Cher Jean-Baptiste,

Félicitations,

hommage amical

Cher Jean-Baptiste,

Félicitations,

hommage amical

(jbhu, Ba'tiste)

Discontinuous Feedback and Nonlinear Systems

Francis Clarke

This is a mathematics talk

This is a mathematics talk

Please remain seated

Outline of the talk

1. **Introduction:**

   dynamic programming,

   nonsmooth analysis

2. **Stability and Lyapunov functions**
3. **Discontinuous feedbacks**
4. **Stabilizing feedback design**

Outline of the talk

1. **Introduction:**
   dynamic programming,
   nonsmooth analysis
2. **Stability and Lyapunov functions**
3. **Discontinuous feedbacks**
4. **Stabilizing feedback design**

Note: detailed tutorial paper

Control theory of ordinary differential equations:

We consider throughout the system, for $x \in R^n$

$$(*) \begin{cases} x'(t) = f\big(x(t), u(t)\big) \\ u(t) \in U \end{cases}$$

Control theory of ordinary differential equations:

We consider throughout the system, for $x \in R^n$

$$(*) \begin{cases} x'(t) = f\big(x(t), u(t)\big) \\ u(t) \in U \end{cases}$$

↑ control $u(\cdot)$ (open-loop)

Control theory of ordinary differential equations:

We consider throughout the system, for $x \in R^n$

$$(*) \begin{cases} x'(t) = f\big(x(t), u(t)\big) \\ u(t) \in U \end{cases}$$

state trajectory $x(\cdot)$

control $u(\cdot)$ (open-loop)

Control theory of ordinary differential equations:

We consider throughout the system, for $x \in R^n$

$$(*) \begin{cases} x'(t) = f\big(x(t), u(t)\big) \\ u(t) \in U \end{cases}$$

state trajectory $x(\cdot)$

control $u(\cdot)$ (open-loop)

Basic hypotheses:

U is compact, f is locally Lipschitz in x

Control theory of ordinary differential equations:

We consider throughout the system, for $x \in R^n$

$$(*) \begin{cases} x'(t) = f\big(x(t), u(t)\big) \\ u(t) \in U \end{cases}$$

state trajectory $x(\cdot)$

control $u(\cdot)$ (open-loop)

Basic hypotheses:

U is compact, f is locally Lipschitz in x

nonlinear

Control theory of ordinary differential equations:

We consider throughout the system, for $x \in R^n$

$$(*) \begin{cases} x'(t) = f\big(x(t), u(t)\big) \\ u(t) \in U \end{cases}$$

state trajectory $x(\cdot)$

control $u(\cdot)$ (open-loop)

Basic hypotheses:

U is compact, f is locally Lipschitz in x

$U \neq \mathbb{R}^m$

nonlinear

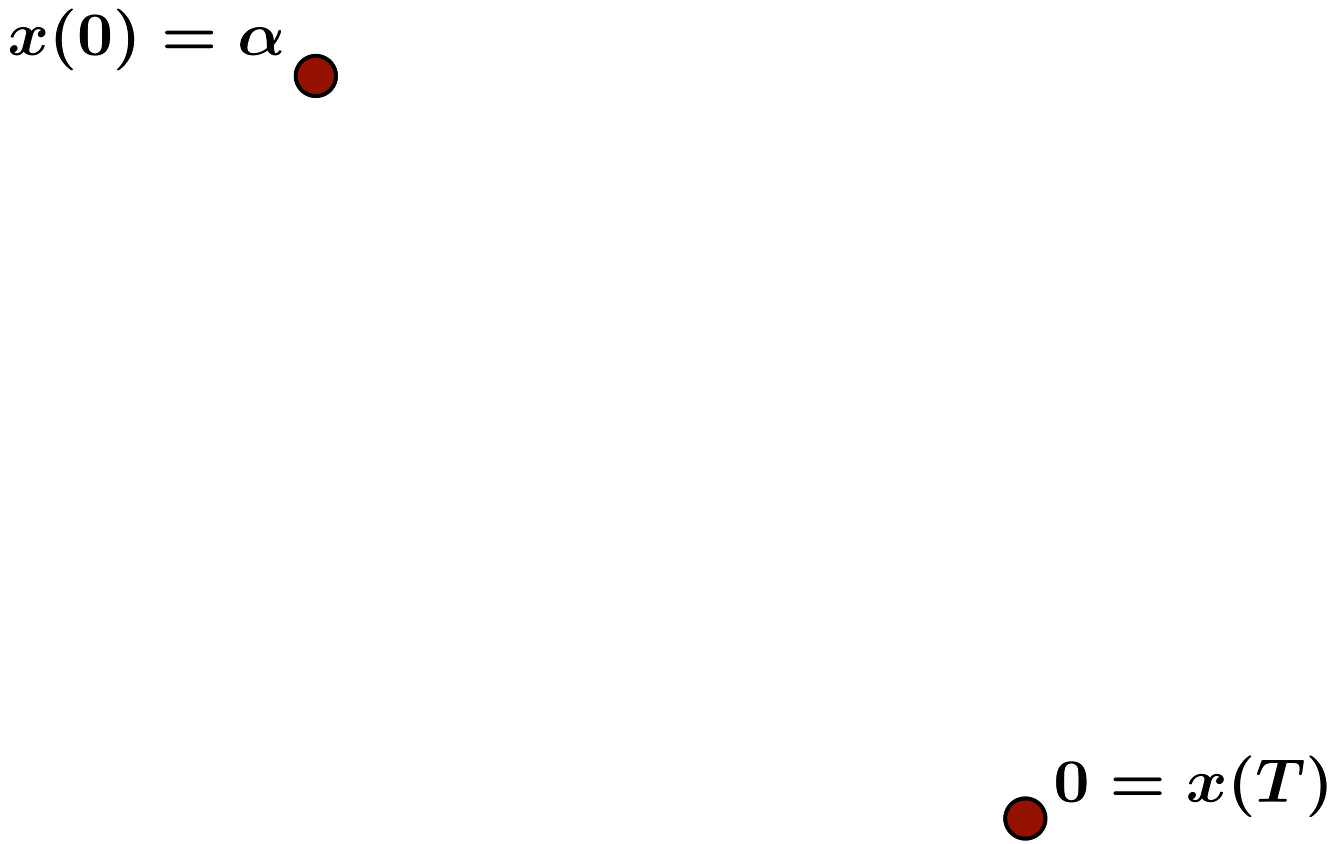
Dynamic Programming

Dynamic Programming

The minimal time function $T(\cdot)$, defined on R^n as follows: $T(\alpha)$ is the least time T such that some trajectory $x(\cdot)$ satisfies $x(0) = \alpha,\ x(T) = 0$.

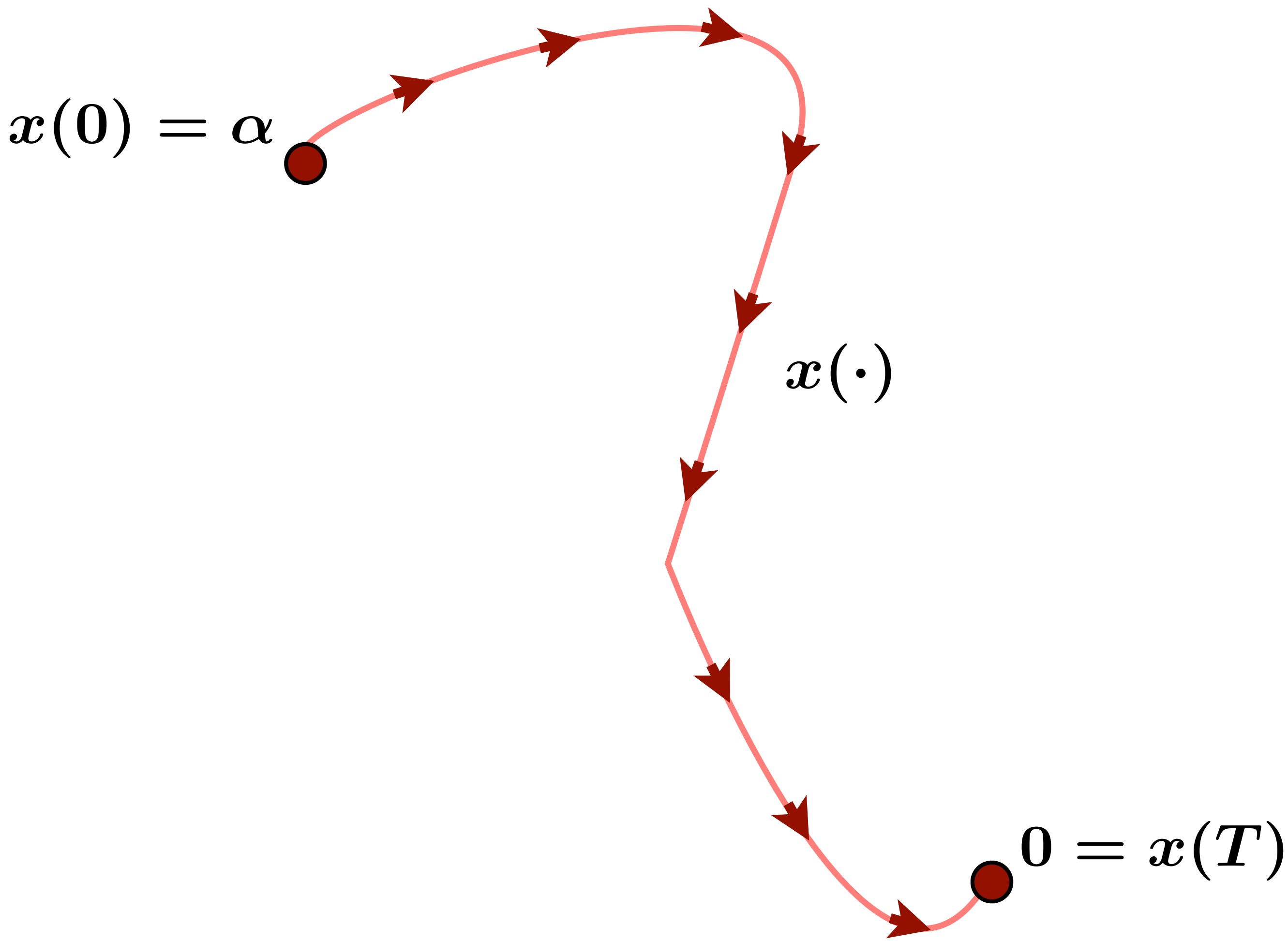
Dynamic Programming

The minimal time function $T(\cdot)$, defined on R^n as follows: $T(\alpha)$ is the least time T such that some trajectory $x(\cdot)$ satisfies $x(0) = \alpha,\ x(T) = 0$.



