Subdifferential and Argmin sets of Functions, Asymptotic Functions and their Legendre-Fenchel Successive conjugates

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Position of the problem

In theory and practise, one disposes of the following scheme

- Primal (initial) setting

$$f : X \to \overline{\mathbb{R}}$$

is a given function, not necessarily convex or lsc; $\operatorname{Argmin} f$ may be empty or not, and may have no specific structure; the Fenchel subdifferential ∂f may not have good properties ...

- Dual setting

$$f^* : X^* \to \overline{\mathbb{R}}$$

is the Legendre-Fenchel conjugate which is convex and weakly lsc, and proper in the most of the interesting cases; the Fenchel subdifferential ∂f^* has many rich properties and, so, one can obtain informations on the function f^* ...

Position of the problem

- Relaxed setting

$$f^{**} : X \to \overline{\mathbb{R}}$$

is the biconjugate function which is also convex and weakly lsc, and satisfies $f^{**} = \overline{\mathrm{co}} f$ in the most of the interesting cases, where $\overline{\mathrm{co}} f$ is the lsc convex hull; $\inf f = \inf f^{**}$,

$$\overline{\mathrm{co}}(\mathrm{Argmin}\, f) \subset \mathrm{Argmin}\, f^{**}; \mathrm{Argmin}\, f^{**} = \partial f^{*}(0).$$

This previous scheme is directly manageable provided that $f \in \Gamma_0(X)$. Indeed, in this case

$$\mathrm{Argmin}\, f = \mathrm{Argmin}\, f^{**},\ \partial f^{*} = (\partial f)^{-1}.$$

Our interest is to find tools which allow the validity of formulas relating ∂f^{*} and ∂f, $\mathrm{Argmin}\, f$ and $\mathrm{Argmin}\, f^{**}$, for general functions which are not necessarily convex or lsc. Roughly speaking, we put two questions

- expressing ∂f^{*} and ∂f^{**} by means of ∂f
- expressing $\mathrm{Argmin}\, f^{**}$ or $\mathrm{Argmin}\, \overline{\mathrm{co}} f$ by means of $\mathrm{Argmin}\, f$?

Outline

- First formulas of the Legendre-Fenchel subdifferential of f^*
  - formulas by means of the ε-subdifferential $\partial_\varepsilon f$ (Hiriart-Urruty - López - Volle)
  - Another enlargement of the Legendre-Fenchel subdifferential: weak subdifferential ∂^w
  - formulas by means of the weak subdifferential ∂^w
  - a couple of example
- Formulas of the Legendre-Fenchel subdifferential and Asymptotic analysis
  - a variant of asymptotic functions: weak asymptotic functions
  - formulas of the Legendre-Fenchel subdifferential of f^* invoking asymptotic terms
- Formulas of the Legendre-Fenchel subdifferential and Argmin set of the biconjugate/lsc convex hull

Notation

In what follows, $(X, X^*, \langle \cdot, \cdot \rangle)$ will be a topological pair of real locally convex spaces. Given a function $f : X \to \overline{\mathbb{R}}$,

- $f^* : X^* \to \overline{\mathbb{R}}$ is the (Legendre-Fenchel) conjugate function

$$f^*(x^*) := \sup\{\langle x, x^* \rangle - f(x), x \in X\}$$

- $f^{**} : X \to \overline{\mathbb{R}}$ is the biconjugate function

$$f^{**}(x^*) := \sup\{\langle x, x^* \rangle - f^*(x^*), x^* \in X^*\}$$

- $\overline{\mathrm{co}} f$ is the lsc convex hull,

$$\mathrm{epi}(\overline{\mathrm{co}} f) = \overline{\mathrm{co}}(\mathrm{epi}\, f)$$

