# Adjoint-Based Inverse Problems for Power-Law Geophysical Free-Surface Flows

**Nathan Martin, Jérôme Monnier**

IMT Toulouse – INSA, 135, avenue de Rangueil, 31400 Toulouse, France

## Context

- **More and more data available** : **radar**, **satellite**, field measurements, ...
  $\Rightarrow$ mostly *Surface Velocities* and *Topography*
- **Control of Non-newtonian flow** : **rheology**, modeling of the **bottom** : *hard to estimate directly*
- **Natural Hazards** : **flood**, **eruption**, **sea-level rise**, ... $\Rightarrow$ improve *predictability* and *risk assessment*

## The Power-Law Stokes Model

- State Equations discretized with Order 2 FEM :

$$\begin{cases} -\mathbf{div}(2\eta(\mathbf{u})\underline{\mathbf{D}}) + \nabla p = \rho \mathbf{g} \ \text{ in } \Omega_t \\ \mathbf{div}(\mathrm{u}) = 0 \ \text{ in } \Omega_t \\ \eta(\mathbf{u}) = \eta_0 \|\underline{\mathbf{D}}\|_F^{\frac{1-n}{n}} \end{cases}$$

- ALE formulation for the time-moving domain $\Omega_t$

## The Adjoint Method

- *Velocity observations* $u^{obs}$ and *input parameters* $\boldsymbol{k}$
- **Cost function** : $j(\boldsymbol{k}) = \int \|Cu^{obs} - u(\boldsymbol{k})\|_2^2 \, d\boldsymbol{x}$
  $+$ *regularization terms*

Measures **misfit of computed state and observations**

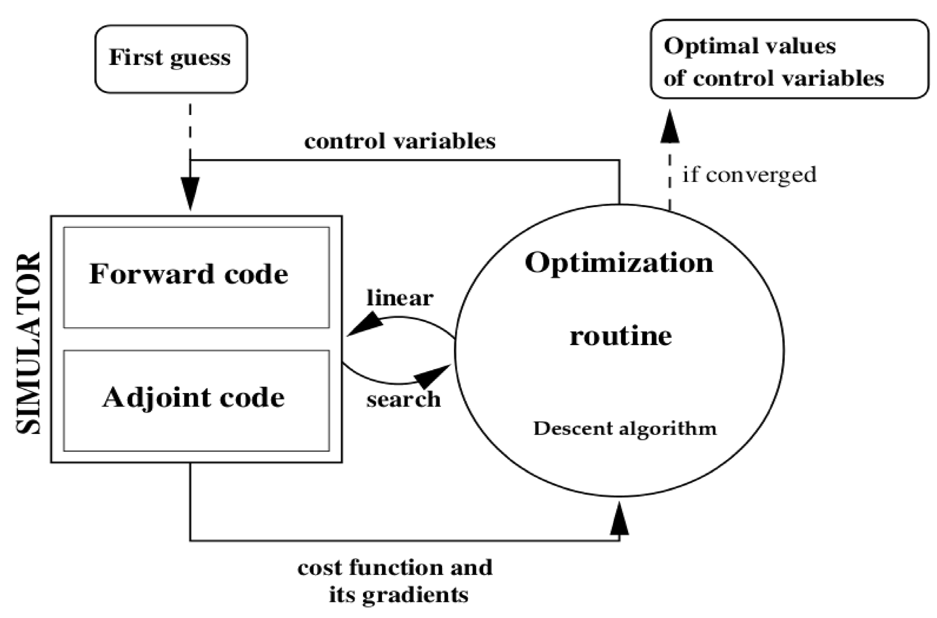
- *Adjoint model* of **Power-Law Stokes problem**
  - obtained using *Algorithmic Differentiation*
  - provides **gradients** $\partial j / \partial \boldsymbol{k}$ for all directions $\delta \boldsymbol{k}$

**It contains all the physics of the direct model + Misfit with the observations**

## Variational Tools

The adjoint model provides two main features :

