Massey products and algebraic curves

Jean Gillibert (joint with Frauke Bleher and Ted Chinburg)

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Double Massey products = cup-products

- Γ: a profinite group
- $\mathbb{Z}/\ell$ with ℓ an odd prime number: Γ-module with trivial action
- $H^1(\Gamma, \mathbb{Z}/\ell) = \mathrm{Hom}(\Gamma, \mathbb{Z}/\ell)$: the first cohomology group

The cup-product is the bilinear alternating map defined by

$$H^1(\Gamma, \mathbb{Z}/\ell) \times H^1(\Gamma, \mathbb{Z}/\ell) \to H^2(\Gamma, \mathbb{Z}/\ell)$$
$$(\chi_1, \chi_2) \mapsto ((g, h) \mapsto \chi_1(g)\chi_2(h)).$$

This cup-product plays an important role in Galois cohomology as well as in étale cohomology (Tate duality, Poincaré duality).

Matrix version of cup-products

Let $U_3(\mathbb{Z}/\ell)$ be the group of 3×3 upper triangular unipotent matrices with coefficients in $\mathbb{Z}/\ell$.

Lemma

$\chi_1 \cup \chi_2 = 0$ *iff there exists a map* $\kappa : \Gamma \to \mathbb{Z}/\ell$ *such that*

$$\begin{pmatrix} 1 & \chi_1 & \kappa \\ 0 & 1 & \chi_2 \\ 0 & 0 & 1 \end{pmatrix} : \Gamma \to U_3(\mathbb{Z}/\ell)$$

is a group homomorphism.

Triple Massey products

Let $U_4(\mathbb{Z}/\ell)$ be the group of 4×4 upper triangular unipotent matrices with coefficients in $\mathbb{Z}/\ell$. Its center Z_4 is the set of matrices whose all entries above the diagonal are zero, except the upper right corner.

Let χ_1, χ_2, χ_3 in $H^1(\Gamma, \mathbb{Z}/\ell)$ such that $\chi_1 \cup \chi_2 = 0 = \chi_2 \cup \chi_3$. Then there exist maps $\kappa_{1,2}$ and $\kappa_{2,3} : \Gamma \to \mathbb{Z}/\ell$ such that

$$\bar{\rho} = \begin{pmatrix} 1 & \chi_1 & \kappa_{1,2} & \square \\ 0 & 1 & \chi_2 & \kappa_{2,3} \\ 0 & 0 & 1 & \chi_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} : \Gamma \to U_4(\mathbb{Z}/\ell)/Z_4$$

is a group homomorphism. In this case, one says that **the triple Massey product** $\langle \chi_1, \chi_2, \chi_3 \rangle$ **is not empty.**

Triple Massey products

One says that **the triple Massey product** $\langle\chi_1,\chi_2,\chi_3\rangle$ **contains zero (or vanishes)** if there exist a choice of $\kappa_{1,2}$ and $\kappa_{2,3}$ for which one can complete the upper right corner of $\bar{\rho}$ into a matrix homomorphism $\rho:\Gamma\to U_4(\mathbb{Z}/\ell)$.

Alternatively, the triple Massey product can be seen as the class in $H^2(\Gamma,\mathbb{Z}/\ell)$ of the 2-cocycle ν defined by

$$\nu(g,h):=\chi_1(g)\kappa_{2,3}(h)+\kappa_{1,2}(g)\chi_3(h).$$

Indeed, a lift of $\bar{\rho}$ exists iff ν is a 2-coboundary.

Adding an arbitrary element of $H^1(\Gamma,\mathbb{Z}/\ell)$ to $\kappa_{1,2}$ or $\kappa_{2,3}$ yields another $\bar{\rho}$. So ν is not uniquely determined by χ_1,χ_2,χ_3. The triple Massey product is in fact **the set** of all possible ν.

Vanishing of triple Massey products

- Ekedhal (1983): there exist a smooth projective surface X over $\mathbb{C}$ and a non-vanishing triple Massey product in the étale cohomology of X.
- Mináč and Tân (2016): if $\Gamma = \mathrm{Gal}(\bar{k}/k)$ for some field k, then triple Massey products contain zero whenever they are not empty.
- Wittenberg and Harpaz (2022): if $\Gamma = \mathrm{Gal}(\bar{k}/k)$ for some number field k, then triple and higher Massey products vanish.

Question: can one find non-vanishing triple Massey products for $\Gamma = \pi_1(X)$ where X is a smooth projective geometrically connected curve over some field k?

The case of curves

Let k be a field (of characteristic $\neq \ell$) and X be curve over k. We let $\bar{X} := X \otimes_k \bar{k}$.

Lemma

Triple (and higher) Massey products for $\pi_1(\bar{X})$ vanish whenever they are not empty.

This is immediate: the $H^2(\bar{X}, \mathbb{Z}/\ell)$ is one-dimensional, so up to modifying the $\kappa_{i,j}$ a lift exists.

Consider the famous exact sequence (split by the choice of a point)

$$1 \to \pi_1(\bar{X}) \to \pi_1(X) \to \mathrm{Gal}(\bar{k}/k) \to 1$$

Triple Massey products vanish for the first and the last term. What happens in the middle? The action of $\mathrm{Gal}(\bar{k}/k)$ on $\pi_1(\bar{X})$ by conjugation is the key to the answer.

Elliptic curves

Let $E = X$ be an elliptic curve defined over k.

- If the ℓ-torsion points of E are rational over k and $\ell > 3$, non-empty triple Massey products always vanish.
- If the ℓ-torsion points of E are not rational over k, then one can construct non-vanishing triple Massey products (for all values of $\ell > 3$) over $k = \mathbb{F}_p$ for some large enough p.
- When $\ell = 3$, we give a complete characterisation of when triple Massey products vanish, depending on the Galois action on 9-torsion points of E.

Hyperelliptic curves

Theorem

Let $\ell > 3$ be a prime number, and let $g > 1$ be an integer. Then there exist infinitely many non-isomorphic pairs (F, X) consisting of a number field F and a smooth genus g hyperelliptic curve X over F having the following two properties:

(i) *The ℓ-torsion points of the Jacobian of X are rational over F.*

(ii) *There is a triple Massey product for $\pi_1(X)$ with coefficients in $\mathbb{Z}/\ell$ that does not vanish.*

The proof relies on topological properties of the moduli space of hyperelliptic curves of genus g, due to Mumford.

Thank you for your attention!