(L^2(\Omega); M_h)$ such that:*

$$\|q - \Pi_h^p q\|_{0,\Omega} \ \leq \ c\ h^s\ \|q\|_{s,\Omega}, \quad \forall q \in H^s(\Omega),\ s \textit{ real},\ \frac{n}{2} - 1 < s \leq l \qquad (4.117)$$

Assumption 8 *(Uniform inf-sup condition). There exists a constant $c > 0$ (c independent of h) such that for each $q_h \in P_h$, there exists a $v_h \in \mathbf{U_h}$ such that:*

$$(q_h, div v_h) \ = \ \|q_h\|_{0,\Omega}^2 \qquad (4.118)$$

$$|v_h|_{1,\Omega} \ \leq \ c\ \|q_h\|_{0,\Omega} \qquad (4.119)$$

where $|.|_{1,\Omega}$ denotes the semi-norm of $H^1(\Omega)$.

Assumption 9 *(Approximation property of W_h). There exists an operator $\Pi_h^T \in \mathcal{L}(H^{1+\beta}(\Omega) \cap W_0(\Omega); W_h)$ and an integer m such that:*

$$\|t - \Pi_h^T t\|_{1,\Omega} \ \leq \ c\ h^\beta\ \|t\|_{1+\beta,\Omega}, \quad \forall t \in H^{1+\beta}(\Omega),\ \beta \textit{ real},\ \frac{n}{2} - 1 < \beta \leq m \quad (4.120)$$

Assumption 10 *(Approximation property of R_h). There exists an operator $\Pi_h^w \in \mathcal{L}(L^2(\partial\Omega); R_h)$ such that:*

$$\|z - \Pi_h^w z\|_{0,\partial\Omega} \ \leq \ c\ h^\gamma\ \|z\|_{\gamma,\partial\Omega}, \quad \forall z \in H^\gamma(\partial\Omega),\ \gamma \textit{ real},\ 0 < \gamma \leq m \qquad (4.121)$$

We denote by $\mathbb{P}_k$ the space of polynomials functions of degree lower or equal to k. Using the interpolation operator of Lagrange $\mathbb{P}_k$, $k \geq 1$, and the L^2-local orthogonal projection operator, the discrete spaces of the classical finite element methods used to approximate a second order elliptic problem satisfy the assumptions 9 and 10. And the classical finite element methods used to approximate the Stokes problem satisfy the assumptions 6, 7 and 8. We detail such finite element methods in the bidimensional case in Section 4.5.

The discrete equations For the sake of simplicity, we suppose that the data T_d on Γ_d is the trace of a function of L_h.

The discrete formulation we consider is the following:

$$\left\{\begin{array}{l} \text{Find } (u_h, p_h, T_h, w_h) \in \mathbf{U_h} \times P_h \times L_h \times R_h \text{ such that:} \\ \forall v_h \in \mathbf{U_h}, \\ \displaystyle\sum_{i=1}^{n} (\nabla u_{hi}, \nabla v_{hi}) + b(u_h, u_h, v_h) - (p_h, div(v_h)) = \sqrt{Gr}(g\varphi(T_h), v_h) \\ \forall q_h \in P_h, \quad (div(u_h), q_h) \ = \ 0 \\ \forall t_h \in W_h, \quad c(u_h; T_h, t_h) \ + \ \displaystyle\int_{\Gamma_f} \Phi_\varphi(T_h, w_h)\ t_h \ ds \ = \ 0 \\ \forall z_h \in R_h, \quad \displaystyle\int_{\partial\Omega} (I - A) w_h z_h \ ds \ = \ \frac{\delta_1}{\delta_2} \int_{\partial\Omega} \varepsilon \ \varphi(T_h)^4 \ z_h \ ds \end{array}\right. \tag{4.122}$$

with the boundary condition

$$T_h \ = \ T_d \text{ on } \Gamma_d \tag{4.123}$$

The equivalent fixed point problem Following Section 4.1, we formulate the discrete problem (4.122)(4.123) as a fixed point problem.

i) We define $I\!S_h : f^* \mapsto -(u_h^*, p_h^*) : \mathbf{H^{-1}}(\Omega) \to \mathbf{U_h} \times P_h$, where (u_h^*, p_h^*) is the solution of the equations:

$$\left\{\begin{array}{l} \forall v_h \in \mathbf{U_h}, \ \displaystyle\sum_{i=1}^{2} (\nabla u_{hi}^*, \nabla v_{hi}) \ - \ (p_h^*, div(v_h)) = (f^*, v_h) \\ \forall q_h \in P_h, \ \ (div(u_h^*), q_h) \ = \ 0 \end{array}\right. \tag{4.124}$$

