SSDB spaces and maximal monotonicity

by

Stephen Simons

<simons@math.ucsb.edu>.

Abstract

We introduce SSDB spaces, which include Hilbert spaces, negative Hilbert spaces and spaces of the form $E \times E^*$, where E is a reflexive real Banach space. We introduce q–positive subsets of a SSDB space, which include monotone subsets of $E \times E^*$, and BC–functions on a SSDB spaces, which include Fitzpatrick functions of monotone multifunctions. We show how convex analysis can be combined with SSDB space theory to obtain and generalize various results on maximally monotone multifunctions on a reflexive Banach space, such as the significant direction of Rockafellar's surjectivity theorem, sufficient conditions for the sum of maximally monotone multifunctions to be maximally monotone, and an abstract Brezis–Browder theorem.

Downloads

You can download files containing related materials from
<www.math.ucsb.edu/~simons/NC.html>.

Outline of lecture

SSDB spaces. The associated quadratic form, q.

q–positive sets, $\mathcal{P}_q(f)$, the intrinsic conjugate $^@$, BC–functions and the surprise result.

Pos–neg theorem.

Theorems on BC–functions.

The bivariate Attouch–Brezis theorems.

The partial episum theorems for BC–functions.

The combination lemma and theorem.

The convex function given by a q–positive set.

The Fitzpatrick function.

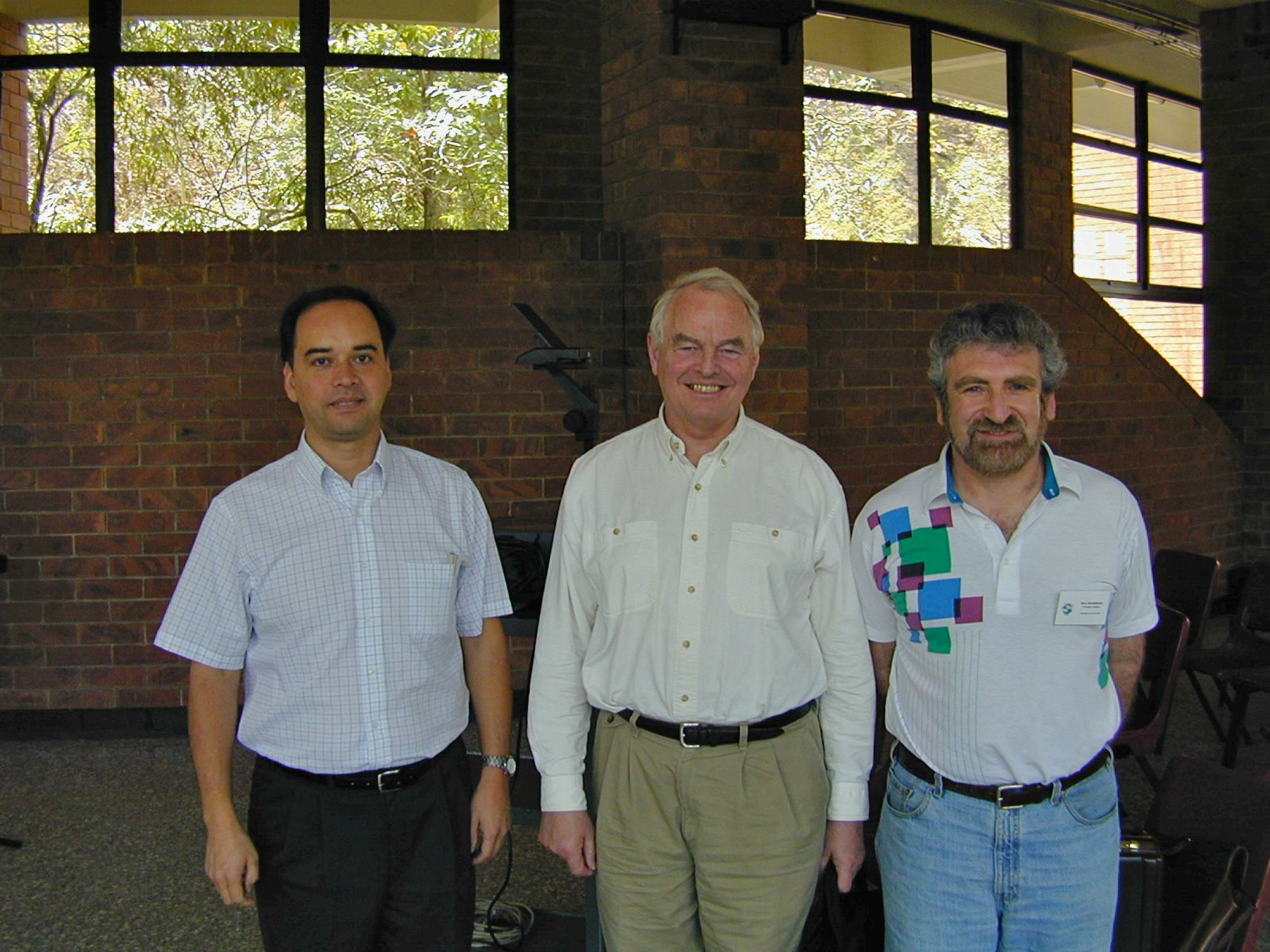
Rockafellar's surjectivity theorem.

The sum of maximally monotone multifunctions on a reflexive Banach space.

q–positive and q–negative sets and subspaces.

Polar subspaces.

Results of Brezis–Browder and Yao.




THERA


A. Seeger
France


VIEWING
DECK


FedEx
Free events
in Millennium Park
all
Summer
long


Jean-Baptiste HIRIART-
URRUTY


J.-B. Hiriart-Urruty

SSDB spaces

We will say that B $\big($more precisely, $\big(B, \lfloor\cdot,\cdot\rfloor, q, \|\cdot\|, \iota\big)\big)$ is a *symmetrically self–dual Banach space (SSDB space)* if B is a nonzero Banach space, $\lfloor\cdot,\cdot\rfloor\colon B \times B \mapsto \mathbb{R}$ is a symmetric bilinear form, the quadratic form q on B is defined by $q(b) := \frac{1}{2}\lfloor b, b\rfloor$ and $\exists$ a linear isometry ι from B onto B^* such that, for all $b, c \in B$, $\langle b, \iota(c)\rangle = \lfloor b, c\rfloor$.

SSDB spaces

We will say that B $\big($more precisely, $\big(B, \lfloor\cdot,\cdot\rfloor, q, \|\cdot\|, \iota\big)\big)$ is a *symmetrically self–dual Banach space (SSDB space)* if B is a nonzero Banach space, $\lfloor\cdot,\cdot\rfloor\colon B \times B \mapsto \mathbb{R}$ is a symmetric bilinear form, the quadratic form q on B is defined by $q(b) := \frac{1}{2}\lfloor b, b\rfloor$ and $\exists$ a linear isometry ι from B onto B^* such that, for all $b, c \in B$, $\langle b, \iota(c)\rangle = \lfloor b, c\rfloor$.

The quadratic form q

We have the parallelogram law:

$$b, c \in B \quad \Longrightarrow \quad \tfrac{1}{2}q(b-c) + \tfrac{1}{2}q(b+c) = q(b) + q(c).$$

SSDB spaces

We will say that B $\big($more precisely, $\big(B, \lfloor\cdot,\cdot\rfloor, q, \|\cdot\|, \iota\big)\big)$ is a *symmetrically self–dual Banach space (SSDB space)* if B is a nonzero Banach space, $\lfloor\cdot,\cdot\rfloor\colon B \times B \mapsto \mathbb{R}$ is a symmetric bilinear form, the quadratic form q on B is defined by $q(b) := \frac{1}{2}\lfloor b, b\rfloor$ and $\exists$ a linear isometry ι from B onto B^* such that, for all $b, c \in B$, $\langle b, \iota(c)\rangle = \lfloor b, c\rfloor$.

The quadratic form q

We have the parallelogram law:

$$b, c \in B \quad \Longrightarrow \quad \tfrac{1}{2}q(b - c) + \tfrac{1}{2}q(b + c) = q(b) + q(c).$$

Examples

(a) If B is a Hilbert space with inner product $(b, c) \mapsto \langle b, c\rangle$ then B is a SSDB space with $\lfloor b, c\rfloor := \langle b, c\rangle$, $q(b) = \frac{1}{2}\|b\|^2$ and $\iota(c) := c$.

(b) If B is a Hilbert space with inner product $(b, c) \mapsto \langle b, c\rangle$ then B is a SSDB space with $\lfloor b, c\rfloor := -\langle b, c\rangle$, $q(b) = -\frac{1}{2}\|b\|^2$ and $\iota(c) := -c$.

(c) $\mathbb{R}^3$ is a SSDB space with $\big\lfloor (b_1, b_2, b_3), (c_1, c_2, c_3)\big\rfloor := b_1c_2 + b_2c_1 + b_3c_3$. Then $q(b_1, b_2, b_3) = b_1b_2 + \frac{1}{2}b_3^2$ and $\iota(c_1, c_2, c_3) := (c_2, c_1, c_3)$.

(d) $\mathbb{R}^3$ is **not** a SSDB space with $\big\lfloor (b_1, b_2, b_3), (c_1, c_2, c_3)\big\rfloor := b_1c_2 + b_2c_3 + b_3c_1$. (The bilinear form $\lfloor\cdot,\cdot\rfloor$ is not symmetric.)

SSDB spaces

We will say that B $\big($more precisely, $\big(B, \lfloor\cdot,\cdot\rfloor, q, \|\cdot\|, \iota\big)\big)$ is a *symmetrically self–dual Banach space (SSDB space)* if B is a nonzero Banach space, $\lfloor\cdot,\cdot\rfloor\colon B \times B \mapsto \mathbb{R}$ is a symmetric bilinear form, the quadratic form q on B is defined by $q(b) := \frac{1}{2}\lfloor b, b\rfloor$ and $\exists$ a linear isometry ι from B onto B^* such that, for all $b, c \in B$, $\langle b, \iota(c)\rangle = \lfloor b, c\rfloor$.

