Geometric Galois module structure and abelian varieties of higher dimension

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Classical Galois module structure

K a number field

L/K a finite Galois extension of K, with group Γ

Normal Basis Theorem: There exists $\alpha \in L$ such that the set $\{\sigma(\alpha)\}_{\sigma \in \Gamma}$ of Galois conjugates of α is a K-basis of L. That is, L is a free rank one $K\Gamma$-module.

$\mathcal{O}_K$ and $\mathcal{O}_L$ the rings of integers of K and L.

Question: What is the structure of $\mathcal{O}_L$ as $\mathcal{O}_K\Gamma$-module ?

Nœther's criterion: $\mathcal{O}_L$ is a locally free $\mathcal{O}_K\Gamma$-module iff L/K is tamely ramified.

The tame case

$\mathrm{Cl}(\mathbb{Z}\Gamma)$ the class group of locally free $\mathbb{Z}\Gamma$-modules
If M is a loc. free $\mathbb{Z}\Gamma$-module, let (M) be its class in $\mathrm{Cl}(\mathbb{Z}\Gamma)$

A. Fröhlich's conjecture, proved by M. J. Taylor:

Theorem (M. J. Taylor, 1981): Suppose L/K is tame, then $2(\mathcal{O}_L) = 0$. Moreover, if the Artin constants of irreducible and symplectic characters of Γ are equal to 1, then $(\mathcal{O}_L) = 0$.

- What can be said about *relative* Galois structure (structure of $\mathcal{O}_L$ as $\mathcal{O}_K\Gamma$-module) ?

- What happens in the case of *wild* extensions ?

Geometric approach of the wild relative case

E an elliptic curve defined over K

$n > 0$ an integer

$P \in E(K)$ a K-rational point of E

Then one defines

$$K(\tfrac{1}{n}P) := \text{the field generated by the coordinates of points } Q \in E(\overline{K}) \text{ such that } nQ = P$$

Suppose the $\overline{K}$-points of $E[n]$ are K-rational. Then, by Kummer theory, $K(\frac{1}{n}P)$ is a Galois extension of K, with group Γ isomorphic to a subgroup of $(\mathbb{Z}/n\mathbb{Z})^2$.

In general, $K(\frac{1}{n}P)/K$ is wild.

M. J. Taylor's class-invariant homomorphism

$\mathcal{E} \to S := Spec(\mathcal{O}_K)$ the Néron model of E

Suppose E has everywhere good reduction. Then $\mathcal{E}$ is an S-abelian scheme and $\mathcal{E}[n]$ is a finite flat S-group scheme.

Martin Taylor introduced in 1988 a homomorphism

$$\psi_n : E(K) = \mathcal{E}(S) \xrightarrow{\partial} H^1(S, \mathcal{E}[n]) \xrightarrow{\pi} \mathrm{Pic}(\mathcal{E}[n]^D)$$

called the *class-invariant homomorphism.*

The map ∂ is the coboundary map of the Kummer exact sequence on $\mathcal{E}$ (for the fppf topology).

The map π has been defined by Waterhouse and measures the Galois structure of $\mathcal{E}[n]$-torsors.

Taylor's conjecture

The following conjecture was made by M. J. Taylor in 1988

Conjecture: Torsion points belong to the kernel of ψ_n.

This conjecture is now a theorem (Srivastav and Taylor, Agboola, Pappas) under the hypothesis that n is coprime to 6.

There are examples where ψ_2 is non-zero on 2-torsion points (Cassou-Noguès and Jehanne).

- What happens if E has bad reduction ?

- What happens if E is replaced by an abelian variety of dimension bigger than 1 ?

Case of semi-stable reduction

$\mathcal{E}[n]$ is not finite in general

Consider a finite flat subgroup G of $\mathcal{E}$

$\mathcal{F}$ the Néron model of $F := E/G_K$

The quotient $\mathcal{E}/G$ is an open subgroup $\mathcal{F}^\Gamma$ of $\mathcal{F}$

So one gets a homomorphism

$$\psi : \mathcal{F}^\Gamma(S) \xrightarrow{\ \partial\ } H^1(S, G) \xrightarrow{\ \pi\ } \mathrm{Pic}(G^D)$$

Theorem (G., 2004): Suppose the order of G is coprime to 6. Then torsion points in $\mathcal{F}^\circ(S)$ belong to the kernel of ψ.