Under the assumptions 6, 7 and 8, the solution (u_h^*, p_h^*) exists and is unique. In addition, we have:

$$|u^* - u_h^*|_{1,\Omega} \ + \ ||p^* - p_h^*||_{0,\Omega} \ \le \ c \ \{ \inf_{v_h \in \mathbf{U_h}} |u^* - v_h|_{1,\Omega} \ + \ \inf_{q_h \in M_h} ||p^* - q_h||_{0,\Omega} \}$$

with c is a constant strictly positive and independent of h. Therefore,

$$|u^* - u_h^*|_{1,\Omega} \ + \ ||p^* - p_h^*||_{0,\Omega} \ \longrightarrow \ 0 \tag{4.125}$$

when h tends to 0 (see [9], Chap II.1.3). Furthermore, under the assumptions 4, 6, 7 and 8,

$$|u^* - u_h^*|_{1,\Omega} \ + \ ||p^* - p_h^*||_{0,\Omega} \ \le \ c \ h^s \ (||u^*||_{1+s,\Omega} + ||p^*||_{s,\Omega}) \tag{4.126}$$

ii) We define the operator:

$$I\!L_h : (h^*, \Phi^*, T_d) \mapsto -T_h^* : H^{-1}(\Omega) \times H^{-\frac{1}{2}}(\Gamma_f) \times H^{\frac{1}{2}}(\Gamma_d) \to L_h$$

where T_h^* is solution of the following equations:

$$\forall t_h \in W_h, \ \ (\nabla T_h^*, \nabla t_h) \ = \ (h^*, t_h) \ - \ \int_{\Gamma_f} \Phi^* \ t_h \ ds \tag{4.127}$$

$$T_h^* \ = \ T_d \text{ on } \Gamma_d \tag{4.128}$$

The existence and uniqueness of T_h^* in L_h is classical, and the Céa's lemma gives:

$$||T^* - T_h^*||_{1,\Omega} \;\leq\; c \inf_{t_h \in W_h} ||T^* - w_h||_{1,\Omega}$$

where c is a constant strictly positive and independent of h.Therefore,

$$||T^* - T_h^*||_{1,\Omega} \;\longrightarrow\; 0 \tag{4.129}$$

when h tends to 0 (see for e.g. [5], Section 18). In addition, under the assumptions 4 and 9,

$$||T^* - T_h^*||_{1,\Omega} \;\leq\; c\; h^\beta\; ||T^*||_{1+\beta,\Omega} \tag{4.130}$$

iii) We define the operator: $I\!E_h : e^* \mapsto -w_h^* : L^2(\partial\Omega) \to R_h$ where w_h^* is solution of the equation:

$$\forall z_h \in R_h, \;\; \mathcal{R}(w_h^*, z_h) \;=\; \int_{\partial\Omega} e_h^*\; z_h\; ds \tag{4.131}$$

The bilinear form $\mathcal{R}$ being defined from $L^2(\partial\Omega) \times L^2(\partial\Omega)$ into $I\!R$ by:

$$\mathcal{R}(w, z) \;=\; \int_{\partial\Omega} (I - A) w\; z\; ds$$

Since A is a contracting operator from $L^\infty(\partial\Omega)$ into itself (see [16]), the mapping $\mathcal{R}$ is L^2-elliptic and the existence and uniqueness of w_h^* follows straightforwardly from the Lax-Milgram theorem. Furthermore, we have:

$$||w^* - w_h^*||_{0,\partial\Omega} \;\longrightarrow\; 0 \tag{4.132}$$

when h tends to 0. In addition, under the assumptions 4 and 10,

$$||w^* - w_h^*||_{0,\partial\Omega} \;\leq\; c\; h^\gamma\; ||w^*||_{\gamma,\partial\Omega} \tag{4.133}$$