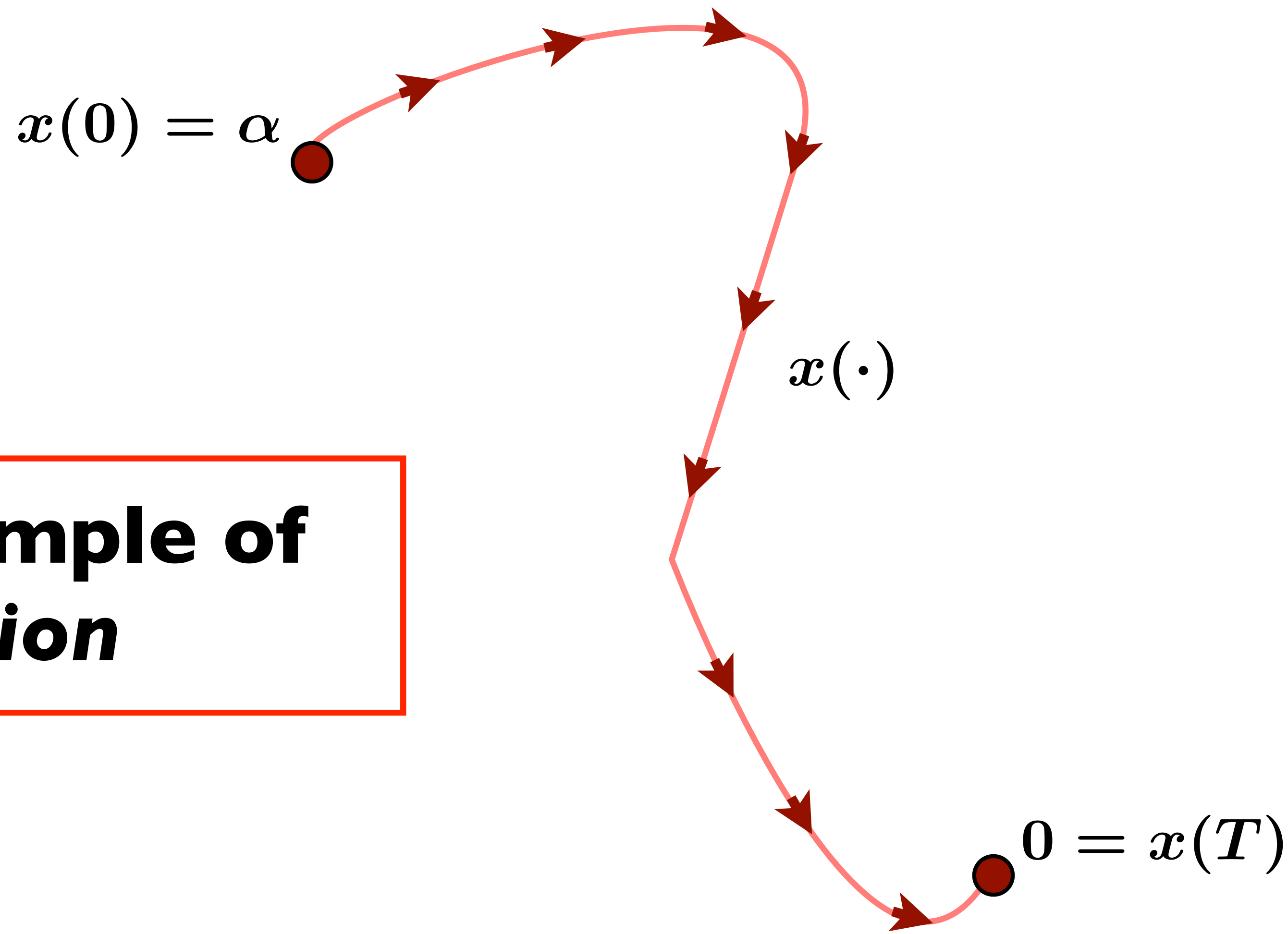
Dynamic Programming

The minimal time function $T(\cdot)$, defined on R^n as follows: $T(\alpha)$ is the least time T such that some trajectory $x(\cdot)$ satisfies $x(0) = \alpha,\ x(T) = 0$.



Dynamic Programming

The minimal time function $T(\cdot)$, defined on R^n as follows: $T(\alpha)$ is the least time T such that some trajectory $x(\cdot)$ satisfies $x(0) = \alpha,\ x(T) = 0$.



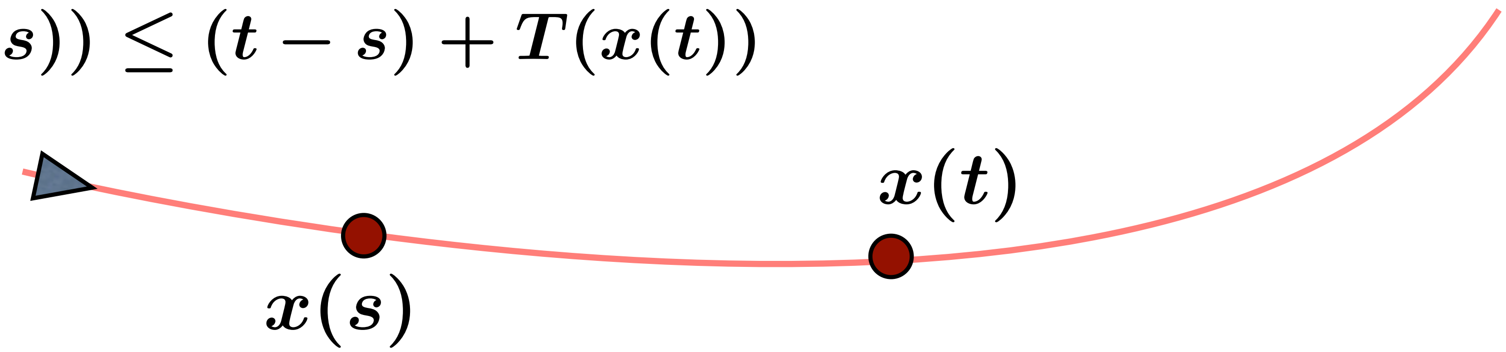
$T(\cdot)$ is an example of a *value function*

The Principle of Optimality establishes some monotonicity properties for $T(\cdot)$:

The **Principle of Optimality** establishes some monotonicity properties for $T(\cdot)$:

A. If $x(\cdot)$ is **any** trajectory, and $0 < s < t$, then

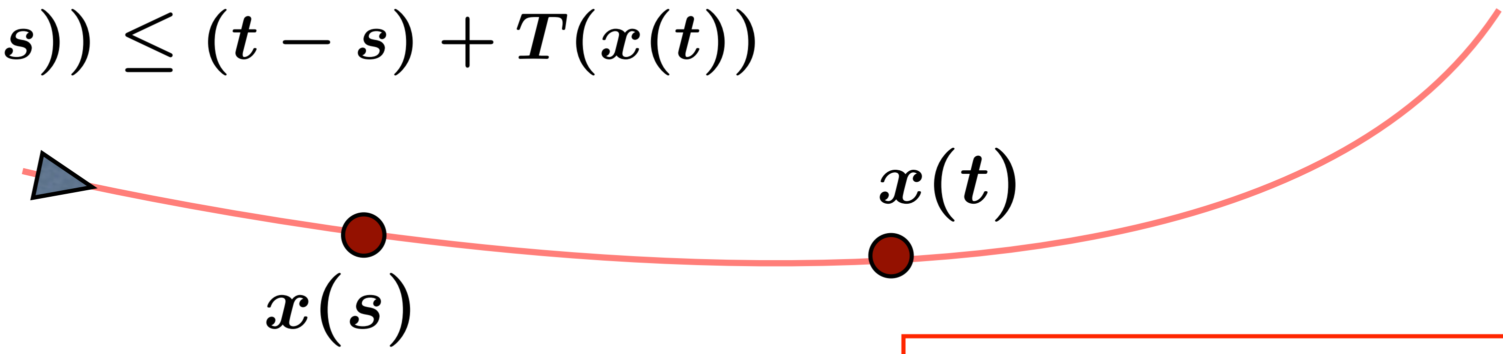
$$T(x(s)) \leq (t-s) + T(x(t))$$



The Principle of Optimality establishes some monotonicity properties for $T(\cdot)$:

A. **If $x(\cdot)$ is any trajectory, and $0 < s < t$, then**

$$T(x(s)) \leq (t - s) + T(x(t))$$

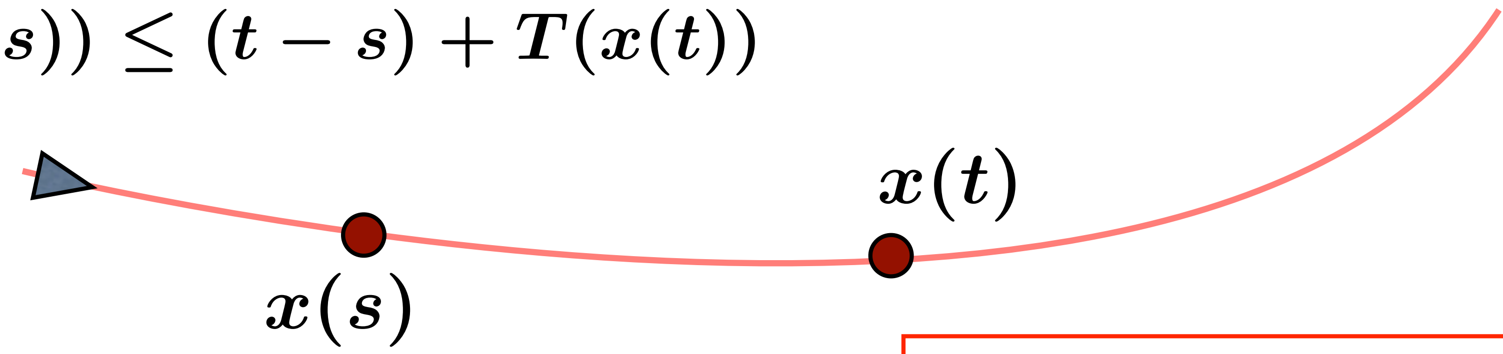


$$\Rightarrow \boxed{s + T(x(s)) \leq t + T(x(t))}$$

The Principle of Optimality establishes some monotonicity properties for $T(\cdot)$:

A. If $x(\cdot)$ is any trajectory, and $0 < s < t$, then

$$T(x(s)) \le (t - s) + T(x(t))$$



$$\Rightarrow \boxed{s + T(x(s)) \le t + T(x(t))}$$

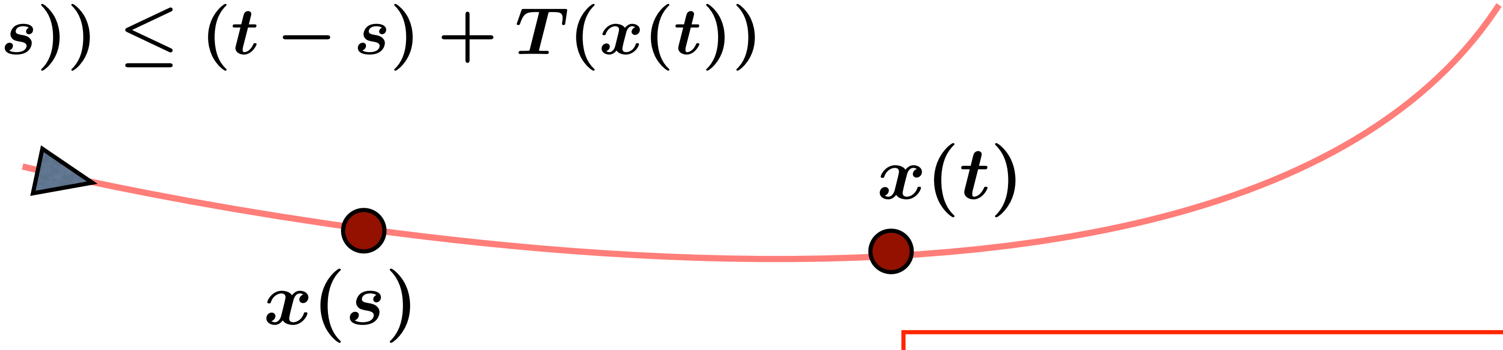
B. If $x(\cdot)$ is an optimal trajectory joining α to 0, then an optimal trajectory from the point $x(t)$ is furnished by the truncation of $x(\cdot)$ to the interval $[t, T(\alpha)]$. Hence $T(x(t)) = T(\alpha) - t$; also

$$T(x(s)) = T(\alpha) - s$$

The **Principle of Optimality** establishes some monotonicity properties for $T(\cdot)$:

