JBHU-ri Omenaldia, October 2010

Control of heat processes: theory and numerics

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JBHU

Basque Colloquium in Applied Maths

11:30 Jean-Baptiste HIRIART-URRUTY, University Paul Sabatier, Toulouse, France

THE ϵ-STRATEGY IN VARIATIONAL ANALYSIS

In this work we discuss variational (or optimization) problems which do not have solutions necessarily, but which do have approximate solutions (or solutions within $\epsilon > 0$). The question we address is: what to do with such ϵ-solutions? We shall see how to recover all the minimizers of the relaxed version of an abstract variational problem in terms of ϵ-minimizers of the original variational problem (specially when the later has no solution).
Applications to two classes of approximation problems in a Hilbert space setting will be shown.

13:00 Björn BIRNIR, University of California, Santa Barbara, CA, USA

EXISTENCE, UNIQUENESS AND STATISTICAL THEORY OF THE STOCHASTIC NAVIER-STOKES EQUATION IN THREE DIMENSIONS

We will discuss the existence of unique rough solution of the Navier-Stokes equation in three dimensions. These solutions are the result of noise that the equation produces at high Reynolds numbers. They also give a unique invariant measure that permits the development of Kolmogorov's statistical theory of turbulence.

October 9, 2009
Room 0.24, Faculty of Science and Technology, UPV/EHU (LEIOA campus)

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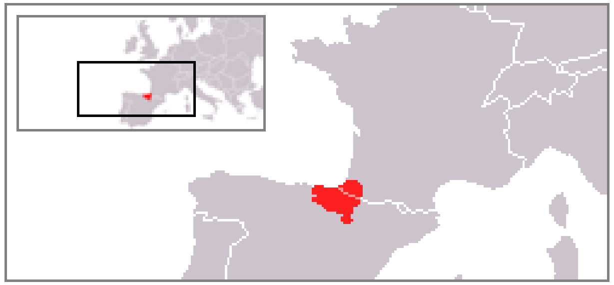


1.- THE CONTROL PROBLEM

Let $n \geq 1$ and $T > 0$, Ω be a simply connected, bounded domain of $\mathbb{R}^n$ with smooth boundary Γ, $Q = (0,T) \times \Omega$ and $\Sigma = (0,T) \times \Gamma$:

$$\begin{cases} u_t - \Delta u = f 1_\omega & \text{in} \quad Q \\ u = 0 & \text{on} \quad \Sigma \\ u(x,0) = u^0(x) & \text{in} \quad \Omega. \end{cases} \tag{1}$$

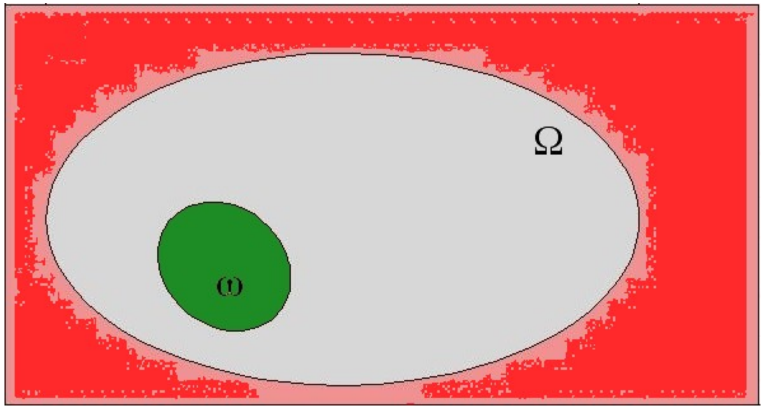
1_ω denotes the characteristic function of the subset ω of Ω where the control is active.

We assume that $u^0 \in L^2(\Omega)$ and $f \in L^2(Q)$ so that (5) admits an unique solution

$$u \in C\left([0,T]\,; L^2(\Omega)\right) \cap L^2\left(0,T; H_0^1(\Omega)\right).$$

$$u = u(x,t) = solution = state,\ f = f(x,t) = control$$

Goal: To produce prescribed deformations on the solution u by means of suitable choices of the control function f.



We consider the *null control* problem. **Objective:**

$$u(T) \equiv 0.$$

2.- APPROXIMATE CONTROL

Null controllability is in fact equivalent to a *quantitative* version of the property of unique continuation for the adjoint system:

$$\begin{cases} \varphi_t + \Delta\varphi = 0 & \text{in} \quad Q \\ \varphi = 0 & \text{on} \quad \Sigma \\ \varphi(x,T) = \varphi^0(x) & \text{in} \quad \Omega. \end{cases} \tag{2}$$

$$\varphi = 0 \text{ in } \omega \times (0,T) \implies \varphi \equiv 0, \text{ i.e. } \varphi^0 \equiv 0. \tag{3}$$

This UCP is a consequence of Holmgren's uniqueness Theorem.