Let us define

$$X_h = \mathbf{U_h} \times P_h \times L_h \times R_h \tag{4.134}$$

We have $X_h \subset X$. We define the operator $I\!\Pi_h : Y \to X_h$ by

$$I\!\Pi_h = \begin{pmatrix} I\!S_h & 0 & 0 \\ 0 & I\!L_h & 0 \\ 0 & 0 & I\!E_h \end{pmatrix}$$

and we define the operator $I\!F_h : X_h \to X_h$ by:

$$I\!F_h(x_h) \;=\; x_h \;+\; I\!\Pi_h I\!G(x_h) \tag{4.135}$$

and we have the

Lemma 8 *The quadruplet* $x_h = (u_h, p_h, T_h, w_h) \in \mathbf{U_h} \times P_h \times L_h \times R_h$ *is solution of (4.122)(4.123) if and only if*

$$I\!F_h(x_h) = 0 \tag{4.136}$$

□

4.4. *Existence, uniqueness and convergence*

In order to prove the existence and uniqueness of x_h, solution of (4.136), and its convergence towards x (x being the solution of (4.93)), we use a discrete implicit function theorem (see F. Brezzi, J. Rappaz and P.A. Raviart [3], see also M. Crouzeix and J. Rappaz [6], J.C. Paumier [15]...). Nevertheless, we do not formulate our problem as a parameterized problem like in the previous references. Then, we recall below this discrete implicit function theorem adapted to a such simplified framework.

Theorem 3 *(Brezzi, Rappaz and Raviart, [3]). Let $x \in X$ be a non singular solution of (4.93). With the notations introduced above, if the operator $I\!G$ defined from X into Y is of class C^2, if $D_x^2 I\!G$ is bounded in any bounded sub-space of X, if there exists Z a Banach space included into Y with continuous injection, such that: $D_x I\!G(\lambda, x) \in \mathcal{L}(X, Z)$ $\forall x \in X$, if*

$$\lim_{h\to 0} ||(I\!T - I\!T_h)(y)||_X \;=\; 0 \qquad \forall y \in Y, \tag{4.137}$$

and

$$\lim_{h\to 0} ||I\!T - I\!T_h||_{\mathcal{L}(Z,X)} \;=\; 0, \tag{4.138}$$

then there exists a real $h_0 > 0$ such that for all $h < h_0$, there exists a non singular solution x_h to (4.136). Furthermore, there exists a constant c independent of h such that:

$$||x - x_h||_X \;\le\; c\; ||(I\!T - I\!T_h)(I\!G(x))||_X \tag{4.139}$$

□

Using Theorem 3, we obtain the main result of this section:

Theorem 4 *If the assumptions 2, 4, 6, 7, 8, 9 and 10 hold, if the viscosity ν, the thermal conductivity λ and the thermal transfer coefficient h are large enough, (assumptions 1, 3 and 5), then there exists a real $h_0 > 0$ such that for all $h < h_0$, there exists a unique solution $(u_h, p_h, T_h, w_h) \in \mathbf{U_h} \times P_h \times L_h \times R_h$ to the discrete problem (4.122)(4.123). Furthermore,*

$$||u - u_h||_{1,\Omega} + ||p - p_h||_{0,\Omega} + ||T - T_h||_{1,\Omega} + ||w - w_h||_{0,\partial\Omega} \;\le\; c\; h^{\alpha} \tag{4.140}$$

where c is a constant independent of h,

$$\alpha = min(s, \beta, \gamma)$$

and $(u, p, T, w) = x \in \mathbf{H^1}(\Omega) \times (L^2(\Omega)/I\!R) \times H^1(\Omega) \times L^\infty(\partial\Omega)$ is the unique solution of (3.16)-(2.11) which satisfies (3.80).

Proof. Let the assumptions 1-10 be satisfied. We prove that the conditions of Theorem 3 are satisfied.

The equation (4.93) has one and only one solution $x = (u, p, T, w) \in X$ and it follows from Proposition 4 that this solution is a non singular solution.

The operator $I\!G$ defined by (4.90) maps X onto Z, it is of class C^2 and $D_x^2 I\!G(x)$ is a bounded operator in any bounded sub-space of X.

The estimate (4.137) follows straightforwardly from the numerical analysis presented previously. As a matter of fact, let $y^* \in Y$, we define $x^* = (u^*, p^*, T^*, w^*) = -I\!T y^*$ and $x_h^* = (u_h^*, p_h^*, T_h^*, w_h^*) = -I\!T_h y^*$. We have (see (4.125), (4.129) and (4.132)):

$$|u^* - u_h^*|_{1,\Omega} \;+\; ||p^* - p_h^*||_{0,\Omega} \;\longrightarrow\; 0$$

$$||T^* - T_h^*||_{1,\Omega} \;\longrightarrow\; 0$$

$$||w^* - w_h^*||_{0,\partial\Omega} \;\longrightarrow\; 0$$

when h tends to 0. Furthermore, since the inclusion $Z \subset Y$ is compact, the estimate (4.138) holds.