A. If $x(\cdot)$ is **any** trajectory, and $0 < s < t$, then

$$T(x(s)) \leq (t - s) + T(x(t))$$



$$\Rightarrow \boxed{s + T(x(s)) \leq t + T(x(t))}$$

B. If $x(\cdot)$ is an **optimal** trajectory joining α to 0, then an optimal trajectory from the point $x(t)$ is furnished by the truncation of $x(\cdot)$ to the interval $[t, T(\alpha)]$. Hence $T(x(t)) = T(\alpha) - t$; also $T(x(s)) = T(\alpha) - s$

$$\Rightarrow \boxed{s + T(x(s)) = t + T(x(t))}$$

Conclusion: $t \mapsto t + T(x(t))$ **is increasing for all** $x(\cdot)$ **,**

constant for optimal $x(\cdot)$

Conclusion: $t \mapsto t + T(x(t))$ **is increasing for all** $x(\cdot)$ **,**

constant for optimal $x(\cdot)$

$$\Rightarrow 1 + \langle \nabla T(x), \underbrace{f(x,u)}_{x'} \rangle \geq 0 \; \forall \, u \in U$$ **(= for** u **optimal)**

Conclusion: $t \mapsto t + T(x(t))$ **is increasing for all** $x(\cdot)$ **,**

constant for optimal $x(\cdot)$

$$\Rightarrow 1 + \langle \nabla T(x), \underbrace{f(x,u)}_{x'} \rangle \geq 0 \; \forall \, u \in U \quad \text{(= for } u \text{ optimal)}$$

$$\Rightarrow \boxed{1 + h(x, \nabla T(x)) = 0} \longleftarrow \textit{Hamilton-Jacobi-Bellman equation}$$

where *h* is the *lower Hamiltonian*

$$h(x,p) := \min_{u \in U} \langle p, f(x,u) \rangle$$

Conclusion: $t \mapsto t + T(x(t))$ **is increasing for all** $x(\cdot)$ **,**

constant for optimal $x(\cdot)$

$$\Rightarrow 1 + \langle \nabla T(x), \underbrace{f(x,u)}_{x'} \rangle \geq 0 \ \forall\, u \in U$$ **(= for** u **optimal)**

$$\Rightarrow \boxed{1 + h(x, \nabla T(x)) = 0}$$ $\longleftarrow$ ***Hamilton-Jacobi-Bellman equation***

where ***h*** **is the** ***lower Hamiltonian***

$$h(x,p) := \min_{u \in U} \langle p, f(x,u) \rangle$$

Suppose we solve the H-J-B equation (with $T(0) = 0$ **).**
How does knowing $T(\cdot)$ **help?**

For each x, let $k(x)$ be a point in U such that

$$1 + \langle \nabla T(x), f(x, k(x)) \rangle = 0$$

(a steepest descent feedback). Then, for any α, the solution to

$$x'(t) = f(x(t), k(x(t)), \;\; x(0) = \alpha$$

is optimal.

For each x, let $k(x)$ be a point in U such that

$$1 + \langle \nabla T(x), f(x, k(x)) \rangle = 0$$

(a steepest descent feedback). Then, for any α, the solution to

$$x'(t) = f(x(t), k(x(t))), \;\; x(0) = \alpha$$

is optimal.

Here's why:

For each x, let $k(x)$ be a point in U such that

$$1 + \langle \nabla T(x), f(x, k(x)) \rangle = 0$$

(a steepest descent feedback). Then, for any α, the solution to

$$x'(t) = f(x(t), k(x(t)), \; x(0) = \alpha$$

is optimal.

Here's why:

$$\frac{d}{dt} T(x(t)) = \langle \nabla T(x(t)), x'(t) \rangle$$

$$= \langle \nabla T(x(t)), f(x(t), k(x(t))) \rangle = -1$$

For each x, let $k(x)$ be a point in U such that

$$1 + \langle \nabla T(x), f(x, k(x)) \rangle = 0$$

(a steepest descent feedback). Then, for any α, the solution to

$$x'(t) = f(x(t), k(x(t)), \;\; x(0) = \alpha$$

is optimal.

Here's why:

$$\frac{d}{dt} T(x(t)) = \langle \nabla T(x(t)), x'(t) \rangle$$

$$= \langle \nabla T(x(t)), f(x(t), k(x(t))) \rangle = -1$$

$$\Rightarrow \; T(x(t)) - T(\alpha) = -t$$

For each x, let $k(x)$ be a point in U such that

$$1 + \langle \nabla T(x), f(x, k(x)) \rangle = 0$$

(a steepest descent feedback). Then, for any α, the solution to

$$x'(t) = f(x(t), k(x(t)), \; x(0) = \alpha$$

is optimal.

Here's why:

$$\frac{d}{dt} T(x(t)) = \langle \nabla T(x(t)), x'(t) \rangle$$

$$= \langle \nabla T(x(t)), f(x(t), k(x(t))) \rangle = -1$$

$\Rightarrow T(x(t)) - T(\alpha) = -t$ **Put** $t = T(\alpha)$

For each x, let $k(x)$ be a point in U such that

$$1 + \langle \nabla T(x), f(x, k(x)) \rangle = 0$$

(a steepest descent feedback). Then, for any α, the solution to

$$x'(t) = f(x(t), k(x(t)), \ x(0) = \alpha$$

is optimal.

Here's why:

$$\frac{d}{dt} T(x(t)) = \langle \nabla T(x(t)), x'(t) \rangle$$

$$= \langle \nabla T(x(t)), f(x(t), k(x(t))) \rangle = -1$$

$$\Rightarrow T(x(t)) - T(\alpha) = -t$$ **Put** $t = T(\alpha)$

$$\Rightarrow T(x(T(\alpha))) = 0$$

For each x, let $k(x)$ be a point in U such that

$$1 + \langle \nabla T(x), f(x, k(x)) \rangle = 0$$

(a steepest descent feedback). Then, for any α, the solution to

$$x'(t) = f(x(t), k(x(t)), \; x(0) = \alpha$$

is optimal.

Here's why:

$$\frac{d}{dt} T(x(t)) = \langle \nabla T(x(t)), x'(t) \rangle$$

$$= \langle \nabla T(x(t)), f(x(t), k(x(t))) \rangle = -1$$

$\Rightarrow \; T(x(t)) - T(\alpha) = -t$ **Put** $t = T(\alpha)$

$\Rightarrow \; T(x(T(\alpha))) = 0$

$\Rightarrow \; x(T(\alpha)) = 0$

For each x, let $k(x)$ be a point in U such that

$$1 + \langle \nabla T(x), f(x, k(x)) \rangle = 0$$

(a steepest descent feedback). Then, for any α, the solution to

$$x'(t) = f(x(t), k(x(t)), \; x(0) = \alpha$$

is optimal.

Here's why:

$$\frac{d}{dt} T(x(t)) = \langle \nabla T(x(t)), x'(t) \rangle$$

$$= \langle \nabla T(x(t)), f(x(t), k(x(t))) \rangle = -1$$

$\Rightarrow \; T(x(t)) - T(\alpha) = -t$ **Put** $t = T(\alpha)$

$\Rightarrow \; T(x(T(\alpha))) = 0$

$\Rightarrow \; x(T(\alpha)) = 0 \Rightarrow x(\cdot)$ **optimal**

For each x, let $k(x)$ be a point in U such that

$$1 + \langle \nabla T(x), f(x, k(x)) \rangle = 0$$

(a steepest descent feedback). Then, for any α, the solution to

$$x'(t) = f(x(t), k(x(t)), \; x(0) = \alpha$$

is optimal.

Here's why:

$$\frac{d}{dt} T(x(t)) = \langle \nabla T(x(t)), x'(t) \rangle$$

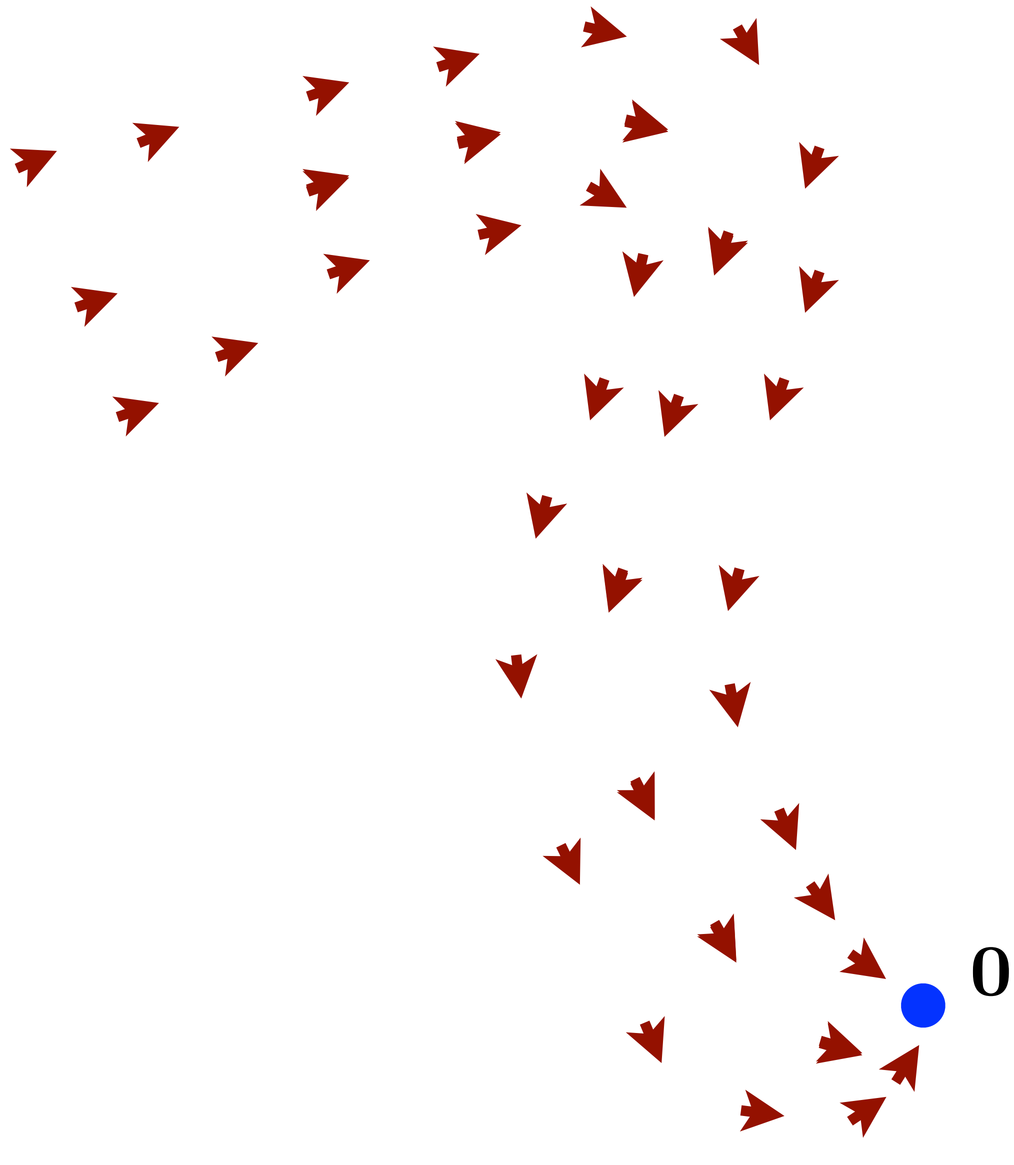
$$= \langle \nabla T(x(t)), f(x(t), k(x(t))) \rangle = -1$$

