ON VARIOUS FORMS OF CONVEX RELAXATION AND THEIR LINK WITH SUBDIFFERENTIAL CALCULUS

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In honor of Jean-Baptiste Hiriart-Urruty

TWO KNOWN FORMULAS FOR THE CLOSED CONVEX RELAXATION

TH1 (Benoist / JBHU) (1996) $h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$

h lsc, bounded below, $\operatorname{int} \operatorname{dom} h^* \neq \emptyset$

$$\operatorname{argmin} h^{**} = \operatorname{co}(\operatorname{argmin} h) + \operatorname{co}(\operatorname{argmin} h_\infty)$$

$$h_\infty(x) = \liminf_{(t,u) \to (0_+, x)} t\, h\left(\frac{u}{t}\right)$$

TH2 (M.A. López / M.V) (2008) X locally convex space, $h : X \to \mathbb{R} \cup \{+\infty\}$

h bounded below

$$\operatorname{argmin} h^{**} = \bigcap_{\substack{\varepsilon > 0 \\ y \in \operatorname{dom} h^*}} \overline{\operatorname{co}}\left(\varepsilon\text{-}\operatorname{argmin} h + y^-\right)$$

$$y^- = \{x \in X : \langle x, y \rangle \leq 0\}$$

$$\operatorname{argmin} h^{**} = \bigcap_{\varepsilon > 0} (\varepsilon\text{-}\operatorname{argmin} h) \top \operatorname{dom} h^*$$

B - RELAXATION

Given $B \subset X^*$ and $h \in \overline{\mathbb{R}}^X$, define h^B as the sup of all the continuous affine minorants $\langle \cdot, y \rangle - z$ of h whose slope y belongs to B, and consider the problem

