From Picard groups of hyperelliptic curves to class groups of quadratic fields

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Elliptic curves over $\mathbb{Q}$

An **elliptic curve** E over a $\mathbb{Q}$ is a non-singular (or smooth) projective curve defined by an equation of the form

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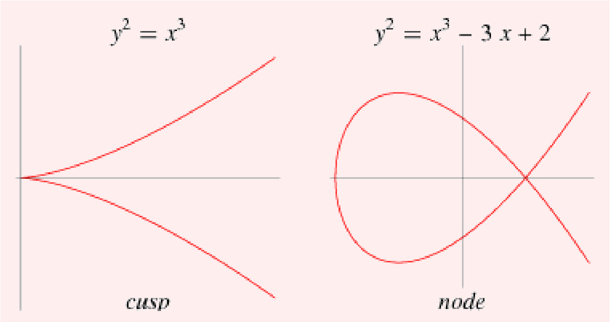
Classical fact:

$$E \text{ is non-singular} \iff x^3 + ax + b \;\text{ has no double root} \\ \iff \Delta \neq 0$$


$y^2 = x^3$
$y^2 = x^3 - 3x + 2$
cusp
node

Why are elliptic curves important?

Short answer: they have a group law, which turns them into an algebraic group

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Detailed answer: there are exactly three types of algebraic groups of dimension one

- $\mathbb{A}^1$ with addition: $\mathbb{G}_a$
- $\mathbb{A}^1 \setminus \{0\}$ with multiplication: $\mathbb{G}_m$
- Elliptic curves

Remark: this classification is over an algebraically closed field. Over an arbitrary field, one has also quadratic twists of $\mathbb{G}_m$

Why are elliptic curves arithmetically important?

Mordell-Weil Theorem (Mordell, 1922): $E(\mathbb{Q})$ is a finitely generated abelian group:

$$E(\mathbb{Q}) \simeq \mathbb{Z}^r \oplus E(\mathbb{Q})_{\text{tors}}$$

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In particular, $\text{rk}_{\mathbb{Z}} E(\mathbb{Q})$ should be bounded...

Reduction of elliptic curves

By change of variables, we may find a Weierstrass equation for E whose coefficients are integers:

$$y^2 = x^3 + ax + b \qquad \text{with } a, b \in \mathbb{Z}$$

Given a prime number p, we say that E has **good reduction** at p if one can find an integral Weierstrass equation such that

$$\Delta \not\equiv 0 \pmod{p}$$

In other terms, the reduction modulo p of the equation is an elliptic curve over $\mathbb{F}_p$.

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Remark: if $p = 2$ or 3 one has to use a more general Weierstrass equation. This explains the -16 in the definition of Δ.

Basic facts on reduction

Bad primes are the divisors of Δ, so there are finitely many.

Fact: An elliptic curve over $\mathbb{Q}$ always has at least one bad place!

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Related fact: there does not exist everywhere unramified extensions of $\mathbb{Q}$.

Remark: if one replaces $\mathbb{Q}$ by an arbitrary number field, then the situation changes for both objects.

Good reduction points

Given $P \in E(\mathbb{Q})$, say that P has **everywhere good reduction** if, for all primes p, the reduction of P mod p is not a singular point on the reduction of E mod p.

Denote by $E^0(\mathbb{Q})$ the set of points with everywhere good reduction.

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Theorem: $E^0(\mathbb{Q})$ is a subgroup of finite index in $E(\mathbb{Q})$, hence

$$\mathrm{rk}_{\mathbb{Z}} E^0(\mathbb{Q}) = \mathrm{rk}_{\mathbb{Z}} E(\mathbb{Q}).$$

In practive, given a point $P \in E(\mathbb{Q})$, there exists some multiple of P which belongs to $E^0(\mathbb{Q})$.