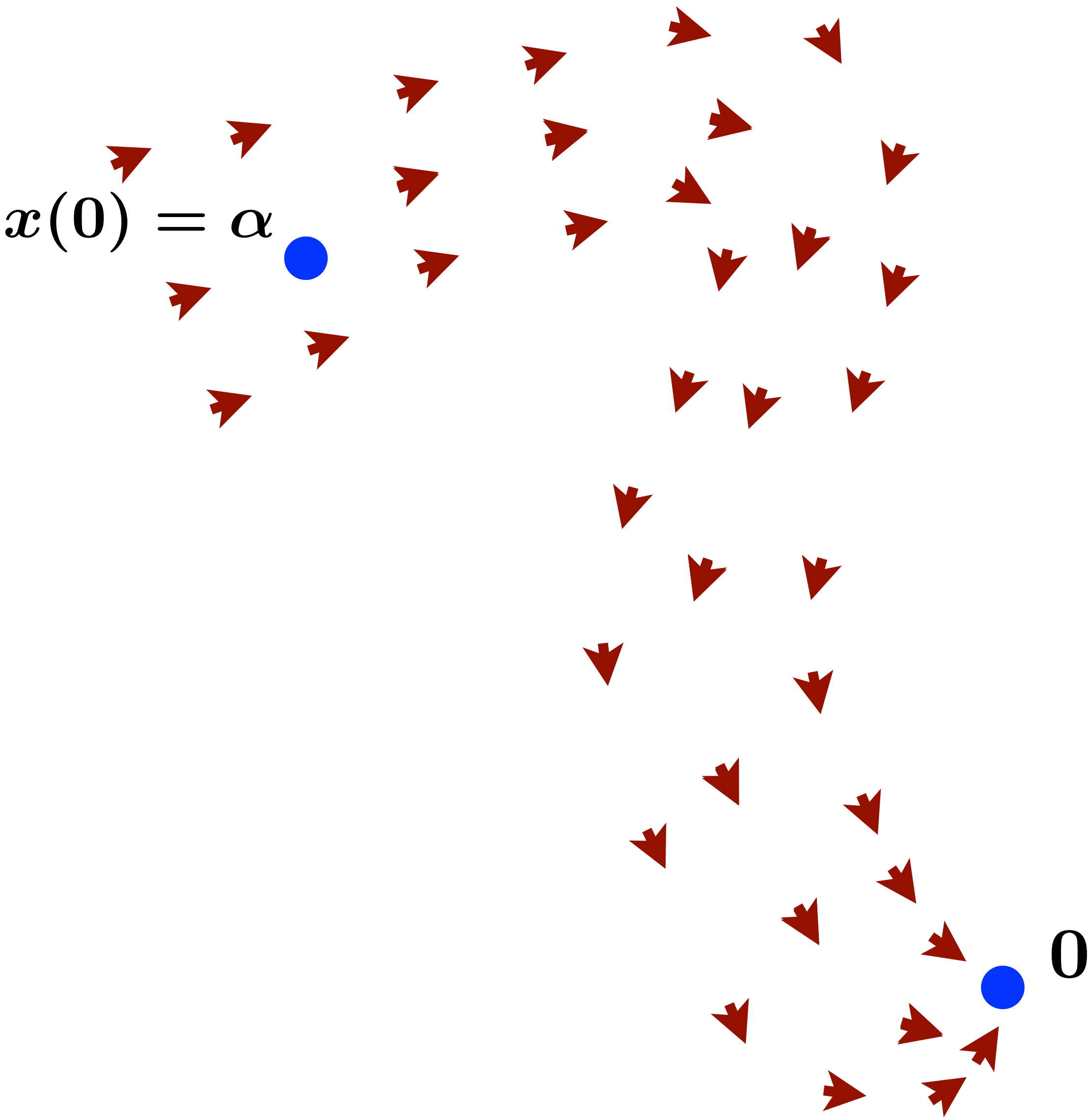
$\Longrightarrow \; T(x(t)) - T(\alpha) = -t$ **Put** $t = T(\alpha)$

$\Longrightarrow \; T(x(T(\alpha))) = 0$

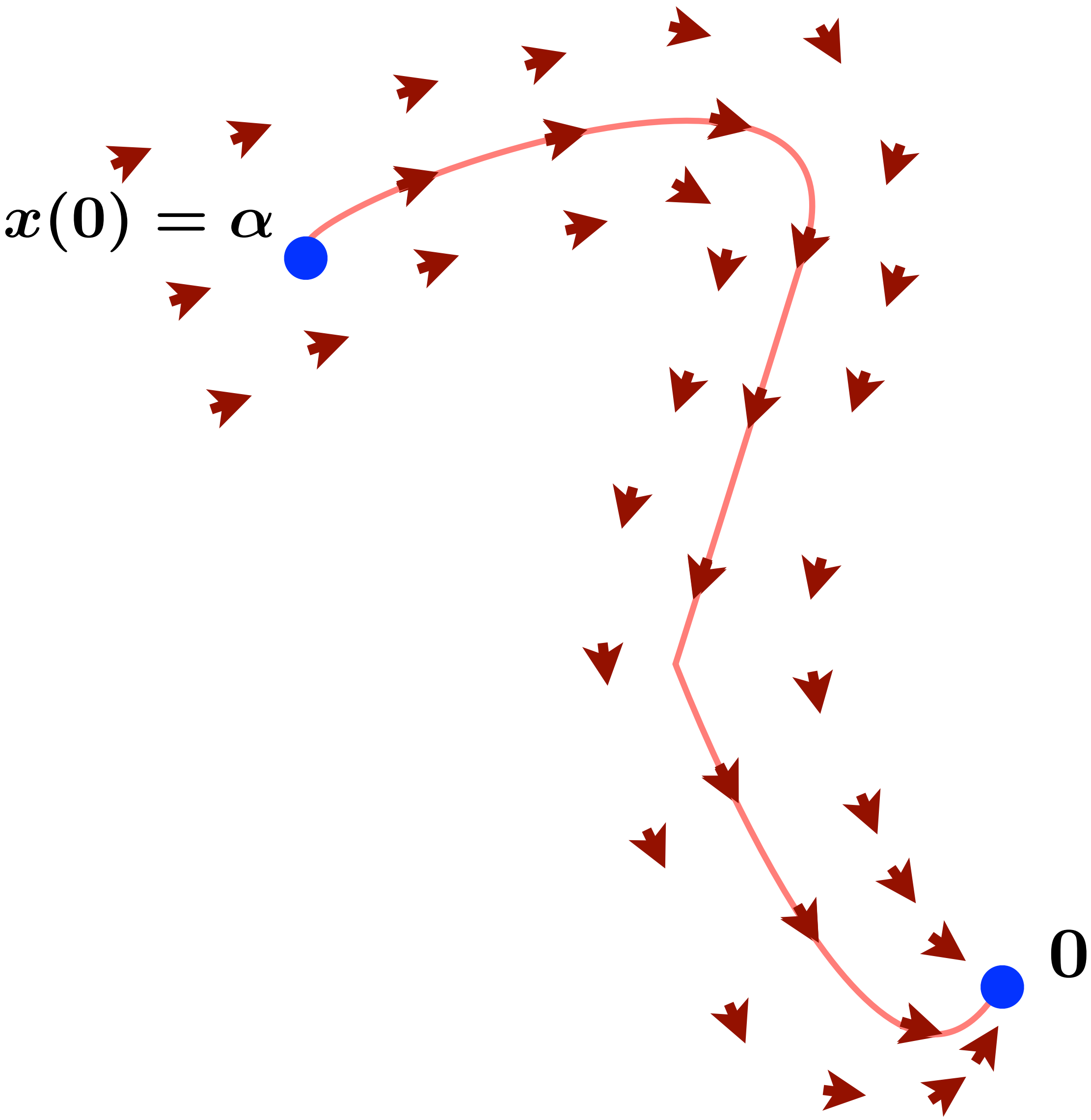
$\Longrightarrow \; x(T(\alpha)) = 0 \Longrightarrow x(\cdot)$ **optimal**

We obtain an *optimal feedback synthesis*


0



$x(0) = \alpha$
$\mathbf{0}$



$x(0) = \alpha$
0

The murder of a beautiful theory by a gang of brutal facts

The murder of a beautiful theory by a gang of brutal facts

Serious difficulties in the dynamic programming approach:

The murder of a beautiful theory by a gang of brutal facts

Serious difficulties in the dynamic programming approach:

- $T(\cdot)$ is **nondifferentiable**;

replace ∇T in monotonicity ?