- $\partial_\varepsilon f : X \rightrightarrows X^*$, $\varepsilon \geq 0$, is the (Legendre-Fenchel) ε-subdifferential of f

$$\partial_\varepsilon f(x) := \{x^* \in X^* \mid f^*(x^*) + f(x) \leq \langle x^*, x \rangle + \varepsilon\};$$

  if $\varepsilon = 0$ we denote $\partial f(x) := \partial_0 f(x)$

Notation

- ε-$\mathrm{Argmin}\, f$ is the set of ε-minimum

$$\varepsilon\text{-}\,\mathrm{Argmin}\, f := \{x \in X : f(x) \leq \inf_X f + \varepsilon\};$$

  if $\varepsilon = 0$ we denote $\mathrm{Argmin}\, f := 0\text{-}\mathrm{Argmin}\, f$
- If $M : X \rightrightarrows X^*$ is a set-valued operator, $M^{-1} : X^* \rightrightarrows X$ denotes its inverse

$$M^{-1}(x^*) := \{x \in X \mid x^* \in Mx\}$$

If $A \subset X$, we denote

- $A^- := \{x^* \in X^* : \langle x^*, x\rangle \leq 0 \ \forall x \in A\}$
- $\mathrm{co}\, A$ and $\overline{\mathrm{co}} A$ are the convex and the closed convex hulls of A, respectively
- $\mathrm{par}\, A$ is the subspace parallel to the affine space $\mathrm{aff}\, A$
- $\mathrm{N}_A(x) := (A - x)^-$ is the normal cone to A at $x \in A$

Formulas via the epsilon-subdifferential

Explicit formulas for ∂f^* have been recently established by M.A. López and M. Volle (see, also, Hiriart-Urruty, M.A. López and M. Volle).The most general one is (if $\mathrm{dom}\, f^* \neq \emptyset$)

$$\partial f^*(x^*) = \bigcap_{\substack{\varepsilon>0 \\ y^* \in \mathrm{dom}\, f^*}} \overline{\mathrm{co}} \left[(\partial_\varepsilon f)^{-1}(x^*) + \{y^* - x^*\}^- \right]$$

Equivalently, it was proved by these authors that

$$\partial f^*(x^*) = \bigcap_{\substack{\varepsilon>0 \\ L \in \widetilde{\mathcal{F}}(f,x^*)}} \overline{\mathrm{co}} \left[(\partial_\varepsilon f)^{-1}(x^*) + \mathrm{N}_{L \cap \mathrm{dom}\, f^*}(x^*) \right]$$

where

$$\widetilde{\mathcal{F}}(f, x^*) := \{ L \subset X^* \mid L \text{ convex}, x^* \in L, \ \mathrm{ri}(\mathrm{cone}(L \cap \mathrm{dom}\, f^*)) \neq \emptyset \}$$

As a consequence of the previous formulas, one gets

$$\mathrm{Argmin}\, \overline{\mathrm{co}} f = \bigcap_{\substack{\varepsilon>0 \\ L \in \widetilde{\mathcal{F}}(f,x^*)}} \overline{\mathrm{co}} \left[\varepsilon\text{- } \mathrm{Argmin}\, f + \mathrm{N}_{L \cap \mathrm{dom}\, f^*}(x^*) \right]$$

Our objectives

- In the first part, we propose another enlargement ∂^w of the Fenchel subdifferential of f (instead of $\partial_\varepsilon f$) in order to obtain new formulas modelled on the formulas above, by Hiriart-Urruty, M. A. López, and M. Volle
- In the second part, we propose an appropriate notion of asymptotic functions in order to rewrite the term $\mathrm{N}_{L\cap \mathrm{dom}\, f^*}(x^*)$ by means of primal objects
- Consequently, we give formulas for $\mathrm{Argmin}\,\overline{\mathrm{co}}f$ by means of $\mathrm{Argmin}\, f$, $\mathrm{Argmin}\, f^\infty$
- This approach allows us to extend some of the results found by Benoist and Hiriart-Urruty, namely

$$\mathrm{Argmin}\,\overline{\mathrm{co}}f = \mathrm{co}(\mathrm{Argmin}\, f) + \mathrm{co}(\mathrm{Argmin}\, f^\infty)$$

$$\partial\overline{\mathrm{co}}f(x) = \bigcap_{0\le i\le n} \partial f(x_i) \bigcap_{1\le j\le n} \partial f^\infty(y_j)$$

for $X = \mathbb{R}^n$, $\mathrm{int}(\mathrm{dom}\, f^*) \neq \emptyset$ (i.e. f is epi-pointed), and f is lsc.

