Galois module structure and log schemes

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Classical Galois module structure

- K a number field
- L/K a finite Galois extension of K, with group Γ

Theorem (Normal Basis)

L is a free $K[\Gamma]$-module of rank 1.

Replace K and L by their rings of integers $\mathcal{O}_K$ and $\mathcal{O}_L$.

Theorem (Nœther's criterion)

$\mathcal{O}_L$ is a projective $\mathcal{O}_K[\Gamma]$-module if and only if L/K is tame.

The tame case

$\mathrm{Cl}(\mathbb{Z}[\Gamma]) := K_0(\mathbb{Z}[\Gamma])/\{\text{free modules}\}$ the locally free classgroup.

If M is a projective $\mathbb{Z}[\Gamma]$-module, let (M) be its class in $\mathrm{Cl}(\mathbb{Z}[\Gamma])$

Fröhlich's conjecture, proved by M. J. Taylor in 1981, states that $(\mathcal{O}_L)$ is 2-torsion, and can be expressed in terms of Artin constants of irreducible and symplectics characters of Γ.

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- What about the **relative** structure ($\mathcal{O}_L$ as $\mathcal{O}_K[\Gamma]$-module) ?
- What happens in the **wild** case ?

Scheme-theoretic setting

- $S := \mathrm{Spec}(\mathcal{O}_K)$ the spectrum of the ring of integers of K
- G a finite flat **commutative** group scheme over S

G-torsors (for the fppf topology S) are "Galois extensions of S with group G".

$$H^1(S, G) := \{\text{isomorphism classes of } G\text{-torsors over } S\}$$

is an abelian group.

It is a subgroup of $H^1(K, G_K) = H^1(Gal(\overline{K}/K), G_K)$.

Ramification of algebras underlying torsors

If $G = \Gamma_S$ is a constant group scheme, then

$$H^1(S, \Gamma_S) = \{\mathrm{Spec}(\mathcal{O}_L) \to S, \text{ where } L/K \text{ is a } K\text{-Galois algebra with group } \Gamma, \text{ everywhere unramified}\}$$

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In the general case, any G-torsor is an order in some finite extension (or algebra) F/K. We know that :

$$\begin{aligned} \text{The places of ramification of } F &\subseteq \text{the places where } G \text{ is not étale} \\ &\subseteq \text{the places dividing the order of } G \end{aligned}$$

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This proves that $H^1(S, G)$ is finite.

Extending torsors

The extension $\mathbb{Q}(\sqrt{3})/\mathbb{Q}$ is a $\mathbb{Z}/2$-torsor over $\mathbb{Q}$.
We try to extend this at the integral level : the map

$$\mathrm{Spec}(\mathbb{Z}[\sqrt{3}, \frac{1}{6}]) \to \mathrm{Spec}(\mathbb{Z}[\frac{1}{6}])$$

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is a $\mathbb{Z}/2$-torsor, but it is not possible to do better, because the extension is ramified at 2 and 3.
Another idea is to replace $\mathbb{Z}/2$ by μ_2, which has the same generic fiber. The map

$$\mathrm{Spec}(\mathbb{Z}[\sqrt{3}, \frac{1}{3}]) \to \mathrm{Spec}(\mathbb{Z}[\frac{1}{3}])$$

is a μ_2-torsor, but this is the best we can do because μ_2 is étale above 3.

What is ramification ?

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Question 1 : what is a tame action of G on a scheme ?

Approaches of tameness

- Grothendieck-Murre, 1971 (via étale topology)
- Childs-Hurley, 1986 (Hopf algebras)
- Chinburg-Erez-Pappas-Taylor, 1996 (schemes)
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Question 2 : Is it possible to get a notion of tameness (for G finite flat) for which tame objects "are" torsors in some topos ?

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Question 2 : Is it possible to get a notion of tameness (for G finite flat) for which tame objects "are" torsors in some topos ?

If G is étale, and if the ramification locus is a normal crossing divisor, then Grothendieck-Murre's answer is YES.