The murder of a beautiful theory by a gang of brutal facts

Serious difficulties in the dynamic programming approach:

- $T(\cdot)$ **is nondifferentiable; replace ∇T in monotonicity ?**
- **Need generalized solutions of H-J-B equation...**

The murder of a beautiful theory by a gang of brutal facts

Serious difficulties in the dynamic programming approach:

- $T(\cdot)$ **is nondifferentiable; replace** ∇T **in monotonicity ?**

- **Need generalized solutions of H-J-B equation...**

- **Even if** $T(\cdot)$ **is smooth, there is no continuous** $k(x)$ **in general: what do we mean by a solution of** $x' = f\big(x, k(x)\big)$ **?**

Resolving these issues has been an ongoing focus of activity for many years; suitable answers now exist, involving:

Resolving these issues has been an ongoing focus of activity for many years; suitable answers now exist, involving:

- ***Nonsmooth analysis***

Resolving these issues has been an ongoing focus of activity for many years; suitable answers now exist, involving:

- ***Nonsmooth analysis***
- ***Generalized solutions of pde's***

Resolving these issues has been an ongoing focus of activity for many years; suitable answers now exist, involving:

- ***Nonsmooth analysis***
- ***Generalized solutions of pde's***
- ***Sample-and-hold analysis of discontinuous feedbacks***

Resolving these issues has been an ongoing focus of activity for many years; suitable answers now exist, involving:

- ***Nonsmooth analysis***
- ***Generalized solutions of pde's***
- ***Sample-and-hold analysis of discontinuous feedbacks***

Next: a quick look at the first two of these

Generalized gradients and proximal normals: a glimpse (Clarke 1972)

Generalized gradients and proximal normals: a glimpse (Clarke 1972)

Let f be Lipschitz (locally): $|f(y)-f(x)| \le K|y-x|$

Define

Generalized gradients and proximal normals: a glimpse (Clarke 1972)

Let f be Lipschitz (locally): |f(y)-f(x)| ≤ K|y-x|

Define

$$f^{\circ}(x;v) = \limsup_{t \downarrow 0, y \to x} \frac{f(y+tv) - f(y)}{t}$$

Generalized gradients and proximal normals: a glimpse (Clarke 1972)

Let f be Lipschitz (locally): |f(y)-f(x)| ≤ K|y-x|

Define

$$f^\circ(x;v) = \limsup_{t\downarrow 0, y\to x} \frac{f(y+tv)-f(y)}{t}$$

$$\partial_C f(x) = \left\{\zeta \in X^* : f^\circ(x;v) \geq \langle \zeta, v\rangle \; \forall v\right\}$$

Generalized gradients and proximal normals: a glimpse (Clarke 1972)

Let f be Lipschitz (locally): |f(y)-f(x)| ≤ K|y-x|

Define

$$f^\circ(x;v) = \limsup_{t\downarrow 0, y\to x} \frac{f(y+tv)-f(y)}{t}$$

generalized gradient

$$\partial_C f(x) = \left\{ \zeta \in X^* : f^\circ(x;v) \geq \langle \zeta, v \rangle \, \forall v \right\}$$

Generalized gradients and proximal normals: a glimpse (Clarke 1972)

Let f be Lipschitz (locally): |f(y)-f(x)| ≤ K|y-x|

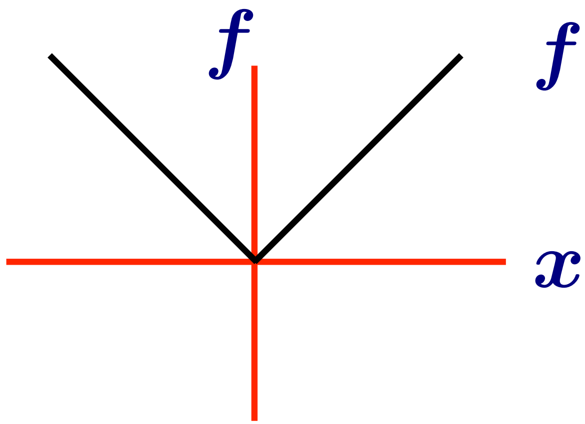
Define

$$f^\circ(x;v) = \limsup_{t\downarrow 0, y\to x} \frac{f(y+tv)-f(y)}{t}$$

generalized gradient

$$\partial_C f(x) = \left\{\zeta \in X^* : f^\circ(x;v) \geq \langle \zeta, v\rangle \ \forall v\right\}$$

Example



$f(x) = |x|$

Generalized gradients and proximal normals: a glimpse (Clarke 1972)

Let f be Lipschitz (locally): |f(y)–f(x)| ≤ K|y–x|

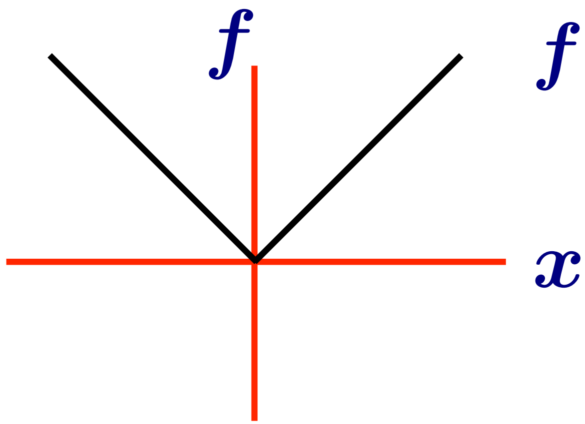
Define

$$f^\circ(x;v) = \limsup_{t\downarrow 0,\, y\to x} \frac{f(y+tv)-f(y)}{t}$$

generalized gradient

$$\partial_C f(x) = \left\{\zeta \in X^* : f^\circ(x;v) \geq \langle \zeta, v\rangle \;\forall v\right\}$$

Example



$f(x) = |x|$

then

$$\partial_C f(0) = [-1, 1]$$

Calculus of generalized gradients

Calculus of generalized gradients

- $\partial_C f(x)$ is compact convex nonempty
-
-
-
-
-
-

Calculus of generalized gradients

- $\partial_C f(x)$ is compact convex nonempty
- $\partial_C f(x) \;=\; \mathbf{co}\,\{\,\lim\limits_{x_i \to x} \nabla f(x_i), x_i \notin \Omega\,\}$
-
-
-
-
-

Calculus of generalized gradients

- $\partial_C f(x)$ is compact convex nonempty
- $\partial_C f(x) = \mathbf{co}\{ \lim\limits_{x_i \to x} \nabla f(x_i), x_i \notin \Omega \}$
- $\partial_C(-f)(x) = -\partial_C f(x)$
-
-
-
-

Calculus of generalized gradients

- $\partial_C f(x)$ is compact convex nonempty
- $\partial_C f(x) = \mathbf{co}\{ \lim_{x_i \to x} \nabla f(x_i), x_i \notin \Omega \}$
- $\partial_C(-f)(x) = -\partial_C f(x)$
- $\partial_C(f+g)(x) \subset \partial_C f(x) + \partial_C g(x)$
-
-
-

Calculus of generalized gradients

- $\partial_C f(x)$ is compact convex nonempty
- $\partial_C f(x) \; = \; \mathbf{co}\{ \lim_{x_i \to x} \nabla f(x_i), x_i \notin \Omega \}$
- $\partial_C \big(-f\big)(x) \; = \; -\partial_C f(x)$
- $\partial_C \big(f + g\big)(x) \; \subset \; \partial_C f(x) + \partial_C g(x)$
- $\partial_C \max_{1 \le i \le n} f_i(x) \; \subset \ldots$
-
-

Calculus of generalized gradients

- $\partial_C f(x)$ is compact convex nonempty

- $\partial_C f(x) = \mathbf{co}\{\lim_{x_i \to x} \nabla f(x_i), x_i \notin \Omega\}$

- $\partial_C(-f)(x) = -\partial_C f(x)$

- $\partial_C(f+g)(x) \subset \partial_C f(x) + \partial_C g(x)$

- $\partial_C \max_{1 \leq i \leq n} f_i(x) \subset \ldots$

- Mean value theorem, inverse functions...

-

Calculus of generalized gradients

- $\partial_C f(x)$ is compact convex nonempty

- $\partial_C f(x) = \mathbf{co}\{ \lim_{x_i \to x} \nabla f(x_i), x_i \notin \Omega \}$

- $\partial_C(-f)(x) = -\partial_C f(x)$

- $\partial_C(f+g)(x) \subset \partial_C f(x) + \partial_C g(x)$

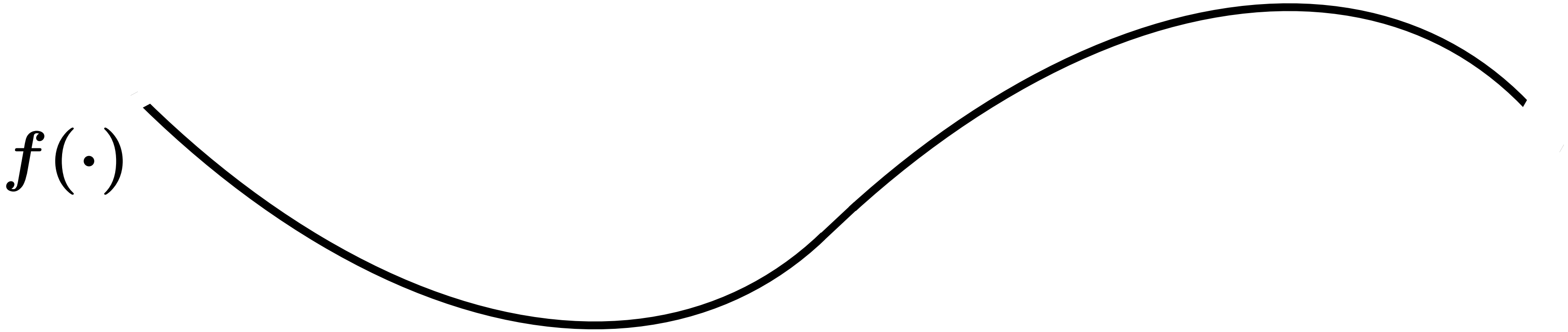
- $\partial_C \max_{1 \leq i \leq n} f_i(x) \subset \ldots$

- Mean value theorem, inverse functions...

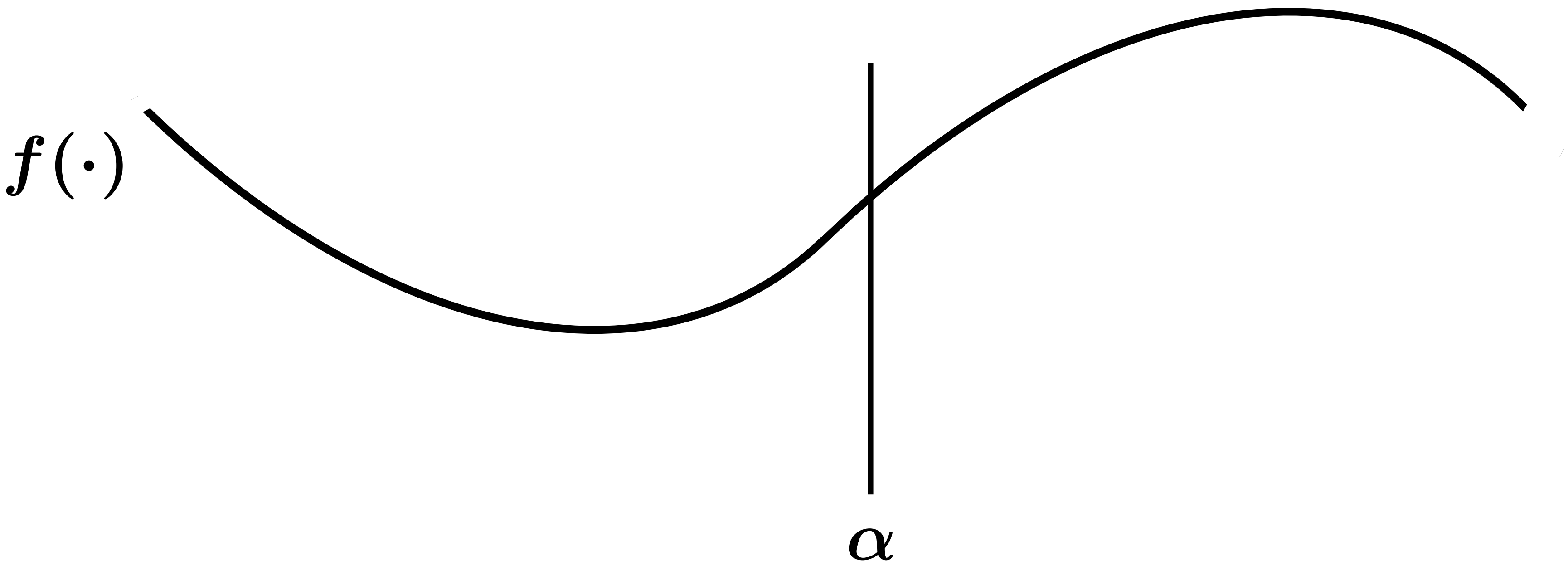
- Tangent vectors and normals to closed sets