Enlargement of the Legendre-Fenchel subdifferential

If $f \in \Gamma_0(X)$, then ∂f^* is completely characterized by ∂f in view of the straightforward relationship

$$\partial f^* = (\partial f)^{-1}$$

But, in general ∂f is too small to build up the whole ∂f^*

Example: let

$$f(x) := e^{-|x|}.$$

Then, direct computations yield

$$f^* = \mathrm{I}_{\{0\}},\ \partial f^*(0) = \mathrm{N}_{\{0\}}(0) = \mathbb{R}$$

$$(\partial f)^{-1}(0) = (\partial f)^{-1}(0) + \mathrm{N}_{\mathrm{dom}\, f^*}(0) = \emptyset.$$

Thus, we need to enlarge the concept of the Fenchel subdifferential by taking into account the geometry of $\mathrm{dom}\, f^*$.

Enlargement of the Legendre-Fenchel subdifferential

Definition:
Given a function $f : X \to \overline{\mathbb{R}}$, a vector $x^* \in X^*$ is said to be a weak (Fenchel-Legendre) subgradient of f at $x \in X$ iff $f^*(x^*) \in \mathbb{R}$, and there exists a net $(x_\gamma) \subset X$ such that

- $\lim \langle x_\gamma - x, y^* \rangle = 0 \;\; \forall y^* \in \overline{\mathrm{par}}\,(\mathrm{dom}\, f^*)$
- $\lim (f(x_\gamma) - \langle x_\gamma, x^* \rangle) = -f^*(x^*)$

The set of such weak subgradients, denoted by $\partial^w f(x)$, is called the weak subdifferential of f at x
In particular, x is a critical point w.r.t. ∂^w if $0 \in \partial^w f(x)$; that is, $\exists (x_\gamma) \subset X$ such that

- $\lim f(x_\gamma) = \inf f \in \mathbb{R}$; that is (x_γ) is a minimizing net
- $\lim \langle x_\gamma - x, y^* \rangle = 0 \;\; \forall y^* \in \overline{\mathrm{par}}\,(\mathrm{dom}\, f^*)$

In other words, if $Y := \overline{\mathrm{par}}\,(\mathrm{dom}\, f^*)$, x is the weak limit (as a point of Y^*) of the minimizing net (x_γ) of f

First properties of the weak subdifferential

- If $f \in \Gamma_0(X)$, then
$$\partial^w f = \partial f$$
- if $\mathrm{int}(\mathrm{dom}\, f^*) \neq \emptyset$, then
$$\partial^w f = \partial(\mathrm{cl}^w f)$$
- If $\mathrm{int}(\mathrm{dom}\, f^*) \neq \emptyset$ and f is weakly lsc, then ∂^w and ∂ coincide
- If $X = \mathbb{R}$, f is lsc, and $\partial f(x) = \{x^*\}$, then
$$\partial^w f(x) = \partial f(x)$$
even if $\mathrm{int}(\mathrm{dom}\, f^*) = \emptyset$
- ∂^w satisfies the following elementary chain rule
$$\partial^w (f + \langle x^*, \cdot \rangle) = \partial^w f + \{x^*\}, \quad x^* \in X^*$$

Illustrative examples

Example: for $f(x) = e^{-|x|}$ we have

$$\operatorname{aff}(\operatorname{dom} f^*) = \{0\} \text{ and } f^*(0) = 0.$$

Then, $x \in (\partial^w f)^{-1}(0)$ if and only if $\exists (x_k)_k \subset \mathbb{R}$ such that

$$f(x_k) - 0x_k = f(x_k) \to -f^*(0) = 0.$$

Therefore, as $\lim_{k\to+\infty} e^{-k} = 0$,

$$\partial f^*(0) = (\partial^w f)^{-1}(0) = \mathbb{R};$$

in other words, $\partial^w f^*(0) = (\partial^w f)^{-1}(0)$.
Example: let

$$f(a) := \sqrt{a} \text{ if } a \geq 0 \;\text{ and } f(a) := +\infty \;\text{ if } a < 0.$$