This is so for all ω and all $T > 0$.

3.- AN ALGORITHM FOR DENSITY

Assume $L : H \to H$ is a linear, bounded operator with dense range.

Then, for all $f \in H$ and $\varepsilon > 0$ there exists $u \in H$ such

$$||Lu - f||_H \leq \varepsilon. \tag{4}$$

Of course the density of the range often happens without the map being surjective. This occurs frequently when looking to the evolution of time-irreversible semigroups and is relevant in control problems (*the system can be steered to a dense set of targets but not to all targets*).

Example: $Lu = G * u$, G being a gaussian.

In practice it is important to have a methodology/algorithm to build the solution u to (4).

Note that, according to Hahn-Banach Theorem, the rank of L is dense if and only if L^*, its adjoint, is injective: $L^*v = 0$ implies $v = 0$.

Consider now the functional

$$J(v) = \frac{1}{2}||L^*v||_H^2 + \varepsilon||v||_H - (f, v)_H.$$

Note that both f and ε in the density property are involved in this definition of J.

If, in addition to the injectivity property, we had,

$$||L^*v||_H^2 \geq \alpha||v||_H^2,$$

then the functional J would be coercive even for $\varepsilon = 0$. But the term added by means of $\varepsilon > 0$ is needed to ensure coercivity under the sole assumption that L^* is injective.

If J achieves its minimum at $\tilde{v}$, then

$$|(L^*(\tilde{v}), L^*v)_H - (f, v)_H| \leq \varepsilon||v||_H.$$

i. e.

$$|(LL^*(\tilde{v}) - f, v)_H| \leq \varepsilon ||v||_H, \text{ i. e., } ||LL^*(\tilde{v}) - f||_H \leq \varepsilon.$$

This means that $u = L^*(\tilde{v})$ is the solution we were looking for.

Does the minimizer of J exist?

$$J(v) = \frac{1}{2}||L^* v||_H^2 + \varepsilon ||v||_H - (f, v)_H.$$

$J : H \to \mathbf{R}$ is continuous and convex in a Hilbert space. It suffices to show coercivity.

We claim that, under the density assumption, or the injectivity of L^*, the functional is coercive in the sense that

$$\underline{\lim}_{||v||_H \to \infty} J(v)/||v||_H \geq \varepsilon.$$

Set $v_j : ||v_j||_H \to \infty$. Normalizing things: $\widehat{v_j} = v_j / ||v_j||_H$ and then

$$J(v_j)/||v_j||_H = \frac{1}{2}||v_j||_H ||L^* \widehat{v_j}||_H^2 + \varepsilon - (f, \widehat{v_j})_H.$$

The delicate case is when $||L^* \widehat{v}_j||_H \to 0$. Then, in the limit, $L^* \widehat{v} = 0$ which implies $\widehat{v} = 0$. This implies weak convergence to zero and thus $(f, \widehat{v_j})_H \to 0$. Consequently,

$$J(v_j)/||v_j||_H \geq \varepsilon - (f, \widehat{v_j})_H \to \varepsilon.$$

Assume now that E is a finite-dimensional subspace of H. Then, for all $f \in H$ and $\varepsilon > 0$ one can find $u \in H$ such that

$$||Lu - f||_H \leq \varepsilon; \quad \pi_E Lu = \pi_E f.$$

Proof: Minimize

$$J(v) = \frac{1}{2}||L^* v||_H^2 + \varepsilon||(1 - \pi_E)v||_H - (f, v)_H.$$

J. L. LIONS y E. ZUAZUA. The cost of controlling unstable systems: The case of boundary controls. J. Anal. Mathématique, LXXIII (1997), 225-249.

3.- NULL CONTROL

The model:

$$\begin{cases} u_t - \Delta u = f1_\omega & \text{in} \quad Q \\ u = 0 & \text{on} \quad \Sigma \\ u(x,0) = u^0(x) & \text{in} \quad \Omega. \end{cases} \tag{5}$$

Objective:

$$u(T) \equiv 0.$$

This corresponds to taking $\varepsilon = 0$ in the approximate control problem above.

The control can be built as follows: Consider the functional

$$J(\varphi^0) = \frac{1}{2}\int_0^T \int_\omega \varphi^2 dx dt + \int_\Omega \varphi(0) u^0 dx. \tag{6}$$

$J : L^2(\Omega) \to \mathbb{R}$ is continuous, and convex.

But, is it coercive?