Galois module structure of torsors

Let G^D be the Cartier dual of G
(W. Waterhouse, 1971). we have a homomorphism

$$\pi : H^1(S, G) \xrightarrow{\sim} \mathrm{Ext}^1(G^D, \mathbf{G}_{\mathrm{m}}) \longrightarrow \mathrm{Pic}(G^D)$$

The first map is an isomorphism deduced from the local-global spectral sequence for Ext^n, the second is the natural one.
One says that π measures the Galois structure of G-torsors.
In the case where $G = \Gamma_S$, the morphism π is given by :

$$\begin{aligned} \pi : H^1(S, \Gamma_S) &\longrightarrow \mathrm{Pic}(\Gamma_S^D) \simeq \mathrm{Cl}(\mathcal{O}_K[\Gamma]) \\ (\mathrm{Spec}(\mathcal{O}_L) \to S) &\longmapsto (\mathcal{O}_L) \end{aligned}$$

(we recover the unramified case of the classical theory).

Galois module structure of tame objects

Let $H^1_{\mathrm{tame}}(K,\Gamma) \subseteq H^1(K,\Gamma)$ be the subgroup of tame extensions. Then, by Nœther's criterion, we have a map (extending π)

$$\mathrm{cl} : H^1_{\mathrm{tame}}(K,\Gamma) \longrightarrow \mathrm{Cl}(\mathcal{O}_K[\Gamma])$$

which in general is **not a morphism** of groups. But the image has been proved to be subgroup by McCulloh.

If Question 2 has a positive answer, we will get a morphism measuring Galois structure in Waterhouse's style, with values in some new "class group".

Log schemes

Fontaine, Illusie, Kato, ...

A log scheme is a scheme endowed with a log structure.

A log structure on a scheme X is a pair (M_X, α) where M_X is a sheaf of (commutative !) monoïds on the étale site of X, and $\alpha : M_X \to \mathcal{O}_X$ is a morphism of sheaves of monoïdes, $\mathcal{O}_X$ being endowed with multiplication law.

We also ask that α induces an isomorphism $\alpha^{-1}(\mathcal{O}_X^*) \simeq \mathcal{O}_X^*$.

The trivial log structure on X is $(\mathcal{O}_X^*, i)$ where $i : \mathcal{O}_X^* \hookrightarrow \mathcal{O}_X$ is the canonical inclusion.

We have a fully faithful functor from the category of schemes into the category of log schemes, sending a scheme to itself endowed with the trivial log structure.

The scheme $X(\log D)$

- X a nœtherian regular scheme
- D a normal crossing divisor on X
- $j : U \subseteq X$ the complement of $D \subseteq X$

The immersion $j : U \to X$ defines a log structure on X, given by

$$M_X = \mathcal{O}_X \cap j_* \mathcal{O}_U^* \longrightarrow \mathcal{O}_X$$

We denote by $X(\log D)$ the log scheme obtained.

Example

- $X = S = \mathrm{Spec}(\mathcal{O}_K)$ as before
- $S^0 := S \backslash \{\text{generic point}\}$ the set of finite places of K
- $D \subseteq S^0$ is any finite set

Log topologies

- Kummer log étale topology ($\mathrm{k\acute{e}t}$)
- Kummer log flat topology (kfl)

The Kummer log étale topology is generated by classical étale covers, and so-called standard Kummer étale covers.

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The standard Kummer flat covers don't have any assumptions on the residue characteristics.

If G is finite étale over X, then as one expects

$$H^1_{\mathrm{kfl}}(X(\log D), G) = H^1_{\mathrm{k\acute{e}t}}(X(\log D), G)$$

Examples of log torsors

(1) If $X = S$ and $G = \Gamma_S$ is a constant group scheme, then

$$H^1_{\text{két}}(S(\log D), \Gamma_S) = \{\, \mathrm{Spec}(\mathcal{O}_L) \to S, \text{ where } L/K \text{ is a } K\text{-Galois algebra with group } \Gamma, \text{ unramified above } U, \text{ and tamely ramified above } D\}$$

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(2) More surprising : if $G = \mu_n$, then

$$H^1_{\text{kfl}}(S(\log D), \mu_n) = H^1_{\text{fppf}}(U, \mu_n)$$

Therefore, we obtain a μ_2-torsor for the log flat topology

$$\mathrm{Spec}(\mathbb{Z}[\sqrt{3}])(\log \sqrt{3}) \to \mathrm{Spec}(\mathbb{Z})(\log 3)$$

extending the torsor $\mathbb{Q}(\sqrt{3})/\mathbb{Q}$.

