On the stability of the optimal value and the optimal set in optimization problems

Joint work with N. Dinh and M.A. Goberna

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Colloque JBHU 2010

Introduction

- Consider the optimization problem

$$\begin{array}{lll} (\mathrm{P}) & \inf & f(x) \\ & \text{s.t.} & f_t(x) \leq 0, \forall t \in T; \\ & & x \in C, \end{array}$$

  where:

  - T is an arbitrary (possibly infinite, possibly empty) index set
  - $\emptyset \neq C \subset X$ is an abstract constraint set, X is a Banach space
  - f, $f_t : X \to \mathbb{R} \cup \{+\infty\}$ for all $t \in T$

- MAIN GOAL: To analyze the stability of the optimal value function and the optimal set mapping of (P), ϑ and say $\mathcal{F}^{opt}$, under different possible types of perturbations of the data preserving the decision space X and the index set T.

Constraints

- In [5] we studied the effect on the set of feasible solutions, i.e. the set of solutions of the *constraint system*

$$\sigma := \{f_t(x) \leq 0, t \in T; \; x \in C\},$$

also represented $\sigma = \{f_t, t \in T; \; C\}$, of perturbing any function f_t, $t \in T$, and possibly the set C, under the condition that these perturbations maintain certain properties of the constraints.

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- The main goal of [5] was to study the stability of the *feasible set mapping* $\mathcal{F} : \Theta_\Diamond \rightrightarrows X$ such that

$$\mathcal{F}(\sigma) \quad = \{x \in X : f_t(x) \leq 0, \forall t \in T;\ x \in C\}.$$

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- If $T \neq \emptyset$, we shall use the *function*

$$g := \sup\{f_t, t \in T\}.$$

Parametric spaces

- We consider *parametric spaces* of the form

$$\Pi_\lozenge = \mathcal{V}_\lozenge \times \Theta_\lozenge,$$

where $\mathcal{V}_\lozenge$ is a particular family of functions $f : X \to \mathbb{R} \cup \{+\infty\}$ and $\Theta_\lozenge$ is a particular family of systems σ.

- The *1st object* analyzed in the present paper is the *optimal value function* $\vartheta : \Pi_\lozenge \to \mathbb{R} \cup \{\pm\infty\}$ defined as follows

$$\pi = (f, \sigma) \in \Pi_\lozenge \Rightarrow \vartheta(\pi) := \inf\{f(x) : x \in \mathcal{F}(\sigma)\} = \inf f(\mathcal{F}(\sigma)).$$

Conventions: $\vartheta(\pi) = +\infty$ if $\mathcal{F}(\sigma) = \emptyset$ (i.e. if $\sigma \notin \mathrm{dom}\,\mathcal{F}$).

- If $\vartheta(\pi) = -\infty$ we say that π is *unbounded*.

Optimal set mapping

- The *2nd* object of this work is the *optimal set mapping* $\mathcal{F}^{opt} : \Pi_\Diamond \rightrightarrows X$

$$\pi = (f, \sigma) \in \Pi_\Diamond \Rightarrow \mathcal{F}^{opt}(\pi) := \{x \in \mathcal{F}(\sigma) : \ f(x) = \vartheta(\pi)\}.$$

- If $\pi \in \mathrm{dom}\, \mathcal{F}^{opt}$ (i.e. $\mathcal{F}^{opt}(\pi) \neq \emptyset$) we say that π is (optimally) *solvable*.
- It is obvious that the stability of ϑ and if $\mathcal{F}^{opt}$ will be greatly influenced by the stability of $\mathcal{F}$, and this why many results in this presentation deal with stability properties of $\mathcal{F}$.

Parametric spaces

- In this presentation we consider only two parameter spaces, namely:

$$\Pi_1 := \left\{ \pi \in \Pi : \begin{array}{l} f \text{ and } f_t, t \in T, \text{ are lsc} \\ C \text{ is closed} \end{array} \right\},$$

$$\Pi_2 := \left\{ \pi \in \Pi_1 : \begin{array}{l} f \text{ and } f_t, t \in T, \text{ are convex} \\ C \text{ is convex} \end{array} \right\},$$

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- If $\pi = (f, \sigma) \in \Pi_2$ both sets are also convex.

