Descent on elliptic curves over function fields

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Elliptic curves over $k(B)$: the Lang-Néron Theorem

Let k be a perfect field of characteristic $\neq 2$, and let B be a smooth projective curve over k, with function field $k(B)$.

An elliptic curve E over $k(B)$ is defined by an equation of the form

$$y^2 = x^3 + a_2 x^2 + a_4 x + a_6$$

where a_2, a_4, a_6 belong to $k(B)$, and $\Delta := \mathrm{Disc}(x^3 + a_2 x^2 + a_4 x + a_6) \neq 0$.

One says E is **constant** if it admits such an equation with a_2, a_4, a_6 constant (*i.e.* elements of k).

Theorem (Lang-Néron 1959)

If E is a non-constant elliptic curve over $k(B)$, then $E(k(B))$ is an abelian group of finite type.

The rank of E over $k(B)$ is by definition the integer r such that

$$E(k(B)) \simeq E(k(B))_{\mathrm{tors}} \oplus \mathbb{Z}^r$$

We denote it by $\operatorname{rk} E(k(B))$.

Igusa's geometric rank bound

- $g :=$ the genus of B,
- $\mathfrak{f}_E :=$ the conductor of E, viewed as a divisor on B.

Igusa (1960) proved that

$$\operatorname{rk} E(\bar{k}(B)) \leq 4g - 4 + \deg(\mathfrak{f}_E).$$

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Toy example: let E be the elliptic curve over $\mathbb{Q}(t)$ defined by

$$y^2 = x^3 + tx^2 + t^2(t^2 + 1).$$

This curve has conductor

$$\mathfrak{f}_E = \{t^2 + 1 = 0\} + \{27(t^2 + 1) + 4t = 0\} + 2 \cdot (\{0\} + \{\infty\})$$

The geometric rank bound yields

$$\operatorname{rk} E(\bar{\mathbb{Q}}(t)) \leq -4 + \deg(\mathfrak{f}_E) = 4,$$

and the rank is actually 4 (the underlying surface is rational). The obvious point $(t, t^2 + t)$ has infinite order. What is the rank over $\mathbb{Q}(t)$?

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This was extended to abelian varieties by Ogg (1962) and Shafarevich (1961).

In fact, what is now known as the Grothendieck-Ogg-Shafarevich formula yields

$$\operatorname{rk} E(\bar{k}(B)) \leq -\chi(B, \mathcal{E}^0[\ell])$$

where $\mathcal{E} \to B$ is the Néron model of E, $\mathcal{E}^0$ is the connected component of the identity of $\mathcal{E}$, and $\chi(B, \mathcal{E}^0[\ell])$ is the étale Euler-Poincaré characteristic of the ℓ-torsion sheaf of $\mathcal{E}^0$.

Arithmetic refinement via 2-descent

Assume that $E(\bar{k}(B))$ has no nontrivial 2-torsion. Then the degree 3 cover $\pi : C \to B$ defined by the equation

$$x^3 + a_2 x^2 + a_4 x + a_6 = 0$$

is geometrically irreducible, and we have a 2-descent map

$$E(k(B)) \longrightarrow H^1(B \setminus \Sigma, \mathcal{E}[2]) \longrightarrow H^1(C \setminus \pi^{-1}(\Sigma), \mu_2)$$

where $\Sigma = (\mathfrak{f}_E)_{\mathrm{red}}$ is the set above which $\mathcal{E} \to B$ has bad reduction.

Using this descent map, we (Levin-G. 2022) obtained the following upper bound

$$\begin{aligned} \operatorname{rk} E(k(B)) \leq & \dim_{\mathbb{F}_2} \operatorname{Pic}(C)[2] - \dim_{\mathbb{F}_2} \operatorname{Pic}(B)[2] + \#\{v \in B, 2 \mid c_v\} \\ & + \#\{v \in B, \text{the fiber type of } \mathcal{E} \text{ at } v \text{ is } \mathrm{I}_{2n}^* \text{ for some } n \geq 0\}, \end{aligned}$$

where c_v denotes the Tamagawa number at some bad place v.

If $k = \bar{k}$, this bound **agrees** with Igusa's one. In general, it is an **arithmetic refinement** of Igusa's one.

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Generalization to abelian varieties with trivial trace (Boudreau-Levin-G. 2023).

Toy example: arithmetic rank bound

The Tamagawa numbers being odd, none of the bad fibers contributes to our bound; therefore our arithmetic bound reads

$$\mathrm{rk}\, E(\mathbb{Q}(t)) \leq \dim_{\mathbb{F}_2} \mathrm{Pic}(C)[2],$$

where the curve C is defined by the equation

$$x^3 + tx^2 + t^2(t^2+1) = 0.$$

Substituting $X = x/t$ and $Y = 2t + X^3 + X^2$, we find another equation for C

$$Y^2 = (X-1)(X^2+2X+2)(X^3+X^2+2).$$

It follows that $\mathrm{Pic}(C)[2] \simeq \mathbb{Z}/2\mathbb{Z}$, hence $\mathrm{rk}\, E(\mathbb{Q}(t)) \leq 1$. Conclusion: $\mathrm{rk} = 1$.

One can go the other way around: given an elliptic curve over $\mathbb{Q}(t)$ with large rank (current record is 18, by Elkies 2004) and odd Tamagawa numbers, one obtains a trigonal curve C such that $\mathrm{Pic}(C)[2]$ is large. We (Levin-G. 2019) used this to build infinitely many cubic fields whose class group contains a subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{11}$.

Upper bound on the number of S-integral points

Let $S \subset B$ be some finite set, and let $R_S \subset k(B)$ be the ring of functions with no poles outside S. We denote by $E(R_S)$ the set of S-integral points of E, which is finite if E is not isotrivial, or if $\mathrm{char}(k) = 0$ and E is non-constant.

Using Riemann-Roch, we give an upper bound on the number of S-integral points of given height having the same image under the 2-descent map. Combined with an upper bound on the height of S-integral points, we (Levin-Hallouin-G. 2024) proved that

$$|E(R_S)| \leq 2^{3|S|h_{\max} - g(C) + 1 + \mathrm{rk}\, E(k(B))} \leq 2^{3|S|h_{\max} - g(C) + 4g - 3 + \deg(\mathfrak{f}_E)}$$

where $$h_{\max} := \begin{cases} 2(2g-2+|S|+\deg(\Delta)) + \frac{h(f)}{2} & \text{if } \mathrm{char}(k) = 0 \\ 3\deg^{in}(j_E)(2g-2+|S|+\deg(\Delta)) + \frac{h(f)}{2} & \text{if } \mathrm{char}(k) > 3 \end{cases}$$

This should be compared with the result by Hindry-Silverman ($\mathrm{char}(k) = 0$)

$$|E(R_S)| \leq \begin{cases} 144\left(10^{7.1}\sqrt{|S|}\right)^{\mathrm{rk}\, E(k(B))} & \text{if } h(E) \geq 2g-2 \\ (4\pi^2(2g-2))^{2/3}\left(10^{7+12g}\sqrt{|S|}\right)^{\mathrm{rk}\, E(k(B))} & \text{if } h(E) < 2g-2 \end{cases}$$

Thank you for your attention!