Restriction of torsors

Let G be a finite flat group scheme over X. Then the restriction map

$$j^* : H^1_{\mathrm{kfl}}(X(\log D), G) \longrightarrow H^1_{\mathrm{fppf}}(U, G_U)$$

is injective.

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is injective.

- What is the image of this map ?

Linearly reductive group schemes

Theorem (J.G., 2011)

Assume G is a linearly reductive finite flat group scheme. Then the restriction map

$$j^* : H^1_{\mathrm{kfl}}(X(\log D), G) \longrightarrow H^1_{\mathrm{fppf}}(U, G_U)$$

is bijective.

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is bijective.

The proof uses the following result (Abramovich-Olsson-Vistoli) : if G is a finite flat linearly reductive group scheme, then locally for the étale topology on X, G sits into an exact sequence

$$1 \longrightarrow \Delta \longrightarrow G \longrightarrow H \longrightarrow 1$$

where Δ is diagonalisable and H is constant of order coprime to the residue characteristics of X.

Link with previous notions of tameness

Theorem (J.G., 2008)

Let G be a commutative finite flat group scheme over X. Let $T \to X(\log D)$ be a G-torsor for the log flat topology. Then

(1) *the action of G on the underlying scheme of T is CEPT-tame.*

(2) *if X is affine, the action is also CH-tame*

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This proves that the image of j^* is contained in the set of algebras that admit an order that is a CH-tame Galois object. One expects that the two sets are in fact equal.

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Theorem (J.G., 2008)

Let G be a commutative finite flat group scheme over X. Let $T \to X(\log D)$ be a G-torsor for the log flat topology. Then

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2. *if X is affine, the action is also CH-tame*

This proves that the image of j^* is contained in the set of algebras that admit an order that is a CH-tame Galois object. One expects that the two sets are in fact equal.

Also, this probably holds for non commutative G !

Galois structure of log flat torsors

Assume again G is commutative finite flat over X affine. Because of CH-tameness, we have a map

$$\mathsf{cl} : H^1_{\mathrm{kpl}}(X(\log D), G^D) \longrightarrow \mathrm{Pic}(G)$$

called "classical Galois structure", which in general is not a morphism.

On the other hand, without assuming X affine, Waterhouse's construction gives us a morphism

$$\pi^{\log} : H^1_{\mathrm{kfl}}(X(\log D), G^D) \longrightarrow H^1_{\mathrm{kfl}}(G, \mathbf{G}_{\mathrm{m}})$$

which measures the "log Galois structure" of log flat torsors.

Galois structure of μ_n-torsors (fppf case)

Using the well-known description :

$$H^1_{\mathrm{fppf}}(S, \mu_n) = \{z \in K^*/(K^*)^n \mid \forall \mathfrak{p} \in S^0, n | v_{\mathfrak{p}}(z)\}$$

Waterhouse's Galois structure morphism is given by

$$\begin{aligned} \pi : H^1_{\mathrm{fppf}}(S, \mu_n) &\longrightarrow \mathrm{Cl}(\mathcal{O}_K) \\ z &\longmapsto \sum_{\mathfrak{p} \in S^0} \frac{v_{\mathfrak{p}}(z)}{n} [\mathfrak{p}] = [\frac{1}{n} \mathrm{div}(z)] \end{aligned}$$

Galois structure of μ_n-torsors (log flat case)

For $\mathfrak{p} \in S^0$, let

$$v_{\mathfrak{p}}(z) = nq_{\mathfrak{p}} + r_{\mathfrak{p}}$$

be the Euclidian division of $v_{\mathfrak{p}}(z)$ by n. We have

$$H^1_{\mathrm{kfl}}(S(\log D), \mu_n) = \{z \in K^*/(K^*)^n \mid \forall \mathfrak{p} \in S^0 \backslash D, n | v_{\mathfrak{p}}(z)\}$$