(P_B) minimize $h^B(x)$ for $x \in X$

which is called the B-relaxation of

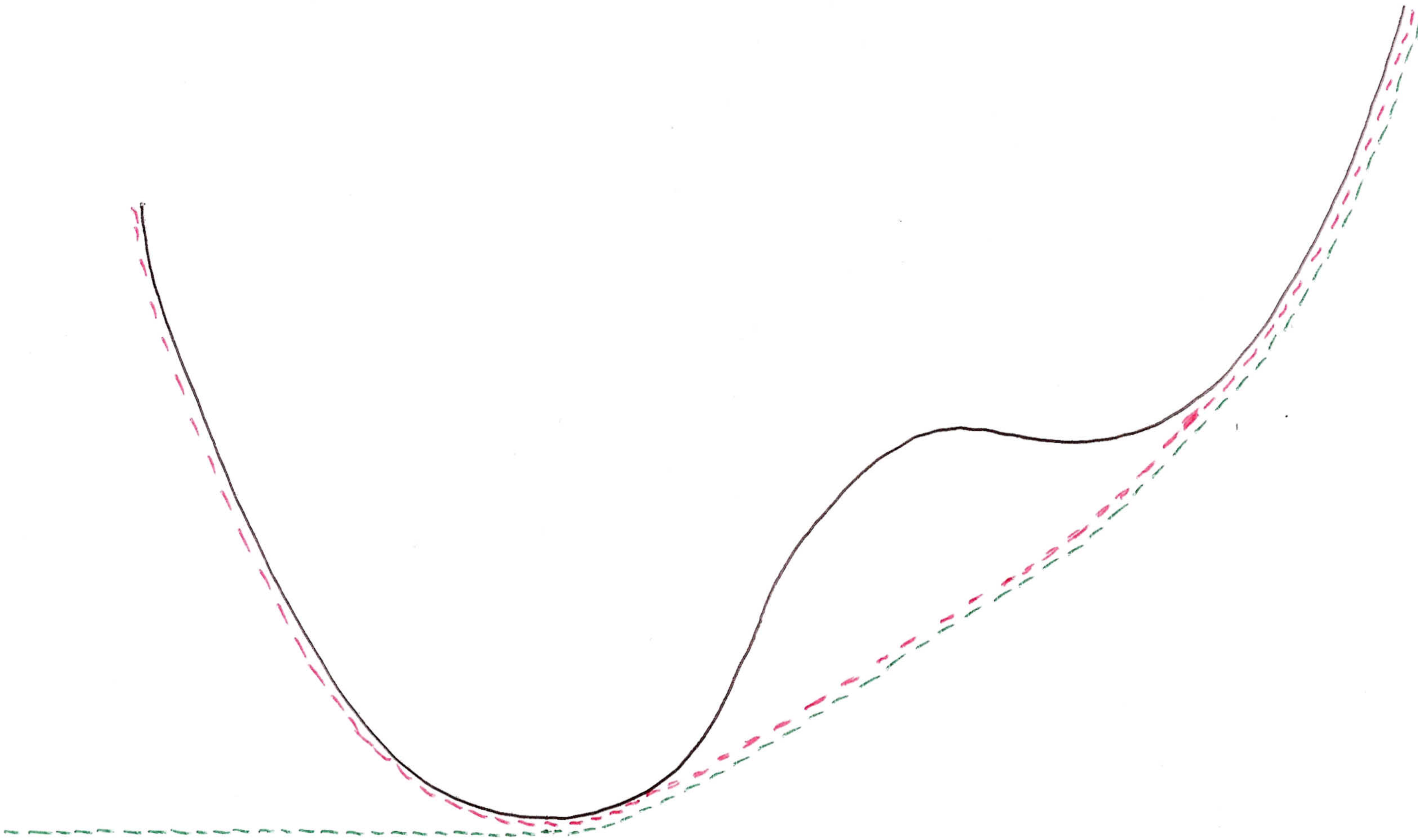
(P) minimize $h(x)$ for $x \in X$

FACT 1 $\quad h^B = (h^* + \iota_B)^*$

FACT 2 $\quad$ If $0 \in B$ then $\inf_X h^B = \inf_X h$ and $\operatorname{argmin} h^B \supset \operatorname{argmin} h$

Question: Is there a formula for $\operatorname{argmin} h^B$ in terms of the ε-approximate minima of h?

Observation: If $B \supset \operatorname{dom} h^*$ we know that $\operatorname{argmin} h^B = \bigcap_{\varepsilon > 0} (\varepsilon\text{-}\operatorname{argmin} h) \top \operatorname{dom} h^*$



h $\overline{\mathrm{co}}\, h$ nondecreasing closed convex hull of h

B-REGULARIZATION: AN EXAMPLE

Let $\emptyset \neq S \subset X$ be a (nonnecessarily convex) cone

Say that $h : X \to \overline{\mathbb{R}}$ is S-nondecreasing if $u - x \in S \Rightarrow h(u) \geq h(x)$

Example $\langle \cdot, y \rangle$ S-nondecreasing $\Leftrightarrow$ $y \in S^+$ (nonnegative polar cone of S)

Assume h is proper and $\operatorname{dom} h^* \cap S^+ \neq \emptyset$

Taking $B = S^+$ we get for h^B the greatest lsc convex S-nondecreasing function minorizing h

FACTS .) h^B is (also) the greatest lsc convex $(\overline{co}\, S)$-nondecreasing function minorizing h

.) h S-nondecreasing $\Rightarrow$ h^{**} $(\overline{co}\, S)$-nondecreasing

.) For any $h \in \Gamma(X)$,
h S-nondecreasing $\Leftrightarrow$ h $(\overline{co}\, S)$-nondecreasing

FROM B-RELAXATION TO SUBDIFFERENTIAL CALCULUS

$B \subset X^*$ $\quad h : X \to \overline{\mathbb{R}}$ proper, bounded below, $h^B = (h^* + L_B)^*$

$$x \in \operatorname{argmin} h^B \iff 0 \in \partial (h^* + L_B)^*(x) \iff x \in \partial (h^* + L_B)^{**}(0)$$