Limit sets

- Let $A_1, A_2, ..., A_n, ..$ be a sequence of nonempty subsets of a first countable Hausdorff space Y. We consider the set of *limit points* of this sequence

$$y \in \operatorname*{Li}_{n\to\infty} A_n \Leftrightarrow \begin{cases} \text{there exist } y_n \in A_n,\ \ n = 1, 2, ..., \\ \text{such that } (y_n)_{n\in\mathbb{N}} \text{ converges to } y \end{cases};$$

and the set of *cluster points*

$$y \in \operatorname*{Ls}_{n\to\infty} A_n \Leftrightarrow \begin{cases} \text{there exist } n_1 < n_2 < ... < n_k ..., \text{ and } y_{n_k} \in A_{n_k} \\ \text{such that } (y_{n_k})_{k\in\mathbb{N}} \text{ converges to } y \end{cases}.$$

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- Clearly $\mathrm{Li}_{n\to\infty} A_n \subset \mathrm{Ls}_{n\to\infty} A_n$ and both sets are closed.
- We say that $A_1, A_2, ..., A_n, ..$ is *Kuratowski-Painlevé* convergent to the closed set A if $\mathrm{Li}_{n\to\infty} A_n = \mathrm{Ls}_{n\to\infty} A_n = A$, and we write then $A = K - \lim_{n\to\infty} A_n$.

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- If both Y and Z are first countable Hausdorff spaces, $\mathcal{S}$ is *closed* at $y \in Y$ if for every pair of sequences $(y_n)_{n \in \mathbb{N}} \subset Y$ and $(z_n)_{n \in \mathbb{N}} \subset Z$ satisfying $z_n \in \mathcal{S}(y_n)$ for all $n \in \mathbb{N}$, $\lim_{n \to \infty} y_n = y$ and $\lim_{n \to \infty} z_n = z$, one has $z \in \mathcal{S}(y)$.

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- $\mathcal{S}$ is said to be *closed* if it is closed at every $y \in Y$. Obviously, $\mathcal{S}$ is closed if and only if $\operatorname{gph} \mathcal{S} := \{(y, z) \in Y \times Z : z \in \mathcal{S}(y)\}$ is closed.

Strong Slater constraint qualification

- We say that $\pi = (f, \sigma)$ (or, equivalently, σ) satisfies the *strong Slater condition* if there exists some $\bar{x} \in \operatorname{int} C$ and some $\rho > 0$ such that $f_t(\bar{x}) < -\rho$ for all $t \in T$ (i.e., $g(\bar{x}) \leq -\rho$).

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- In such a case, $\bar{x}$ is called *strong Slater* (SS) *point* of π (or σ) with associated constant ρ.

Metrics for functions and sets

In order to define a suitable topology on the parameter spaces $\Pi_\Diamond$ we proceed in two steps. Let us start with the *1st step*.

- We equip the space $\mathcal{V}$ of all functions of the form $f : X \to \mathbb{R} \cup \{+\infty\}$ with the topology of uniform convergence on bounded sets of X.
- It is well known that a compatible metric for this topology is given by

$$d(f, h) := \sum_{k=1}^{+\infty} 2^{-k} \min\{1, \sup_{\|x\| \leq k} |f(x) - h(x)|\}.$$

  Here, by convention, we understand that
  $(+\infty) - (+\infty) = 0, \ |-\infty| = |+\infty| = +\infty.$
- Let $f, f_n \in \mathcal{V}, \ n = 1, 2, ...$ Then $d(f_n, f) \to 0$ if and only if the sequence $f_1, f_{2,}..., f_n, ..$*converges uniformly* to f *on the bounded sets* of X.
- The function spaces $\mathcal{V}_1 := \{f \in \mathcal{V} : f \text{ is lsc}\}$ and $\mathcal{V}_2 := \{f \in \mathcal{V}_1 : f \text{ is convex}\}$, with the metric d, are *complete* metric spaces.