SSDB spaces

We will say that B $\big($more precisely, $(B, \lfloor\cdot,\cdot\rfloor, q, \|\cdot\|, \iota)\big)$ is a *symmetrically self–dual Banach space (SSDB space)* if B is a nonzero Banach space, $\lfloor\cdot,\cdot\rfloor\colon B \times B \mapsto \mathbb{R}$ is a symmetric bilinear form, the quadratic form q on B is defined by $q(b) := \frac{1}{2}\lfloor b, b\rfloor$ and $\exists$ a linear isometry ι from B onto B^* such that, for all $b, c \in B$, $\langle b, \iota(c)\rangle = \lfloor b, c\rfloor$.

Another example

(e) Let E be a nonzero reflexive Banach space and $B := E \times E^*$ under the norm

$$\big\|(x, x^*)\big\| := \sqrt{\|x\|^2 + \|x^*\|^2}.$$

Let $\big(E \times E^*, \|\cdot\|\big)^* = \big(E^* \times E, \|\cdot\|\big)$, with $\big\|((y^*, y)\big\| := \sqrt{\|y^*\|^2 + \|y\|^2}$ and $\big\langle (x, x^*), (y^*, y)\big\rangle := \langle x, y^*\rangle + \langle y, x^*\rangle$. $\forall\ (x, x^*),\ (y, y^*) \in B$, let

$$\big\lfloor (x, x^*), (y, y^*)\big\rfloor := \langle x, y^*\rangle + \langle y, x^*\rangle.$$

Then B is a SSDB space,

$$q(b) = \langle x, x^*\rangle$$

and

$$\iota(y, y^*) := (y^*, y).$$

Any finite dimensional SSDB space of this form must have even dimension. Thus odd dimensional cases of the examples considered on the previous slide cannot be of this form.

SSDB spaces

We will say that B $\big($more precisely, $(B, \lfloor\cdot,\cdot\rfloor, q, \|\cdot\|, \iota)\big)$ is a *symmetrically self–dual Banach space (SSDB space)* if B is a nonzero Banach space, $\lfloor\cdot,\cdot\rfloor\colon B \times B \mapsto \mathbb{R}$ is a symmetric bilinear form, the quadratic form q on B is defined by $q(b) := \frac{1}{2}\lfloor b, b\rfloor$ and $\exists$ a linear isometry ι from B onto B^* such that, for all $b, c \in B$, $\langle b, \iota(c)\rangle = \lfloor b, c\rfloor$.

Another example

(e) Let E be a nonzero reflexive Banach space and $B := E \times E^*$ under the norm

$$\big\|(x, x^*)\big\| := \sqrt{\|x\|^2 + \|x^*\|^2}.$$

Let $\big(E \times E^*, \|\cdot\|\big)^* = \big(E^* \times E, \|\cdot\|\big)$, with $\big\|((y^*, y)\big\| := \sqrt{\|y^*\|^2 + \|y\|^2}$ and $\big\langle (x, x^*), (y^*, y)\big\rangle := \langle x, y^*\rangle + \langle y, x^*\rangle$. $\forall\ (x, x^*),\ (y, y^*) \in B$, let

$$\big\lfloor (x, x^*), (y, y^*)\big\rfloor := \langle x, y^*\rangle + \langle y, x^*\rangle.$$

Then B is a SSDB space,

$$q(b) = \langle x, x^*\rangle$$

and

$$\iota(y, y^*) := (y^*, y).$$

Any finite dimensional SSDB space of this form must have even dimension. Thus odd dimensional cases of the examples considered on the previous slide cannot be of this form.

- From now on, $B = \big(B, \lfloor\cdot,\cdot\rfloor, q, \|\cdot\|, \iota\big)$ will always be a SSDB space.

q–positive sets

Let $A \subset B$. We say that A is *q–positive* if $A \neq \emptyset$ and

$$b, c \in A \Longrightarrow q(b - c) \geq 0.$$

Examples

(a) B is a Hilbert space with $q(b) = \frac{1}{2}\|b\|^2$: every nonempty subset of B is q–positive.

(b) B is a Hilbert space with $q(b) = -\frac{1}{2}\|b\|^2$: the q–positive subsets of B are the singletons.

(e) E is a nonzero reflexive Banach space, $B := E \times E^*$ and, $\forall\ (x, x^*) \in B$, $q(x, x^*) = \langle x, x^* \rangle$. Let $\emptyset \neq A \subset B$. Then A is q–positive when

$$(x, x^*), (y, y^*) \in A \quad \Longrightarrow \quad \langle x - y, x^* - y^* \rangle \geq 0.$$

That is to say,

$$A \text{ is } q\text{–positive} \iff A \text{ is a monotone subset of } E \times E^*.$$

q–positive sets

Let $A \subset B$. We say that A is *q–positive* if $A \neq \emptyset$ and

$$b, c \in A \Longrightarrow q(b - c) \geq 0.$$

Examples

(a) B is a Hilbert space with $q(b) = \frac{1}{2}\|b\|^2$: every nonempty subset of B is q–positive.

(b) B is a Hilbert space with $q(b) = -\frac{1}{2}\|b\|^2$: the q–positive subsets of B are the singletons.

(e) E is a nonzero reflexive Banach space, $B := E \times E^*$ and, $\forall\ (x, x^*) \in B$, $q(x, x^*) = \langle x, x^* \rangle$. Let $\emptyset \neq A \subset B$. Then A is q–positive when

$$(x, x^*), (y, y^*) \in A \quad \Longrightarrow \quad \langle x - y, x^* - y^* \rangle \geq 0.$$

That is to say,

$$A \text{ is } q\text{–positive} \iff A \text{ is a monotone subset of } E \times E^*.$$

General notation

- Let X be a vector space and $f\colon X \mapsto]-\infty, \infty]$. Then $\operatorname{dom} f := \{x \in X\colon f(x) \in \mathbb{R}\}$.
- f is *proper* if $\operatorname{dom} f \neq \emptyset$.
- $\mathcal{PC}(X)$ is the set of all proper convex functions $f\colon X \mapsto]-\infty, \infty]$.
- If X is a Banach space, $\mathcal{PCLSC}(X) := \{f \in \mathcal{PC}(X)\colon f \text{ is lower semicontinuous}\}$.

The q–positive set given by a convex function

Let $f \in \mathcal{PC}(B)$ *and* $f \geq q$ *on* B. *Let* $\mathcal{P}_q(f) := \big\{b \in B\colon\ f(b) = q(b)\big\}$. *If* $\mathcal{P}_q(f) \neq \emptyset$ *then* $\mathcal{P}_q(f)$ *is a q–positive subset of* B.

The q–positive set given by a convex function

Let $f \in \mathcal{PC}(B)$ and $f \geq q$ on B. Let $\mathcal{P}_q(f) := \{b \in B\colon\ f(b) = q(b)\}$. If $\mathcal{P}_q(f) \neq \emptyset$ then $\mathcal{P}_q(f)$ is a q–positive subset of B.

Proof. Let $b, c \in \mathcal{P}_q(f)$. Then, from the parallelogram law, the quadraticity of q, and the convexity of f,

$$\begin{aligned}\tfrac{1}{2}q(b-c) = q(b) + q(c) - \tfrac{1}{2}q(b+c) &= q(b) + q(c) - 2q\big(\tfrac{1}{2}(b+c)\big)\\ &\geq f(b) + f(c) - 2f\big(\tfrac{1}{2}(b+c)\big) \geq 0.\end{aligned}$$

□

The q–positive set given by a convex function

Let $f \in \mathcal{PC}(B)$ and $f \ge q$ on B. Let $\mathcal{P}_q(f) := \big\{b \in B\colon\ f(b) = q(b)\big\}$. If $\mathcal{P}_q(f) \ne \emptyset$ then $\mathcal{P}_q(f)$ is a q–positive subset of B.

Proof. Let $b, c \in \mathcal{P}_q(f)$. Then, from the parallelogram law, the quadraticity of q, and the convexity of f,

$$\tfrac{1}{2}q(b-c) = q(b) + q(c) - \tfrac{1}{2}q(b+c) = q(b) + q(c) - 2q\big(\tfrac{1}{2}(b+c)\big)$$
$$\ge f(b) + f(c) - 2f\big(\tfrac{1}{2}(b+c)\big) \ge 0. \qquad \square$$

• If $f \in \mathcal{PC}(B)$, we write $f^{@}$ for the conjugate of f with respect to the pairing $\lfloor\cdot,\cdot\rfloor$. That is to say, $\forall\ c \in B$,

$$f^{@}(c) := \sup\nolimits_B\big[\lfloor\cdot, c\rfloor - f\big] = \sup\nolimits_B\big[\langle\cdot, \iota(c)\rangle - f\big] = f^*\big(\iota(c)\big).$$

The q–positive set given by a convex function

Let $f \in \mathcal{PC}(B)$ and $f \geq q$ on B. Let $\mathcal{P}_q(f) := \{b \in B\colon\ f(b) = q(b)\}$. If $\mathcal{P}_q(f) \neq \emptyset$ then $\mathcal{P}_q(f)$ is a q–positive subset of B.

Proof. Let $b, c \in \mathcal{P}_q(f)$. Then, from the parallelogram law, the quadraticity of q, and the convexity of f,

$$\begin{aligned}\tfrac{1}{2}q(b-c) = q(b) + q(c) - \tfrac{1}{2}q(b+c) &= q(b) + q(c) - 2q\big(\tfrac{1}{2}(b+c)\big)\\ &\geq f(b) + f(c) - 2f\big(\tfrac{1}{2}(b+c)\big) \geq 0.\end{aligned}$$ □

• If $f \in \mathcal{PC}(B)$, we write $f^@$ for the conjugate of f with respect to the pairing $\lfloor\cdot,\cdot\rfloor$. That is to say, $\forall\ c \in B$,

$$f^@(c) := \sup\nolimits_B \big[\lfloor\cdot, c\rfloor - f\big] = \sup\nolimits_B \big[\langle\cdot, \iota(c)\rangle - f\big] = f^*\big(\iota(c)\big).$$

• Let $f \in \mathcal{PC}(B)$. f is a BC–function if

$$b \in B \quad\Longrightarrow\quad f^@(b) \geq f(b) \geq q(b). \qquad (\star)$$

"BC" stands for "bigger conjugate".