FACT $\quad 0 \in B \Rightarrow (h^* + L_B)^{**}(0) = (h^* + L_B)(0) = h^*(0)$

CONSEQUENCE $\quad \operatorname{argmin} h^B = \partial (h^* + L_B)(0)$ whenever $0 \in B$

We are now concerned with the problem of finding

$$\partial (f + L_D)(x)$$

where $f = g^* \in \Gamma(X)$, $D \subset X$, $x \in D \cap \operatorname{dom} f$

Rq 1 The set D is not necessarily convex

Rq 2 Our aim is to express $\partial (f + L_D)(x)$ in terms of any function $g : X^* \to \overline{\mathbb{R}}$, not necessarily convex, such that $g^* = f$

TOOL 1 MULTIMAPS $(\partial_\varepsilon g)^{-1}$

Given $g : X^* \to \overline{\mathbb{R}}$, $\varepsilon \geq 0$, define $M_\varepsilon g : X \rightrightarrows X^*$ as $(\partial_\varepsilon g)^{-1}$: for any $(x,y) \in X \times X^*$

$$x \in \partial_\varepsilon g(y) \iff y \in M_\varepsilon g(x) \iff y \in \varepsilon\text{-}\operatorname{argmin}(g - \langle x, \cdot \rangle)$$

For $\varepsilon = 0$ we set $Mg(x) := M_0 g(x) = \operatorname{argmin}(g - \langle x, \cdot \rangle)$ $\quad \partial_0 g = \partial g$

INTEREST: CALCULUS RULES ON $M_\varepsilon g$ IN THE ABSENCE OF CONVEXITY

Two very simple examples

1) Let $g(y) = \min(g_1(y), g_2(y))$ and $x \in X$ s.t. $g_1^*(x) = g_2^*(x) \in \mathbb{R}$. Then

$$M_\varepsilon g(x) = M_\varepsilon g_1(x) \cup M_\varepsilon g_2(x)$$

2) Assume the infimal convolution $g_1 \square g_2$ is exact. Then

$$M(g_1 \square g_2)(x) = Mg_1(x) + Mg_2(x) \quad \forall x \in X$$

TOOL 2 : OPERATION $\top$

For any $E \subset X^*$, $F \subset X$, define $E \top F = \bigcap_{u \in F} \overline{co}(E + u^-)$, $\phi \top \phi = X^*$,
where $u^- = \{ y \in X^* : \langle u, y \rangle \leq 0 \}$.

FACTS

- $E \top F$ is the intersection of all the half-spaces $[\langle u, \cdot \rangle \leq z]$ containing E and whose slope u belongs to F
- $E \top F = \{ y \in X^* : \langle u, y \rangle \leq L^*_E(u), \forall u \in F \}$
- For each $F \subset X$, the map $X^* \supset E \mapsto E \top F$ is a closure operator: extensive, isotone, idempotent
- For each $E \subset X^*$, the map $X \supset F \mapsto E \top F$ is a polarity:

$$E \top (\bigcup_i F_i) = \bigcap_i E \top F_i$$

Dual version $\quad A \subset X$, $B \subset X^*$ $\quad A \top B = \bigcap_{y \in B} \overline{co}(A + y^-)$

SOME FORMULAS FOR $E \top F$

Notation $M^\circ = [l^*_M \leq 1]$ (resp. $M^- = [l^*_M \leq 0]$) polar set (resp. polar cone) of $M \subset X$ or $M \subset X^*$, $b(M) = \operatorname{dom} l^*_M$ barrier cone of M, $\operatorname{cone} M = \mathbb{R}_+ M$

PR1 $\overline{co}(E + F^-) \subset \bigcap_{\varepsilon > 0} \overline{co}(E + \varepsilon F^\circ) \subset E \top F \subset (E^\circ \cap \operatorname{cone} F)^\circ \quad \forall E \subset X^*, \forall F \subset E$

PR2 cone F convex, $b(E) \cap \operatorname{ri} \operatorname{cone} F \neq \phi \Rightarrow E \top F = \overline{co}(E + F^-)$

