Descent on elliptic surfaces and arithmetic bounds for the Mordell-Weil rank

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The Lang-Néron theorem for elliptic curves over $k(t)$

Let k be a field of characteristic $\neq 2$ or 3.

An elliptic curve E over $k(t)$ can be defined by an equation of the form

$$y^2 = x^3 + a(t)x + b(t)$$

where $a(t)$ and $b(t)$ belong to $k[t]$, and $\Delta(t) := 4a(t)^3 + 27b(t)^2 \neq 0$.

We say that E is **constant** if it admits an equation as above where $a(t)$ and $b(t)$ belong to k, *i.e.* are constant polynomials.

Theorem (Lang-Néron)

If E is a non-constant elliptic curve over $k(t)$, then $E(k(t))$ is a finitely generated abelian group.

The geometric rank bound

The rank of E over $k(t)$ is by definition the integer r such that

$$E(k(t)) \simeq E(k(t))_{\text{tors}} \oplus \mathbb{Z}^r.$$

We denote it by $\operatorname{rk} E(k(t))$.

Theorem (Geometric, or Igusa's, rank bound)

$$\operatorname{rk} E(\bar{k}(t)) \leq \deg(\mathfrak{f}_E) - 4$$

where $\mathfrak{f}_E$ denotes the conductor of E, a divisor on $\mathbb{P}^1$ that we shall now describe.

Let $\mathcal{E}$ be the (minimal) smooth projective **surface** over k defined by

$$y^2 = x^3 + a(t)x + b(t).$$

This surface is a fibration $\mathcal{E} \to \mathbb{P}^1$ in elliptic curves, via the t-coordinate map. The fiber of $\mathcal{E}$ at some $t_0 \in \mathbb{P}^1(\bar{k})$ is just the curve over $\bar{k}$ obtained by letting $t = t_0$.

The fiber of $\mathcal{E} \to \mathbb{P}^1$ at some root t_0 of $\Delta(t)$ is singular, so it is not an elliptic curve anymore. There are two cases:

- multiplicative type: $x^3 + a(t_0)x + b(t_0)$ has a double root;
- additive type: $x^3 + a(t_0)x + b(t_0)$ has a triple root.

The **conductor** of E is the divisor $\mathfrak{f}_E \subset \mathbb{P}^1$ defined by

$$\mathfrak{f}_E := \sum_{\text{multiplicative } t_0} t_0 + \sum_{\text{additive } t_0} 2 \cdot t_0$$

Geometric rank bound: a toy example

Let $\beta \in \mathbb{Q}^*$ and let E be the elliptic curve over $\mathbb{Q}(t)$ defined by

$$y^2 = x^3 + t^2(t^2 + \beta).$$

This curve has additive fibers at $t = 0$, at $t^2 + \beta = 0$, and at $t = \infty$.

Its conductor is

$$\mathfrak{f}_E = 2 \cdot (\{0\} + \{t^2 + \beta = 0\} + \{\infty\})$$

which has degree 8.

The geometric rank bound yields

$$\operatorname{rk} E(\bar{\mathbb{Q}}(t)) \leq \deg(\mathfrak{f}_E) - 4 = 4.$$

A question by Ulmer

In 2004, Ulmer asks the following question:

Does there exists a refinement of the geometric rank bound for elliptic curves over $k(t)$, where k is a non-algebraically closed field?

In our toy example, can we find a better bound for rk $E(\mathbb{Q}(t))$, depending on the *arithmetic* of the curve E?

Under mild hypotheses, we shall give a (quite elementary) answer to this question, by adapting to the function field case classical 2-descent arguments, which have been extensively used for bounding the rank of elliptic curves over number fields.

Arithmetic rank bound via 2-descent

The 2-torsion multisection $\mathcal{E}[2]$ of the elliptic surface $\mathcal{E}$ with equation $y^2 = x^3 + a(t)x + b(t)$ can be described as

$$\mathcal{E}[2] = \mathbb{P}^1 \cup C$$

where $\mathbb{P}^1$ is the zero section of $\mathcal{E}$, and C is the smooth projective curve over k defined by the affine equation

$$x^3 + a(t)x + b(t) = 0.$$

Theorem (Levin-G, 2018)

Assume that $E(\bar{k}(t))[2] = 0$. Then C is geometrically integral and

$$\begin{aligned} \operatorname{rk} E(k(t)) \leq & \dim_{\mathbb{F}_2} \operatorname{Pic}(C)[2] + \#\{v \in \mathbb{P}^1, 2 \mid c_v\} \\ & + \#\{v \in \mathbb{P}^1, \textit{the fiber type of } \mathcal{E} \textit{ at } v \textit{ is } \mathrm{I}_{2n}^* \textit{ for some } n \geq 0\}, \end{aligned}$$

where c_v denotes the Tamagawa number at some bad place v, i.e. the number of irreducible components of multiplicity 1 in the fiber of $\mathcal{E}$ at v.