The q–positive set given by a convex function

Let $f \in \mathcal{PC}(B)$ and $f \geq q$ on B. Let $\mathcal{P}_q(f) := \{b \in B\colon\ f(b) = q(b)\}$. If $\mathcal{P}_q(f) \neq \emptyset$ then $\mathcal{P}_q(f)$ is a q–positive subset of B.

Proof. Let $b, c \in \mathcal{P}_q(f)$. Then, from the parallelogram law, the quadraticity of q, and the convexity of f,

$$\tfrac{1}{2}q(b-c) = q(b) + q(c) - \tfrac{1}{2}q(b+c) = q(b) + q(c) - 2q\big(\tfrac{1}{2}(b+c)\big)$$
$$\geq f(b) + f(c) - 2f\big(\tfrac{1}{2}(b+c)\big) \geq 0. \qquad \square$$

• If $f \in \mathcal{PC}(B)$, we write $f^{@}$ for the conjugate of f with respect to the pairing $\lfloor \cdot, \cdot \rfloor$. That is to say, $\forall\ c \in B$,

$$f^{@}(c) := \sup_B \big[\lfloor \cdot, c \rfloor - f\big] = \sup_B \big[\langle \cdot, \iota(c) \rangle - f\big] = f^*\big(\iota(c)\big).$$

• Let $f \in \mathcal{PC}(B)$. f is a BC–function if

$$b \in B \quad \Longrightarrow \quad f^{@}(b) \geq f(b) \geq q(b). \qquad (\star)$$

"BC" stands for "bigger conjugate".

Surprise result

Let $f \in \mathcal{PC}(B)$ be a BC–function. Then $\mathcal{P}_q(f^{@}) = \mathcal{P}_q(f)$.

- Let $f \in \mathcal{PC}(B)$. f is a BC–function if

$$b \in B \quad \Longrightarrow \quad f^{@}(b) \geq f(b) \geq q(b). \tag{$\star$}$$

"BC" stands for "bigger conjugate".

Surprise result

Let $f \in \mathcal{PC}(B)$ *be a BC–function. Then* $\mathcal{P}_q(f^{@}) = \mathcal{P}_q(f)$.

Proof. This proof uses a differentiability argument. Details can be found in the material on the web. Go to:

<www.math.ucsb.edu/~simons/NC.html>.

- If $f \in \mathcal{PC}(B)$ and $c \in B$, we define $f_c := f(\cdot + c) - \lfloor \cdot, c \rfloor - q(c)$. Clearly, $f_c \in \mathcal{PC}(B)$.

Translation lemma

(a) $(f_c)^{@} = (f^{@})_c$. *In view of this we write* $f_c^{@}$ *for both these function.*
(b) Let $b, d \in B$. *Then* $f_c(b) + f_c^{@}(d) - \lfloor b, d \rfloor = f(b + c) + f^{@}(d + c) - \lfloor b + c, d + c \rfloor$.
(c) $\mathcal{P}_q(f_c) = \mathcal{P}_q(f) - c$ *and* $\operatorname{dom} f_c = \operatorname{dom} f - c$.
(d) If $f \geq q$ *on* B *then* $f_c \geq q$ *on* B.
(e) Let $f \in \mathcal{PC}(B)$ *be a BC–function and* $c \in B$. *Then* f_c *is a BC–function.*

Proof. This is routine. Details can be found in the material on the web. Go to <www.math.ucsb.edu/~simons/NC.html>.

- Let $f \in \mathcal{PC}(B)$. Recall that f is a BC–function if

$$b \in B \quad \Longrightarrow \quad f^{@}(b) \geq f(b) \geq q(b). \tag{$\star$}$$

- Let $g \in \mathcal{PC}(B)$. g is a TBC–function if

$$b \in B \quad \Longrightarrow \quad g^{@}(-b) \geq g(b) \geq -q(b). \tag{☾}$$

"T" stands for "twisted". In this case, we write $\quad \mathcal{N}_q(g) := \big\{b \in B\colon\ g(b) = -q(b)\big\}.$

- Let $f \in \mathcal{PC}(B)$. Recall that f is a BC–function if

$$b \in B \quad \Longrightarrow \quad f^{@}(b) \geq f(b) \geq q(b). \tag{☆}$$

- Let $g \in \mathcal{PC}(B)$. g is a TBC–function if

$$b \in B \quad \Longrightarrow \quad g^{@}(-b) \geq g(b) \geq -q(b). \tag{☾}$$

"T" stands for "twisted". In this case, we write $\mathcal{N}_q(g) := \{b \in B\colon\ g(b) = -q(b)\}$.

> **Pos–neg theorem**
>
> *Let $f \in \mathcal{PC}(B)$ be a BC–function and $g\colon B \mapsto \mathbb{R}$ be a continuous TBC–function.*
> *Then* $$\mathcal{P}_q(f) - \mathcal{N}_q(g) = B.$$

Proof. Let $c \in B$. Since f_c is a BC–function, it follows from (☆) and (☾) that

$$b \in B \quad \Longrightarrow \quad f_c(b) + g(b) \geq q(b) - q(b) = 0.$$

Thus Rockafellar's version of the Fenchel duality theorem gives $a \in B$ such that

$${f_c}^{@}(a) + g^{@}(-a) \leq 0.$$

From (☆) and (☾) again,

$$f_c(a) + g(a) \leq 0 = q(a) - q(a).$$

From (☆) and (☾) for a third time, $f_c(a) = q(a)$ and $g(a) = -q(a)$, that is to say,

$$a \in \mathcal{P}_q(f_c) = \mathcal{P}_q(f) - c \quad \text{and} \quad a \in \mathcal{N}_q(g).$$

But then

$$c = (c + a) - a \in \mathcal{P}_q(f) - \mathcal{N}_q(g). \qquad \square$$

Pos–neg theorem

Let $f \in \mathcal{PC}(B)$ be a BC–function and $g\colon B \mapsto \mathbb{R}$ be a continuous TBC–function.
Then
$$\mathcal{P}_q(f) - \mathcal{N}_q(g) = B.$$

Theorem on BC–functions on B

Let $f \in \mathcal{PC}(B)$ be a BC–function and $g_0 := \frac{1}{2}\|\cdot\|^2$ on B. Then
$$\mathcal{P}_q(f) - \mathcal{N}_q(g_0) = B$$
and $\mathcal{P}_q(f)$ is maximally q–positive (in the obvious sense).

Proof. For all $b \in B$, $g_0{}^{@}(-b) = \frac{1}{2}\|-b\|^2 = \frac{1}{2}\|b\|^2 = g_0(b) = \frac{1}{2}\|b\|^2 \ge -\frac{1}{2}\lfloor b, b\rfloor = -q(b)$, and so g_0 is a TBC–function. The pos–neg theorem now gives
$$\mathcal{P}_q(f) - \mathcal{N}_q(g_0) = B.$$
Now suppose that $b \in B$ and $\mathcal{P}_q(f) \cup \{b\}$ is q–positive. From the above,
$$\exists\ a \in \mathcal{P}_q(f) \quad \text{such that} \quad a - b \in \mathcal{N}_q(g_0).$$
Thus
$$\tfrac{1}{2}\|a - b\|^2 = -q(a - b).$$
Since $\mathcal{P}_q(f) \cup \{b\}$ is q–positive, $q(a-b) \ge 0$, and so $\frac{1}{2}\|a-b\|^2 \le 0$, from which
$$b = a \in \mathcal{P}_q(f).$$
□

Theorem on BC–functions on B

Let $f \in \mathcal{PC}(B)$ *be a BC–function and* $g_0 := \frac{1}{2}\|\cdot\|^2$ *on* B. *Then*

$$\mathcal{P}_q(f) - \mathcal{N}_q(g_0) = B$$

and $\mathcal{P}_q(f)$ *is maximally* q*–positive (in the obvious sense).*

More on Example (e)

- From now on, E is a nonzero **reflexive** Banach space.
- Consider the SSDB space $E \times E^*$, so that $q(x, x^*) = \langle x, x^* \rangle$. Let $\emptyset \neq A \subset E \times E^*$.

We know already that

$$A \text{ is } q\text{–positive} \iff A \text{ is a monotone subset of } E \times E^*.$$

It follows that

$$A \text{ is maximally } q\text{–positive} \iff A \text{ is a maximally monotone subset of } E \times E^*.$$

Theorem on BC–functions on B

Let $f \in \mathcal{PC}(B)$ *be a BC–function and* $g_0 := \frac{1}{2}\|\cdot\|^2$ *on* B. *Then*

$$\mathcal{P}_q(f) - \mathcal{N}_q(g_0) = B$$

and $\mathcal{P}_q(f)$ *is maximally q–positive (in the obvious sense).*

More on Example (e)

- From now on, E is a nonzero **reflexive** Banach space.
- Consider the SSDB space $E \times E^*$, so that $q(x, x^*) = \langle x, x^* \rangle$. Let $\emptyset \neq A \subset E \times E^*$.

We know already that

$$A \text{ is } q\text{–positive} \iff A \text{ is a monotone subset of } E \times E^*.$$

It follows that

$$A \text{ is maximally } q\text{–positive} \iff A \text{ is a maximally monotone subset of } E \times E^*.$$

- If $g_0 := \frac{1}{2}\|\cdot\|^2$ on $E \times E^*$ then

$$(x, x^*) \in \mathcal{N}_q(g_0) \iff \tfrac{1}{2}\|x\|^2 + \tfrac{1}{2}\|x^*\|^2 = -\langle x, x^* \rangle \iff (x, x^*) \in G(-J),$$

where $J \colon E \rightrightarrows E^*$ is the duality map.