$b(E) \cap \operatorname{cone} F$ convex, $\operatorname{ri}(b(E) \cap \operatorname{cone} F) \neq \phi \Rightarrow E \top F = \overline{co}(E + (b(E) \cap F)^-)$

cone F convex, $F \subset b(E)$, $\dim X < \infty \Rightarrow E \top F = \overline{co}(E + F^-)$

PR3 cone F convex closed $\Rightarrow E \top F = \overline{co}(E + F^-)$

$b(E) \cap \operatorname{cone} F$ convex closed $\Rightarrow E \top F = \overline{co}(E + (b(E) \cap F)^-)$

PR4 cone F convex and, either, l^*_E finite and continuous at a point of cone F, or, $b(E) \cap \operatorname{int}(\operatorname{cone} F) \neq \phi$, $\Rightarrow E \top F = (\overline{co} E) + F^-$

PR5 $0 \in F$ closed convex $\Rightarrow \bigcap_{\varepsilon > 0} \overline{co}(E + \varepsilon F^\circ) = E \top F$

$\top$ AND THE POLAR OF AN INTERSECTION

PR 6 (a) For any sets $C, D \subset X$ such that $0 \in C = \overline{co}\, C$ and D is a (non necessarily convex) cone, one has

$$(C \cap D)^{\circ} = C^{\circ} \top D$$

(b) For any closed convex cone C, any D, one has

$$(C \cap D)^{-} = C^{-} \top D$$

COR 1 (a) Assume $0 \in C = \overline{co}\, C$, D is a convex cone, and either $\operatorname{cone} C \cap \operatorname{ri} D \neq \emptyset$ or D is closed. Then

$$(C \cap D)^{\circ} = \operatorname{cl}(C^{\circ} + D^{\circ})$$

(b) If C is a closed convex cone, $\operatorname{cone} D$ is convex, and either $C \cap \operatorname{ri} \operatorname{cone} D \neq \emptyset$ or $\operatorname{cone} D$ is closed, then

$$(C \cap D)^{-} = \operatorname{cl}(C^{-} + D^{-})$$

THE USE OF $\top$ IN SUBDIFFERENTIAL CALCULUS

FACTS
$$\begin{cases} \forall A \subset X, \ \forall B \subset X^*, B \neq \emptyset: & A \top B = \partial\left(L_A^* + L_{\operatorname{cone} B}\right)(0) \\ \text{If } 0 \in B \text{ then} & A \top B = \partial\left(L_A^* + L_B\right)(0) \end{cases}$$

PR7 Let $q \in \Gamma(X)$ be sublinear, and $0 \in D \subset X$. Then

$$\partial(q + L_D)(0) = \partial q(0) \top D$$

If, moreover, cone D is convex and, either, cone D is closed or $\operatorname{dom} q \cap \operatorname{ri} \operatorname{cone} D \neq \emptyset$, then

$$\partial(q + L_D)(0) = \operatorname{cl}\left(\partial q(0) + N(D,0)\right)$$

where $N(D,0) = D^-$ is the normal cone of D at 0

$\top$ AND SUBDIFFERENTIAL CALCULUS CONTINUED...

TH 1 For any $f \in \overline{\mathbb{R}}^X$, $D \subset X$, $x \in D \cap f^{-1}(\mathbb{R})$, $\alpha \geq 0$, one has

$$\partial_\alpha f(x) \top (D - x) \subset \bigcap_{\varepsilon > \alpha} \left(\partial_\varepsilon f(x) \top (D - x)\right) \subset \bigcap_{\varepsilon > \alpha} \left(\partial_\varepsilon f(x) \top (D \cap \operatorname{dom} f - x)\right) \subset \partial_\alpha (f + \iota_D)(x)$$

If $f \in \Gamma(X)$ and $D - x$ is a (non necessarily convex) cone, then

$$\partial_\varepsilon (f + \iota_D)(x) = \partial_\varepsilon f(x) \top (D - x) \quad \forall \varepsilon > 0$$