The classical Galois structure map is

$$\begin{aligned} \mathrm{cl} : H^1_{\mathrm{kfl}}(S, \mu_n) &\longrightarrow \mathrm{Cl}(\mathcal{O}_K) \\ z &\longmapsto \sum_{\mathfrak{p} \in S^0} q_{\mathfrak{p}}[\mathfrak{p}] \end{aligned}$$

On the other hand, one computes that

$$H^1_{\mathrm{kfl}}(S(\log D), \mathbf{G}_{\mathrm{m}}) = (\mathrm{Div}(U) \oplus \bigoplus_{\mathfrak{p} \in D} \mathbb{Q}.[\mathfrak{p}]) / \operatorname{DivPrinc}(S)$$

(divisors with rational coefficients above D modulo usual principal divisors)

The log flat Galois structure morphism is given by

$$\begin{aligned} \pi^{\log} : H^1_{\mathrm{kfl}}(S, \mu_n) &\longrightarrow H^1_{\mathrm{kfl}}(S(\log D), \mathbf{G}_{\mathrm{m}}) \\ z &\longmapsto \sum_{\mathfrak{p} \in S^0} q_{\mathfrak{p}}[\mathfrak{p}] + r_{\mathfrak{p}}[\frac{1}{n}\mathfrak{p}] = [\frac{1}{n}\operatorname{div}(z)] \end{aligned}$$

Building torsors from isogenies

- $\phi_K : A_K \to B_K$ an isogeny between abelian varieties over K
- $G_K \subseteq A_K$ the kernel of ϕ_K (a finite subgroup scheme of A_K).

We have an exact sequence

$$0 \longrightarrow G_K \longrightarrow A_K \xrightarrow{\phi_K} B_K \longrightarrow 0$$

and a dual sequence

$$0 \longrightarrow G_K^D \longrightarrow B_K^t \xrightarrow{\phi_K^t} A_K^t \longrightarrow 0$$

The cobundary of this sequence is

$$\delta_K : A_K^t(K) \longrightarrow H^1(K, G_K^D)$$

Let $P \in A_K^t(K)$ a point. Then $\delta_K(P)$ is the spectrum of some K-algebra, and we would like to compute the Galois module structure of its ring of integers.

Good reduction case

- $\mathcal{A}$, $\mathcal{A}^t$, $\mathcal{B}$, $\mathcal{B}^t$ the Néron models of A_K, A_K^t, B_K, B_K^t
- Let $\phi : \mathcal{A} \to \mathcal{B}$ and $\phi^t : \mathcal{B}^t \to \mathcal{A}^t$ be the morphisms extending ϕ_K and ϕ_K^t
- Assume A_K has everywhere good reduction.

Then $\mathcal{A}$ is an S-abelian scheme, and $G := \ker(\phi)$ is a finite flat subgroup of $\mathcal{A}$. Moreover, we have exact sequences

$$0 \longrightarrow G \longrightarrow \mathcal{A} \xrightarrow{\phi} \mathcal{B} \longrightarrow 0$$

$$0 \longrightarrow G^D \longrightarrow \mathcal{B}^t \xrightarrow{\phi^t} \mathcal{A}^t \longrightarrow 0$$

By composing the cobundary of the last sequence with π we obtain the class-invariant homomorphism (M. J. Taylor, 1988)

$$\psi : A_K^t(K) = \mathcal{A}^t(S) \xrightarrow{\delta} H^1(S, G^D) \xrightarrow{\pi} \mathrm{Pic}(G)$$

So any $P \in A_K^t(K)$ gives rise to a G^D-torsor.