Theorem on BC–functions on B

Let $f \in \mathcal{PC}(B)$ *be a BC–function and* $g_0 := \frac{1}{2}\|\cdot\|^2$ *on* B. *Then*

$$\mathcal{P}_q(f) - \mathcal{N}_q(g_0) = B$$

and $\mathcal{P}_q(f)$ *is maximally q–positive (in the obvious sense).*

More on Example (e)

- From now on, E is a nonzero **reflexive** Banach space.
- Consider the SSDB space $E \times E^*$, so that $q(x, x^*) = \langle x, x^* \rangle$. Let $\emptyset \neq A \subset E \times E^*$.

We know already that

$$A \text{ is } q\text{–positive} \iff A \text{ is a monotone subset of } E \times E^*.$$

It follows that

$$A \text{ is maximally } q\text{–positive} \iff A \text{ is a maximally monotone subset of } E \times E^*.$$

- If $g_0 := \frac{1}{2}\|\cdot\|^2$ on $E \times E^*$ then

$$(x, x^*) \in \mathcal{N}_q(g_0) \iff \tfrac{1}{2}\|x\|^2 + \tfrac{1}{2}\|x^*\|^2 = -\langle x, x^* \rangle \iff (x, x^*) \in G(-J),$$

where $J\colon E \rightrightarrows E^*$ is the duality map.

Theorem on BC–functions on $E \times E^*$

Let $f \in \mathcal{PC}(E \times E^*)$ *be a BC–function. Then* $\mathcal{P}_q(f)$ *is maximally monotone. Further,* $\mathcal{P}_q(f^{@}) = \mathcal{P}_q(f)$ *and* $\mathcal{P}_q(f) - G(-J) = E \times E^*$.

Theorem on BC–functions on $E \times E^*$

Let $f \in \mathcal{PC}(E \times E^*)$ be a BC–function. Then $\mathcal{P}_q(f)$ is maximally monotone. Further, $\mathcal{P}_q(f^@) = \mathcal{P}_q(f)$ and $\mathcal{P}_q(f) - G(-J) = E \times E^*$.

A nice example

Let $h \in \mathcal{PCLSC}(E)$. Define $f \in \mathcal{PCLSC}(E \times E^*)$ by $f(x, x^*) := h(x) + h^*(x^*)$. It is easily seen that $f^@ = f$. Furthermore, from the Fenchel–Young inequality,

$$f(x, x^*) = h(x) + h^*(x^*) \geq \langle x, x^* \rangle = q(x, x^*).$$

Thus f is a BC–function. It now follows from the theorem on BC–functions on $E \times E^*$ that $\mathcal{P}_q(f)$ is maximally monotone. But

$$\begin{aligned} (x, x^*) \in \mathcal{P}_q(f) &\iff f(x, x^*) = \langle x, x^* \rangle \\ &\iff h(x) + h^*(x^*) = \langle x, x^* \rangle \\ &\iff x^* \in \partial h(x). \end{aligned}$$

So we have proved that

if $h \in \mathcal{PCLSC}(E)$ then ∂h is maximally monotone.

Theorem on BC–functions on $E \times E^*$

Let $f \in \mathcal{PC}(E \times E^*)$ be a BC–function. Then $\mathcal{P}_q(f)$ is maximally monotone. Further, $\mathcal{P}_q(f^@) = \mathcal{P}_q(f)$ and $\mathcal{P}_q(f) - G(-J) = E \times E^*$.

A nice example

Let $h \in \mathcal{PCLSC}(E)$. Define $f \in \mathcal{PCLSC}(E \times E^*)$ by $f(x, x^*) := h(x) + h^*(x^*)$. It is easily seen that $f^@ = f$. Furthermore, from the Fenchel–Young inequality,

$$f(x, x^*) = h(x) + h^*(x^*) \geq \langle x, x^* \rangle = q(x, x^*).$$

Thus f is a BC–function. It now follows from the theorem on BC–functions on $E \times E^*$ that $\mathcal{P}_q(f)$ is maximally monotone. But

$$\begin{aligned} (x, x^*) \in \mathcal{P}_q(f) &\iff f(x, x^*) = \langle x, x^* \rangle \\ &\iff h(x) + h^*(x^*) = \langle x, x^* \rangle \\ &\iff x^* \in \partial h(x). \end{aligned}$$

So we have proved that

if $h \in \mathcal{PCLSC}(E)$ then ∂h is maximally monotone.

- Remember that we are assuming that E is reflexive.

The vanilla Attouch–Brezis theorem.

Let $f, g \in \mathcal{PCLSC}(E)$, $f + g \geq 0$ *on* E *and* $\bigcup_{\lambda>0} \lambda\big[\operatorname{dom} f - \operatorname{dom} g\big] = E$. *Then* $\exists\ z^* \in E^*$ *such that* $f^*(-z^*) + g^*(z^*) \leq 0$.

The vanilla Attouch–Brezis theorem.

Let $f, g \in \mathcal{PCLSC}(E)$, $f + g \geq 0$ *on* E *and* $\bigcup_{\lambda>0} \lambda\big[\operatorname{dom} f - \operatorname{dom} g\big] = E$. *Then* $\exists\ z^* \in E^*$ *such that* $f^*(-z^*) + g^*(z^*) \leq 0$.

- If X and Y are nonempty sets, define $\pi_1\colon\ X \times Y \mapsto X$ by $\pi_1(x, y) := x$.

The bivariate Attouch–Brezis theorem

Let $f, g \in \mathcal{PCLSC}(E \times E^*)$,

$$\bigcup_{\lambda>0} \lambda\big[\pi_1 \operatorname{dom} f - \pi_1 \operatorname{dom} g\big] = E$$

and, $\forall\ (x, x^*) \in E \times E^*$,

$$h(x, x^*) := \inf\big\{f(x, s^*) + g(x, t^*)\colon\ s^*, t^* \in E^*,\ s^* + t^* = x^*\big\} > -\infty.$$

Then, $\forall\ (x, x^*) \in E \times E^*$,

$$h^{@}(x, x^*) = \min\big\{f^{@}(x, s^*) + g^{@}(x, t^*)\colon\ s^*, t^* \in E^*,\ s^* + t^* = x^*\big\}.$$

The vanilla Attouch–Brezis theorem.

Let $f, g \in \mathcal{PCLSC}(E)$, $f + g \geq 0$ *on* E and $\bigcup_{\lambda>0} \lambda\big[\operatorname{dom} f - \operatorname{dom} g\big] = E$. *Then* $\exists\ z^* \in E^*$ *such that* $f^*(-z^*) + g^*(z^*) \leq 0$.

• If X and Y are nonempty sets, define $\pi_1\colon X \times Y \mapsto X$ by $\pi_1(x, y) := x$.

The bivariate Attouch–Brezis theorem

Let $f, g \in \mathcal{PCLSC}(E \times E^*)$,

$$\bigcup_{\lambda>0} \lambda\big[\pi_1 \operatorname{dom} f - \pi_1 \operatorname{dom} g\big] = E$$

and, $\forall\ (x, x^*) \in E \times E^*$,

$$h(x, x^*) := \inf\big\{f(x, s^*) + g(x, t^*)\colon\ s^*, t^* \in E^*,\ s^* + t^* = x^*\big\} > -\infty.$$

Then, $\forall\ (x, x^*) \in E \times E^*$,

$$h^@(x, x^*) = \min\big\{f^@(x, s^*) + g^@(x, t^*)\colon\ s^*, t^* \in E^*,\ s^* + t^* = x^*\big\}.$$

• The hypothesis is that $h(x, \cdot)$ is the inf–convolution of $f(x, \cdot)$ and $g(x, \cdot)$, and the conclusion is that $h^@(x, \cdot)$ is the exact inf–convolution of $f^@(x, \cdot)$ and $g^@(x, \cdot)$.
• The results on this slide are true even if E is not reflexive.

The partial episum theorem for BC–functions

Let $f, g \in \mathcal{PCLSC}(E \times E^*)$ *be BC–functions,* $\bigcup_{\lambda>0} \lambda\big[\pi_1 \operatorname{dom} f - \pi_1 \operatorname{dom} g\big] = E$ *and,* $\forall\ (x, x^*) \in E \times E^*$,

$$h(x, x^*) := \inf\big\{f(x, s^*) + g(x, t^*)\colon\ s^*,\, t^* \in E^*,\ s^* + t^* = x^*\big\}.$$

Then, $\forall\ (x, x^*) \in E \times E^*$,

$$h^{@}(x, x^*) = \min\big\{f^{@}(x, s^*) + g^{@}(x, t^*)\colon\ s^*, t^* \in E^*,\ s^* + t^* = x^*\big\},$$

and

h is a BC–function.

The partial episum theorem for BC–functions

Let $f, g \in \mathcal{PCLSC}(E \times E^*)$ *be BC–functions,* $\bigcup_{\lambda>0} \lambda\big[\pi_1 \operatorname{dom} f - \pi_1 \operatorname{dom} g\big] = E$ *and,* $\forall\ (x, x^*) \in E \times E^*$,

$$h(x, x^*) := \inf\big\{f(x, s^*) + g(x, t^*)\colon\ s^*,\, t^* \in E^*,\ s^* + t^* = x^*\big\}.$$

Then, $\forall\ (x, x^*) \in E \times E^*$,

$$h^{@}(x, x^*) = \min\big\{f^{@}(x, s^*) + g^{@}(x, t^*)\colon\ s^*, t^* \in E^*,\ s^* + t^* = x^*\big\},$$

and

h *is a BC–function.*

Proof. This follows from the bivariate Attouch–Brezis theorem. Details can be found in the material on the web. Go to

<www.math.ucsb.edu/~simons/NC.html>.