- **Sensitivity Analysis**
  - Quantify the propagation of perturbations of $\boldsymbol{k}$ through the model
  - Local around a given point $\boldsymbol{k_0}$
  - Better understanding of the physics of the model
- **Data Assimilation**



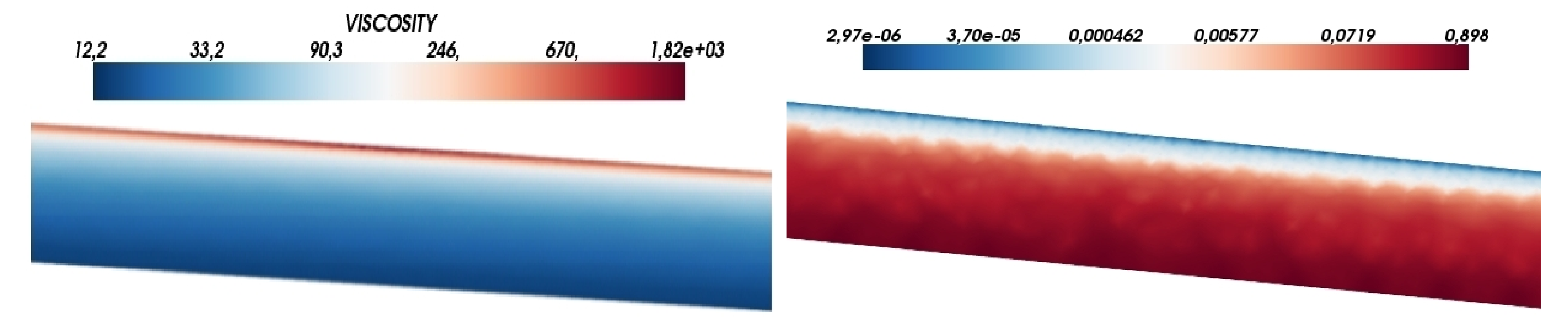

Optimal control problem generally *ill-posed*
$\Rightarrow$ Regularization based on *a priori* physical knowledge

## Reliability of Velocity Surface Observations (geophysical context)

- Typical remote-sensed data in geophysic
- Power-Law description $\Rightarrow$ Infinite viscosity at free surface

*Are such observations reliable to infer rheological parameters and/or friction at bottom using a power-law description ?*

- Sensitivity to a locally defined power-law index $n$ on a uniform stationnary flow :




(a) Computed Viscosity field with $n=3$ (b) Log-Scale Sensitivity $\frac{\partial j}{\partial n}(n_0)$

FIGURE: Resulting viscosity field subject to a power-law description and related sensitivity to the power-law exponent in the case of a uniform stationnary flow. The viscosity field shows an infinite viscosity at surface upper-bounded by the mesh cell size. The sensitivity has been computed around a state $n_0 = 2$ using observations $u^{obs}$ computed with $n=3$. It is plotted using a log-scale and it decreases from bottom to surface . We observe a boundary layer of zero gradient at surface corresponding to the boundary layer of infinite viscosity. The gradient $\partial j / \partial \eta_0$ computed in the same situation shows identical results

*Uniform flow and simple shear-rate distribution* $\Rightarrow$ *Gradient* $\frac{\partial j}{\partial n}$ *vanishing at free surface*

- High correlation between viscosity and sensitivity to the rheological exponent $n$
  $\Rightarrow$ the rheology at high shear-rate is dominant
- Surface velocities **insensitive to an eventual regularization** of the viscosity at surface (*e.g. cut-off on the shear rate* or *yield stress*)