Therefore, we can apply Theorem 3. Let $x_h = (u_h, p_h, T_h, w_h) \in X_h$ be the unique solution of the discrete problem, the estimate (4.139) gives:

$$\begin{aligned} &||u - u_h||_{1,\Omega} + ||p - p_h||_{0,\Omega} + ||T - T_h||_{1,\Omega} + ||w - w_h||_{0,\partial\Omega} \\ \le\; & c\; ||u^* - u_h^*||_{1,\Omega} + ||p^* - p_h^*||_{0,\Omega} + ||T^* - T_h^*||_{1,\Omega} + ||w^* - w_h^*||_{0,\partial\Omega} \end{aligned} \tag{4.141}$$

where $(u^*, p^*, T^*, w^*) = x^* = -\Pi T z$ and $(u_h^*, p_h^*, T_h^*, w_h^*) = x_h^* = -\Pi_h z$, $z = \mathbb{G}(x) \in Z$. Finally, the estimate (4.140) is a direct consequence of (4.141), (4.126), (4.130) and (4.133).

□

4.5. *Application to some finite element methods*

In this section, we consider the bidimensional case ($\Omega \subset \mathbb{R}^2$) and we present some finite element methods which fulfill the assumptions 6-10. Let s, β and γ be the regularity coefficients defined in Assumption 4 and let $\alpha = min(s, \beta, \gamma)$. Let us suppose that the couple of finite element spaces (U_h, M_h) satisfies the inf-sup condition (Assumption 8). Then, Theorem 4 suggests us to choose the finite element spaces U_h, M_h, W_h and R_h in order to have the assumptions 6, 7, 9 and 10 satisfied for the integers l and m such that: $\alpha \leq l \leq \alpha + 1$ and $\alpha \leq m \leq \alpha + 1$. We propose four combinations of finite element methods which lead to two first order methods and to two second order methods.

The "mini" finite element / $\mathbb{P}_1$ Lagrange - $\mathbb{P}_0$ piecewise. We approximate the Navier-Stokes equations using the "mini" finite element, namely:

$$\begin{aligned}\mathbf{U_h} \;=\; & \{v_h,\ v_h \in (C^0(\bar{\Omega}))^2,\ v_h|_T = \varphi_h + \mu\ \lambda_1\ \lambda_2\ \lambda_3,\ \forall T \in \mathcal{T}_h,\ \varphi_h \in (\mathbb{P}_1)^2,\\ & \mu \text{ a constant of } \mathbb{R}^2,\ v_h = 0 \text{ on } \partial\Omega\}\end{aligned}$$

where the scalar functions λ_j, $j = 1, 2, 3$, denote the barycentric coordinates.

$$P_h = \{q_h \in C^0(\bar{\Omega}),\ q_h|_T \in \mathbb{P}_1,\ \forall T \in \mathcal{T}_h,\ q_h(dof_0) = 0\}$$

where dof_0 is the first degree of freedom of the pressure.
For the thermal equations, we choose a $\mathbb{P}_1$ Lagrange - $\mathbb{P}_0$ piecewise scheme type:

$$W_h = \{t_h \in C^0(\bar{\Omega}),\ t_h|_T \in \mathbb{P}_1,\ \forall T \in \mathcal{T}_h,\ t_h = 0 \text{ on } \Gamma_d\} \tag{4.142}$$

$$R_h = \{z_h,\ z_h|_{\partial T} \in \mathbb{P}_0,\ \forall \partial T \in \partial\Omega,\ T \in \mathcal{T}_h\} \tag{4.143}$$