Theorem on BC–functions on $E \times E^*$

Let $f \in \mathcal{PC}(E \times E^*)$ *be a BC–function. Then* $\mathcal{P}_q(f)$ *is maximally monotone. Further,* $\mathcal{P}_q(f^@) = \mathcal{P}_q(f)$ *and* $\mathcal{P}_q(f) - G(-J) = E \times E^*$.

The partial episum theorem for BC–functions

Let $f, g \in \mathcal{PCLSC}(E \times E^*)$ *be BC–functions,* $\bigcup_{\lambda>0} \lambda\big[\pi_1 \operatorname{dom} f - \pi_1 \operatorname{dom} g\big] = E$ *and,* $\forall\ (x, x^*) \in E \times E^*$,

$$h(x, x^*) := \inf\big\{f(x, s^*) + g(x, t^*)\colon\ s^*,\, t^* \in E^*,\ s^* + t^* = x^*\big\}.$$

Then, $\forall\ (x, x^*) \in E \times E^*$,

$$h^@(x, x^*) = \min\big\{f^@(x, s^*) + g^@(x, t^*)\colon\ s^*, t^* \in E^*,\ s^* + t^* = x^*\big\},$$

and

h *is a BC–function.*

Theorem on BC–functions on $E \times E^*$

Let $f \in \mathcal{PC}(E \times E^*)$ be a BC–function. Then $\mathcal{P}_q(f)$ is maximally monotone. Further, $\mathcal{P}_q(f^@) = \mathcal{P}_q(f)$ and $\mathcal{P}_q(f) - G(-J) = E \times E^*$.

The partial episum theorem for BC–functions

Let $f, g \in \mathcal{PCLSC}(E \times E^*)$ be BC–functions, $\bigcup_{\lambda > 0} \lambda\big[\pi_1 \operatorname{dom} f - \pi_1 \operatorname{dom} g\big] = E$ and, $\forall\ (x, x^*) \in E \times E^*$,

$$h(x, x^*) := \inf \big\{ f(x, s^*) + g(x, t^*) \colon\ s^*,\ t^* \in E^*,\ s^* + t^* = x^* \big\}.$$

Then, $\forall\ (x, x^*) \in E \times E^*$,

$$h^@(x, x^*) = \min \big\{ f^@(x, s^*) + g^@(x, t^*) \colon\ s^*, t^* \in E^*,\ s^* + t^* = x^* \big\},$$

and

h is a BC–function.

Combination lemma

Under the conditions above,

$\mathcal{P}_q(h^@)$ is maximally monotone.

Combination lemma

Let $f, g \in \mathcal{PCLSC}(E \times E^*)$ *be BC–functions,* $\bigcup_{\lambda>0} \lambda\big[\pi_1 \operatorname{dom} f - \pi_1 \operatorname{dom} g\big] = E$ *and,* $\forall\ (x, x^*) \in E \times E^*$,

$$h(x, x^*) := \inf\big\{f(x, s^*) + g(x, t^*)\colon\ s^*,\, t^* \in E^*,\ s^* + t^* = x^*\big\}.$$

Then, $\forall\ (x, x^*) \in E \times E^*$,

$$h^{@}(x, x^*) = \min\big\{f^{@}(x, s^*) + g^{@}(x, t^*)\colon\ s^*, t^* \in E^*,\ s^* + t^* = x^*\big\},$$

and

$$\mathcal{P}_q(h^{@}) \quad \textit{is maximally monotone.}$$

Combination lemma

Let $f, g \in \mathcal{PCLSC}(E \times E^*)$ *be BC–functions,* $\bigcup_{\lambda>0} \lambda\big[\pi_1 \operatorname{dom} f - \pi_1 \operatorname{dom} g\big] = E$ *and,* $\forall\, (x, x^*) \in E \times E^*$,

$$h(x, x^*) := \inf \big\{ f(x, s^*) + g(x, t^*)\colon\ s^*,\, t^* \in E^*,\ s^* + t^* = x^* \big\}.$$

Then, $\forall\, (x, x^*) \in E \times E^*$,

$$h^@(x, x^*) = \min \big\{ f^@(x, s^*) + g^@(x, t^*)\colon\ s^*, t^* \in E^*,\ s^* + t^* = x^* \big\},$$

and

$$\mathcal{P}_q(h^@) \quad \textit{is maximally monotone.}$$

Note from the form of $h^@$ above and the surprise result that

$(x, x^*) \in \mathcal{P}_q(h^@)$

$\iff \exists\ s^*,\, t^* \in E^*$ such that $(x, s^*) \in \mathcal{P}_q(f^@)$, $(x, t^*) \in \mathcal{P}_q(g^@)$ and $s^* + t^* = x^*$

$\iff \exists\ s^*,\, t^* \in E^*$ such that $(x, s^*) \in \mathcal{P}_q(f)$, $(x, t^*) \in \mathcal{P}_q(g)$ and $s^* + t^* = x^*$.

Combination lemma

Let $f, g \in \mathcal{PCLSC}(E \times E^*)$ *be BC–functions,* $\bigcup_{\lambda>0} \lambda\big[\pi_1 \operatorname{dom} f - \pi_1 \operatorname{dom} g\big] = E$ *and,* $\forall\ (x, x^*) \in E \times E^*$,

$$h(x, x^*) := \inf\big\{f(x, s^*) + g(x, t^*):\ s^*,\, t^* \in E^*,\ s^* + t^* = x^*\big\}.$$

Then, $\forall\ (x, x^*) \in E \times E^*$,

$$h^@(x, x^*) = \min\big\{f^@(x, s^*) + g^@(x, t^*):\ s^*, t^* \in E^*,\ s^* + t^* = x^*\big\},$$

and

$$\mathcal{P}_q(h^@) \quad \textit{is maximally monotone.}$$

Note from the form of $h^@$ above and the surprise result that

$$(x,x^*) \in \mathcal{P}_q(h^@)$$

$$\iff \exists\ s^*,\, t^* \in E^* \text{ such that } (x, s^*) \in \mathcal{P}_q(f^@),\ (x, t^*) \in \mathcal{P}_q(g^@) \text{ and } s^* + t^* = x^*$$

$$\iff \exists\ s^*,\, t^* \in E^* \text{ such that } (x, s^*) \in \mathcal{P}_q(f),\ (x, t^*) \in \mathcal{P}_q(g) \text{ and } s^* + t^* = x^*.$$

Combination theorem

Let $f, g \in \mathcal{PCLSC}(E \times E^*)$ *be BC–functions, and* $\bigcup_{\lambda>0} \lambda\big[\pi_1 \operatorname{dom} f - \pi_1 \operatorname{dom} g\big] = E$. *Then* $\big\{(x, s^*{+}t^*):\ (x, s^*) \in \mathcal{P}_q(f),\ (x, t^*) \in \mathcal{P}_q(g)\big\}$ *is a maximally monotone subset of* $E \times E^*$.

The convex function given by a q–positive set

Let A be a q–positive subset of B. We define $\Phi_A\colon\ B \mapsto \left]-\infty, \infty\right]$ by

$$\Phi_A(b) := \sup\nolimits_A \left[\lfloor b, \cdot\rfloor - q\right] = q(b) - \inf q(A - b).$$

- $\Phi_A = q$ on A and $\Phi_A \in \mathcal{PC}(B)$.
- Let $c \in B$. Then

$$\Phi_A{}^{@}(c) = \sup\nolimits_B \left[\lfloor \cdot, c\rfloor - \Phi_A\right] \geq \sup\nolimits_A \left[\lfloor c, \cdot\rfloor - \Phi_A\right] = \sup\nolimits_A \left[\lfloor c, \cdot\rfloor - q\right] = \Phi_A(c).$$

- We have:

$$A \text{ maximally } q\text{–positive} \quad \Longrightarrow \quad \Phi_A \geq q \text{ on } B \quad \text{and} \quad \mathcal{P}_q(\Phi_A) = A.$$

The convex function given by a q–positive set

Let A be a q–positive subset of B. We define $\Phi_A\colon\ B \mapsto \,]-\infty, \infty]$ by

$$\Phi_A(b) := \sup\nolimits_A \big[\lfloor b, \cdot\rfloor - q\big] = q(b) - \inf q(A - b).$$

- $\Phi_A = q$ on A and $\Phi_A \in \mathcal{PC}(B)$.
- Let $c \in B$. Then

$$\Phi_A{}^{@}(c) = \sup\nolimits_B \big[\lfloor \cdot, c\rfloor - \Phi_A\big] \geq \sup\nolimits_A \big[\lfloor c, \cdot\rfloor - \Phi_A\big] = \sup\nolimits_A \big[\lfloor c, \cdot\rfloor - q\big] = \Phi_A(c).$$

- We have:

$$A \text{ maximally } q\text{–positive} \quad \Longrightarrow \quad \Phi_A \geq q \text{ on } B \quad \text{and} \quad \mathcal{P}_q(\Phi_A) = A.$$

The convex function given by a maximally q–positive set

Let A be a maximally q–positive subset of B. Then Φ_A is a BC–function, and so

$$\mathcal{P}_q(\Phi_A{}^{@}) = \mathcal{P}_q(\Phi_A) = A.$$

The convex function given by a maximally q–positive set

Let A be a maximally q–positive subset of B. Then Φ_A is a BC–function, and so

$$\mathcal{P}_q({\Phi_A}^{@}) = \mathcal{P}_q(\Phi_A) = A.$$

The Fitzpatrick function

Let $S\colon E \rightrightarrows E^*$ be maximally monotone. Let $G(S)$ be the maximally monotone set $\big\{(x,x^*) \in E \times E^*\colon x^* \in Sx\big\}$. We define the Fitzpatrick function φ_S associated with S by

$$\varphi_S(x,x^*) := \Phi_{G(S)}(x,x^*) = \sup_{(s,s^*)\in G(S)} \big[\langle x,s^*\rangle + \langle s,x^*\rangle - \langle s,s^*\rangle\big].$$