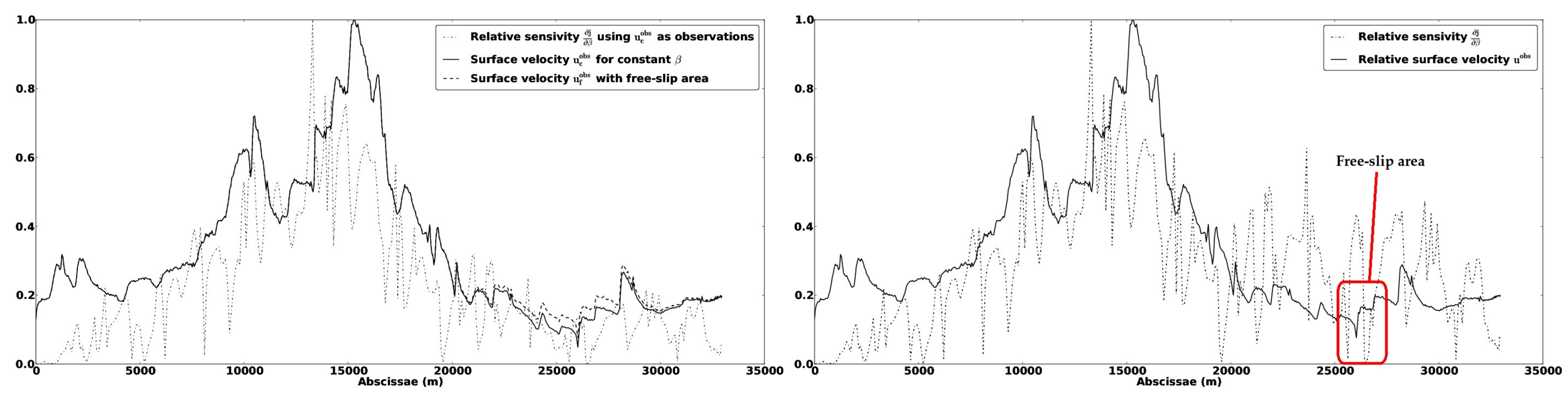
**Confirm that surface velocities are relevant observations to characterize the flow in term of rheology**

*N.B.* The context of a quasi-uniform stationnary flow is *typical in laboratory experiment*. It shows a *simple sensitivity* field and has a *reproductible behaviour*.

## Sensitivity to the friction coefficient $\beta$ and free-slip area (glaciological context)

- *Unmeasurable quantity* representing a *major uncertainty* in glaciological modeling
- Represents *many physical processes* (roughness of the bedrock, pressure of subglacial lake, etc.)

*Sensitivity analysis provides a better understanding of the underlying physics of the friction modeling*




(a) At bottom : constant intermediate friction everywhere (b) At bottom : free-slip area between $x=25$km and $x=27$km and constant intermediate friction elsewhere

FIGURE: Sensitivity $\partial j / \partial \beta$ and surface velocity observations. Left : both surface velocity (with and without the local free-slip) are plotted to point out the minor difference between them. Right : In the presence of a local free-slip zone, a higher sensitivity area appears. The gradient has been computed at point $\beta_0 = 10\beta$ in both plots.

- *Sensitivity follows the variations of surface velocity* (natural since high velocity $\rightarrow$ high misfit $\rightarrow$ high gradient)
  $\Rightarrow$ Surface velocities well-suited as observations to infer the friction coefficient
- *Free-slip zone almost invisible in term of velocity surface (since local) but clearly highlighted in term of sensitivity*

**The gradients of the fluid model help to see beyond the filtering and the non-local behaviour of the transmission of the basal movement to the surface**

## The regularization term : a crucial component of the cost function

The *definition of the regularization* term maybe a *great difficulty in data assimilation*