Let us notice that the pressure is continuous $\mathbb{P}_1$ piecewise in $\bar{\Omega}$ and the radiosity is discontinuous $\mathbb{P}_0$ piecewise on $\partial\Omega$. The degrees of freedom of the unknowns are indicated in Fig.1.
With the spaces $\mathbf{U_h}$ and P_h defined as above, Assumption 8 holds (see [9], Chap. II, Lemma 4.1) and the assumptions 6 and 7 hold for $l \leq 1$. Concerning Assumption 7, the proof is done in [23], Theorem 4.1., and concerning Assumption 6, the proof is done in [9], Chap. II, Lemma 2.4, when $s \in \mathbb{N}$ and it is done in Proposition 5 at the end of the present paper when $s \in \mathbb{R}^{*+}$.
In others respects, using the interpolation operator of Lagrange $\mathbb{P}_1$ and the L^2-local orthogonal projection operator, the assumptions 9 and 10 are satisfied for $m \leq 1$ (see [23], Theorem 4.1, for Assumption 9 and [9], Lemma A.5 for Assumption 10). Then, it follows from Theorem 4 that this method is a first order method:

$$||u - u_h||_{1,\Omega} + ||p - p_h||_{0,\Omega} + ||T - T_h||_{1,\Omega} + ||w - w_h||_{0,\partial\Omega} \;=\; O(h^{min(\alpha,1)})$$

The Crouzeix-Raviart element / $\mathbb{P}_2$ Lagrange - $\mathbb{P}_1$ Lagrange. The finite element we use for the Navier-Stokes equations is the Crouzeix-Raviart element:

$$\begin{aligned}\mathbf{U_h} \;=\; & \{v_h,\ v_h \in (C^0(\bar{\Omega}))^2,\ v_h|_T = \varphi_h + \mu\ \lambda_1\ \lambda_2\ \lambda_3,\ \forall T \in \mathcal{T}_h,\ \varphi_h \in (\mathbb{P}_2)^2,\\ & \mu \text{ a constant of } \mathbb{R}^2,\ v_h = 0 \text{ on } \partial\Omega\}\end{aligned}$$

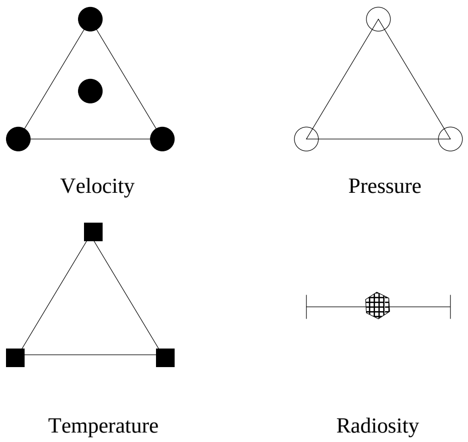

Figure 1: The "mini" finite element / $\mathbb{P}_1$ Lagrange - $\mathbb{P}_0$ piecewise.

$$P_h = \{q_h,\ q_h|_T \in \mathbb{P}_1,\ \forall T \in \mathcal{T}_h,\ q_h(dof_0) = 0\}$$

The pressure is discontinuous $\mathbb{P}_1$ piecewise. The spaces U_h and P_h satisfy Assumption 8 (see [9], Chap. II, Lemma 2.6) and the assumptions 6 and 7 are satisfied for $l \leq 2$ (see [9], Chap I Lemma A.5, Chap. II Lemma 2.4., and Proposition 5 of the present paper).
We use for the thermal equations a scheme of $\mathbb{P}_2$ Lagrange - $\mathbb{P}_1$ Lagrange type:

$$W_h = \{t_h \in C^0(\bar{\Omega}),\ t_h|_T \in \mathbb{P}_2,\ \forall T \in \mathcal{T}_h,\ t_h = 0 \text{ on } \Gamma_d\} \tag{4.144}$$

$$R_h = \{z_h \in C^0(\partial\bar{\Omega}),\ z_h|_{\partial T} \in \mathbb{P}_1,\ \forall \partial T \in \partial\Omega,\ T \in \mathcal{T}_h\} \tag{4.145}$$

The radiosity is continuous $\mathbb{P}_1$ piecewise. As the previous case, using the interpolation operator of Lagrange (of degree 2) and the L^2-local orthogonal projection operator, the assumption 9 and 10 are satisfied for $m \leq 2$.
The degrees of freedom of the unknowns are indicated in Fig.2. It follows from Theorem 4 that this method is a second order method:

$$||u - u_h||_{1,\Omega} + ||p - p_h||_{0,\Omega} + ||T - T_h||_{1,\Omega} + ||w - w_h||_{0,\partial\Omega} \; = \; O(h^{min(\alpha,2)})$$