If, moreover, $D - x$ is a <u>convex</u> cone and, either, $\operatorname{dom} f \cap \operatorname{ri} D \neq \emptyset$ or D is closed, then

$$\partial_\varepsilon (f + \iota_D)(x) = \operatorname{cl}\left(\partial_\varepsilon f(x) + N(D, x)\right) \quad \forall \varepsilon > 0$$

$$\partial (f + \iota_D)(x) = \bigcap_{\varepsilon > 0} \operatorname{cl}\left(\partial_\varepsilon f(x) + N(D, x)\right)$$

MORE ON T AND EXACT SUBDIFFERENTIAL CALCULUS

Definition A set $D \subset X$ is said to be pseudo-starshaped at $x \in D$ if (equivalently)

$$\forall u \in D, \quad \forall \lambda \in (0,1), \exists \mu \in (0,\lambda) \text{ s.t. } \quad x + \mu(u-x) \in D$$

$$\forall u \in D, \exists (\lambda)_{n \in \mathbb{N}} \subset (0,+\infty) \text{ s.t. } \lim_{n \to \infty} \lambda_n = 0 \text{ and } x + \lambda_n(u-x) \in D, \forall n \in \mathbb{N}$$

$D - x$ is included in the radial tangent cone of D at x

Notation Given $\varepsilon \geq 0$, $g : X^* \to \overline{\mathbb{R}}$, and $x \in X$ s.t. $g^*(x) \in \mathbb{R}$ we set

$$M_\varepsilon g(x) = \varepsilon\text{-argmin}(g - \langle x, \cdot \rangle) = (\partial_\varepsilon g)^{-1}(x)$$

TH2 Let $D \subset X$, $g : X^* \to \overline{\mathbb{R}}$, $f = g^*$, and $x \in D$ s.t. $f(x) \in \mathbb{R}$.

Assume that $D \cap \operatorname{dom} f$ is pseudo-starshaped at x. Then

$$\partial(f + \iota_D)(x) = \bigcap_{\varepsilon > 0} \left(M_\varepsilon g(x) \, T \, (D \cap \operatorname{dom} f - x) \right)$$

$$= \bigcap_{\varepsilon > 0} \left(\partial_\varepsilon f(x) \, T \, (D - x) \right)$$

COR 2 Let $f \in \Gamma(X)$, $D \subset X$, $x \in D \cap \operatorname{dom} f$.

Assume $D \cap \operatorname{dom} f$ is pseudo-starshaped at x,

$\operatorname{cone}(D - x)$ is convex,

and either $\operatorname{cone}(D-x)$ is closed or $\operatorname{cone}(\operatorname{dom} f - x) \cap \operatorname{ri} \operatorname{cone}(D-x) \neq \phi$.

We then have

$$\partial(f + L_D)(x) = \bigcap_{\varepsilon > 0} \operatorname{cl}\left(\partial_\varepsilon f(x) + N(D, x)\right)$$

COR 3 Assume $f \in \Gamma(X)$, D convex, $\operatorname{dom} f \cap \operatorname{ri} D \neq \phi$.

Then,

$$\partial(f + L_D)(x) = \bigcap_{\varepsilon > 0} \operatorname{cl}\left(\partial_\varepsilon f(x) + N(D, x)\right) \quad \forall x \in X$$

T AND EXACT SUBDIFFERENTIAL CALCULUS CONTINUED...