Combining this with the result above, we obtain:

The convex function given by a maximally q–positive set

Let A be a maximally q–positive subset of B. Then Φ_A is a BC–function, and so

$$\mathcal{P}_q({\Phi_A}^{@}) = \mathcal{P}_q(\Phi_A) = A.$$

The Fitzpatrick function

Let $S\colon E \rightrightarrows E^*$ be maximally monotone. Let $G(S)$ be the maximally monotone set $\{(x, x^*) \in E \times E^*\colon x^* \in Sx\}$. We define the Fitzpatrick function φ_S associated with S by

$$\varphi_S(x, x^*) := \Phi_{G(S)}(x, x^*) = \sup_{(s,s^*)\in G(S)} \big[\langle x, s^*\rangle + \langle s, x^*\rangle - \langle s, s^*\rangle\big].$$

Combining this with the result above, we obtain:

Theorem on the Fitzpatrick function

Let $S\colon E \rightrightarrows E^*$ be maximally monotone. Then

$$\varphi_S \text{ is a BC–function} \quad \text{and} \quad \mathcal{P}_q({\varphi_S}^{@}) = \mathcal{P}_q(\varphi_S) = G(S).$$

The convex function given by a maximally q–positive set

Let A be a maximally q–positive subset of B. Then Φ_A is a BC–function, and so

$$\mathcal{P}_q({\Phi_A}^{@}) = \mathcal{P}_q(\Phi_A) = A.$$

The Fitzpatrick function

Let $S\colon E \rightrightarrows E^*$ be maximally monotone. Let $G(S)$ be the maximally monotone set $\{(x,x^*) \in E \times E^*\colon x^* \in Sx\}$. We define the Fitzpatrick function φ_S associated with S by

$$\varphi_S(x,x^*) := \Phi_{G(S)}(x,x^*) = \sup_{(s,s^*)\in G(S)} \big[\langle x,s^*\rangle + \langle s,x^*\rangle - \langle s,s^*\rangle\big].$$

Combining this with the result above, we obtain:

Theorem on the Fitzpatrick function

Let $S\colon E \rightrightarrows E^*$ be maximally monotone. Then

$$\varphi_S \text{ is a BC–function} \quad \text{and} \quad \mathcal{P}_q({\varphi_S}^{@}) = \mathcal{P}_q(\varphi_S) = G(S). \qquad (🔑)$$

Lemma on D and π_1

Let $S\colon E \rightrightarrows E^*$ be maximally monotone and $D(S) := \{x \in E\colon Sx \ne \emptyset\}$. Then

$$D(S) \subset \pi_1 \mathrm{dom}\,\varphi_S.$$

Proof. From (🔑), $G(S) = \mathcal{P}_q(\varphi_S) \subset \mathrm{dom}\,\varphi_S$, thus $D(S) = \pi_1 G(S) \subset \pi_1 \mathrm{dom}\,\varphi_S$. $\square$

Theorem on the Fitzpatrick function

Let $S\colon E \rightrightarrows E^*$ be maximally monotone. Then

$$\varphi_S \text{ is a BC–function} \quad \text{and} \quad \mathcal{P}_q({\varphi_S}^{@}) = \mathcal{P}_q(\varphi_S) = G(S).$$

Theorem on BC–functions on $E \times E^*$

Let $f \in \mathcal{PC}(E \times E^*)$ be a BC–function. Then $\mathcal{P}_q(f)$ is maximally monotone. Further, $\mathcal{P}_q(f^{@}) = \mathcal{P}_q(f)$ and $\mathcal{P}_q(f) - G(-J) = E \times E^*$.

- If $S, T\colon E \rightrightarrows E^*$ then, $\forall x \in E$, $(S+T)x := \{x^* + y^*\colon x^* \in Sx,\ y^* \in Tx\}$.

Theorem on the Fitzpatrick function

Let $S\colon E \rightrightarrows E^*$ be maximally monotone. Then

$$\varphi_S \text{ is a BC–function} \quad \text{and} \quad \mathcal{P}_q({\varphi_S}^@) = \mathcal{P}_q(\varphi_S) = G(S).$$

Theorem on BC–functions on $E \times E^*$

Let $f \in \mathcal{PC}(E \times E^*)$ be a BC–function. Then $\mathcal{P}_q(f)$ is maximally monotone. Further, $\mathcal{P}_q(f^@) = \mathcal{P}_q(f)$ and $\mathcal{P}_q(f) - G(-J) = E \times E^*$.

- If $S, T\colon E \rightrightarrows E^*$ then, $\forall x \in E$, $(S+T)x := \{x^* + y^*\colon x^* \in Sx,\ y^* \in Tx\}$.

Rockafellar's surjectivity theorem

Let $S\colon E \rightrightarrows E^*$ be maximally monotone. Then $(S+J)(E) = E^*$.

Proof. Let y^* be an arbitrary element of E^*. From the theorem on the Fitzpatrick function,

$$\varphi_S \text{ is a BC–function} \quad \text{and} \quad \mathcal{P}_q({\varphi_S}^@) = \mathcal{P}_q(\varphi_S) = G(S).$$

Thus, taking $f = \varphi_S$ in the theorem on BC–functions on $E \times E^*$, $\exists\ (s, s^*) \in G(S)$ and $(x, x^*) \in G(J)$ such that $(0, y^*) = (s, s^*) - (x, -x^*)$. But then $x = s$ and so $y^* = s^* + x^* \in (S+J)s$. □

Theorem on the Fitzpatrick function

Let $S\colon E \rightrightarrows E^*$ be maximally monotone. Then

$$\varphi_S \text{ is a BC–function} \quad \text{and} \quad \mathcal{P}_q({\varphi_S}^{@}) = \mathcal{P}_q(\varphi_S) = G(S).$$

Combination theorem

Let $f, g \in \mathcal{PCLSC}(E \times E^*)$ be BC–functions, and $\bigcup_{\lambda>0} \lambda\big[\pi_1 \operatorname{dom} f - \pi_1 \operatorname{dom} g\big] = E$. Then $\big\{(x, s^*{+}t^*)\colon (x, s^*) \in \mathcal{P}_q(f),\ (x, t^*) \in \mathcal{P}_q(g)\big\}$ is a maximally monotone subset of $E \times E^*$.

Theorem on the Fitzpatrick function

Let $S\colon E \rightrightarrows E^*$ be maximally monotone. Then

$$\varphi_S \text{ is a BC–function} \quad \text{and} \quad \mathcal{P}_q({\varphi_S}^{@}) = \mathcal{P}_q(\varphi_S) = G(S).$$

Combination theorem

Let $f, g \in \mathcal{PCLSC}(E \times E^*)$ be BC–functions, and $\bigcup_{\lambda>0} \lambda\big[\pi_1 \operatorname{dom} f - \pi_1 \operatorname{dom} g\big] = E$. Then $\big\{(x, s^*{+}t^*)\colon (x, s^*) \in \mathcal{P}_q(f),\ (x, t^*) \in \mathcal{P}_q(g)\big\}$ is a maximally monotone subset of $E \times E^*$.

We now prove:

The sum theorem

Let $S, T\colon E \rightrightarrows E^*$ be maximally monotone and

$$\bigcup_{\lambda>0} \lambda\big[\pi_1 \operatorname{dom} \varphi_S - \pi_1 \operatorname{dom} \varphi_T\big] = E.$$

Then $S + T$ *is maximally monotone.*

Proof. We have: φ_S and φ_T are BC–functions, $\mathcal{P}_q(\varphi_S) = G(S)$ and $\mathcal{P}_q(\varphi_T) = G(T)$. From the combination theorem, $\big\{(x, s^* + t^*)\colon (x, s^*) \in G(S),\ (x, t^*) \in G(T)\big\}$ is a maximally monotone subset of $E \times E^*$, that is to say,

$$G(S + T) \text{ is a maximally monotone subset of } E \times E^*. \qquad \square$$

The sum theorem

Let $S, T\colon\ E \rightrightarrows E^*$ *be maximally monotone and*

$$\bigcup_{\lambda>0} \lambda\big[\pi_1\,\mathrm{dom}\,\varphi_S - \pi_1\,\mathrm{dom}\,\varphi_T\big] = E.$$

Then $S + T$ *is maximally monotone.*

Lemma on D and π_1

Let $S\colon\ E \rightrightarrows E^*$ *be maximally monotone and* $D(S) := \{x \in E\colon\ Sx \neq \emptyset\}$. *Then*

$$D(S) \subset \pi_1\mathrm{dom}\,\varphi_S.$$

The sum theorem

Let $S, T\colon\ E \rightrightarrows E^*$ *be maximally monotone and*

$$\bigcup_{\lambda>0} \lambda\big[\pi_1\,\mathrm{dom}\,\varphi_S - \pi_1\,\mathrm{dom}\,\varphi_T\big] = E.$$

Then $S + T$ *is maximally monotone.*

Lemma on D and π_1

Let $S\colon\ E \rightrightarrows E^*$ *be maximally monotone and* $D(S) := \{x \in E\colon\ Sx \neq \emptyset\}$. *Then*

$$D(S) \subset \pi_1 \mathrm{dom}\,\varphi_S.$$

Thus:

The sum corollary

Let $S, T\colon\ E \rightrightarrows E^*$ *be maximally monotone and* $\bigcup_{\lambda>0} \lambda\big[D(S) - D(T)\big] = E$. *Then*

$S + T$ *is maximally monotone.*

The sum theorem

Let $S, T\colon\ E \rightrightarrows E^*$ *be maximally monotone and*

$$\bigcup_{\lambda>0} \lambda\big[\pi_1 \operatorname{dom} \varphi_S - \pi_1 \operatorname{dom} \varphi_T\big] = E.$$