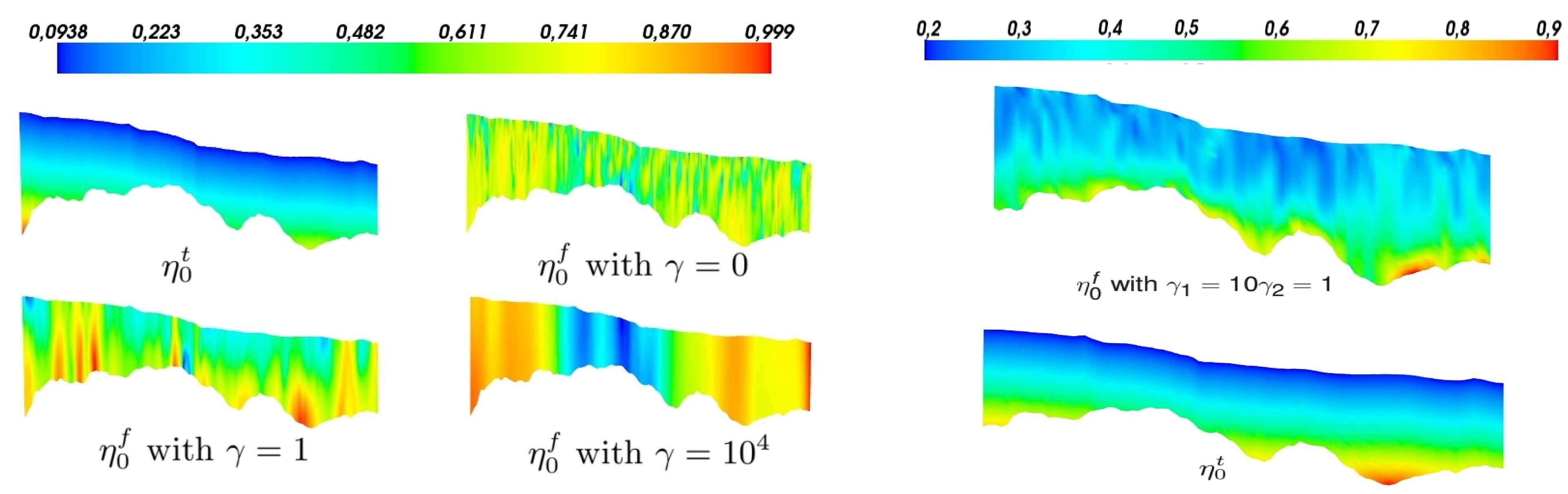
- Cost function : $j(\eta_0) = \int_{\Gamma_s} \|u_s^{obs} - u_s(\eta_0)\|_2^2 \, d\boldsymbol{x}$
  $+\gamma_1 \int_{\Gamma_s} \|\partial_x \eta_0\|_2^2 \, ds + \gamma_2 \int_{\Gamma_s} \|\partial_z \eta_0\|_2^2 \, ds$

**The choice of $\gamma_1$ and $\gamma_2$ is crucial** since Ill-posed problem $\Leftrightarrow$ equifinality on the system

$\Downarrow$

*The trust region on the identified parameter is strongly defined by the regularization term*

Every user should lean on the sensitivity analysis to understand and improve the contents of the cost function to infer the expected "true state".




(a) Target consistency field and Identification Results for different values of $\gamma_1 = \gamma_2 = \gamma$ (b) Target consistency and Identification Results with $\gamma_1 \neq \gamma_2$

FIGURE: The left plot demonstrates the impossibility to identify the target consistency when chosing an isotropic regularization. The target has been computed from a thermal law using a stationnary linear temperature profile leading to a layered consistency $\eta_0$ decreasing from bottom to surface. This target does not behave isotropically in term of spatial gradient. Yet, the cost has reach a close-to-zero value. The equifinality of the system does not lead to the expected solution. As we can see on the right, a regularization that consider independently the horizontal and the vertical gradient leads to a solution more consistent with the expected one.

**References :** ***Four-field finite element solver for viscoplastic free-surface flows and variational sensitivity analysis*** **- Submitted**
***Some inverse problems for geophysical flows addressed using variational methods*** **- Submitted**

**http://www-gmm.insa-toulouse.fr/ monnier/dassflow – {nathan.martin, jerome.monnier}@insa-toulouse.fr**