Then, $\operatorname{dom} f^* = (-\infty, 0]$ and so

$$\partial^w f(0) = \mathbb{R}_-;\; \partial^w f(a) = \emptyset \text{ if } a \neq 0.$$

Formulas for the subdifferential of the conjugate

Theorem: for every $x^* \in X^*$ we have the formula

$$\partial f^*(x^*) = \bigcap_{L \in \mathcal{F}(f,x^*)} \overline{\mathrm{co}} \left\{ (\partial^w (f \Box \sigma_L))^{-1}(x^*) + \mathrm{N}_{L \cap \mathrm{dom}\, f^*}(x^*) \right\}$$

where

$$\mathcal{F}(f,x^*) = \{L \in \mathcal{F}(f) \mid x^* \in L\} \text{ and}$$

$$\begin{aligned} \mathcal{F}(f) \quad = \quad & \{L \subset X^* \text{ closed and convex} \mid \mathrm{ri}(L \cap \mathrm{dom}\, f^*) \neq \emptyset, \\ & \qquad f^*_{|\mathrm{ri}(L \cap \mathrm{dom}\, f^*)} \text{ is finite and continuous}\} \end{aligned}$$

The same formula holds if instead of $\mathcal{F}(f)$ we take the set

$$\{L \subset X^* \mid L \text{ is a finite-dimensional linear subspace containing } x^*\}$$

or, equivalently,

$$\{L := x^* + \mathbb{R}(y^* - x^*),\ y^* \in \mathrm{dom}\, f^*\}$$

Special cases

- If $X^* \in \mathcal{F}(f)$, then

$$\partial f^*(x^*) \subset \overline{\mathrm{co}}\left\{(\partial^w f)^{-1}(x^*) + \mathrm{N}_{\mathrm{dom}\, f^*}(x^*)\right\}$$

As the inverse inclusion always holds, we obtain

$$\partial f^*(x^*) = \overline{\mathrm{co}}\left\{(\partial^w f)^{-1}(x^*) + \mathrm{N}_{\mathrm{dom}\, f^*}(x^*)\right\}$$

- The last formula obviously holds if $X = \mathbb{R}^n$.
- If $X^* \in \mathcal{F}(f)$ and f is positively homogeneous, and f^* is proper, then

$$\partial f^*(x^*) = \overline{\mathrm{co}}\left\{(\partial^w f)^{-1}(x^*)\right\}$$

- If f satisfies

$$f(x) \leq \liminf\{f(y), \quad \langle y - x, y^*\rangle \to 0 \ \ \forall y^* \in \overline{\mathrm{par}}\,(\mathrm{dom}\, f^*)\}$$

then $\partial^w f$ and ∂f coincide and, so, the formulas above hold with ∂f instead of $\partial^w f$

Special cases

- If $\operatorname{int}(\operatorname{dom} f^*) \neq \emptyset$ and f^* is continuous in $\operatorname{int}(\operatorname{dom} f^*)$, then

$$\partial f^*(x^*) = \mathrm{N}_{\operatorname{dom} f^*}(x^*) + \overline{\mathrm{co}}\left\{(\partial^w f)^{-1}(x^*)\right\}$$

  thus if f is, in addition, weakly lsc, then

$$\partial f^*(x^*) = \mathrm{N}_{\operatorname{dom} f^*}(x^*) + \overline{\mathrm{co}}\left\{(\partial f)^{-1}(x^*)\right\}$$

- The previous formulas hold if $\operatorname{int}(\operatorname{dom} f^*) \neq \emptyset$ and X^* is a barrelled space (in particular, a Banach space)
- If $X = \mathbb{R}^n$ and $\operatorname{int}(\operatorname{dom} f^*) \neq \emptyset$, then

$$\partial f^*(x^*) = \mathrm{N}_{\operatorname{dom} f^*}(x^*) + \mathrm{co}\left\{(\partial^w f)^{-1}(x^*)\right\}$$

  If f is, in addition, lsc, then

$$\partial f^*(x^*) = \mathrm{N}_{\operatorname{dom} f^*}(x^*) + \mathrm{co}\left\{(\partial f)^{-1}(x^*)\right\}$$