PR8 Let $f : X \to \overline{\mathbb{R}}$ be convex and such that $f'(x,\cdot)$ is lsc and proper at a given point $x \in f^{-1}(\mathbb{R})$. For any set $D \subset X$, $x \in D$, one has

$$\partial(f + \iota_D)(x) = \partial f(x) \,\mathrm{T}\, (D - x)$$

If, moreover, cone$(D-x)$ is convex and, either, cone$(D-x)$ is closed or cone$(\mathrm{dom} f - x) \cap \mathrm{ri\,cone}(D-x) \neq \emptyset$, then

$$\partial(f + \iota_D)(x) = \mathrm{cl}\left(\partial f(x) + N(D,x)\right)$$

Remarks. • $f'(x,\cdot)$ is lsc and proper whenever

f is finite and continuous at x

f is polyhedral and $x \in \mathrm{dom} f$

X is Banach, $f \in \Gamma(X)$, and cone$(\mathrm{dom} f - x)$ is a closed linear space

• If $f = \iota_A$ with $x \in A$ convex, then $f'(x,\cdot)$ lsc means that cone$(A - x)$ is closed

USING $M_\varepsilon g = (\partial_\varepsilon g)^{-1}$ WITH g NONCONVEX

I arbitrary set, $(g_i)_{i\in I} \subset \overline{\mathbb{R}}^{X^*}$, $g = \inf_{i\in I} g_i$, $f = g^* = \sup_{i\in I} g_i^*$

Assume $f(x)\in\mathbb{R}$, $D\subset X$, and $D\cap \operatorname{dom} f$ pseudo-starshaped at x. By TH2:

$$\partial(f+\iota_D)(x) = \bigcap_{\varepsilon>0}\left(M_\varepsilon g(x)\,T\,(D\cap\operatorname{dom} f - x)\right)$$

Now

$$M_{\varepsilon/2}\, g(x) \subset \bigcup_{i\in I_\varepsilon(x)} M_\varepsilon g_i(x) \subset M_{2\varepsilon}\, g(x)$$

where

$$I_\varepsilon(x) = \{ i\in I : g_i^*(x) \geq g^*(x) - \varepsilon \}$$

and we can state

TH3 $$\partial(f+\iota_D)(x) = \bigcap_{\varepsilon>0}\left(\left(\bigcup_{i\in I_\varepsilon(x)} M_\varepsilon g_i(x)\right) T\,(D\cap\operatorname{dom} f - x)\right)$$

COR 4 Let $(f_i)_{i\in I} \subset \Gamma(X)$, $f = \sup_{i\in I} f_i$, $D \subset X$.

Assume that $D \cap \operatorname{dom} f$ is pseudo-starshaped at a given point x. We then have

$$\partial(f + \iota_D)(x) = \bigcap_{\varepsilon>0} \Big(\Big(\bigcup_{i \in I_\varepsilon(x)} \partial_\varepsilon f_i(x) \Big)^{\top} (D \cap \operatorname{dom} f - x) \Big)$$

where $I_\varepsilon(x) = \{ i \in I : f_i(x) \geq f(x) - \varepsilon \}$

If the index set I is finite, then

$$\partial(f + \iota_D)(x) = \bigcap_{\varepsilon>0} \Big(\Big(\bigcup_{i \in I(x)} \partial_\varepsilon f_i(x) \Big)^{\top} (D - x) \Big)$$

where $I(x) = \{ i \in I : f_i(x) = f(x) \}$

Remark Taking $D = X$ we recover known results, for instance: M. A. López and M. V. (2008), or Brøndsted (1972)...

DIRECTIONAL DERIVATIVES

COR 5 $(f_i)_{i\in I} \subset \Gamma(X)$ $\quad D \subset X$ $\quad f = \sup_{i\in I} f_i$

Assume $D \cap \operatorname{dom} f$ is pseudo-starshaped at x. Then

$$y \in \partial(f + \iota_D)(x) \iff \sup_{i\in I_\varepsilon(x)} (f_i)'_\varepsilon(x,d) \geq \langle d, y\rangle, \forall \varepsilon > 0, \forall d \in \operatorname{cone}(D \cap \operatorname{dom} f - x)$$

In particular

$$x \text{ minimizes } f \text{ on } D \iff \sup_{i\in I_\varepsilon(x)} (f_i)'_\varepsilon(x,d) \geq 0, \forall \varepsilon > 0, \forall d \in \operatorname{cone}(D \cap \operatorname{dom} f - x)$$