The Hood-Taylor element / $\mathbb{P}_2$ Lagrange - $\mathbb{P}_1$ Lagrange. We use the Hood-Taylor element in order to approximate the Navier-Stokes equations:

$$\mathbf{U_h} = \{v_h \in (C^0(\bar{\Omega}))^2,\ v_h|_T \in (\mathbb{P}_2)^2,\ \forall T \in \mathcal{T}_h,\ v_h = 0 \text{ on } \partial\Omega\}$$

$$P_h = \{q_h \in C^0(\bar{\Omega}),\ q_h|_T \in \mathbb{P}_1,\ \forall T \in \mathcal{T}_h,\ q_h(dof_0) = 0\}$$

For the thermal equations, the spaces W_h and R_h are defined respectively by (4.144) and (4.145). All the approximated quantities are continuous.
The spaces U_h and P_h satisfy Assumption 8 (see [9], Chap. II, Corollary 4.1) and using the interpolation operator of Lagrange $\mathbb{P}_1$ and $\mathbb{P}_2$, the assumptions 6, 7, 9 and 10 are satisfied for l and $m \leq 2$ (see [23], Theorem 4.1). The degrees of freedom of the unknowns are indicated in Fig.3 and Theorem 4 gives:

$$||u - u_h||_{1,\Omega} + ||p - p_h||_{0,\Omega} + ||T - T_h||_{1,\Omega} + ||w - w_h||_{0,\partial\Omega} \; = \; O(h^{min(\alpha,2)})$$

This method is a second order method.

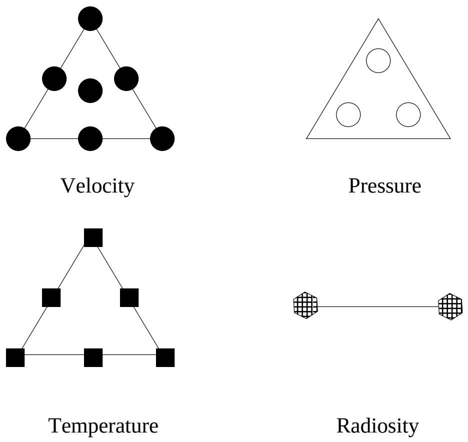

Figure 2: The Crouzeix-Raviart element / $\mathbb{P}_2$ Lagrange - $\mathbb{P}_1$ Lagrange.

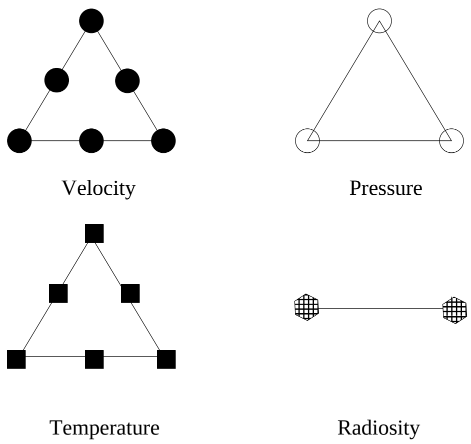

Figure 3: The Hood-Taylor element / $\mathbb{P}_2$ Lagrange - $\mathbb{P}_1$ Lagrange.

The ($I\!\!P_1$ iso $I\!\!P_2$)-$I\!\!P_1$ element / $I\!\!P_1$ Lagrange - $I\!\!P_0$ piecewise. The last finite element method we study is based on a classical variant of the Hood-Taylor element: the ($I\!\!P_1$ iso $I\!\!P_2$)-$I\!\!P_1$ element. The pressure is continuous $I\!\!P_1$ piecewise, while the velocity components are $I\!\!P_1$ piecewise over the four sub-triangles defined as indicated in Fig.4. The spaces $\mathbf{U_h}$ and P_h are defined as follows:

$$\mathbf{U_h} = \{v_h \in (C^0(\bar{\Omega}))^2, \ v_h|_{\tilde{T}} \in (I\!\!P_1)^2, \ \forall \tilde{T} \in \mathcal{T}_{\frac{h}{2}}, \ v_h = 0 \text{ on } \partial\Omega\}$$

where $(\mathcal{T}_{\frac{h}{2}})$ denotes the triangulation obtained from $(\mathcal{T}_h)$ by dividing each triangle into four sub-triangles as in Fig.4.

$$P_h = \{q_h \in C^0(\bar{\Omega}), \ q_h|_T \in I\!\!P_1, \ \forall T \in \mathcal{T}_h, \ q_h(dof_0) = 0\}$$

