METRIC REGULARITY AND CONDITIONING IN OPTIMIZATION

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Talk given at the

International Workshop on Variational Analysis, Optimization and Applications
tribute to Jean-Baptiste Hiriart-Urruty

Bayonne, France, October 2010

Based on joint work with Javier Peña (Carnegie Mellon) and Vera Roshchina (Évora)

MATRIX GAMES

Solve the matrix game equilibrium problem (MG):

$$\min_{x \in \Delta_m} \max_{y \in \Delta_n} x^\top A y = \max_{y \in \Delta_n} \min_{x \in \Delta_m} x^\top A y$$

where A is a given $m \times n$ matrix and where the m-simplex

$$\Delta_m := \left\{ x \in \mathbb{R}^m \,\middle|\, \sum_{i=1}^{m} x_i = 1,\ x \geq 0 \right\}$$

describes the set of mixed strategies for the x-player (Player 1); similarly for Player 2.

This problem is known as: to find a Nash equilibrium of a two-person zero-sum matric game.

NONSMOOTH CONVEX OPTIMIZATION

Consider the cost function

$$F(x,y) := \max\left\{x^\top Av - u^\top Ay \,\middle|\, (u,v) \in \Delta_m \times \Delta_n\right\}$$

and the constrained optimization problem

$$\text{minimize} \quad F(x,y) \quad \text{subject to} \quad (x,y) \in \Delta_m \times \Delta_n$$

equivalent to the initial matrix game (MG).

Then Nash equilibrium $(\bar{x},\bar{y})$ for (MG) corresponds to $F(\bar{x},\bar{y}) = 0$ and an ε-equilibrium to (MG) corresponds to

$$F(\bar{x},\bar{y}) < \varepsilon, \quad \varepsilon > 0$$

CONDITION MEASURE OF A SMOOTHING ALGORITHM

It was shown by Peña et al. (2010) that an iterative version of Nesterov's first-order smoothing algorithm computes an ε-equilibrium point for (MG) in $\mathcal{O}(\|A\|\kappa(A)\,\mathrm{ln}(1/\varepsilon))$ iterations, where $\kappa(A)$ is a condition measure of (MG) defined by

$$\kappa(A) := \inf\left\{\kappa \geq 0 \,\middle|\, \mathrm{dist}\Big((x,y);S\Big) \leq \kappa F(x,y) \ \text{ as } \ (x,y) \in \Delta_m \times \Delta_n\right\}$$

with the solution set S to (MG) given by

$$S := \left\{(\bar{x},\bar{y}) \in \Delta_m \times \Delta_n \,\middle|\, F(\bar{x},\bar{y}) = 0\right\} = F^{-1}(0) \cap (\Delta_m \times \Delta_n)$$

Note that Peña's complexity bound is exponentially better than that of Nesterov $\mathcal{O}(1/\varepsilon)$ while no explicit formula or upper bound of the condition measure $\kappa(A)$ was given.

OUR MAIN GOALS AND RESULTS

• precisely relate the condition measure $\kappa(A)$ to the exact bound of metric regularity for an associated set-valued mapping

• express this exact regularity bound via the subdifferential of F and the normal cone to $\Delta_m \times \Delta_n$ and then compute the latter constructions in terms of the initial data of (MG)

• derive an exact formula for evaluating $\kappa(A)$, which is a key step towards performing further complexity analysis of the algorithm

METRIC REGULARITY

DEFINITION. A set-valued mapping $G\colon I\!R^n \rightrightarrows I\!R^m$ is metrically regular around $(\bar{x}, \bar{z}) \in \operatorname{gph} G$ with modulus $\mu \geq 0$ if there exist neighborhoods U of $\bar{x}$ and V of $\bar{z}$ such that

$$\operatorname{dist}\left(x; G^{-1}(z)\right) \leq \mu \operatorname{dist}\left(z; G(x)\right) \quad \text{for all} \quad x \in U \quad \text{and} \quad z \in V.$$

The infimum of $\mu \geq 0$ over all (μ, U, V) for which the latter holds is called the exact regularity bound of G around $(\bar{x}, \bar{z})$ and is denoted by $\operatorname{reg} G(\bar{x}, \bar{z})$.

NORMALS AND CODERIVATIVES

Given $\Omega \subset I\!R^n$, the Euclidean projector to Ω is

$$\Pi(x;\Omega) := \left\{ y \in \Omega \,\middle|\, \|y - x\| = \mathrm{dist}\,(x;\Omega) \right\}$$

The normal cone to Ω at $\bar{x} \in \Omega$ is

$$N(\bar{x};\Omega) := \Big\{ v \in I\!R^n \,\Big|\; \exists\, x_k \to \bar{x},\ y_k \in \Pi(x_k;\Omega),\ \lambda_k \ge 0 \\ \text{such that } \lambda_k(x_k - y_k) \to v \Big\}$$

Given $G\colon I\!R^n \rightrightarrows I\!R^m$, the coderivative of G at $(\bar{x},\bar{y}) \in \mathrm{gph}\, G$ is

$$D^*G(\bar{x},\bar{y})(u) := \left\{ v \in I\!R^n \,\middle|\, (v,-u) \in N\Big((\bar{x},\bar{y});\mathrm{gph}\, G\Big) \right\}$$

For smooth $G\colon I\!R^n \to I\!R^m$ we have

$$D^*G(\bar{x})(u) = \left\{ \nabla G(\bar{x})^T u \right\}, \quad u \in I\!R^m$$