Distances between sets

- In the space of closed sets in X we shall consider the *Attouch-Wets topology*, which is the inherited topology from the one considered in $\mathcal{V}_1$ under the identification

$$C \longleftrightarrow d_C(\cdot),$$

with $d_C(x) = \inf_{c \in C} \|x - c\|$.

- The sequence of nonempty closed sets $(C_n)_{n\in\mathbb{N}}$ *converges in the sense of Attouch-Wets* to the nonempty closed set C if the sequence of functions $(d_{C_n})_{n\in\mathbb{N}}$ converges to d_C uniformly on the bounded sets of X.
- This topology is compatible with the distance

$$\widetilde{d}(C, D) := \sum_{k=1}^{+\infty} 2^{-k} \min \left\{ 1, \sup_{\|x\| \leq k} |d_C(x) - d_D(x)| \right\},$$

i.e. $\widetilde{d}(C, D) = d(d_C, d_D)$.

More on convergence of sets

- The space of all closed sets in X equipped with this distance $\widetilde{d}$ becomes a complete metric space.
- Because X is Banach, we have that if the sequence $(d_{C_n})_{n\in\mathbb{N}}$ converges uniformly on bounded sets of X to a continuous function f, there exists a nonempty closed set C such that $f = d_C$.
- The sequence of nonempty closed sets $(C_n)_{n\in\mathbb{N}}$ converges in Attouch-Wets sense to the nonempty closed C if and only if

$$\forall k \in \mathbb{N} : \lim_{n\to\infty} \max\{e(C_n \cap k\mathbb{B}, C), e(C \cap k\mathbb{B}, C_n)\} = 0,$$

where

$$e(A, B) := \sup_{a\in A} d_B(a) = \inf\{\alpha > 0 : B + \alpha\mathbb{B} \supset A\},$$

and $\mathbb{B} := \{x \in X : \|x\| \le 1\}$.

Our metric

- Given $\pi = (f, \{f_t, t \in T; C\})$, $\pi' = (f', \{f'_t, t \in T; C'\}) \in \Pi$, we define

$$\mathbf{d}(\pi, \pi') := \max\{d(f, f'), \sup_{t \in T} d(f_t, f'_t), \widetilde{d}(C, C')\}. \qquad (1)$$

If $T = \emptyset$, we take $\sup_{t \in T} d(f_t, f'_t) = 0$.

Theorem

$(\Pi_i, \mathbf{d})$, $i = 1, 2$, *are complete metric spaces.*

Some preliminary results

Lemma

Let C be a closed set in X, $x_0 \in \operatorname{int} C$, and consider $\varepsilon > 0$ such that $x_0 + \varepsilon\mathbb{B} \subset C$. Then there is $\rho > 0$ such that

$$\widetilde{d}(C, C') < \rho \implies (x_0 + \varepsilon\mathbb{B}) \cap C' \neq \emptyset.$$

Lemma

Consider $\pi = (f; \{f_t, t \in T; C\}) \in \Pi_1$ and suppose that the marginal function $g = \sup_{t\in T} f_t$ is usc (and so, continuous). If $\hat{x}$ is an SS-point of π, then there exists $\varepsilon > 0$ such that

$$x \in \hat{x} + \varepsilon\mathbb{B} \text{ and } d(\pi, \pi') < \varepsilon \Longrightarrow g'(x) < 0,$$

with $\pi' = (f'; \{f'_t, t \in T; C'\}) \in \Pi_1$ and $g' := \sup_{t\in T} f'_t$.