If yes, the minimizer $\widehat{\varphi}^0$ exists and the control

$$f = \widehat{\varphi}$$

where $\widehat{\varphi}$ is the solution of the adjoint system corresponding to the minimizer is the control such that

$$u(T) \equiv 0.$$

For coercivity the following observability inequality is needed:

$$\| \varphi(0) \|^2_{L^2(\Omega)} \leq C \int_0^T \int_\omega \varphi^2 dx dt, \quad \forall \varphi^0 \in L^2(\Omega). \tag{7}$$

This estimate was proved by Fursikov and Imanuvilov (1996) using Carleman inequalities.

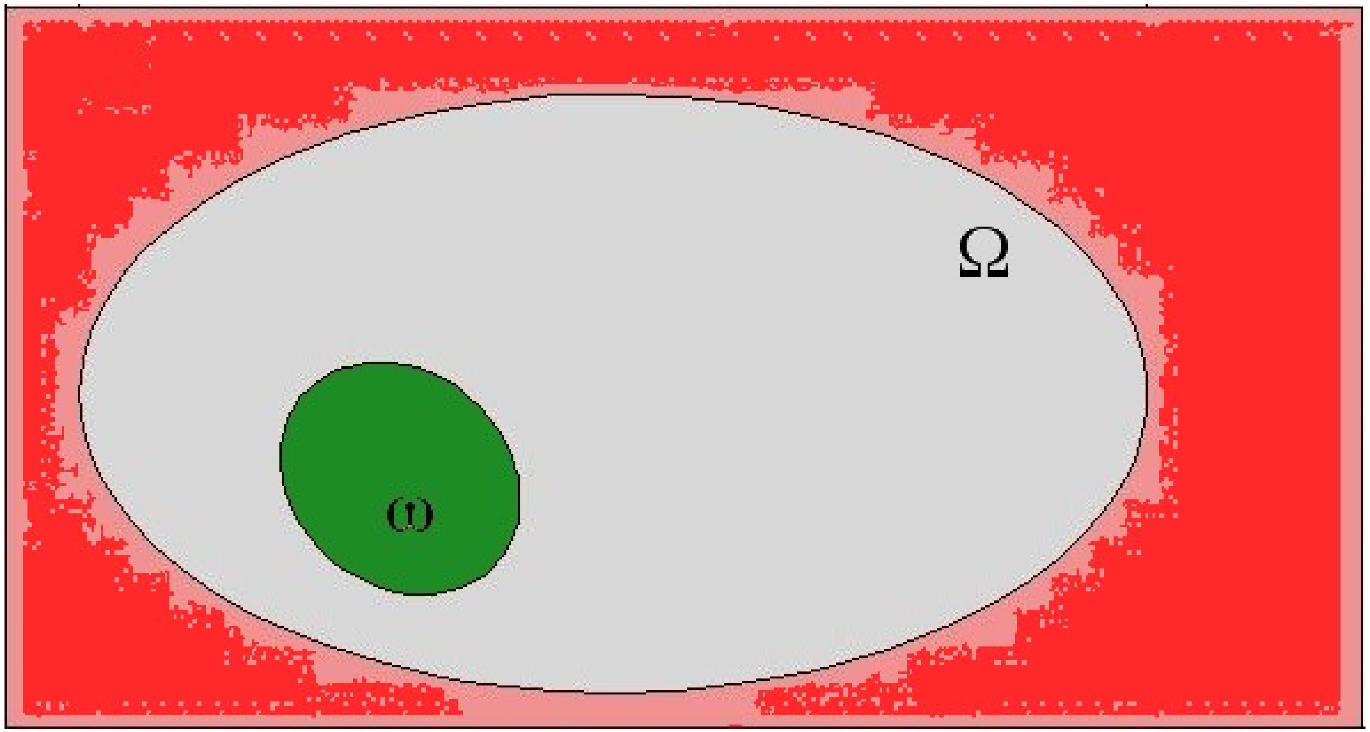
In view of this inequality the null control can be obtained by minimizing

$$J_0(\varphi^0) = \frac{1}{2} \int_0^T \int_\omega \varphi^2 dx dt + \int_\Omega \varphi(0) u^0 dx \tag{8}$$

in the space

$$\mathcal{H} = \{\varphi^0 : \text{sol. of the adjoint system s. t. } ||\varphi^0||_{\mathcal{H}} = \left[\int_0^T \int_\omega \varphi^2 dx dt\right]^{1/2} < \infty\}.$$

What about $\mathcal{H}$?