Then $S + T$ *is maximally monotone.*

Lemma on D and π_1

Let $S\colon\ E \rightrightarrows E^*$ *be maximally monotone and* $D(S) := \{x \in E\colon\ Sx \neq \emptyset\}$. *Then*

$$D(S) \subset \pi_1 \operatorname{dom} \varphi_S.$$

Thus:

The sum corollary

Let $S, T\colon\ E \rightrightarrows E^*$ *be maximally monotone and* $\bigcup_{\lambda>0} \lambda\big[D(S) - D(T)\big] = E$. *Then*

$S + T$ *is maximally monotone.*

In particular,

Rockafellar's sum theorem

Let $S, T\colon\ E \rightrightarrows E^*$ *be maximally monotone and* $D(S) \cap \operatorname{int} D(T) \neq \emptyset$. *Then*

$S + T$ *is maximally monotone.*

Dom–neg theorem

Let $f \in \mathcal{PCLSC}(B)$ *and, whenever* $b, d \in B$,

$$f(b) + f^{@}(d) = \lfloor b, d \rfloor \quad \Longrightarrow \quad q(b) + q(d) \le f(b) + f^{@}(d).$$

Then $$\operatorname{dom} f - \mathcal{N}_q(g_0) = B.$$

Dom–neg theorem

Let $f \in \mathcal{PCLSC}(B)$ *and, whenever* $b, d \in B$*,*

$$f(b) + f^{@}(d) = \lfloor b, d \rfloor \quad \Longrightarrow \quad q(b) + q(d) \le f(b) + f^{@}(d).$$

Then $$\operatorname{dom} f - \mathcal{N}_q(g_0) = B.$$

Proof. Let $c \in B$. Since $f_c + g_0$ is coercive and $w(B, B^*)$–lower semicontinuous, and B is reflexive, $\exists\ b \in B$ such that

$$(f_c + g_0)(b) = \min_B \left[f_c + g_0\right].$$

Since g_0 is continuous, Rockafellar's sum formula implies that

$$\partial f_c(b) + \partial g_0(b) \ni 0.$$

One can show that

$$c \in \operatorname{dom} f - b \quad \text{and} \quad b \in \mathcal{N}_q(g_0).$$

Consequently,

$$c \in \operatorname{dom} f - \mathcal{N}_q(g_0).$$

More details can be found in the material on the web. Go to:

<www.math.ucsb.edu/~simons/NC.html>.

□

SSDB spaces

We will say that B $\big($more precisely, $\big(B, \lfloor\cdot,\cdot\rfloor, q, \|\cdot\|, \iota\big)\big)$ is a *symmetrically self–dual Banach space (SSDB space)* if B is a nonzero Banach space, $\lfloor\cdot,\cdot\rfloor\colon B \times B \mapsto \mathbb{R}$ is a symmetric bilinear form, the quadratic form q on B is defined by $q(b) := \frac{1}{2}\lfloor b, b\rfloor$ and $\exists$ a linear isometry ι from B onto B^* such that, for all $b, c \in B$, $\langle b, \iota(c)\rangle = \lfloor b, c\rfloor$.

SSDB spaces

We will say that B $\big($more precisely, $\big(B, \lfloor\cdot,\cdot\rfloor, q, \|\cdot\|, \iota\big)\big)$ is a *symmetrically self–dual Banach space* (*SSDB space*) if B is a nonzero Banach space, $\lfloor\cdot,\cdot\rfloor\colon B \times B \mapsto \mathbb{R}$ is a symmetric bilinear form, the quadratic form q on B is defined by $q(b) := \frac{1}{2}\lfloor b, b\rfloor$ and $\exists$ a linear isometry ι from B onto B^* such that, for all $b, c \in B$, $\langle b, \iota(c)\rangle = \lfloor b, c\rfloor$.

• If $\big(B, \lfloor\cdot,\cdot\rfloor, q, \|\cdot\|, \iota\big)$ is a SSDB space then so also is $\big(B, -\lfloor\cdot,\cdot\rfloor, -q, \|\cdot\|, -\iota\big)$. If "$B$" represents $\big(B, \lfloor\cdot,\cdot\rfloor, q, \|\cdot\|, \iota\big)$, then "$B^-$" represents $\big(B, -\lfloor\cdot,\cdot\rfloor, -q, \|\cdot\|, -\iota\big)$.

SSDB spaces

We will say that B $\big($more precisely, $\big(B, \lfloor\cdot,\cdot\rfloor, q, \|\cdot\|, \iota\big)\big)$ is a *symmetrically self–dual Banach space* (*SSDB space*) if B is a nonzero Banach space, $\lfloor\cdot,\cdot\rfloor\colon B \times B \mapsto \mathbb{R}$ is a symmetric bilinear form, the quadratic form q on B is defined by $q(b) := \frac{1}{2}\lfloor b, b\rfloor$ and $\exists$ a linear isometry ι from B onto B^* such that, for all $b, c \in B$, $\langle b, \iota(c)\rangle = \lfloor b, c\rfloor$.

• If $\big(B, \lfloor\cdot,\cdot\rfloor, q, \|\cdot\|, \iota\big)$ is a SSDB space then so also is $\big(B, -\lfloor\cdot,\cdot\rfloor, -q, \|\cdot\|, -\iota\big)$. If “$B$” represents $\big(B, \lfloor\cdot,\cdot\rfloor, q, \|\cdot\|, \iota\big)$, then “$B^-$” represents $\big(B, -\lfloor\cdot,\cdot\rfloor, -q, \|\cdot\|, -\iota\big)$.

q–positive sets

Let $A \subset B$. We say that A is *q–positive* if $A \neq \emptyset$ and

$$b, c \in A \Longrightarrow q(b - c) \geq 0.$$

SSDB spaces

We will say that B $\big($more precisely, $\big(B, \lfloor\cdot,\cdot\rfloor, q, \|\cdot\|, \iota\big)\big)$ is a *symmetrically self–dual Banach space (SSDB space)* if B is a nonzero Banach space, $\lfloor\cdot,\cdot\rfloor\colon B \times B \mapsto \mathbb{R}$ is a symmetric bilinear form, the quadratic form q on B is defined by $q(b) := \frac{1}{2}\lfloor b, b\rfloor$ and $\exists$ a linear isometry ι from B onto B^* such that, for all $b, c \in B$, $\langle b, \iota(c)\rangle = \lfloor b, c\rfloor$.

• If $\big(B, \lfloor\cdot,\cdot\rfloor, q, \|\cdot\|, \iota\big)$ is a SSDB space then so also is $\big(B, -\lfloor\cdot,\cdot\rfloor, -q, \|\cdot\|, -\iota\big)$. If "$B$" represents $\big(B, \lfloor\cdot,\cdot\rfloor, q, \|\cdot\|, \iota\big)$, then "$B^-$" represents $\big(B, -\lfloor\cdot,\cdot\rfloor, -q, \|\cdot\|, -\iota\big)$.

q–positive sets

Let $A \subset B$. We say that A is *q–positive* if $A \neq \emptyset$ and

$$b, c \in A \Longrightarrow q(b - c) \geq 0.$$

q–negative sets

Let $A \subset B$. We say that A is *q–negative* if $A \neq \emptyset$ and

$$b, c \in A \Longrightarrow q(b - c) \leq 0.$$

Obviously, A is q–negative exactly when A is $(-q)$–positive.

SSDB spaces

We will say that B (more precisely, $(B, \lfloor\cdot,\cdot\rfloor, q, \|\cdot\|, \iota)$) is a *symmetrically self–dual Banach space (SSDB space)* if B is a nonzero Banach space, $\lfloor\cdot,\cdot\rfloor\colon B\times B \mapsto \mathbb{R}$ is a symmetric bilinear form, the quadratic form q on B is defined by $q(b) := \frac{1}{2}\lfloor b, b\rfloor$ and $\exists$ a linear isometry ι from B onto B^* such that, for all $b, c \in B$, $\langle b, \iota(c)\rangle = \lfloor b, c\rfloor$.

- If $(B, \lfloor\cdot,\cdot\rfloor, q, \|\cdot\|, \iota)$ is a SSDB space then so also is $(B, -\lfloor\cdot,\cdot\rfloor, -q, \|\cdot\|, -\iota)$. If "$B$" represents $(B, \lfloor\cdot,\cdot\rfloor, q, \|\cdot\|, \iota)$, then "$B^-$" represents $(B, -\lfloor\cdot,\cdot\rfloor, -q, \|\cdot\|, -\iota)$.

q–positive sets

Let $A \subset B$. We say that A is *q–positive* if $A \neq \emptyset$ and

$$b, c \in A \Longrightarrow q(b - c) \geq 0.$$

q–negative sets

Let $A \subset B$. We say that A is *q–negative* if $A \neq \emptyset$ and

$$b, c \in A \Longrightarrow q(b - c) \leq 0.$$

Obviously, A is q–negative exactly when A is $(-q)$–positive.

Polar subspace

Let A be a linear subspace of a SSDB space B. Then A^0 is the linear subspace $\big\{b \in B\colon\ \lfloor A, b\rfloor = \{0\}\big\}$ of B.

— SSDB spaces and maximal monotonicity —

- Let A be a maximally q–positive subset of a SSDB space B. Then

$$b \in B \quad \Longrightarrow \quad \inf q(A - b) \le 0.$$ (🌳)

(This is equivalent to the statement that $\Phi_A \ge q$ on B.)

• Let A be a maximally q–positive subset of a SSDB space B. Then

$$b \in B \quad \Longrightarrow \quad \inf q(A - b) \le 0. \qquad (🌳)$$

(This is equivalent to the statement that $\Phi_A \ge q$ on B.)

Initial result on polarity

Let A be a maximally q–positive linear subspace of a SSDB space B. Then A^0 is q–negative.