Example 1

Given a subset $C \subset H$, we define $f : H \to \mathbb{R} \cup \{+\infty\}$ as

$$f(x) := \frac{1}{2}\|x\|^2 + \mathrm{I}_C(x)$$

so that $\mathrm{cl}^w f(x) = \frac{1}{2}\liminf_{y \rightharpoonup x}\|y\|^2 + \mathrm{I}_{\overline{C}^w}(x)$. By Asplund's formula

$$(\mathrm{cl}^w f)^*(x^*) = f^*(x^*) = \frac{1}{2}(\|x^*\|^2 - d_C^2(x^*))$$

Hence, $\mathrm{dom}\, f^* = H$ and f^* is continuous on H.Therefore,

$$\begin{aligned} \partial f^*(x^*) &= \overline{\mathrm{co}}\left\{(\partial(\mathrm{cl}^w f))^{-1}(x^*)\right\} \\ &= \overline{\mathrm{co}}\{x \in \overline{C}^w \mid \exists\ C \ni x_k \rightharpoonup x,\ \lim_k \|x_k - x^*\| = d_C(x^*)\} \end{aligned}$$

Consequences: $\partial f^*(x^*) = \overline{\mathrm{co}}\{\pi_C(x^*)\}$ in each one of the following cases:

- If C is weakly closed or approximately compact
- If C is approximately convex; that is, $\limsup_{y \to x} \frac{d(y,C) - d(x,C)}{\|x-y\|} = 1$

Example 2

Given a bounded set $C \subset H$, we consider the function $f : H \to \mathbb{R} \cup \{+\infty\}$ defined by

$$f(x) := -\frac{1}{2}\|x\|^2 + \mathrm{I}_{-C}(x)$$

so that the conjugate given by

$$f^*(x^*) = \frac{1}{2}(\sup_{y \in C} \|y - x^*\|^2 - \|x^*\|^2)$$

is continuous on H. Consequently,

$$\begin{aligned} \partial f^*(x^*) &= -\overline{\mathrm{co}}\{(\partial^w f)^{-1}(x^*)\} \\ &= -\overline{\mathrm{co}}\{x \in C \mid \exists C \ni x_k \rightharpoonup x \text{ s.t. } \lim_k \|x_k - x^*\| = \sup_{y \in C} \|y - x^*\|\} \end{aligned}$$

Consequences:

- if H is finite-dimensional, then $\partial f^*(x^*) = -\,\mathrm{co}\{F_{\overline{C}}(x^*)\}$ where $F_{\overline{C}}(x^*)$ is the set of furthest points in $\overline{C}$ from x^*
- If C is approximately compact, then $\partial f^*(x^*) = -\overline{\mathrm{co}}\{F_{\overline{C}}(x^*)\}$

Asymptotic functions

The previous formulas for ∂f^* includes the term $\mathrm{N}_{\mathrm{dom}\, f^*}(x^*)$ which requires explicit knowledge of $\mathrm{dom}\, f^*$. Our objective in this part, is to provide formulas where $\mathrm{N}_{\mathrm{dom}\, f^*}(x^*)$ is replaced by a primal object, namely some appropriate asymptotic function

Definition:
Given a function $f : X \to \overline{\mathbb{R}}$ such that $\mathrm{dom}\, f^* \neq \emptyset$, we call weak asymptotic function of f the function $f^\infty : X \to \overline{\mathbb{R}}$ defined by

$$f^\infty(x) := \liminf\{sf\left(s^{-1}y\right) - \langle z^*, y - x\rangle : s \to 0^+, \\ \langle y - x, y^*\rangle \to 0 \ \ \forall y^* \in \overline{\mathrm{par}}(\mathrm{dom}\, f^*)\},$$

where z^* is any vector in $\mathrm{cl}(\mathrm{dom}\, f^*)$.