Of course,

$$C_1||\varphi(0)||_{L^2(\Omega)} \leq ||\varphi^0||_{\mathcal{H}} \leq C_2||\varphi^0||_{L^2(\Omega)}$$

but there is a gap of exponential order in the two norms of the left and right hand side terms of these inequalities:

$$C_1 \sum_{j\geq 1} e^{-\lambda_j T}|\hat{\varphi}_j^0|^2 \leq ||\varphi^0||_{\mathcal{H}} \leq C_2 \sum_{j\geq 1} |\hat{\varphi}_j^0|^2.$$

Accordingly

$$L^2(\Omega) \subset \mathcal{H} \subset H^{-\infty}(\Omega).$$

As we shall see, **this will make the effective numerical approximation issue hard.**

The Carleman inequalities due to Fursikov & Imanuvilov, 1996 yields, as observed by E. Fernández-Cara & E. Zuazua, 2000, the following observability estimate for the solutions of the heat equation:

$$\int_0^\infty \int_\Omega e^{\frac{-A}{(T-t)}} \varphi^2 dx dt \leq C \int_0^\infty \int_\omega \varphi^2 dx dt.$$

Conversely:

$$\int_0^\infty \int_\omega \varphi^2 dx dt \leq C ||\varphi^0||^2_{H^{-1}(\Omega)}.$$

This is another evidence of the lack of understanding of the space $\mathcal{H}$ that leads to the ill-posedness of the minimization problem.

4.- THE KANNAI TRANSFORM

Kannai transform allows transfering the results we have obtained for the wave equation to other models and in particular to the heat equation (Y. Kannai, 1977; K. D. Phung, 2001; L. Miller, 2004)

$$e^{t\Delta}\varphi = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{-s^2/4t} W(s)ds$$

where $W(x,s)$ solves the corresponding wave equation with data $(\varphi, 0)$.

$$W_{ss} - \Delta W = 0 \quad + \quad K_t - K_{ss} = 0 \quad \rightarrow \quad U_t - \Delta U = 0,$$

$$W_{ss} - \Delta W = 0 \quad + \quad iK_t - K_{ss} = 0 \quad \rightarrow \quad iU_t - \Delta U = 0.$$

This can be actually applied in a more general abstract context ($U_t + AU = 0$) but not when the equation has time-dependent coefficients.

This can also be used in the context of control:

[Control of the wave equation in Ω]

$$+$$

[$1-d$ controlled fundamental solution of the heat equation]

$$\Longrightarrow$$

[Control of the heat equation in Ω].

5.- EFFECTIVE NUMERICAL APPROXIMATION OF CONTROLS*

*A. MÜNCH y E. ZUAZUA. Numerical approximation of null controls for the heat equation through transmutation, J. Inverse Problems, to appear.

Recall that for the continuous heat equation the null control was obtained by minimizing

$$J(\varphi^0) = \frac{1}{2}\int_0^T \int_\omega \varphi^2 dx dt + \int_\Omega \varphi(0) u^0 dx \tag{9}$$

in the space $\mathcal{H}$ and that

$$C_1 ||\varphi(0)||_{L^2(\Omega)} \leq ||\varphi^0||_{\mathcal{H}} \leq C_2 ||\varphi^0||_{L^2(\Omega)}$$

Warning! We are dealing with a severely ill-posed problem.

Let us analyze how the functional J behaves when restricted to V_M, the space generated by the first M eigenfunctions of the Laplacian.

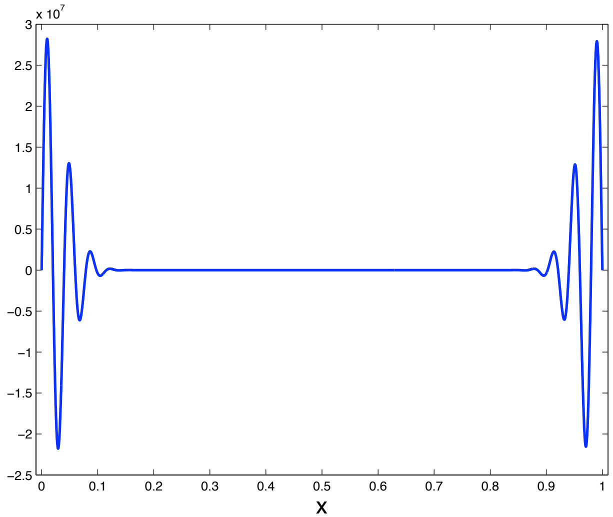
Descent algorithms when applied to J over V_M turn out to be very slow, and this is due to the very bad conditioning of the corresponding quadratic form.

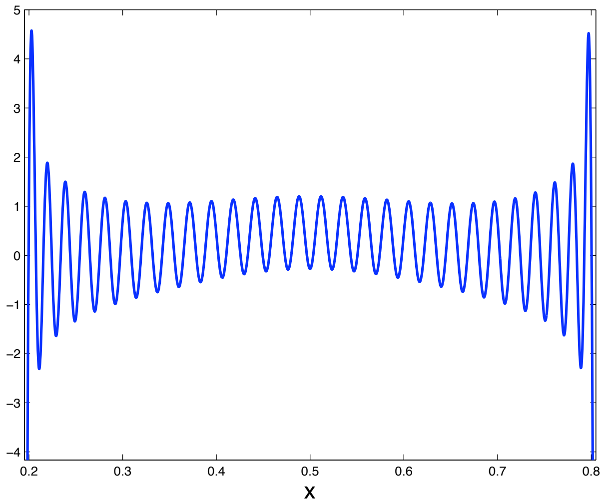
Condition number with respect to M for various $\omega \subset \Omega$ and $\omega = \Omega = (0,1)$: $T = 1$.