- Consider a convex set C with $0 \in \operatorname{int} C \neq \emptyset$, and the associated *Minkovski gauge function* defined as
$$p_C(x) := \inf\{\lambda \geq 0 \mid x \in \lambda C\},$$
and for any positive real number μ, define a set
$$C_\mu := \{x \in X \mid p_C(x) \leq \mu\}.$$
Given $\varepsilon > 0$, there exists $\mu \in]0,1[$ such that
$$\widetilde{d}(C, C_\mu) \leq \varepsilon.$$
- The system σ is said to be *Tuy regular* if there exists $\varepsilon > 0$ such that for any $u \in \mathbb{R}^T$ and for any nonempty convex set $C' \subset X$ satisfying $\max\{\sup_{t \in T} |u_t|, \ \widetilde{d}(C, C')\} < \varepsilon$, the system $\sigma' = \{f_t(x) - u_t \leq 0, t \in T; \ x \in C'\} \in \operatorname{dom} \mathcal{F}$.

The last definition is inspired in a similar one of H. Tuy ([3]).

Theorem

The feasible set mapping $\mathcal{F}$ is closed on Θ_i, $i = 1, 2$.

Theorem

Let $\sigma = \{f_t, t \in T; C\} \in \Theta_1$ with $T \neq \emptyset$, and consider the following statements:
(i) $\mathcal{F}$ is lsc at σ;
(ii) $\sigma \in \operatorname{int} \operatorname{dom} \mathcal{F}$;
(iii) σ is Tuy regular;
(iv) σ satisfies the strong Slater condition;
(v) $\mathcal{F}(\sigma)$ is the closure of the set of SS points of σ.
Then, $(i) \Rightarrow (ii) \Rightarrow (iii)$ and $(v) \Rightarrow (iv)$. Moreover, if C is convex, and $\operatorname{int} C \neq \emptyset$, then $(i) \Rightarrow (v)$ and $(iii) \Rightarrow (iv)$.
If, in addition, $\sigma \in \Theta_2$ and $g = \sup_{t \in T} f_t$ is usc, then all the statements $(i) - (v)$ are equivalent.

Metric regularity of $\mathcal{F}^{-1}$

- This property has important consequences in the overall stability of a system σ, as well as in the sensitivity analysis of perturbed systems, affecting even the numerical complexity of the algorithms conceived for finding a solution of the system.
- Many authors ([Aubin84], [Ausl84], [Com90], [JuThi90], [KlaHenr98], [KlKu85], [Rob75,76], [ZoKur79], etc.) have investigated this property and explored the relationship of this property with standard constraint qualification as Mangasarian-Fromovitz CQ, Slater CQ, Robinson CQ, etc.
- For instance, in [KlaHenr98] the relationships among the metric regularity, the metric regularity with respect to RHS perturbations, and the extended Mangasarian-Fromowitz CQ are established in a non-convex differentiable setting.

Let us remember the definition of metric regularity in our specific setting:

Definition

$\mathcal{F}^{-1}$ is said to be *metrically regular at* $(x,\sigma) \in \operatorname{gph} \mathcal{F}^{-1}$ if there exist real numbers $\varepsilon, \delta > 0$ and $\kappa \geq 0$ such that

$$\left. \begin{array}{l} \mathbf{d}(\sigma,\sigma') < \delta \\ \|x - x'\| < \varepsilon \end{array} \right\} \Rightarrow d(x',\mathcal{F}(\sigma')) \leq \kappa \mathbf{d}(\sigma',\mathcal{F}^{-1}(x')). \tag{2}$$

This inequality is specially useful if the residual $\mathbf{d}(\sigma',\mathcal{F}^{-1}(x'))$ can be easily computed.

The existence of an abstract constraint set makes the computation of $\mathbf{d}(\sigma',\mathcal{F}^{-1}(x'))$ very difficult. In fact, if $\sigma' = \{f_t', t \in T, C'\}$ we have

$$\mathbf{d}(\sigma',\mathcal{F}^{-1}(x')) = \max\left\{ \left[g'(x')\right]_+, \widetilde{d}(C',\mathcal{C}_{x'}(X)) \right\}, \tag{3}$$

where $\mathcal{C}_{x'}(X)$ is the family of all the closed convex sets $C \subset X$ such that $x' \in C$, and

$$\widetilde{d}(C',\mathcal{C}_{x'}(X)) = \inf\left\{ \widetilde{d}(C',C) : \ C \in \mathcal{C}_{x'}(X)) \right\}.$$

Nevertheless, when we assume that C is the whole space X, the property makes sense. In fact, if C is constantly equal to X and $\sigma' = \{f'_t, t \in T\}$, it is straightforward that

$$\mathbf{d}(\sigma', \mathcal{F}^{-1}(x')) = \left[\sup_{t\in T} f'_t(x')\right]_+ \equiv \left[g'(x')\right]_+,$$

where $g' = \sup_{t\in T} f'_t$ and $[\alpha]_+ := \max\{\alpha, 0\}$.