From elliptic curves to class groups

Mazur-Tate's class group pairing: let K be a number field. Then we have a bilinear map of abelian groups

$$\begin{aligned} E^0(\mathbb{Q}) \times E(K) &\longrightarrow \mathrm{Cl}(K) \\ (P, Q) &\longmapsto \langle P, Q \rangle^{\mathrm{cl}} \end{aligned}$$

Barry Mazur and John Tate, *Canonical height pairings via biextensions*, Arithmetic and geometry, Vol. I, Progr. Math., vol. 35, Birkhäuser Boston, Boston, MA, 1983

Definition of the class group pairing

A point $P \in E(\mathbb{Q})$ gives rise to a degree zero line bundle L_P on E. If P belongs to E^0, then L_P entends (uniquely) into a line bundle $\mathcal{L}_P$ on the minimal regular model $\mathcal{X} \to \mathrm{Spec}(\mathbb{Z})$ of E.

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On the other hand, a point $Q \in E(K)$ gives rise to an integral section $Q : \mathrm{Spec}(\mathcal{O}_K) \to \mathcal{X}$, and we let

$$\langle P, Q \rangle^{\mathrm{cl}} := Q^* \mathcal{L}_P$$

which belongs to $\mathrm{Pic}(\mathrm{Spec}(\mathcal{O}_K)) = \mathrm{Cl}(K)$.

Relation with work of Buell

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This was proved later by Call in his PhD thesis (1986).

More recently, a new proof of this result appeared in

Duncan Buell and Gregory Call, *Class pairings and isogenies on elliptic curves*, J. Number Theory **167** (2016).

Which classes can be built from one given curve?

Given $Q \in E(K)$, the class group pairing induces a group morphism

$$\langle -, Q \rangle^{\mathrm{cl}} : E^0(\mathbb{Q}) \longrightarrow \mathrm{Cl}(K)$$

Question: given a field K, is it possible to find a curve E and a point $Q \in E(K)$ such that $\langle -, Q \rangle^{\mathrm{cl}}$ is surjective?

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Remark: if $E^0(\mathbb{Q}) \to \mathrm{Cl}(K)$ is surjective, then

$$\begin{aligned} \mathrm{rk}_{\mathbb{Z}} E(\mathbb{Q}) &\geq \dim_2 \mathrm{Cl}(K)[2] - \dim_2 E^0(\mathbb{Q})_{\mathrm{tors}} \\ &\geq \dim_2 \mathrm{Cl}(K)[2] - 2. \end{aligned}$$

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There exist quadratic fields K for which $\dim_2 \mathrm{Cl}(K)[2]$ is arbitrarily large. So the surjectivity implies the existence of elliptic curves over $\mathbb{Q}$ whose rank is arbitrarily large. Not likely!

Question: is the pairing non-degenerate?

Question: More precisely, given $P \in E^0(\mathbb{Q})$, does there exist some field K and some $Q \in E(K)$ such that

$$\langle P, Q \rangle^{\mathrm{cl}} \neq 0 \text{ ?}$$

This question was asked (in a more general setting) by Agboola and Pappas in 2000.

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This question was asked (in a more general setting) by Agboola and Pappas in 2000.

Theorem (G.–Levin, 2012): Yes if P is a torsion point.
More precisely, one can find infinitely many Q defined over imaginary quadratic fields K such that the map $\langle -, Q \rangle^{\mathrm{cl}}$ is injective on $E^0(\mathbb{Q})_{\mathrm{tors}}$

In fact, our result holds for “imaginary” hyperelliptic curves over $\mathbb{Q}$, *i.e.* curves defined by equations of the form

$$y^2 = f(x)$$

where $f \in \mathbb{Q}[x]$ is a monic square-free polynomial of odd degree.

Ingredients of our proof: Kummer theory and Hilbert’s irreducibility theorem.

For points of infinite order, we need another strategy!