	$M = 10$	$M = 20$	$M = 40$	$M = 80$
$\omega = (0.2, 0.8)$	9.05×10^2	1.65×10^5	1.66×10^9	6.96×10^{16}
$\omega = (0.5, 0.8)$	3.57×10^5	3.81×10^{10}	7.31×10^{18}	$\geq 10^{20}$
$\omega = (0.7, 0.8)$	1.82×10^7	2.40×10^{14}	$\geq 10^{20}$	$\geq 10^{20}$
$\omega = (0, 1)$	8.61×10^1	3.44×10^2	1.33×10^3	5.51×10^3

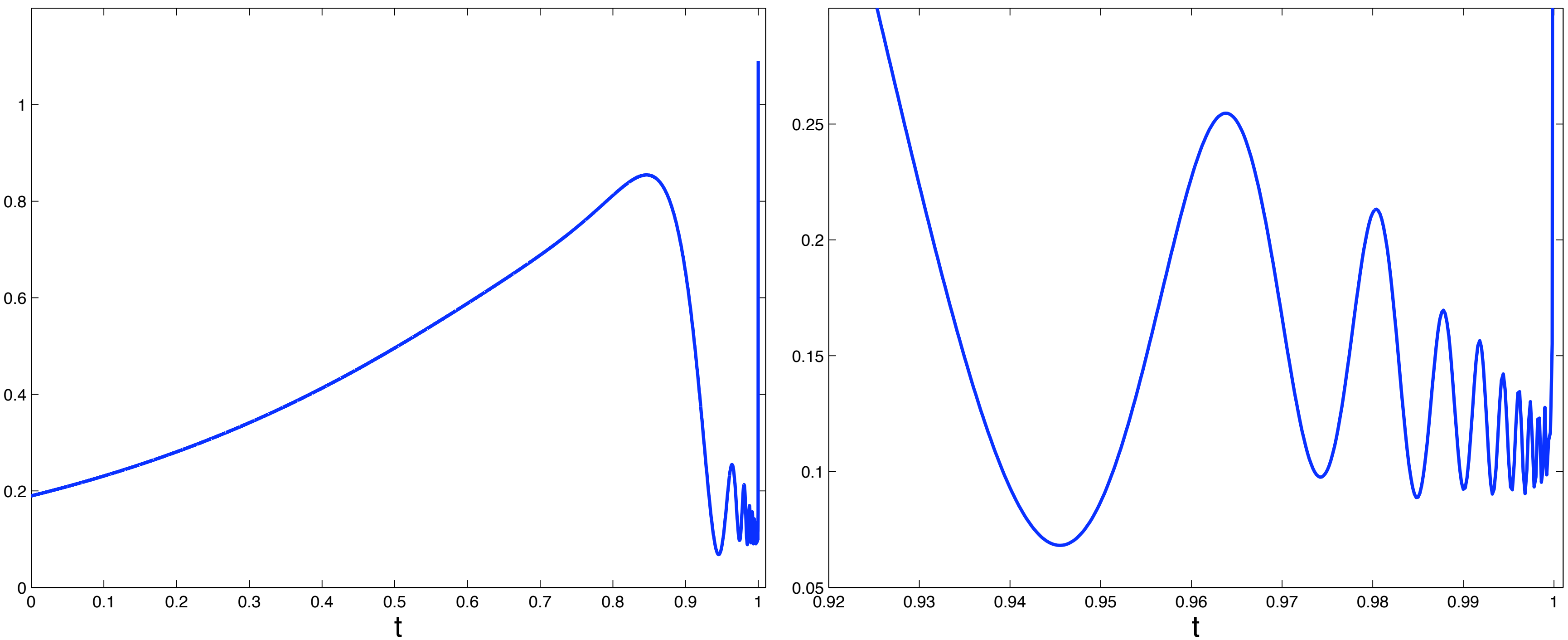
As a consequence of this, even if the control is in L^2, the data φ^0 of the adjoint system at time T (which is surely in $\mathcal{H}$) tend not to be in any reasonable space, thus making computations very hard.

$T = 1$, $\omega = (0.2, 0.8)$: $\varphi^{0,M}$ for $M = 80$ on Ω (**Left**) and on ω (**Right**).





$T = 1$, $\omega = (0.2, 0.8)$: $\|\varphi^M(\cdot, x)\mathcal{X}_\omega(x)\|_{L^2(\Omega)}$ for $M = 80$ on $[0, T]$ (**Left**) and on $[0.92T, T]$ (**Right**).