CODERIVATIVE CHARACTERIZATION OF METRIC REGULARITY

THEOREM [Mor84]. Let a set-valued mapping $G\colon I\!R^n \rightrightarrows I\!R^m$ be closed-graph around $(\bar{x},\bar{y}) \in \operatorname{gph} G$. Then G is metrically regular around $(\bar{x},\bar{y})$ if and only if

$$\ker D^*G(\bar{x},\bar{y}) = \{0\}$$

Furthermore, we compute the exact regularity bound

$$\operatorname{reg} G(\bar{x},\bar{y}) = \|D^*G^{-1}(\bar{y},\bar{x})\| = \|D^*G(\bar{x},\bar{y})^{-1}\|$$

where the norm of a positively homogeneous set-valued mapping $P\colon I\!R^n \rightrightarrows I\!R^m$ is defined by

$$\|P\| := \sup\Big\{\|y\| \;\Big|\; y \in P(x),\ \|x\| \le 1\Big\}$$

CONDITION MEASURE VIA EXACT REGULARITY BOUND

Define $\Phi\colon I\!\!R^{m+n} \rightrightarrows I\!\!R$ by

$$\Phi(x,y) := \begin{cases} \big[F(x,y),\infty) & \text{if } (x,y) \in \Delta_m \times \Delta_n, \\ \emptyset & \text{otherwise} \end{cases}$$

THEOREM. Assume that $(\Delta_m \times \Delta_n) \setminus S \neq \emptyset$ for the solution set S. Then we have the precise relationship

$$\kappa(A) = \sup_{(x,y)\in(\Delta_m\times\Delta_n)\setminus S} \operatorname{reg}\Phi\Big((x,y),F(x,y)\Big)$$

between the condition measure $\kappa(A)$ of the algorithm and the exact regularity bound of the mapping Φ

INDEX SETS

Let a_i as $i = 1, \ldots, n$ and $-b_k^\top$ as $k = 1, \ldots, m$ stand for the columns and the rows of the matrix A. By e_j, $j = 1, \ldots, m+n$, denote the unit vectors in $\mathbb{R}^{m+n}$, i.e.,

$$(e_j)_l = 0 \text{ for all } l \neq j \text{ and } (e_j)_j = 1 \text{ as } j = 1, \ldots, m+n$$

For a positive integer p, let $\mathbf{1}_p := \begin{bmatrix} 1 & \ldots & 1 \end{bmatrix} \in \mathbb{R}^p$. Finally a feasible point $(x, y) \in \Delta_m \times \Delta_n$, form the index sets by

$$\begin{cases} I(x) := \left\{ \bar{\imath} \in \{1, \ldots, n\} \,\middle|\, a_{\bar{\imath}}^\top x = \max\limits_{i \in \{1,\ldots,n\}} a_i^\top x \right\} \\ K(y) := \left\{ \bar{k} \in \{1, \ldots, m\} \,\middle|\, b_{\bar{k}}^\top y = \max\limits_{k \in \{1,\ldots,m\}} b_k^\top y \right\} \\ J(x,y) := \left\{ j \in \{1, \ldots, m\} \,\middle|\, x_j = 0 \right\} \cup \left\{ j = m+p \,\middle|\, y_p = 0 \right\} \end{cases}$$

COMPUTING THE EXACT BOUND OF METRIC REGULARITY

THEOREM. For any $(x,y) \in (\Delta_m \times \Delta_n) \setminus S$ the exact regularity bound of Φ around $((x,y), F(x,y))$ admits the representation

$$\text{reg}\,\Phi\Big((x,y), F(x,y)\Big) = \frac{1}{\text{dist}\Big(0; \partial F(x,y) + N_{\Delta_m \times \Delta_n}(x,y)\Big)}$$

Furthermore, we have the precise computing formulas

$$\partial F(x,y) = \text{co}\Big\{(a_i, b_k) \in I\!R^m \times I\!R^n \Big|\, i \in I(x),\ k \in K(y)\Big\}$$

$$N_{\Delta_m \times \Delta_n}(x,y) = \text{span}\,\{\mathbf{1}_m\} \times \text{span}\,\{\mathbf{1}_n\} - \text{cone}\Big[\text{co}\Big\{e_j \Big|\, j \in J(x,y)\Big\}\Big]$$

COMPUTING THE CONDITION MEASURE

THEOREM. Let $(\Delta_m \times \Delta_n) \setminus S \neq \emptyset$. Then the condition measure $k(A)$ of the algorithm is computed by

$$\kappa(A) = \sup_{(x,y)\in(\Delta_m \times \Delta_n)\setminus S} \Big[\ \text{dist}\Big(0; \text{co}\left\{(a_i, b_k)\Big|\ i \in I(x),\ k \in K(y)\right\}$$

$$+\text{span}\,\{\mathbf{1}_m\} \times \text{span}\,\{\mathbf{1}_n\}$$

$$-\text{cone}\left[\text{co}\left\{e_j\Big|\ j \in J(x,y)\right\}\right]\Big)\Big]^{-1}$$

SOME FUTURE RESEARCH

- average case analysis of $\kappa(A)$ algorithm
- singling out classes of well-conditioned problem
- preconditioning issues
- extensions to sequential games

$$\min_{x \in Q_1} \max_{y \in Q_2} x^\top A y = \max_{y \in Q_2} \min_{x \in Q_1} x^\top A y$$

where Q_1 and Q_2 are treeplexes

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