The positively homogeneous function f^∞ is well defined since the right-hand side does not depend on the choice of z^* in $\mathrm{cl}(\mathrm{dom}\, f^*)$

Some properties of the asymptotic functions

- If $f \in \Gamma_0(X)$, then f^∞ coincides with the recession function in the sense of convex analysis; that is,

$$f^\infty(x) = \sup_{t>0} \frac{f(y+tx) - f(y)}{t}, \quad y \in \operatorname{dom} f.$$

- f^∞ coincides with the asymptotic function in the sense of Debreu-Dedieu when the topology in X is $\sigma(X, X^*)$

$$f^\infty(x) = \liminf_{s \to 0^+,\ y \rightharpoonup x} sf(s^{-1}y),$$

  if one of the following hold
  - $\operatorname{int}(\operatorname{dom} f^*) \neq \emptyset$
  - $\operatorname{dom} f^* = \{z^*\}$
  - $X = \mathbb{R}$ and $\operatorname{dom} f^* \neq \emptyset$
- If $X^* \in \mathcal{F}(f)$, then $(f^\infty)^\infty = f^\infty$ and $\partial^w f^\infty = \partial f^\infty$

Relationship between asymptotic and recession functions

It is known (Debreu, Benoist - Hiriart-Urruty) that if $X = \mathbb{R}^n$, $\operatorname{int}(\operatorname{dom} f^*) \neq \emptyset$, and f is lsc, then the lsc convex hull of f^∞ is the recession function of the (proper lsc convex) function $\overline{\operatorname{co}} f$; that is,

$$\overline{\operatorname{co}} f^\infty = (\overline{\operatorname{co}} f)^\infty$$

We show that this property also holds for the functions $(f \Box \sigma_L)^\infty$:

Theorem: Let $f : X \to \overline{\mathbb{R}}$ be a given function. Then, for every $L \in \mathcal{F}(f)$,

$$(\overline{\operatorname{co}}(f \Box \sigma_L))^\infty = \overline{\operatorname{co}}\left((f \Box \sigma_L)^\infty\right)$$

Consequently,

- $((f \Box \sigma_L)^\infty)^* = \mathrm{I}_{\operatorname{cl}(L \cap \operatorname{dom} f^*))}$
- $\operatorname{dom}\left((f \Box \sigma_L)^\infty\right)^* = \operatorname{cl}(L \cap \operatorname{dom} f^*)$
- $X^* \in \mathcal{F}((f \Box \sigma_L)^\infty)$

Subdifferential of the conjugate via weak subdifferentials

From the previous formulas,

$$\partial f^*(x^*) = \bigcap_{L \in \mathcal{F}(f,x^*)} \overline{\mathrm{co}} \left\{ (\partial^w (f \Box \sigma_L))^{-1}(x^*) + \mathrm{N}_{L \cap \mathrm{dom}\, f^*}(x^*) \right\}$$

But,

$$\mathrm{N}_{\mathrm{dom}(f\Box\sigma_L)^*}(x^*) = \partial \mathrm{I}_{\overline{\mathrm{dom}(\overline{\mathrm{co}}(f\Box\sigma_L))^*}}(x^*) = \partial \left[(\overline{\mathrm{co}}(f\Box\sigma_L))^{\infty} \right]^* (x^*)$$

and $(\overline{\mathrm{co}}(f\Box\sigma_L))^{\infty} = \overline{\mathrm{co}}\left((f\Box\sigma_L)^{\infty}\right)$. Then,

$$\mathrm{N}_{L \cap \mathrm{dom}\, f^*}(x^*) = \partial \left[\overline{\mathrm{co}}\left((f\Box\sigma_L)^{\infty}\right) \right]^* (x^*) = \partial((f\Box\sigma_L)^{\infty})^*(x^*).$$

Since $(f\Box\sigma_L)^{\infty}$ is positively homogeneous, and $X^* \in \mathcal{F}((f\Box\sigma_L)^{\infty})$, we obtain

$$\partial((f\Box\sigma_L)^{\infty})^*(x^*) = \overline{\mathrm{co}} \left\{ (\partial(f\Box\sigma_L)^{\infty})^{-1}(x^*) \right\}$$

Theorem: we have that

$$\partial f^*(x^*) = \bigcap_{L \in \mathcal{F}(f,x^*)} \overline{\mathrm{co}} \left\{ (\partial^w (f \Box \sigma_L))^{-1}(x^*) + (\partial(f\Box\sigma_L)^{\infty})^{-1}(x^*) \right\}$$