The spaces U_h and P_h satisfy Assumption 8 (see [9], Chap. II, Lemma 4.2) and using the interpolation operator of Lagrange $I\!\!P_1$, the assumptions 6 and 7 are satisfied for $l \leq 1$. The spaces W_h and R_h are defined respectively by (4.142) and (4.143) and the assumptions 9 and 10 are satisfied for $m \leq 1$.
The degrees of freedom and the sub-triangulation are indicated in Fig.4. Theorem 4 gives:

$$||u - u_h||_{1,\Omega} + ||p - p_h||_{0,\Omega} + ||T - T_h||_{1,\Omega} + ||w - w_h||_{0,\partial\Omega} \ = \ O(h^{min(\alpha,1)})$$

and this method is a first order method.

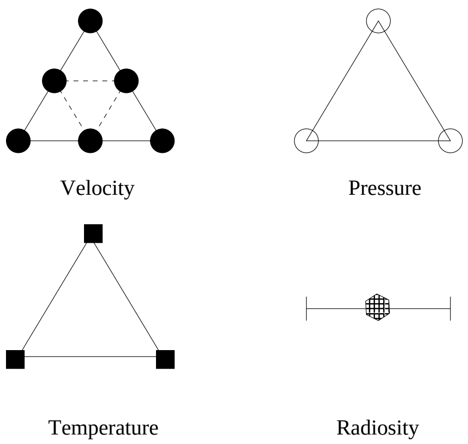

Figure 4: The ($I\!\!P_1$ iso $I\!\!P_2$)-$I\!\!P_1$ element / $I\!\!P_2$ Lagrange - $I\!\!P_1$ Lagrange.

Appendix

In this appendix, we consider the bidimensional case ($\Omega \subset I\!\!R^2$) and we prove an estimate of a projection error in finite element spaces containing bubble functions. We denote by k an integer, $k = 1$ or 2, and we define:

$$\begin{aligned} U_h^k \ &= \ \{v_h, \ v_h \in C^0(\bar{\Omega}), \ v_h|_T = \varphi_h + \mu \ \lambda_1 \ \lambda_2 \ \lambda_3, \ \forall T \in \mathcal{T}_h, \ \varphi_h \in I\!\!P_k, \\ &\quad \ \mu \text{ a constant of } I\!\!R, \ v_h = 0 \text{ on } \partial\Omega\} \end{aligned}$$

Let us recall that U_h^k is a subspace of $H_0^1(\Omega)$ and equipped with the semi-norm of $H^1(\Omega)$, it is a Hilbert space. The degrees of freedom of a function belonging to U_h^k

are indicated in Fig.1 if $k = 1$ and in Fig.2 if $k = 2$.
We define P_h the orthogonal projection operator from $H_0^1(\Omega)$ into U_h^k as follows:

$$\int_\Omega \nabla(v - P_h v)\nabla v_h dx = 0 \qquad \forall v_h \in U_h^k \tag{4.146}$$

We denote by T any triangle of $\mathcal{T}_h$, by $(a_{i,T})$, $i = 1,2,3$, its vertices, by $(a_{i,T})$, $i = 4,5,6$, the midpoint of its sides and by $a_{0,T}$ its center of gravity. We define r_T^k, $k = 1$ or 2, the local interpolation operators as follows. For $k = 1$,

$$r_T^1(v) \in \mathbb{P}_1 \oplus span\{\lambda_1\lambda_2\lambda_3\}$$

$$r_T^1(v(a_{i,T})) = v(a_{i,T}) \quad i = 1,2,3$$

$$r_T^1(v(a_{0,T})) = v(a_{0,T})$$

The function v belongs to $\mathcal{C}^0(\bar{T})$. For $k = 2$,

$$r_T^2(v) \in \mathbb{P}_2 \oplus span\{\lambda_1\lambda_2\lambda_3\}$$

$$r_T^2(v(a_{i,T})) = v(a_{i,T}) \quad i = 1,2,3$$

$$r_T^2(v(a_{i,T})) = v(a_{i,T}) \quad i = 4,5,6$$

$$r_T^2(v(a_{0,T})) = v(a_{0,T})$$

Then, we define the interpolation operator r_h^k by:

$$\forall v \in \mathcal{C}^0(\bar{\Omega}), \quad r_h^k v|_T \ = \ r_T^k v \quad \forall T \in \mathcal{T}_h$$

We state an estimate of $|v - P_h v|_{1,\Omega}$ for $v \in H^s(\Omega)$, $s > 0$. The result is proved in [9], Chap. II Lemma 2.4, when $s \in \mathbb{N}$. Using an inequality of interpolation in fractional order Sobolev spaces due to J.L. Lions and J. Peetre [12], we extend this result to the case $s \in \mathbb{R}^{*+}$.