Work in progress: Using logarithmic flat cohomology, one can extend the coboundary map ∂ to the whole group $\mathcal{F}(S)$.

Abelian varieties of higher dimension

What happens to Taylor's conjecture if we replace the elliptic curve E considered above by an abelian variety ?

Function field case:

Theorem (Pappas, 1998): Let r and ℓ be two distinct primes. There exists a smooth affine curve C over a finite field of characteristic r, and a C-abelian scheme A of (relative) dimension 2, with a point P of order ℓ such that $\psi_\ell(P) \neq 0$.

No similar result is known if we replace C by the spectrum of the ring of integers of a number field.

Counterexamples over a number field

Theorem (G., 2006): Let

1. $p \geq 3$ a prime number,
2. K a quadratic imaginary field such that $\mathrm{Cl}(\mathcal{O}_K)[p] \neq 0$,
3. H the Hilbert class field of K,
4. E a K-elliptic curve, having good reduction above p, such that $E(H)$ is a finite group, and $E(K)[p] \neq 0$.

Then there exists a K-abelian variety A of dimension $[H : K]$, whose Néron model contains $\mu_{p,S}$ as finite subgroup and such that the associated class-invariant homomorphism is non-zero on some p-torsion point.

Sketch of proof

The point of order p in E generates a subgroup $(\mathbb{Z}/p\mathbb{Z})_K \subseteq E$

Let $F := E/(\mathbb{Z}/p\mathbb{Z})_K$ and $\mathcal{F}$ the Néron model of F

By duality of elliptic curves, $\mu_{p,K}$ is a subgroup of F. As F has good reduction above p, one proves that $\mu_{p,S}$ is a subgroup of $\mathcal{F}$.

Let $A := Res_{H/K}(F_H)$, with Néron model $\mathcal{A} := Res_{T/S}(\mathcal{F}_{/T})$

(where $T = Spec(\mathcal{O}_H)$ and $\mathcal{F}_{/T}$ is the Néron model of F over T).

We have immersions

$$\mu_{p,S} \longrightarrow \mathcal{F} \longrightarrow Res_{T/S}(\mathcal{F}_T) \longrightarrow Res_{T/S}(\mathcal{F}_{/T}) = \mathcal{A}$$

so $\mu_{p,S}$ is a finite flat subgroup of $\mathcal{A}$

Now, consider the exact sequence

$$0 \longrightarrow \mu_{p,S} \longrightarrow \mathcal{A} \longrightarrow B \longrightarrow 0$$

It gives rise to an exact sequence of cohomology

$$B(S) \xrightarrow{\ \partial\ } H^1(S, \mu_{p,S}) \xrightarrow{\ q\ } H^1(S, \mathcal{A})$$

and on the other hand,

$$H^1(S, \mathcal{A}) = H^1(S, Res_{T/S}(\mathcal{F}_{/T})) = H^1(T, \mathcal{F}_{/T})$$

so the map q is equal to the composite of the maps

$$H^1(S, \mu_{p,S}) \xrightarrow{\ b\ } H^1(T, \mu_{p,T}) \longrightarrow H^1(T, \mathcal{F}_{/T})$$

and one checks that the base change map b is zero (this is due to the fact that the base change map $\mathrm{Cl}(\mathcal{O}_K) \to \mathrm{Cl}(\mathcal{O}_H)$ is zero).

So $q = 0$, which means that ∂ is surjective.

Moreover, the Galois structure morphism for $\mu_{p,S}$ is roughly given by the (surjective) map

$$H^1(S, \mu_{p,S}) \longrightarrow H^1(S, \mathbf{G}_{\mathrm{m}})[p] = \mathrm{Cl}(\mathcal{O}_K)[p]$$

so in this special case, the class-invariant homomorphism ψ is surjective.

Finally we remark that

$$\mathrm{rank}(B(S)) = \mathrm{rank}(\mathcal{A}(S)) = \mathrm{rank}(F(H)) = \mathrm{rank}(E(H)) = 0$$

so $B(S)$ is torsion, which concludes the proof.