Theorem

Let $\mathcal{F} : \Theta_\Diamond \rightrightarrows X$ and $(x, \sigma) \in \operatorname{gph} \mathcal{F}^{-1}$ with $\sigma = \{f_t, t \in T\}$, where $\Theta_\Diamond$ is the set of parameters whose constraint set is X and f_t is convex for all $t \in T$. Then the following statements are true:
(i) If $g = \sup_{t\in T} f_t$ is usc at x, and $\mathcal{F}^{-1}$ is metrically regular at (x, σ), then $\mathcal{F}$ is lsc at σ.
(ii) If X is a Hilbert space, and $\mathcal{F}$ is lsc at σ, then $\mathcal{F}^{-1}$ is metrically regular at (x, σ).

Upper semicontinuity of the optimal value function

We now study the upper semicontinuity of the optimal value function ϑ.

Theorem

Let $\pi = (f, \sigma) \in \Pi_1$. The following statements hold.
(i) If $\mathcal{F}$ is lsc at σ then ϑ is usc at π provided that f is usc.
(ii) If ϑ is usc at π then $\mathcal{F}$ is lsc at σ provided that the functions f_t, $t \in T$, are convex, C is convex (i.e., if $\sigma \in \Theta_2$), $\operatorname{int} C \neq \emptyset$, and the corresponding marginal function $g = \sup_{t \in T} f_t$ is usc.

Lower semicontinuity of the optimal value function

Consider the *sublevel sets mapping* $\mathcal{L} : \Pi_{\lozenge} \times \mathbb{R} \rightrightarrows X$:

$$\mathcal{L}(\pi, \lambda) := \{x \in \mathcal{F}(\sigma) : \ f(x) \leq \lambda\}, \text{ with } \pi = (f, \sigma).$$

Theorem

The mapping $\mathcal{L}$ is closed at any point $(\pi, \lambda) \in \Pi_1 \times \mathbb{R}$.

Definition

Let Y and Z be two top. spaces and $\mathcal{S} : Y \rightrightarrows Z$. We say that $\mathcal{S}$ is *uniformly compact-bounded* at $y_0 \in Y$ if $\exists$ a compact set $K \subset Z$ and a neighborhood V of y_0 such that $y \in V \Longrightarrow \mathcal{S}(y) \subset K$.

Theorem

(a) If $\mathcal{L}$ is uniformly compact-bounded at $(\pi, \vartheta(\pi))$ with $\pi \in \Pi_1$, then ϑ is lsc at π.
(b) Suppose that $X = \mathbb{R}^n$, and $\pi \in \Pi_2$. If $\mathcal{F}^{opt}(\pi)$ is a nonempty compact set, then $\mathcal{L}$ is uniformly compact-bounded at $(\pi, \vartheta(\pi))$.

Optimal set mapping

This section starts with a sufficient condition for the closedness of $\mathcal{F}^{opt}$.

Theorem

Consider $\pi = (f, \sigma) \in \Pi_1$ such that f is usc and $\mathcal{F}$ is lsc at σ. Then $\mathcal{F}^{opt}$ is closed at π.

Theorem

Consider $\pi = (f, \sigma) \in \Pi_1$ such that f is usc, $\mathcal{F}$ is lsc at σ, and $\mathcal{L}$ is uniformly compact-bounded at $(\pi, \vartheta(\pi))$. Then, ϑ is continuous at π and $\mathcal{F}^{opt}$ is usc at π.

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