ICCGNFRT 2017

Debopam Chakraborty gave a talk in which he explicitely constructs ideal classes of order 2 over biquadratic fields from points on elliptic curves (joint work with Anupam Saikia).

Then he mentions a paper by Ragnar Soleng according to which one can build from points of infinite order ideal classes whose order is arbitrarily large!

I immediately looked for a copy of Soleng's paper.

Ragnar Soleng, *Homomorphisms from the group of rational points on elliptic curves to class groups of quadratic number fields*, J. Number Theory **46** (1994).

Soleng's result

Without refering to previous constructions, Soleng defines a family of maps $E^0(\mathbb{Q}) \to \mathrm{Cl}(K)$ using the language of quadratic forms.

His construction is the same than Buell's one, so according to Call his maps are $\langle -, Q\rangle^{\mathrm{cl}}$ for some Q.

Theorem (Soleng, 1994): Let E be an elliptic curve over $\mathbb{Q}$, and let $P \in E^0(\mathbb{Q})$ be a point of infinite order. Then there exists a sequence $(Q_n)_{n\in\mathbb{N}}$ of points defined over quadratic imaginary fields such that the order of $\langle P, Q_n\rangle^{\mathrm{cl}}$ is unbounded when $n \to \infty$.

The proof is less than one page long!

Soleng's setting

Consider an integral Weierstrass equation of E

$$y^2 = f(x) \qquad f \in \mathbb{Z}[x], \text{ monic of degree 3.}$$

Any rational point $P \in E(\mathbb{Q}) \setminus \{0\}$ can be written as

$$P = \left(\frac{A}{e^2}, \frac{B}{e^3} \right)$$

with A, B, e in $\mathbb{Z}$ such that

$$\gcd(A, e) = \gcd(B, e) = 1.$$

Remark: If P belongs to $E^0(\mathbb{Q})$, then $\gcd(A, 2B, e) = 1$.

Soleng's definition of the map

Fix $n \in \mathbb{Z}$, and let $Q_n := (n, \sqrt{f(n)})$. If $f(n)$ is not a square, then Q_n is a quadratic point on E.

Soleng's construction

$$P = \left(\frac{A}{e^2}, \frac{B}{e^3}\right) \rightsquigarrow F_n := \left[(ne^2 - A), 2kB, \frac{k^2B^2 - f(n)}{ne^2 - A}\right]$$

where k is some integer such that $ke^3 \equiv 1 \pmod{ne^2 - A}$.

This binary quadratic form F_n has discriminant $4f(n)$.

The condition $\gcd(A, 2B, e) = 1$ implies that F_n is primitive, hence defines a class $\mathrm{cl}(F_n)$ in $\mathrm{Cl}(\mathbb{Z}[\sqrt{f(n)}])$.

Soleng's proof

$\langle P, Q_n\rangle^{\mathrm{cl}}$ is the image of $\mathrm{cl}(F_n)$ by the natural map

$$\mathrm{Cl}(\mathbb{Z}[\sqrt{f(n)}]) \longrightarrow \mathrm{Cl}(\mathbb{Q}(\sqrt{f(n)}))$$

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Assume $f(n)$ is squarefree, and < -3. Then the kernel of this map has order ≤ 3.

Theorem (Hooley, 1967): there exist infinitely many values of n such that $f(n)$ is square-free.

So it suffices to prove the result for the map

$$E^0(\mathbb{Q}) \to \mathrm{Cl}(\mathbb{Z}[\sqrt{f(n)}]);\, P \mapsto \mathrm{cl}(F_n)$$

Soleng's proof, continued

Lemma: When $n \to -\infty$, the form F_n is not equivalent to the identity form $[1, 0, -f(n)]$.

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The proof is based on the following idea: if two binary quadratic forms over $\mathbb{Z}$ have small coefficients in X^2 compared to their (negative) discriminant, then they are not equivalent.