Let $g(C)$ be the genus of C; then $\dim_{\mathbb{F}_2} \operatorname{Pic}(C \otimes \bar{k})[2] = 2g(C)$, hence over $\bar{k}$ our bound is the following

$$\operatorname{rk} E(\bar{k}(S)) \leq 2g(C) + \#\{v \in \mathbb{P}^1(\bar{k}), 2 \mid c_v\} \\ + \#\{v \in \mathbb{P}^1(\bar{k}), \text{the fiber type of } \mathcal{E} \text{ at } v \text{ is } \mathrm{I}_{2n}^* \text{ for some } n \geq 0\}.$$

The ramification points of $t : C \to \mathbb{P}^1$ are in the support of $\mathfrak{f}_E$.

The ramification type at v depends on the fiber type at v, and on the parity of the Tamagawa number c_v.

When computing $g(C)$ by the Riemann-Hurwitz formula, we find that the quantity above is equal to

$$\deg(\mathfrak{f}_E) - 4$$

hence we recover the geometric rank bound over $\bar{k}$.

Toy example: arithmetic rank bound

Let $\beta \in \mathbb{Q}^*$ and let E be the elliptic curve over $\mathbb{Q}(t)$ defined by

$$y^2 = x^3 + t^2(t^2 + \beta)$$

The singular fibers of this curve are additive of the following type

	$t = 0$	$t^2 + \beta = 0$	$t = \infty$
fiber type	IV	II	IV
Tamagawa number	3	1	3

The Tamagawa numbers being odd, none of the bad fibers contributes to our bound; therefore

$$\operatorname{rk} E(\mathbb{Q}(t)) \leq \dim_{\mathbb{F}_2} \operatorname{Pic}(C)[2].$$

We shall now compute the right-hand side.

One checks that the curve C defined by the equation

$$x^3 + t^2(t^2 + \beta) = 0$$

is a hyperelliptic curve of genus 2, with equation

$$Y^2 = X^6 - 4\beta.$$

Let s be the number of irreducible factors of $X^6 - 4\beta$ over $\mathbb{Q}$; then

$$\dim_{\mathbb{F}_2} \mathrm{Pic}(C)[2] \leq \begin{cases} s-1 & \text{if all factors of } X^6 - 4\beta \text{ have even degree} \\ s-2 & \text{otherwise} \end{cases}$$

Here are some examples:

β	1	2	16
factorization type of $X^6 - 4\beta$	$[3,3]$	$[2,4]$	$[1,1,2,2]$
rank bound over $\mathbb{Q}(t)$	0	1	2

Sketch of proof

The arguments mimic classical 2-descent: étale Kummer theory over the Néron (group scheme) model $\mathcal{E}$ yields an injective map

$$\mathcal{E}^{2\Phi}(\mathbb{P}^1)/2\mathcal{E}(\mathbb{P}^1) \hookrightarrow H^1(\mathbb{P}^1, \mathcal{E}[2])$$

where $\mathcal{E}^{2\Phi}(\mathbb{P}^1)/2\mathcal{E}(\mathbb{P}^1)$ is a subgroup of $E(k(t))/2E(k(t))$ of finite index; the $\mathbb{F}_2$-dimension of the cokernel is bounded above by

$$\#\{v \in \mathbb{P}^1, 2 \mid c_v\} + \#\{v \in \mathbb{P}^1, \text{the fiber type of } \mathcal{E} \text{ at } v \text{ is } \mathrm{I}_{2n}^* \text{ for some } n \geq 0\}.$$

On the other hand, using the fact that C is geometrically integral, one proves that $H^1(\mathbb{P}^1, \mathcal{E}[2])$, which is a kind of geometric Selmer group, is a subgroup of $\mathrm{Pic}(C)[2]$.

Another family of arithmetic bounds

Let $p \geq 3$ be a prime number, $p \neq \mathrm{char}(k)$. By performing p-descent we obtain the following:

Theorem

Assume that the action of $\mathrm{Gal}(\overline{k(t)}/\bar{k}(t))$ *on* $E[p]$ *is irreducible, and let* C_p *be the complement of the zero section in* $\mathcal{E}[p]$*. Then:*

$$\mathrm{rk}\, E(k(t)) \leq \dim_{\mathbb{F}_p} \mathrm{Pic}(C_p)[p] + \#\{v \in \mathbb{P}^1, p \mid c_v\}.$$

If k is algebraically closed, all these bounds are weaker than the geometric rank bound. But in general, it may (and does) happen that one of these bounds improves on the geometric one.

We also obtain similar statements for elliptic curves over general function fields, i.e. we can replace $\mathbb{P}^1$ by any smooth projective geometrically integral curve.

A conjecture by Silverman

In 2004, Silverman conjectured that, if k is a number field, there exists a constant c_k such that, for non-isotrivial elliptic curves $E/k(t)$,

$$\operatorname{rk} E(k(t)) \leq c_k \frac{\deg(\mathfrak{f}_E)}{\log \deg(\mathfrak{f}_E)}.$$

In fact, Brumer proved in 1992 that this holds when k is a finite field.

We now ask the following: does there exist a constant $c_{p,d,k}$ such that, for curves C/k of genus g admitting a morphism of degree d to $\mathbb{P}^1$,

$$\dim_{\mathbb{F}_p} \operatorname{Pic}(C)[p] \leq c_{p,d,k} \frac{g}{\log g}.$$

If this question has a positive answer for one prime p, then, provided one can control the Tamagawa numbers, one could prove Silverman's conjecture for a large family of elliptic curves.

Thank you for your attention!