Geometric description of ψ

We have a commutative diagram

$$\begin{tikzcd}
\mathcal{A}^t(S) \arrow[r, "\sim"] \arrow[d, "\delta"'] & \mathrm{Ext}^1(\mathcal{A}, \mathbf{G}_{\mathrm{m}}) \arrow[r] \arrow[d] & \mathrm{Pic}(\mathcal{A}) \arrow[d] \\
H^1(S, G^D) \arrow[r, "\sim"] & \mathrm{Ext}^1(G, \mathbf{G}_{\mathrm{m}}) \arrow[r] & \mathrm{Pic}(G)
\end{tikzcd}$$

The composition of the maps from $\mathcal{A}^t(S)$ to $\mathrm{Pic}(G)$ is equal to ψ.

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There are examples where 2-torsion points are not in the kernel of ψ (Bley-Klebel, Cassou-Noguès-Jehanne).

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- What happens when A_K has bad reduction ?

Bad reduction case

Assume A_K has semi-stable reduction at places dividing the order of G_K.

Then $G := \ker(\phi)$ is a quasi-finite flat subscheme of $\mathcal{A}$, not finite in general.

We now make the assumption that G is finite (this is the case, for example, if G_K is a constant group scheme over K).

In order to construct ψ, we want to use the geometric description. But what can be said about duality of Néron models ?

Let $\mathcal{A}^{t,\circ}$ the connected component of $\mathcal{A}^t$.

Theorem (Grothendieck)

There exists a unique biextension W of $(\mathcal{A}, \mathcal{A}^{t,\circ})$ by $\mathbf{G}_{\mathrm{m}}$ extending Weil's biextension.

This biextension W gives us a morphism

$$\gamma : \mathcal{A}^{t,\circ}(S) \longrightarrow \mathrm{Ext}^1(\mathcal{A}, \mathbf{G}_{\mathrm{m}})$$

We now get a morphism ψ by composing maps in the diagram

$$\begin{tikzcd} \mathcal{A}^{t,\circ}(S) \arrow[r, "\gamma"] & \mathrm{Ext}^1(\mathcal{A}, \mathbf{G}_{\mathrm{m}}) \arrow[r] \arrow[d] & \mathrm{Pic}(\mathcal{A}) \arrow[d] \\ & \mathrm{Ext}^1(G, \mathbf{G}_{\mathrm{m}}) \arrow[r] & \mathrm{Pic}(G) \end{tikzcd}$$

This is a generalisation of previous constructions.

$\mathcal{A}^{t,\circ}(S)$ is the subgroup of $A_K^t(K)$ of "points with everywhere good reduction".

If P is such a point, we have proved that $\delta_K(P)$ can be extended into a G^D-torsor, even when $\mathcal{A}$ has bad reduction.

Taylor's conjecture is still true in this context, at least if $\mathcal{A}$ is semistable.

Theorem (J. G., 2004)

If A_K is a semistable elliptic curve, and if the order of G is coprime to 6, then torsion points in $\mathcal{A}^{t,\circ}(S)$ belong to the kernel of ψ.

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- What happens for a point $P \notin \mathcal{A}^{t,\circ}(S)$?

Lifting δ_K with log flat topology

If we want to define ψ in the log flat context, we only need une more ingredient.

Let D be the set of places of bad reduction of $\mathcal{A}$.

Theorem

There exists a unique biextension $W^{\log}$ of $(\mathcal{A}, \mathcal{A}^t)$ by $\mathbf{G}_{\mathrm{m}}$ for the log flat topology on $S(\log D)$, extending Weil's biextension.

Thus, we get a map lifting δ_K

$$\mathcal{A}^t(S) \simeq \mathrm{Ext}^1_{\mathrm{kfl}}(\mathcal{A}, \mathbf{G}_{\mathrm{m}}) \longrightarrow \mathrm{Ext}^1_{\mathrm{kfl}}(G, \mathbf{G}_{\mathrm{m}}) \simeq H^1_{\mathrm{kfl}}(S(\log D), G^D)$$

This means that, for all $P \in A^t_K(K)$, the torsor $\delta_K(P)$ extends into a G^D-torsor for the log flat topology on $S(\log D)$.

Class invariant homomorphism and log flat topology

We have two maps extending ψ : the classical one

$$\psi^{\mathrm{cl}} : \mathcal{A}^t(S) \longrightarrow \mathrm{Pic}(G)$$

and the log one :

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But none of them is expected to satisfy Taylor's conjecture !