The proximal approach

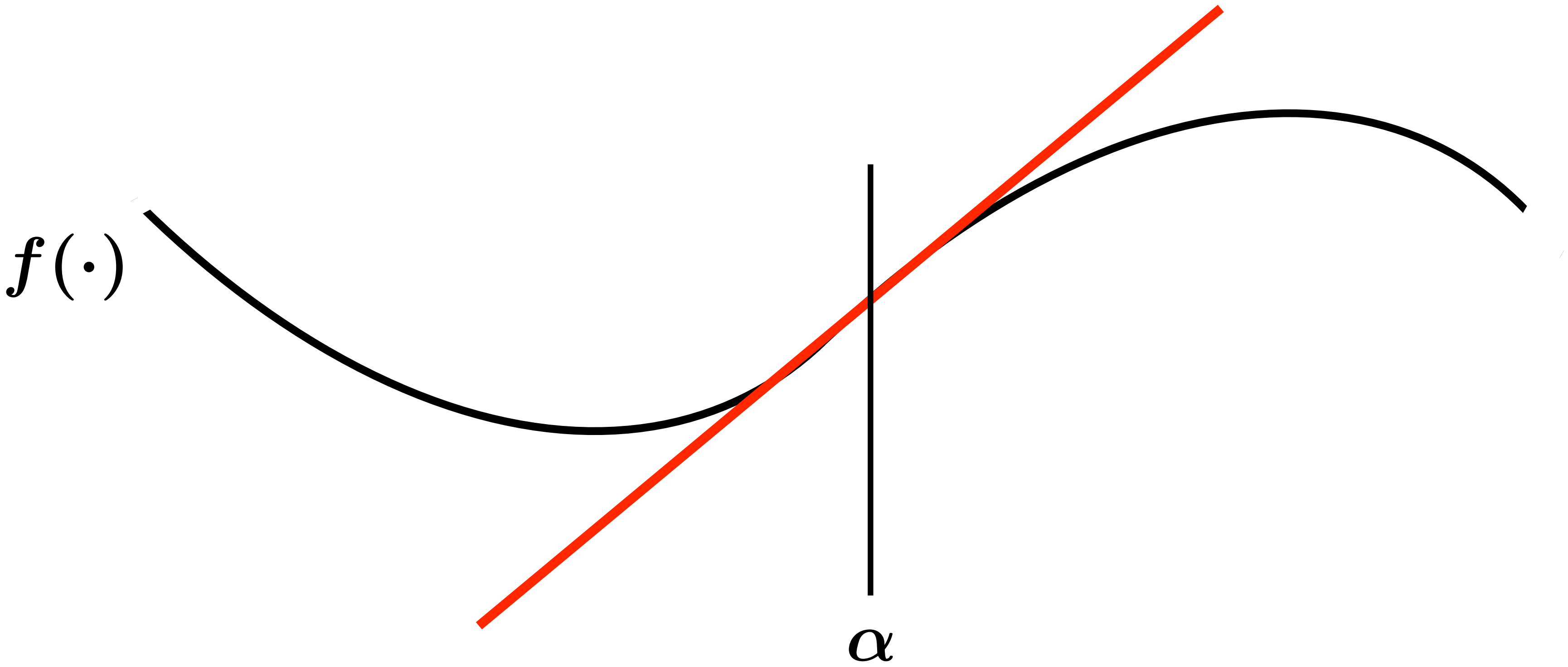
The proximal approach



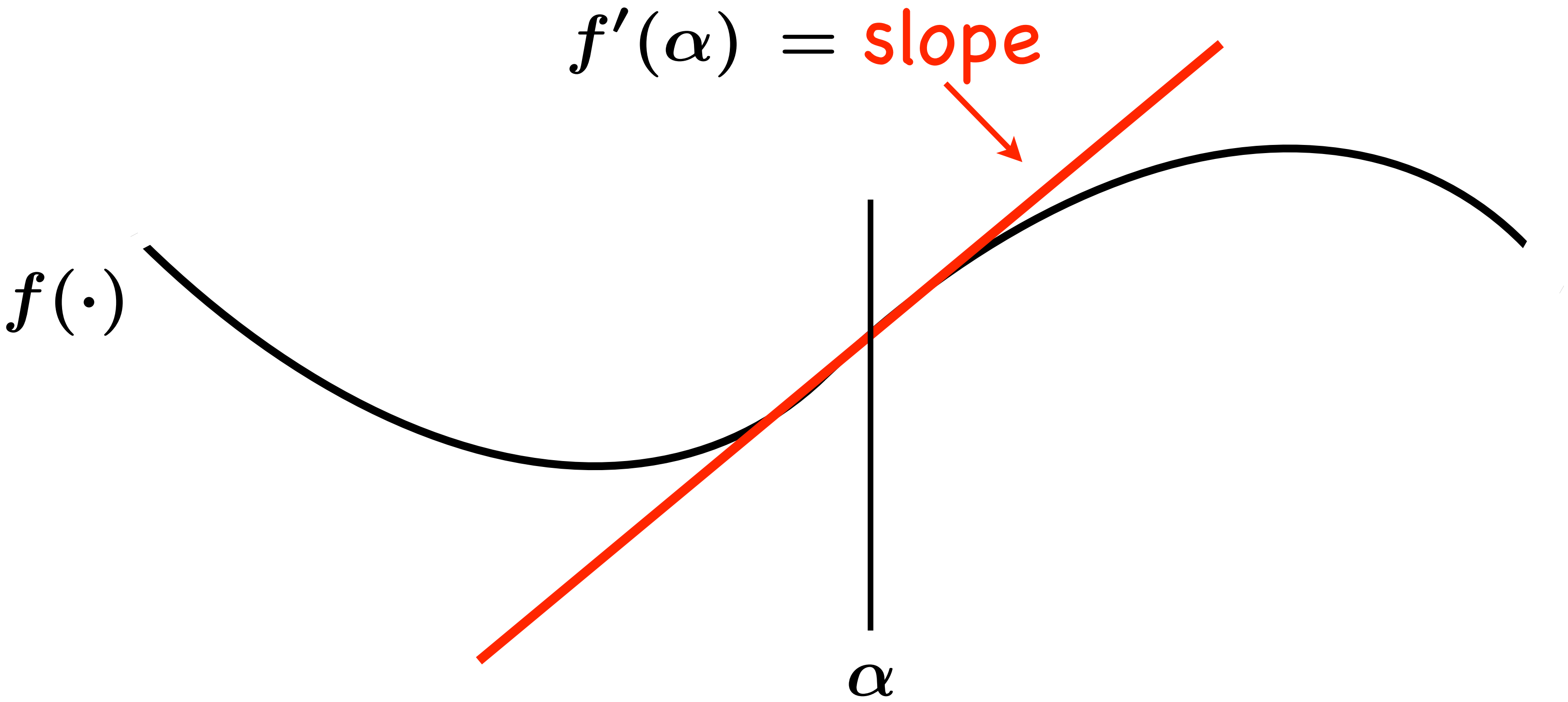
The proximal approach

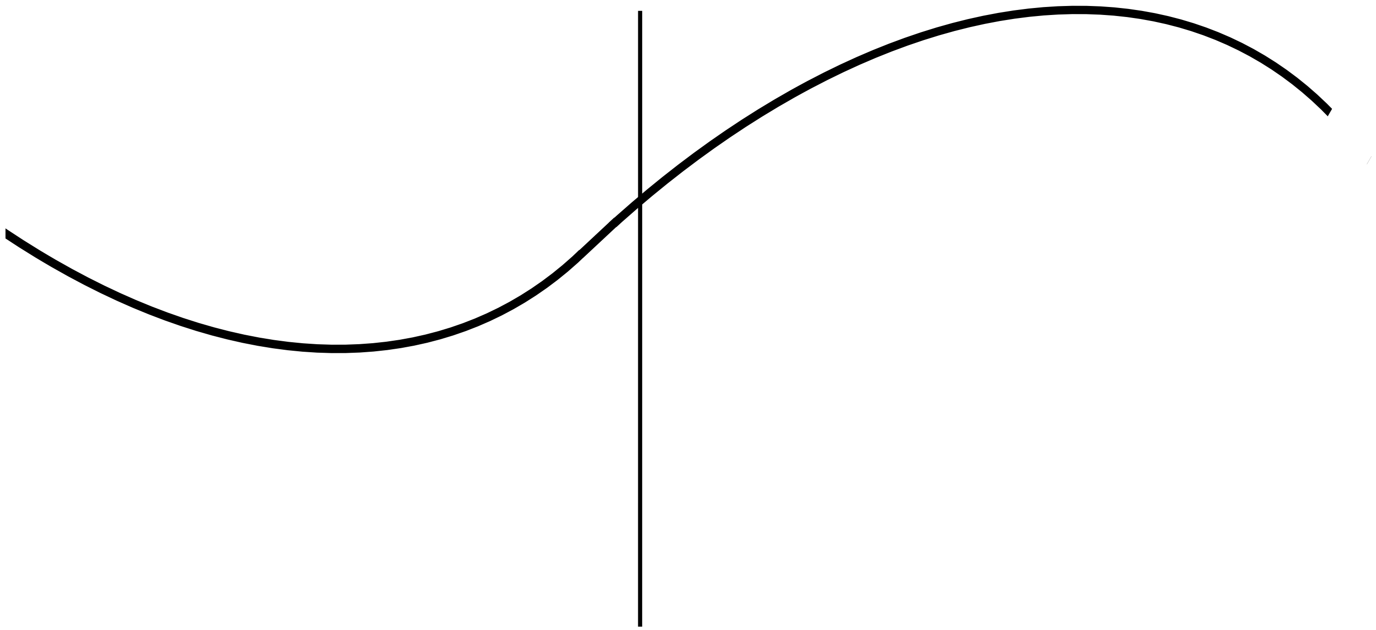


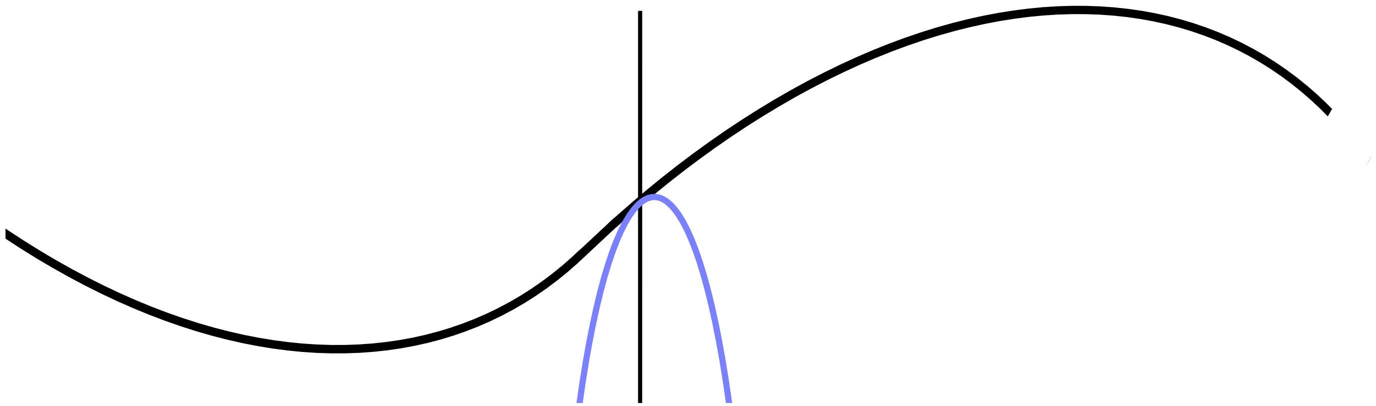
The proximal approach

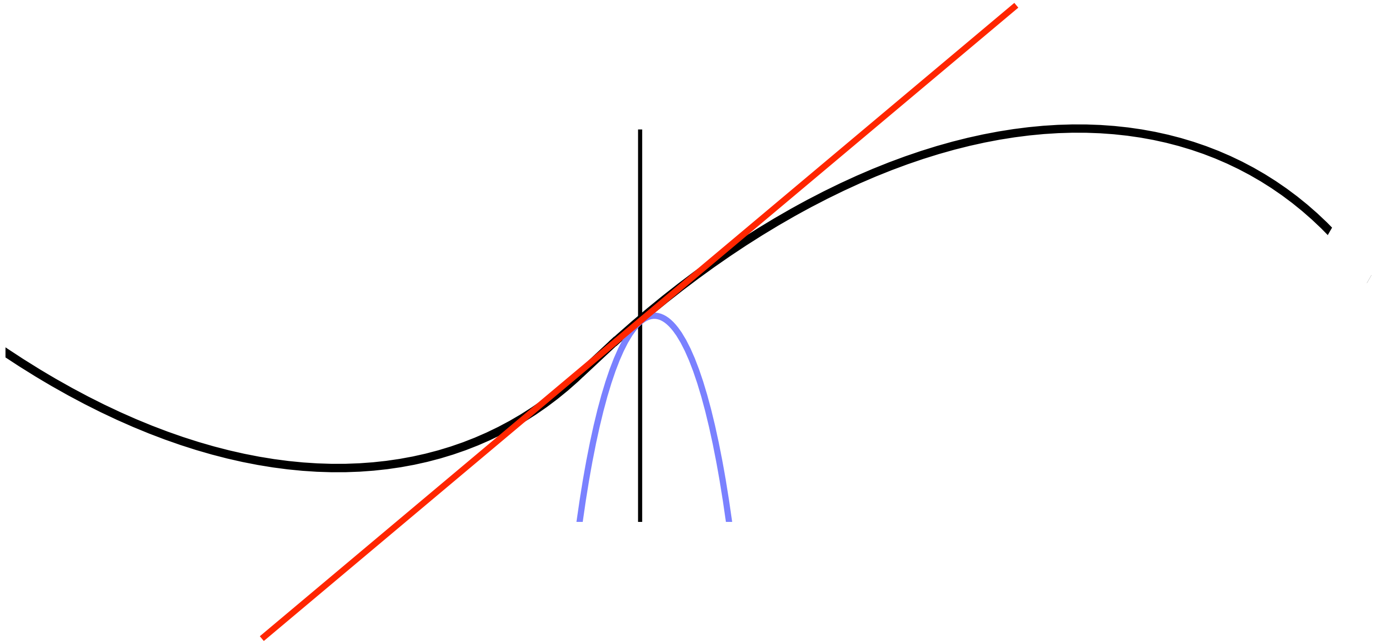


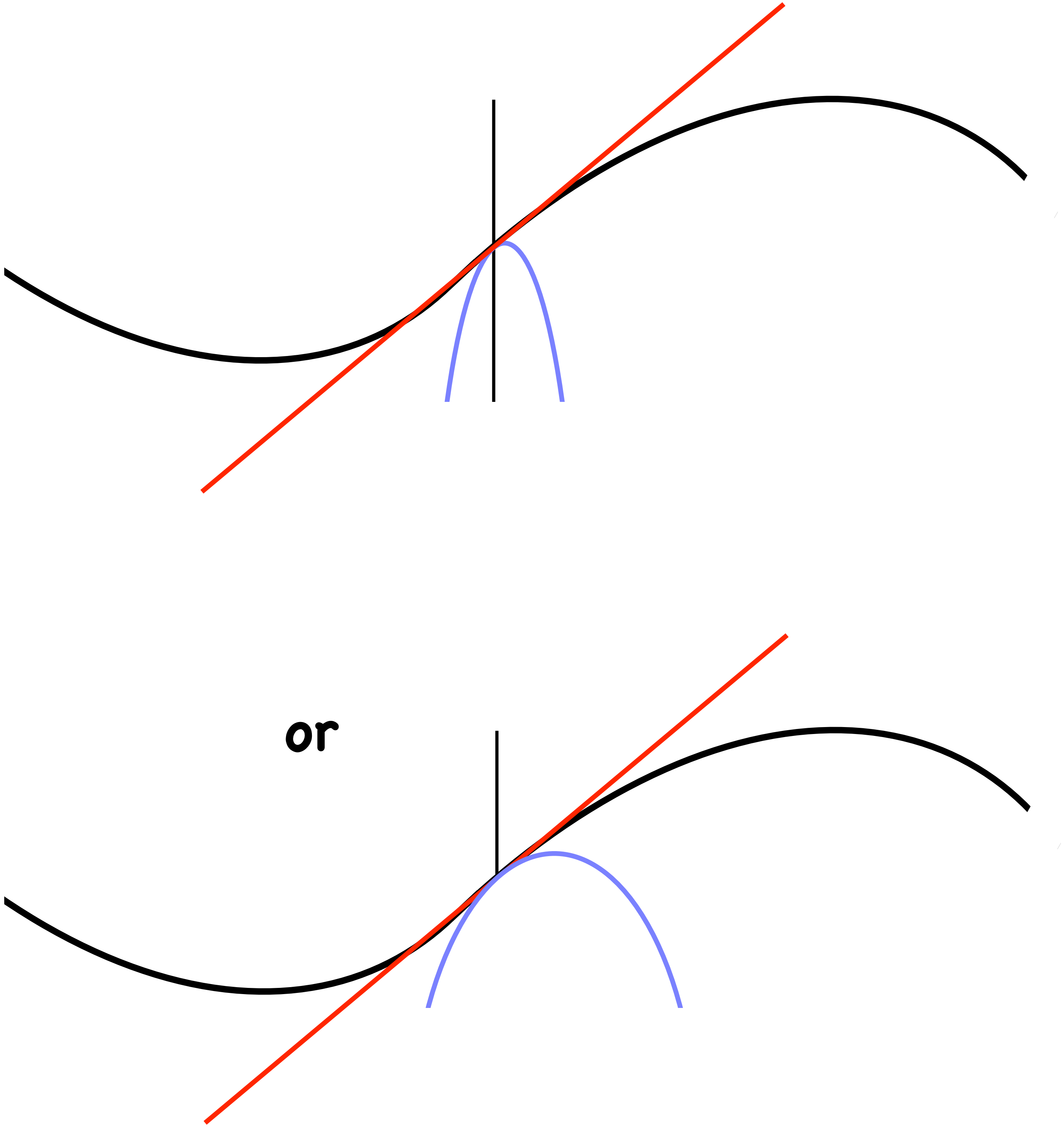