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$$F_n := \left[(ne^2 - A), 2kB, \frac{k^2B^2 - f(n)}{ne^2 - A}\right]$$

When $n \to -\infty$, $ne^2 - A$ (linear in n) is small compared to the discriminant $4f(n)$ (cubic in n). $\square$

The hyperelliptic case

It is tempting to generalize Soleng's proof when replacing f by a monic, square-free polynomial of odd degree.

This means that we are considering the imaginary hyperelliptic curve C defined by

$$y^2 = f(x)$$

The genus of C is $g(C) := (\deg(f) - 1)/2$.

Hooley's result cannot be extended to arbitrary degrees, so we make the following assumption:

Hypothesis: f is the product of polynomials of degree ≤ 3.

Line bundles on hyperelliptic curves

We replace the elliptic curve E by the Jacobian variety J of C, which is an abelian variety over $\mathbb{Q}$.

Points on J are degree zero divisor classes, or line bundles, on C.

We have a subgroup $J^0(\mathbb{Q}) \subset J(\mathbb{Q})$ consisting of points with everywhere good reduction, and a class group pairing

$$J^0(\mathbb{Q}) \times C(K) \longrightarrow \mathrm{Cl}(K)$$

Question: is there an explicit description of $J(\mathbb{Q})$?

Mumford's representation

Every element in $J(\mathbb{Q}) = \mathrm{Pic}^0(C)$ can be uniquely represented by a quadratic form $[u, 2v, w]$ over $\mathbb{Q}[x]$, with discriminant $4f$, where:

(1) u is monic;

(2) $\deg v < \deg u \leq g(C)$.

In this correspondence, the quadratic form $F = [u, 2v, w]$ corresponds to the divisor

$$D_F := \mathrm{div}(u) \cap \mathrm{div}(y - v) = \sum_{i=1}^{r} P_i - r \cdot \infty$$

where $P_i = (x_i, y_i)$, the x_i are the roots of u, and $v(x_i) = y_i$.

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People doing cryptography with hyperelliptic curves over finite fields use this all the time, refeering to Cantor's paper!

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Compare this to Soleng's construction:

$$P \rightsquigarrow F_n := \left[(ne^2 - A), 2kB, \frac{k^2B^2 - f(n)}{ne^2 - A}\right]$$

where k is the inverse of e^3 modulo $(ne^2 - A)$.

Conclusion

By chasing denominators, q.f. over $\mathbb{Q}[x] \rightsquigarrow$ q.f. over $\mathbb{Z}[x]$.

A natural generalization of Soleng's construction is to consider the specialization map

$$\begin{aligned} \{\text{q.f. over } \mathbb{Z}[x] \text{ with disc. } 4f\} &\longrightarrow \{\text{q.f. over } \mathbb{Z} \text{ with disc. } 4f(n)\} \\ [u, 2v, w] &\longmapsto [u(n), 2v(n), w(n)] \end{aligned}$$

This map is just $L_P \mapsto Q_n^* L_P$ (specialisation on line bundles along the section Q_n).

One recovers Mazur-Tate's definition of the class group pairing.

At the same time, one obtains a new proof of the fact that Soleng's and Buell's constructions coincide with the class group pairing.

Classical analogy

algebraic geometry	number theory
curve $C/\mathbb{Q}$	number field $K/\mathbb{Q}$
with $\phi : C \to \mathbb{P}^1$ of degree 2	with $[K : \mathbb{Q}] = 2$
Jacobian $J(C) := \mathrm{Pic}^0(C)$	Class group $\mathrm{Cl}(K)$

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Melanie Wood, 2011: generalisation to arbitrary double covers of schemes! This involves *sheaves* of quadratic forms.

Further directions

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- Replace the hyperelliptic curve C by a **trigonal** curve

$$C \to \mathbb{P}^1 \qquad \text{of degree } 3$$

The work of Bhargava on *higher composition laws* allows to represent divisor classes on C by binary **cubic** forms.

Thank you for your attention!