Proposition 5 *Let k be an integer, $k = 1$ or 2, and σ a real, $0 < \sigma \leq 1$, then the orthogonal projection operator P_h defined by (4.146) satisfies:*

$$|v - P_h v|_{1,\Omega} \ \leq \ c \ h^{k-1+\sigma} ||v||_{k+\sigma,\Omega} \qquad \forall v \in H^{k+\sigma}(\Omega) \tag{4.147}$$

with c is a constant independent of h and $|.|_{1,\Omega}$ denotes the semi-norm of $H^1(\Omega)$.

Proof. Let us recall the inequality of interpolation of [12]. There exists a positive constant c such that for all $t > 0$, and for all v in $H^{k+\sigma}(\Omega)$, $0 < \sigma \leq 1$, there exists $v_1 \in H^k(\Omega)$ and $v_2 \in H^{k+1}(\Omega)$ such that:

$$v \ = \ v_1 \ + \ v_2$$

$$||v_1||_{k,\Omega} \ \leq \ c \ t^\sigma \ ||v||_{k+\sigma,\Omega} \tag{4.148}$$

$$||v_2||_{k+1,\Omega} \ \leq \ c \ t^{\sigma-1} \ ||v||_{k+\sigma,\Omega} \tag{4.149}$$

We apply the result above. Let $v \in H^{k+\sigma}(\Omega)$, there exists $v_1 \in H^k(\Omega)$ and $v_2 \in H^{k+1}(\Omega)$ such that $v = v_1 + v_2$. Hence,

$$|v - P_h v|_{1,\Omega} \ \leq \ |v_1 - P_h v_1|_{1,\Omega} \ + \ |v_2 - P_h v_2|_{1,\Omega}$$

We bound above the two terms of the right part. By this end, we consider separately the cases $k=1$ and $k=2$.
Case $k=1$. We have

$$|v_1 - P_h v_1|_{1,\Omega} \leq |v_1|_{1,\Omega}$$

and

$$|v_2 - P_h v_2|_{1,\Omega} \leq |v_2 - r_h^1 v_2|_{1,\Omega}$$

Since $v_2 \in H^2(\Omega)$, we have (see [9], Chap II, Lemma 2.4):

$$|v_2 - r_h^1 v_2|_{1,\Omega} \leq c\ h\ \|v_2\|_{2,\Omega}$$

Case $k=2$. We have $v_1 \in H^2(\Omega)$ and $v_2 \in H^3(\Omega)$. It follows from [9], Chap II, Lemma 2.4, that:

$$\begin{aligned} |v_1 - P_h v_1|_{1,\Omega} &\leq |v_1 - r_h^2 v_1|_{1,\Omega} \\ &\leq c\ h\ \|v_1\|_{2,\Omega} \end{aligned}$$

and

$$\begin{aligned} |v_2 - P_h v_2|_{1,\Omega} &\leq |v_2 - r_h^2 v_2|_{1,\Omega} \\ &\leq c\ h^2\ \|v_2\|_{3,\Omega} \end{aligned}$$

Therefore, in both case we obtain:

$$|v - P_h v|_{1,\Omega} \leq c\ (h^{k-1}\|v_1\|_{k,\Omega} + h^k\|v_2\|_{k+1,\Omega})$$

Using (4.148) and (4.149), we obtain:

$$|v - P_h v|_{1,\Omega} \leq c\ h^{k-1}(t^\sigma\|v\|_{k+\sigma,\Omega} + t^{\sigma-1}h\|v\|_{k+\sigma,\Omega})$$

We write $t = h^\alpha,\ \alpha \in \mathbb{R}$, and we obtain:

$$|v - P_h v|_{1,\Omega} \leq c\ h^{k-1+\alpha\sigma}(1 + h^{1-\alpha})\|v\|_{k+\sigma,\Omega}$$

The optimal coefficient α is 1, which gives the result. □

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