Proof. If $p \in A^0$ then $\inf q(A - p) = \inf q(A) + q(p) = q(p)$, and so (🌳) gives $q(p) \le 0$. If now $b, c \in A^0$ then $b - c \in A^0$ and so $q(b - c) \le 0$. Thus A^0 is q–negative.$\square$

— SSDB spaces and maximal monotonicity —

- Let A be a maximally q–positive subset of a SSDB space B. Then

$$b \in B \quad \Longrightarrow \quad \inf q(A - b) \le 0. \qquad (🌳)$$

(This is equivalent to the statement that $\Phi_A \ge q$ on B.)

Initial result on polarity

Let A be a maximally q–positive linear subspace of a SSDB space B. Then A^0 is q–negative.

Proof. If $p \in A^0$ then $\inf q(A - p) = \inf q(A) + q(p) = q(p)$, and so (🌳) gives $q(p) \le 0$. If now $b, c \in A^0$ then $b - c \in A^0$ and so $q(b - c) \le 0$. Thus A^0 is q–negative.$\square$

We will prove the following

Converse result

Let A be a closed q–positive linear subspace of a SSDB space B and A^0 be q–negative. Then A is maximally q–positive.

- Let A be a maximally q–positive subset of a SSDB space B. Then

$$b \in B \quad \Longrightarrow \quad \inf q(A - b) \le 0. \qquad (🌳)$$

(This is equivalent to the statement that $\Phi_A \ge q$ on B.)

Initial result on polarity

Let A be a maximally q–positive linear subspace of a SSDB space B. Then A^0 is q–negative.

Proof. If $p \in A^0$ then $\inf q(A - p) = \inf q(A) + q(p) = q(p)$, and so (🌳) gives $q(p) \le 0$. If now $b, c \in A^0$ then $b - c \in A^0$ and so $q(b - c) \le 0$. Thus A^0 is q–negative.□

We will prove the following

Converse result

Let A be a closed q–positive linear subspace of a SSDB space B and A^0 be q–negative. Then A is maximally q–positive.

Our proof of the converse result depends on the function q_A, which we now introduce.

The function q_A

Let A be a closed q–positive linear subspace of a SSDB space B. Define $q_A\colon B \to]-\infty,\infty]$ by $q_A := q$ on A and $q_A := \infty$ on $B \setminus A$. Then $q_A \in \mathcal{PCLSC}(B)$, and

$$q_A(b) + {q_A}^{@}(d) = \lfloor b, d \rfloor \quad \Longrightarrow \quad b - d \in A^0.$$

Proof. These results follow easily from the definitions. The third assertion uses a differentiability argument. □

Dom–neg theorem

Let $f \in \mathcal{PCLSC}(B)$ *and, whenever* $b, d \in B$,

$$f(b) + f^{@}(d) = \lfloor b, d \rfloor \quad \Longrightarrow \quad q(b) + q(d) \le f(b) + f^{@}(d).$$

Then $$\operatorname{dom} f - \mathcal{N}_q(g_0) = B.$$

The function q_A

Let A *be a closed q–positive linear subspace of a SSDB space* B. *Define* $q_A\colon B \to \left]-\infty, \infty\right]$ *by* $q_A := q$ *on* A *and* $q_A := \infty$ *on* $B \setminus A$. *Then* $q_A \in \mathcal{PCLSC}(B)$, *and*

$$q_A(b) + {q_A}^{@}(d) = \lfloor b, d \rfloor \quad \Longrightarrow \quad b - d \in A^0.$$

Converse result

Let A *be a closed q–positive linear subspace of a SSDB space* B *and* A^0 *be q–negative. Then* A *is maximally q–positive.*

Dom–neg theorem

Let $f \in \mathcal{PCLSC}(B)$ *and, whenever* $b, d \in B$,

$$f(b) + f^{@}(d) = \lfloor b, d \rfloor \quad \Longrightarrow \quad q(b) + q(d) \le f(b) + f^{@}(d).$$

Then $$\operatorname{dom} f - \mathcal{N}_q(g_0) = B.$$

The function q_A

Let A *be a closed* *q–positive* *linear subspace of a* *SSDB space* B. *Define* $q_A\colon B \to \,]-\infty, \infty]$ *by* $q_A := q$ *on* A *and* $q_A := \infty$ *on* $B \setminus A$. *Then* $q_A \in \mathcal{PCLSC}(B)$, *and*

$$q_A(b) + {q_A}^{@}(d) = \lfloor b, d \rfloor \quad \Longrightarrow \quad b - d \in A^0.$$

Converse result

Let A *be a closed* *q–positive* *linear subspace of a* *SSDB space* B *and* A^0 *be* *q–negative*. *Then* A *is maximally* *q–positive*.

Proof. It is clear that

$$q_A(b) + {q_A}^{@}(d) = \lfloor b, d \rfloor \Longrightarrow q(b - d) \le 0 \Longrightarrow q(b) + q(d) \le \lfloor b, d \rfloor = q_A(b) + {q_A}^{@}(d).$$

It now follows easily from the Dom–neg theorem with $f = q_A$ that

$$A - \mathcal{N}_q(g_0) = B. \qquad (𓅃)$$

Now suppose that $c \in B$ and $A \cup \{c\}$ is q–positive. From (𓅃), $\exists\ a \in A$ such that $a - c \in \mathcal{N}_q(g_0)$. Thus $\frac{1}{2}\|a - c\|^2 = -q(a - c)$. Since $A \cup \{c\}$ is q–positive, $q(a - c) \ge 0$, and so $\frac{1}{2}\|a - c\|^2 \le 0$, from which $c = a \in A$. $\square$

• If $\big(B, \lfloor\cdot,\cdot\rfloor, q, \|\cdot\|, \iota\big)$ is a SSDB space then so also is $\big(B, -\lfloor\cdot,\cdot\rfloor, -q, \|\cdot\|, -\iota\big)$. If "$B$" represents $\big(B, \lfloor\cdot,\cdot\rfloor, q, \|\cdot\|, \iota\big)$, then "$B^-$" represents $\big(B, -\lfloor\cdot,\cdot\rfloor, -q, \|\cdot\|, -\iota\big)$.

q–negative sets

Let $A \subset B$. We say that A is *q–negative* if $A \neq \emptyset$ and

$$b, c \in A \Longrightarrow q(b - c) \le 0.$$

Obviously, A is q–negative exactly when A is $(-q)$–positive.

Main theorem on polar subspaces

Let A be a norm–closed q–positive linear subspace of a SSDB space B. Then:

(a) *A is maximally q–positive $\iff$ A^0 is q–negative.*

(b) *A is maximally q–positive $\iff$ A^0 is maximally q–negative.*

Proof. (a) is immediate from the "initial result" and the "converse result".

($\Longleftarrow$) in (b) is immediate from the "converse result".

We now prove ($\Longrightarrow$) in (b). Let A be maximally q–positive. From the "initial result", A^0 is q–negative, and it only remains to prove the maximality.

So A^0 is $(-q)$–positive and A is (maximally) $(-q)$–negative. Since A is norm–closed, $A = (A^0)^0$. Thus $(A^0)^0$ is $(-q)$–negative. From the "converse result" with B replaced by B^-, q replaced by $-q$ and A replaced by A^0, A^0 is maximally $(-q)$–positive, that is to say, maximally q–negative, which completes the proof of (b). □

• Let E be a nonzero reflexive Banach space and A be a monotone linear subspace of $E \times E^*$. Then the linear subspace A^* of $E \times E^*$ is defined by:

$$(x, x^*) \in A^* \iff \text{for all } (a, a^*) \in A,\ \langle x, a^* \rangle = \langle a, x^* \rangle.$$

• Let E be a nonzero reflexive Banach space and A be a monotone linear subspace of $E \times E^*$. Then the linear subspace A^* of $E \times E^*$ is defined by:

$$(x, x^*) \in A^* \iff \text{for all } (a, a^*) \in A,\ \langle x, a^* \rangle = \langle a, x^* \rangle.$$

• Our final result is immediate from the "main theorem" on the previous slide. (a) was proved by Brezis–Browder and (b) by Yao.

Results of Brezis–Browder and Yao

Let E be a nonzero reflexive Banach space with topological dual E^* and A be a norm–closed monotone linear subspace of $E \times E^*$. Then:

(a) *A is maximally monotone if, and only if, A^* is monotone*

(b) *A is maximally monotone if, and only if, A^* is maximally monotone.*

— SSDB spaces and maximal monotonicity —

SSDB spaces and maximal monotonicity
Download in .ps format (223K)
Download in .pdf format (152K)

A Brezis-Browder theorem for SSDB spaces
Download in .ps format (116K)
Download in .pdf format (72K)

Banach SSD spaces and classes of monotone sets
Download in .ps format (388K)
Download in .pdf format (284K)

Quadrivariate versions of the Attouch-Brezis theorem and strong representability
Download in .ps format (236K)
Download in .pdf format (146K)

From Hahn-Banach to monotonicity *Lecture Notes in Mathematics* **1693**, second edition (2008), Springer--Verlag.
Download table of contents and introduction in .ps format (1.2M)
Download table of contents and introduction in .pdf format (247K)

Go to Stephen Simons's home page

SSDB spaces and maximal monotonicity

by

Stephen Simons

<simons@math.ucsb.edu>.

Abstract

We introduce SSDB spaces, which include Hilbert spaces, negative Hilbert spaces and spaces of the form $E \times E^*$, where E is a reflexive real Banach space. We introduce q–positive subsets of a SSDB space, which include monotone subsets of $E \times E^*$, and BC–functions on a SSDB spaces, which include Fitzpatrick functions of monotone multifunctions. We show how convex analysis can be combined with SSDB space theory to obtain and generalize various results on maximally monotone multifunctions on a reflexive Banach space, such as the significant direction of Rockafellar's surjectivity theorem, sufficient conditions for the sum of maximally monotone multifunctions to be maximally monotone, and an abstract Brezis–Browder theorem.

Downloads

You can download files containing related materials from
<www.math.ucsb.edu/~simons/NC.html>.