Several remedies have been derived in the literature, starting with the pioneering work by R. Glowinski and J. L. Lions.[†] One of them is based on Tychonnoff regularization. It consists on adding a regularizing term to the functional to be minimized (or its discrete version):

$$J_0(\varphi^0) = \frac{1}{2}\int_0^T \int_\omega \varphi^2 dxdt + \int_\Omega \varphi(0) u^0 dx. \tag{10}$$

Namely:

$$J_0^\varepsilon(\varphi^0) = \frac{1}{2}\int_0^T \int_\omega \varphi^2 dxdt + \frac{\varepsilon}{2}||\varphi^0||_{L^2}^2 + \int_\Omega \varphi(0) u^0 dx. \tag{11}$$

[†]R. Glowinski and J.L. Lions, *Exact and approximate controllability for distributed parameter systems*, Acta Numerica, 1996.

One can prove that, whenever the minimizer of the original functional J belongs to L^2, then the regularized controls converge polynomially as ε tends to zero[‡]

But, as the numerical experiments show, it is very unlikely that the minimizer lies in L^2.

In fact a recent result of S. Micu & E. Z. [§]shows that, in $1-d$, the minimizer φ^0 is never in L^2.

[‡]J. P. Puel, A Nonstandard Approach to a data assimilation problem and Tychonov regularization revisited, SIAM J. Control Optim. Vol. 48, No. 2, pp. 1089–1111

[§]S. Micu & E. Z. Regularity issues for the null-controllability of the linear 1-d heat equation, preprint, 2010.

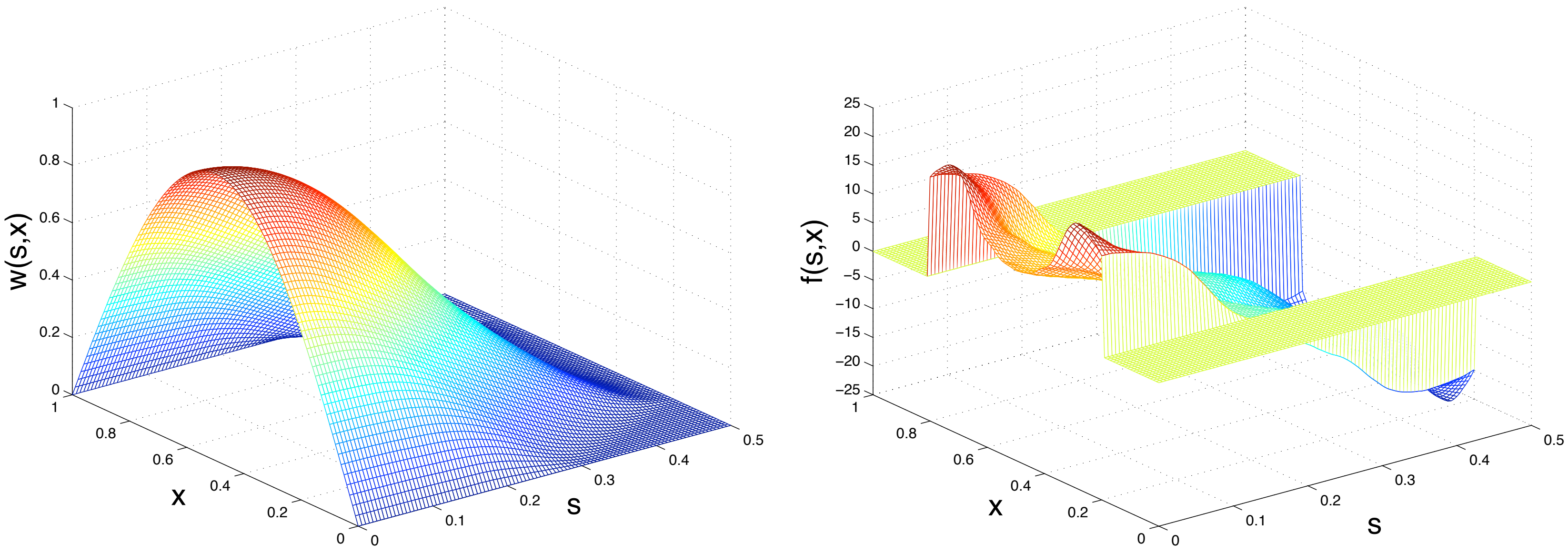
In a recent paper in collaboration with A. Münch we propose a different strategy based on the following facts:

- A lot of work has been done to build efficient algorithms to compute exact controls for the wave equation.