Some particular cases

- if $X^* \in \mathcal{F}(f)$, then

$$\partial f^*(x^*) = \overline{\mathrm{co}}\left\{(\partial^w f)^{-1}(x^*) + (\partial f^\infty)^{-1}(x^*)\right\}$$

- if $X^* \in \mathcal{F}(f)$ and $\mathrm{int}(\mathrm{dom}\, f^*) \neq \emptyset$, then

$$\begin{aligned}\partial f^*(x^*) &= \overline{\mathrm{co}}\left\{(\partial^w f)^{-1}(x^*)\right\} + \overline{\mathrm{co}}\left\{(\partial f^\infty)^{-1}(x^*)\right\} \\ &= \overline{\mathrm{co}}\left\{(\partial f)^{-1}(x^*)\right\} + \overline{\mathrm{co}}\left\{(\partial f^\infty)^{-1}(x^*)\right\}\ (f \text{ wlsc})\end{aligned}$$

- if $X = \mathbb{R}^n$ and $\mathrm{int}(\mathrm{dom}\, f^*) \neq \emptyset$, then

$$\begin{aligned}\partial f^*(x^*) &= \mathrm{co}\left\{(\partial^w f)^{-1}(x^*)\right\} + \mathrm{co}\left\{(\partial f^\infty)^{-1}(x^*)\right\} \\ &= \mathrm{co}\left\{(\partial f)^{-1}(x^*)\right\} + \mathrm{co}\left\{(\partial f^\infty)^{-1}(x^*)\right\}\ (f \text{ lsc})\end{aligned}$$

Going back to the epsilon-subdifferential

Similarly, we obtain

Theorem: Given a function $f : X \to \overline{\mathbb{R}}$, for every $x^* \in X^*$ we have that

$$\partial f^*(x^*) = \bigcap_{\substack{\varepsilon > 0 \\ L \in \mathcal{F}(f, x^*)}} \overline{\mathrm{co}} \left\{ (\partial_\varepsilon f)^{-1}(x^*) + (\partial (f \Box \sigma_L)^\infty)^{-1}(x^*) \right\}$$

Moreover,

- if $X^* \in \mathcal{F}(f)$, then

$$\partial f^*(x^*) = \bigcap_{\varepsilon > 0} \overline{\mathrm{co}} \left\{ (\partial_\varepsilon f)^{-1}(x^*) + (\partial f^\infty)^{-1}(x^*) \right\}$$

- if $X^* \in \mathcal{F}(f)$ and $\mathrm{int}(\mathrm{dom}\, f^*) \neq \emptyset$, then

$$\begin{aligned} \partial f^*(x^*) &= \overline{\mathrm{co}} \left\{ (\partial f^\infty)^{-1}(x^*) \right\} + \bigcap_{\varepsilon > 0} \overline{\mathrm{co}} \left\{ (\partial_\varepsilon f)^{-1}(x^*) \right\} \\ &= \mathrm{co} \left\{ (\partial f^\infty)^{-1}(x^*) \right\} + \bigcap_{\varepsilon > 0} \overline{\mathrm{co}} \left\{ (\partial_\varepsilon f)^{-1}(x^*) \right\} \ (X = \mathbb{R}^n) \end{aligned}$$

The lsc convex hull via weak subdifferentials

In view of the relationship $\operatorname{Argmin} f^{**} = \partial f^*(\theta)$, we obtain

Theorem: For every function $f : X \to \overline{\mathbb{R}}$ with proper conjugate, we have

$$\operatorname{Argmin} f^{**} = \bigcap_{L \in \mathcal{F}(f, x^*)} \overline{\operatorname{co}} \left\{ (\partial^w (f \Box \sigma_L))^{-1}(\theta) + \operatorname{Argmin}(f \Box \sigma_L)^\infty \right\}$$

Consequently,

- if $X^* \in \mathcal{F}(f)$, then

$$\operatorname{Argmin} f^{**} = \overline{\operatorname{co}} \left\{ (\partial^w f)^{-1}(\theta) + \operatorname{Argmin} f^\infty \right\}$$

