On the Chevalley-Bass number of a field

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Sums of units

In his study of exponential Diophantine equations (Mordell-Lang conjecture for algebraic tori), Michel Laurent proved the following:

Proposition

Let K be a finitely generated field of characteristic 0, and let

$$\Gamma := \{x \in \overline{K}^\times \mid x^n \in K^\times \text{ for some } n > 0\}.$$

Let $\alpha_1, \dots, \alpha_s$ be elements of Γ such that $\alpha_1 + \cdots + \alpha_s = 1$ and no proper subsum of $\alpha_1 + \cdots + \alpha_s$ vanishes. Then there exist an integer b, depending only on K, and roots of unity $\xi_1, \dots, \xi_s$ in $\overline{K}^\times$ such that $\alpha_i^b \xi_i$ belongs to K.

The constant b is an integer which kills the Galois cohomology groups

$$H^1(\mathrm{Gal}(K(\zeta_n)/K), \mu_n)$$

when n runs through all positive integers.

Laurent's proof

Let n be an integer such that $\alpha_1^n, \dots, \alpha_s^n$ all belong to K.

Let $\chi_1, \dots, \chi_s$ be the corresponding Galois cocycles $\mathrm{Gal}(\overline{K}/K) \to \mu_n$. Then the relation $\alpha_1 + \dots + \alpha_s = 1$ becomes

$$\alpha_1\chi_1 + \dots + \alpha_s\chi_s = 1.$$

Over the field $K(\zeta_n)$, these cocycles become characters $\mathrm{Gal}(\overline{K}/K(\zeta_n)) \to \mu_n$. By Artin's theorem on linear independance of characters, one deduces that $\chi_1 = \dots = \chi_s = 1$ (because no proper subsum of $\alpha_1 + \dots + \alpha_s$ vanishes).

This means that the α_i belong to $K(\zeta_n)$.

By the inflation-restriction exact sequence

$$0 \longrightarrow H^1(\mathrm{Gal}(K(\zeta_n)/K), \mu_n) \longrightarrow K^\times/(K^\times)^n \longrightarrow K(\zeta_n)^\times/(K(\zeta_n)^\times)^n$$

the (class of the) α_i^n belong to the kernel of restriction.

Since b kills $H^1(\mathrm{Gal}(K(\zeta_n)/K), \mu_n)$, the α_i^{bn} belong to $(K^\times)^n$.

The Chevalley-Bass Theorem

In order to work out the value of b, while avoiding the cohomological calculation, Yuri Bilu derives from work of Chevalley and Bass the following.

Theorem

Let K be a finitely generated field of characteristic 0. Then there exists a positive integer Λ such that

$$\text{for any positive integer } n, \quad (K(\zeta_{\Lambda n})^\times)^{\Lambda n} \cap K^\times \subseteq (K^\times)^n.$$

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Bilu gives estimates for Λ (in terms of the discriminant of K) and asks

- What are the possibles values of Λ? Can Λ be equal to 1?
- Is there an algorithm to compute Λ?

For example, if $K = \mathbb{Q}$ then Λ is a multiple of 4 because $(1+i)^4 = -4$ belongs to $\mathbb{Q}$ but -4 is not a square in $\mathbb{Q}$.

Our main result

Let $K_{\text{ab}}/\mathbb{Q}$ denote the maximal abelian subextension of $K/\mathbb{Q}$. Since K is finitely generated, it has finite degree.

Theorem

Let λ be the number of roots of unity contained in K, let f be the conductor of $K_{\text{ab}}/\mathbb{Q}$, and let

$$f' := \begin{cases} \prod_{p\mid\lambda} p^{\operatorname{ord}_p(f)} & \text{if } 4 \mid f \\ 4\prod_{p\mid\lambda} p^{\operatorname{ord}_p(f)} & \text{if } f \text{ is odd.} \end{cases}$$

Then the Chevalley-Bass number Λ of K satisfies

$$\operatorname{lcm}\left(4, \lambda, \frac{f'}{\lambda}\right) \mid \Lambda \mid f'.$$

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- $\mathbb{Q}$ and $\mathbb{Q}(i)$ have Chevalley-Bass number $\Lambda = 4$.
- more generally, if r is odd then $\mathbb{Q}(\zeta_r)$ and $\mathbb{Q}(\zeta_{4r})$ have $\Lambda = 4r$.
- the possible Chevalley-Bass numbers are the multiples of 4.

An algorithm which computes the Chevalley-Bass number

The Chevalley-Bass number Λ satisfies the following property: given a prime number p, $\mathrm{ord}_p(\Lambda)$ is the smallest integer j such that, for every integer $n > 0$, the map

$$H^1(\mathrm{Gal}(K(\zeta_{p^j n})/K), \mu_{p^j}) \to H^1(\mathrm{Gal}(K(\zeta_{p^j n})/K), \mu_{p^{j+\mathrm{ord}_p(n)}})$$

induced by the inclusion $\mu_{p^j} \subseteq \mu_{p^{j+\mathrm{ord}_p(n)}}$, is surjective.

We deduce from this an algorithm which, given $K_{\mathrm{ab}}/\mathbb{Q}$, computes Λ.

Using this algorithm, we prove that any integer in the range of our theorem occurs as the Chevalley-Bass of some suitably chosen field.

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Example: Let p and m be odd primes such that $p^3 \mid m-1$, and pick γ of order p^3 in $(\mathbb{Z}/m\mathbb{Z})^\times$. Then the field fixed by the cyclic subgroup generated by $(1+p^2, \gamma) \in (\mathbb{Z}/p^4\mathbb{Z})^\times \times (\mathbb{Z}/m\mathbb{Z})^\times$ has $\lambda = 2p^2$ roots of unity, conductor $f = p^4 m$, and Chevalley-Bass number $\Lambda = 4p^3$.

Laurent's strategy is back!

While proving our main result, the following shows-up:

Proposition

The cohomology group $H^1(\mathrm{Gal}(K(\zeta_n)/K), \mu_n)$ *is killed by* λ*.*

So, Laurent's constant b is equal to λ, the number of roots of unity in K.

This also applies to the cases considered by Bilu and his collaborators (twisted rational zeros of linear recurrence sequences).

Conclusion: the Chevalley-Bass number is a finer invariant, which is interesting in its own but is not necessary to make this argument work.

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Thank you for your attention!