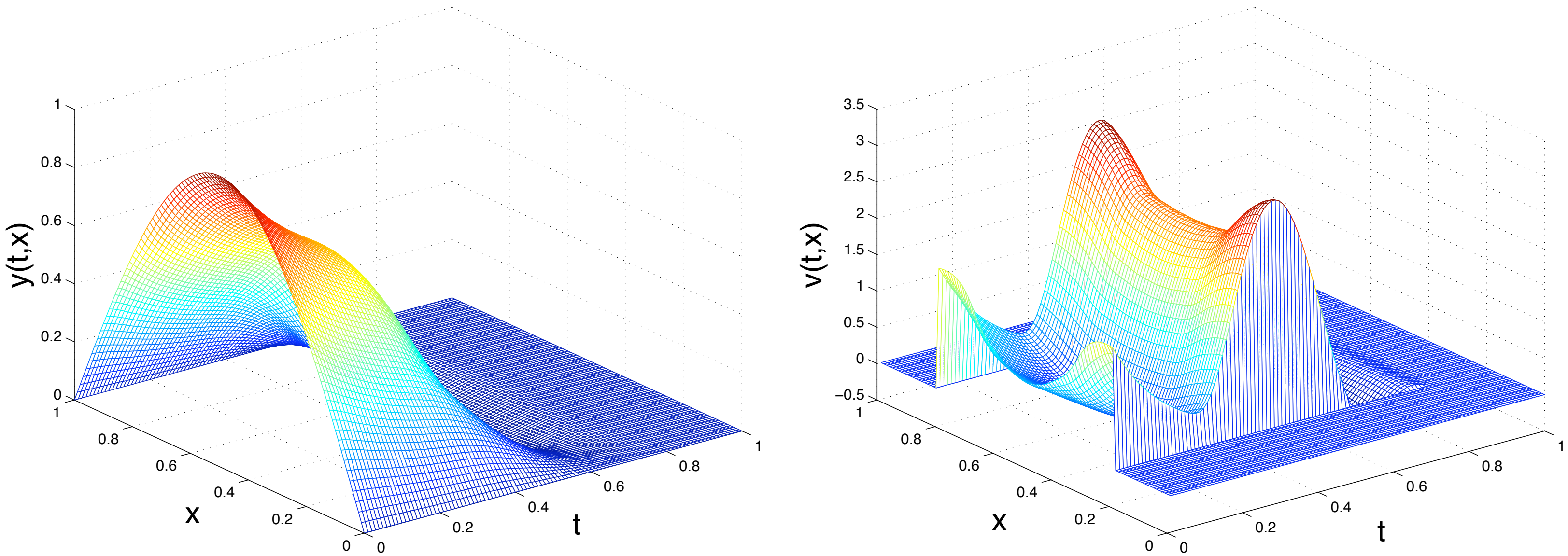
- The Kannai transform allows to construct the control of the heat equation by convolution of the wave one with a $1-d$ heat kernel.

The method is laborious to be developped numerically but turns out to be efficient.

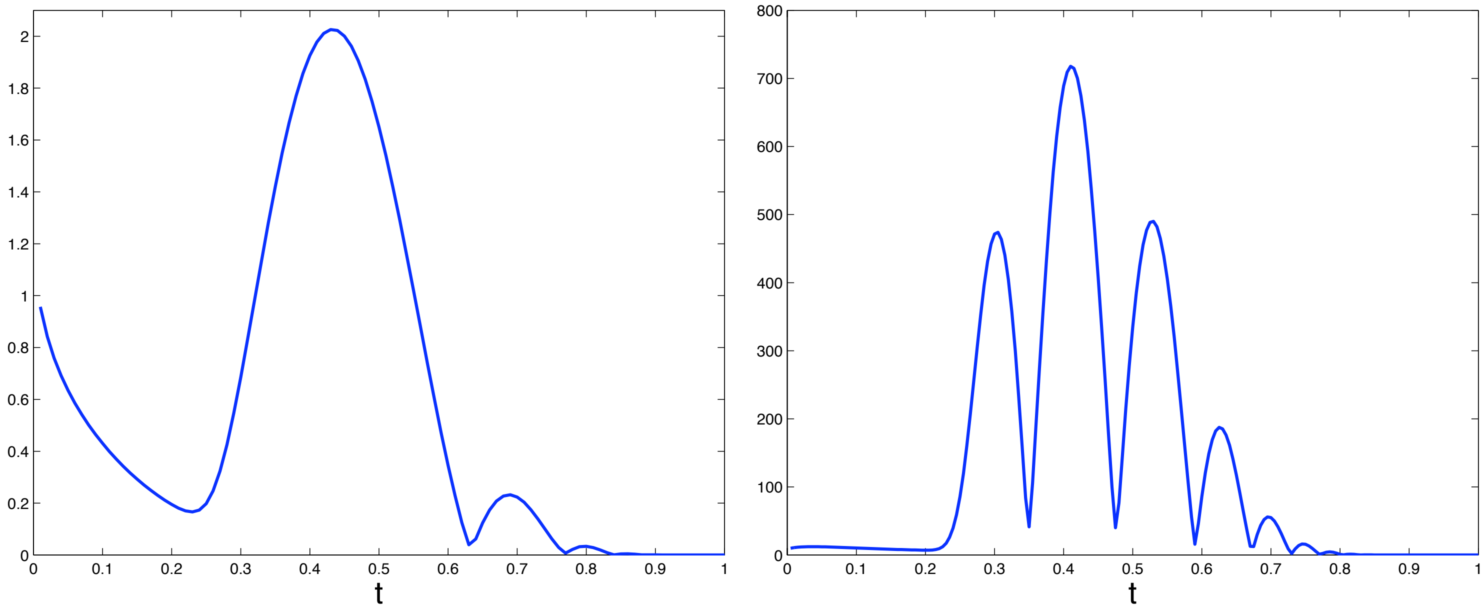
Wave equation: $y_0(x) = \sin(\pi x), L = 0.5,\ \omega = (0.2, 0.8)$. Controlled wave solution w (**Left**) and the corresponding control f on $(0, L) \times \Omega$ (**Right**).