- if $X^* \in \mathcal{F}(f)$ and $\operatorname{int}(\operatorname{dom} f^*) \neq \emptyset$, then

$$\begin{aligned} \operatorname{Argmin} f^{**} &= \overline{\operatorname{co}} \left\{ (\partial^w f)^{-1}(\theta) \right\} + \overline{\operatorname{co}} \left\{ \operatorname{Argmin} f^\infty \right\} \\ &= \overline{\operatorname{co}} \left\{ \operatorname{Argmin} f \right\} + \overline{\operatorname{co}} \left\{ \operatorname{Argmin} f^\infty \right\} \ \ (f \text{ wlsc}) \\ &= \operatorname{co} \left\{ \operatorname{Argmin} f \right\} + \operatorname{co} \left\{ \operatorname{Argmin} f^\infty \right\} \ \ (X = \mathbb{R}^n,\ f \text{ lsc}) \end{aligned}$$

Remark: the last formula is due to Benoist and Hiriart-Urruty

The lsc convex hull via epsilon-minima

Theorem: Given a function $f : X \to \overline{\mathbb{R}}$ having a proper conjugate, we have

$$\operatorname{Argmin} f^{**} = \bigcap_{\substack{\varepsilon>0 \\ L\in\mathcal{F}(f,x^*)}} \overline{\operatorname{co}} \{\varepsilon\text{-}\operatorname{Argmin}(f\Box\sigma_L) + \operatorname{Argmin}(f\Box\sigma_L)^{\infty}\}$$

Consequently,

- if $X^* \in \mathcal{F}(f)$, then

$$\operatorname{Argmin} f^{**} = \bigcap_{\varepsilon>0} \overline{\operatorname{co}} \{\varepsilon\text{-}\operatorname{Argmin} f + \operatorname{Argmin} f^{\infty}\}$$

- if $X^* \in \mathcal{F}(f)$ and $\operatorname{int}(\operatorname{dom} f^*) \neq \emptyset$, then

$$\begin{aligned} \operatorname{Argmin} f^{**} &= \overline{\operatorname{co}} \{\operatorname{Argmin} f^{\infty}\} + \bigcap_{\varepsilon>0} \overline{\operatorname{co}} \{\varepsilon\text{-}\operatorname{Argmin} f\} \\ &= \operatorname{co} \{\operatorname{Argmin} f^{\infty}\} + \bigcap_{\varepsilon>0} \overline{\operatorname{co}} \{\varepsilon\text{-}\operatorname{Argmin} f\} \quad (X = \mathbb{R}^n) \end{aligned}$$

Subdifferential of the lsc convex hull

For simplicity, we suppose that $X = \mathbb{R}^n$, $\mathrm{int}(\mathrm{dom}\, f^*) \neq \emptyset$, and f is lsc (this is the setting of Benoist - Hiriart-Urruty) so that

$$\partial f^*(x^*) = \mathrm{co}\left\{(\partial f)^{-1}(x^*)\right\} + \mathrm{co}\left\{(\partial f^\infty)^{-1}(x^*)\right\}$$

If $x^* \in \partial f^{**}(x)$, then $x \in \partial f^*(x^*)$ and, so, there are $\lambda_0, \cdots, \lambda_n, \mu_1, \cdots, \mu_n \geq 0$, with $\lambda_0 + \cdots + \lambda_n = 1$, $x_0, \cdots, x_n \in (\partial f)^{-1}(x^*)$, and $y_1, \cdots, y_n \in (\partial f^\infty)^{-1}(x^*)$ such that

$$x = \lambda_0 x_0 + \cdots + \lambda_n x_n + \mu_1 y_1 + \cdots + \mu_n y_n$$

Then,

$$x^* \in \bigcap_{0 \leq i \leq n} \partial f(x_i) \bigcap_{1 \leq j \leq n} \partial f^\infty(y_j)$$

Theorem (Benoist - Hiriart-Urruty): For every $x \in X$,

$$\partial f^{**}(x) = \bigcap_{0 \leq i \leq n} \partial f(x_i) \bigcap_{1 \leq j \leq n} \partial f^\infty(y_j)$$

where $x = \lambda_0 x_0 + \cdots + \lambda_n x_n + \mu_1 y_1 + \cdots + \mu_n y_n$

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Thank you very much