Taking $D = X$ we get

COR 6 $\quad \iota^*_{\partial f(x)}(d) \leq \inf_{\varepsilon>0} \sup_{i\in I_\varepsilon(x)} (f_i)'_\varepsilon(x,d) \leq f'(x,d) \quad \forall d \in \operatorname{cone}(\operatorname{dom} f - x)$

If, moreover, $f'(x,\cdot)$ is lsc proper, then

$$f'(x,d) = \inf_{\varepsilon>0} \sup_{i\in I_\varepsilon(x)} (f_i)'_\varepsilon(x,d) \qquad \forall d \in \operatorname{cone}(\operatorname{dom} f - x)$$

USING T, $Mg = (\partial g)^{-1}$, AND COMPACITY

TH 4 Let $g : X^* \to \overline{\mathbb{R}}$, $f = g^*$, $D \subset X$, and $x \in f^{-1}(\mathbb{R})$. Assume

1) $D \cap \operatorname{dom} f$ is pseudo-starshaped at x
2) there is a topology ν on X^*, compatible with the pairing (X^*, X), and such that $g - \langle x, \cdot \rangle$ admits a ν-relatively compact nonvoid sublevel set.

We then have

$$\partial(f + \iota_D)(x) = \Big(\bigcap_{\varepsilon > 0} \nu\text{-}\mathrm{cl}\, M_\varepsilon g(x) \Big) T (D \cap \operatorname{dom} f - x) \neq \emptyset$$

If, moreover, g is ν-lsc, then

$$\partial(f + \iota_D)(x) = Mg(x)\, T\, (D \cap \operatorname{dom} f - x)$$

If, in addition, $\operatorname{cone}(D \cap \operatorname{dom} f - x) = X$, then

$$\partial(f + \iota_D)(x) = \overline{\mathrm{co}}\, Mg(x)$$

BACK TO B-RELAXATION

Given $h : X \to \overline{\mathbb{R}}$ and $B \subset X^*$, define

$$h^B = \sup\{\langle \cdot, y\rangle - r : y \in B, r \in \mathbb{R}, \langle \cdot, y\rangle - r \le h\}$$

Recall that if $0 \in B$ one has

$$\operatorname{argmin} h^B = \partial(h^* + \iota_B)(0)$$

From TH2 we get

TH5 Let $h : X \to \overline{\mathbb{R}}$ be proper and bounded below, $B \subset X^*$.
Assume that $B \cap \operatorname{dom} h^*$ is pseudo-starshaped at 0. Then,

$$\operatorname{argmin} h^B = \bigcap_{\varepsilon > 0} \left((\varepsilon\text{-}\operatorname{argmin} h)^{\top} (B \cap \operatorname{dom} h^*) \right)$$

$$= \bigcap_{\varepsilon > 0} \left((\varepsilon\text{-}\operatorname{argmin} h^{**})^{\top} B \right)$$

B-RELAXATION AND COMPACITY

Applying (the dual form of) TH4 we get

TH6 Let $h : X \to \overline{\mathbb{R}}$ be proper, bounded below, and let $B \subset X^*$. Assume

1) $B \cap \operatorname{dom} h^*$ is pseudo-starshaped at 0
2) there is a topology ϑ on X, compatible with the pairing (X, X^*), and such that h admits a ϑ-relatively compact nonvoid sublevel set

Then we have

$$\operatorname{argmin} h^B = \Big(\bigcap_{\varepsilon > 0} \vartheta\text{-}\mathrm{cl}(\varepsilon\text{-}\operatorname{argmin} h)\Big) \top \big(B \cap \operatorname{dom} h^*\big) \neq \emptyset$$

If, moreover, h is ϑ-lsc, then

$$\operatorname{argmin} h^B = (\operatorname{argmin} h) \top (B \cap \operatorname{dom} h^*)$$

B-RELAXATION IN BANACH SPACES

TH7 Let X Banach, $h : X \to \overline{\mathbb{R}}$ proper, and $B \subset X^*$. Assume