$y_0(x) = \sin(\pi x), T = 1, c = 1/10$ and $(\delta, \alpha) = (T/5, 1)$. Controlled heat solution y (**Left**) and corresponding transmuted control v on $(0, T) \times \Omega$ (**Right**).



$L^2(\omega)$-norm of the control v vs time t for $(y_0(x), T, c) = (\sin(\pi x), 1, 1/10)$ (**Left**) and $(y_0(x), T, c) = (\sin(3\pi x), 1, 1/5)$ (**Right**).



Some open problems:

- Are Kannai controls optimal in a way?

- Kannai can not be applied in more general situations, nonlinear problems, for instance.

- What's the optimal algorithm for building controls?

- What to learn from the theory of ill-posed problems and their cures?

Some references:

- E. ZUAZUA, “Controllability and Observability of Partial Differential Equations: Some results and open problems”, in *Handbook of Differential Equations: Evolutionary Equations, vol. 3*, C. M. Dafermos and E. Feireisl eds., Elsevier Science, 2006, pp. 527-621.

- E. ZUAZUA. Propagation, observation, and control of waves approximated by finite difference methods. SIAM Review, 47 (2) (2005), 197-243.

- A. MÜNCH y E. ZUAZUA. Numerical approximation of null controls for the heat equation through transmutation, J. Inverse Problems, to appear.

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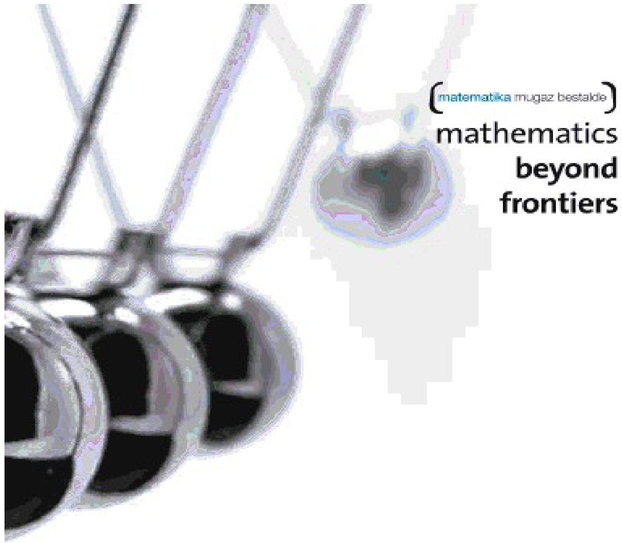

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Ruper Ordorika: 37 Galdera Mugaz Bestalde Dudan Kontaktu Bakarrari (Bernardo Atxaga)

37 Questions à mon seul contact de l'autre côté de la frontière

Esaidan, zoriontsuak al zarete mugaz bestaldeko biztanleak?

Stp, dis moi si les gens de l'autre côté de la frontière sont heureux ?

Mugaz bestaldean, hostoek ematen al diete babesa fruituei? Ba al dago marrubirik? Arrain abisalek ba al dute aurresentipenik eguzkiaz?

De l'autre côté de la frontière, la feuille protège-t-elle le fruit?

Y-a-t' il des fraises?

Les poissons abyssaux ont ils l'appréhension du soleil?

Asko al dira, asko al zarete mugaz bestaldeko erresuma hartan? Egunero kaletik ikusten dudan jende hau, han bizi al da?

Les habitants de l'autre côté de la frontière êtes-vous, sont-ils nombreux?

Les gens que je vois tous les jours dans la rue, habitent-ils là bas?

ZORIONAK JBHU, ... mugaz bestaldeko lagun eta kidea!