1) $B \cap \operatorname{dom} h^*$ is pseudo-starshaped at 0
2) $\operatorname{cone}(B \cap \operatorname{dom} h^*)$ is convex
3) $\exists t > \inf_X h : [h \leq t]$ is relatively compact

Then

$$\operatorname{argmin} h^B = \left(\overline{\operatorname{co}} \bigcap_{\varepsilon > 0} \overline{\varepsilon\text{-}\operatorname{argmin} h}\right) + (B \cap \operatorname{dom} h^*)^-$$

$$= \left(\bigcap_{\varepsilon > 0} \overline{\operatorname{co}}(\varepsilon\text{-}\operatorname{argmin} h)\right) + (B \cap \operatorname{dom} h^*)^-$$

If, moreover, h is lsc, then

$$\operatorname{argmin} h^B = \overline{\operatorname{co}}(\operatorname{argmin} h) + (B \cap \operatorname{dom} h^*)^-$$

B-RELAXATION AND WEAK TOPOLOGY

TH8 Let X reflescive, $h : X \to \overline{\mathbb{R}}$ proper, and $B \subset X^*$. Assume

1) $B \cap \operatorname{dom} h^*$ is pseudo-starshaped at 0
2) $\operatorname{cone}(B \cap \operatorname{dom} h^*)$ is convex
3) $\exists t > \inf_X h : [h \leq t]$ is bounded

Then

$$\operatorname{argmin} h^B = \left(\overline{\operatorname{co}} \bigcap_{\varepsilon > 0} w\text{-}cl(\varepsilon\text{-}\operatorname{argmin} h)\right) + (B \cap \operatorname{dom} h^*)^-$$

$$= \left(\bigcap_{\varepsilon > 0} \overline{\operatorname{co}}(\varepsilon\text{-}\operatorname{argmin} h)\right) + (B \cap \operatorname{dom} h^*)^-$$

If, moreover, h is weakly lsc, then

$$\operatorname{argmin} h^B = \overline{\operatorname{co}}(\operatorname{argmin} h) + (B \cap \operatorname{dom} h^*)^-$$

References

[1] J.Benoist, J.-B.Hiriart-Urruty, What is the subdifferential of the closed convex hull of a function? SIAM J. Math. Anal. 27 (1996) 6 1661-1679

[2] R.Correa, A.Hantoute, New formulas for the Fenchel subdifferential of the conjugate function, to appear in SVVA

[3] A.Hantoute, M.A.Lopez, C.Zalinescu, Subdifferential calculus rules in convex analysis: a unifying approach via pointwise supremum functions, SIAM J. Opt. 19 (2008) 863-882

[4] J.-B.Hiriart-Urruty, R.R.Phelps, Subdifferential calculus using epsilon subdifferentials, J.Funct. Anal. 118 (1993) 154-166

[5] J.-B.Hiriart-Urruty, M.Moussaoui, A.Seeger, M.Volle, Subdifferential calculus without qualifications conditions, using approximate subdifferentials: a survey, Nonlinear Anal. 24 (1995) 1727-1754

[6] J.-B.Hiriart-Urruty, M.A.Lopez, M.Volle, The epsilon strategy in variational analysis: illustration with the closed convex hull of a function, to appear in Rev.Mat.Iber.

[7] J.-B.Hiriart-Urruty, M.Volle, « Adequate » weakly lower-semicontinuous functions on reflexive Banach spaces are necessarily strictly convex, submitted

[8] M.A.Lopez, M.Volle, A formula for the optimal solution of a relaxed minimization problem. Applications to subdifferential calculus, J. Conv. Anal. 17 (2010) 1057-1075

[9] M.A.Lopez, M.Volle, On the subdifferential of the supremum of an arbitrary family of extended real-valued functions, to appear in RACSAM

[10] M.Volle, A primal-dual operation on sets linked with closed convex relaxation processes, submitted