The proximal approach

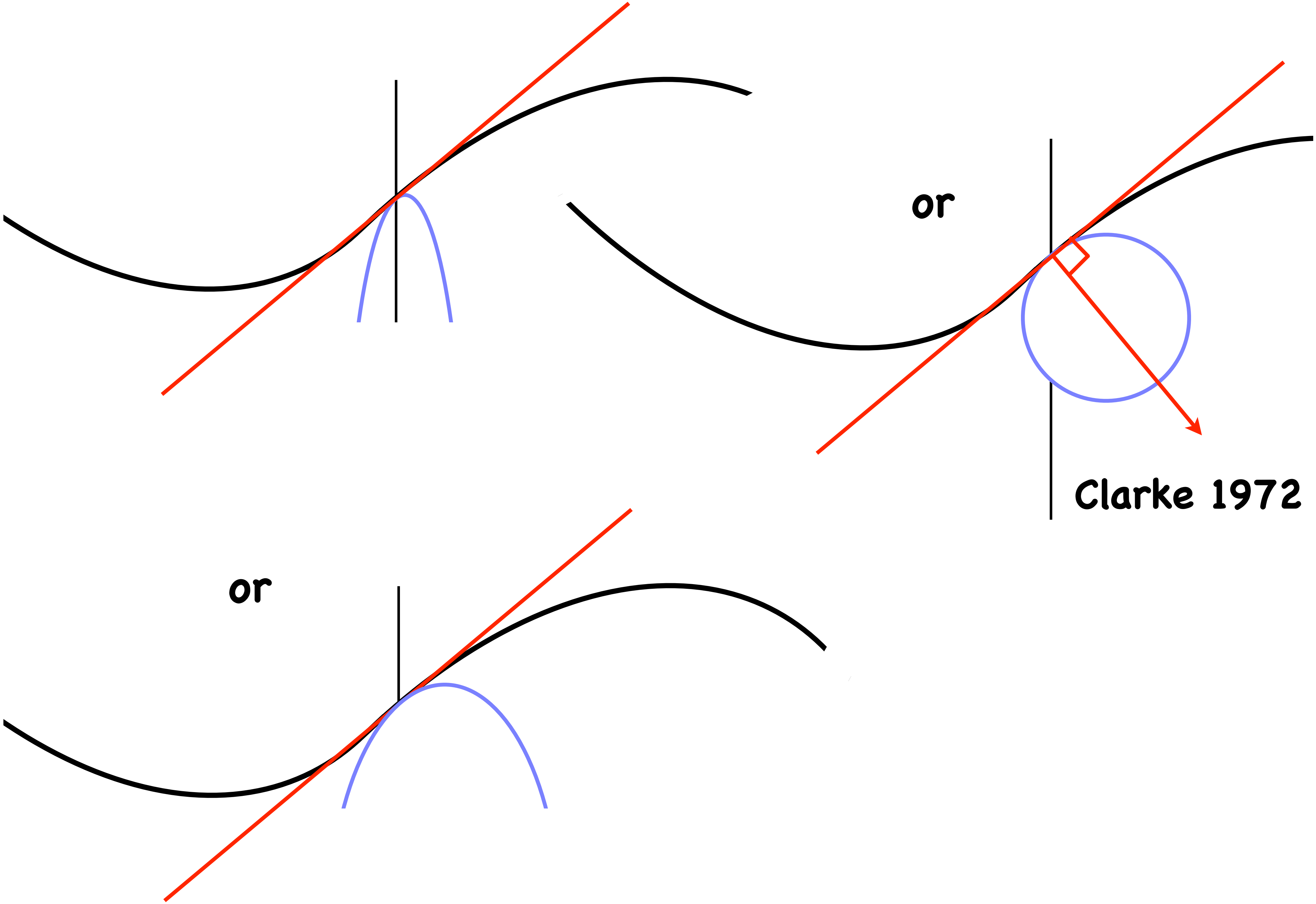


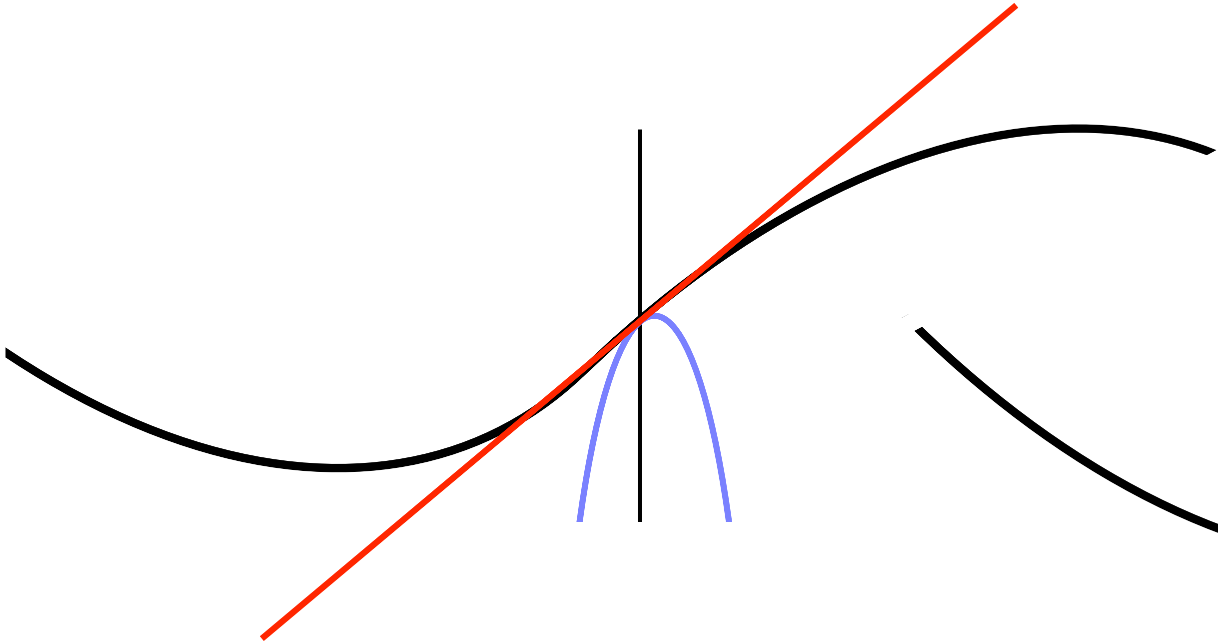




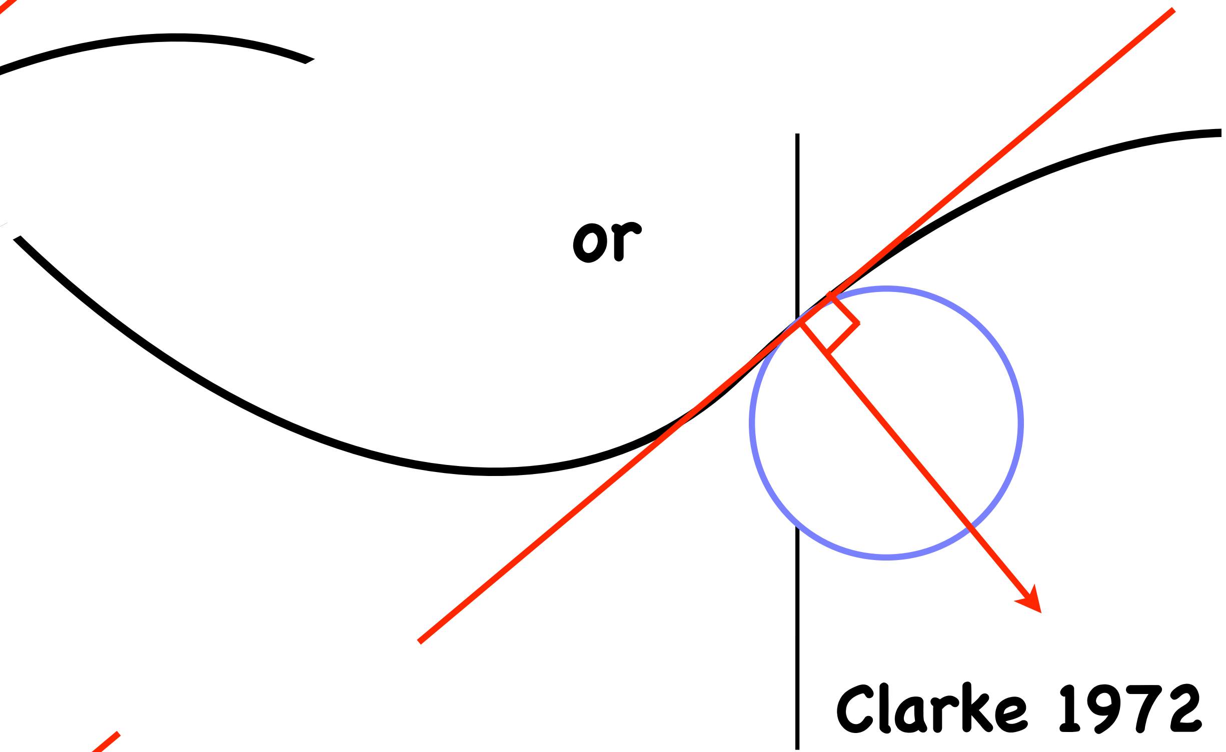



or

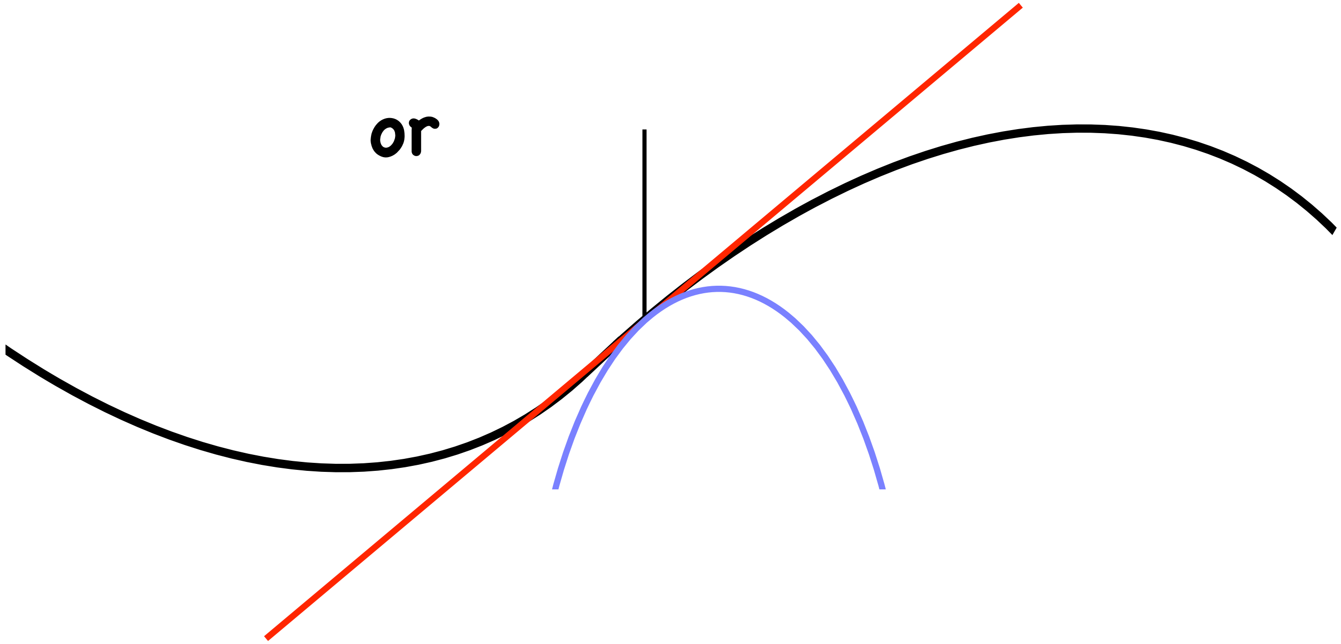

or
Clarke 1972
or



or



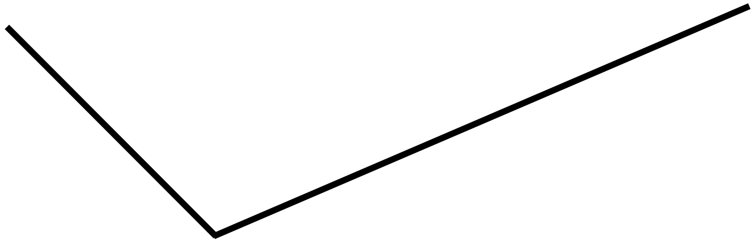
or



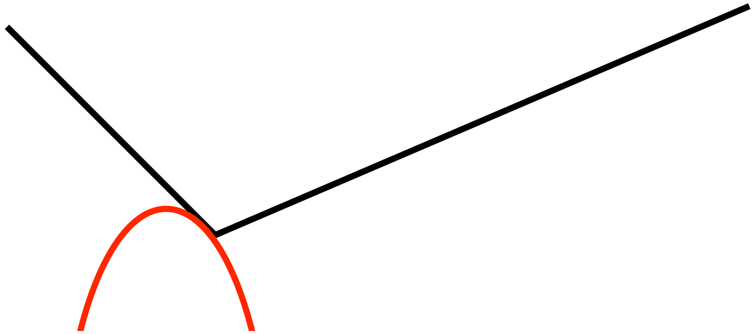
We can apply the `local lower-approximation by parabolas' idea to nonsmooth (lsc) functions

The set of all 'contact slopes' of lower locally supporting parabolas is the **proximal subdifferential** $\partial_P f(\alpha)$

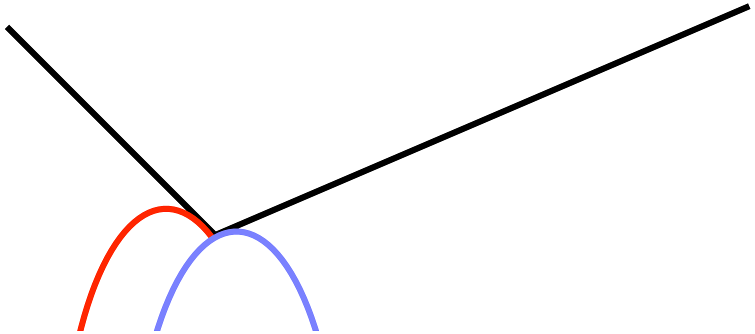
The set of all 'contact slopes' of lower locally supporting parabolas is the **proximal subdifferential** $\partial_P f(\alpha)$



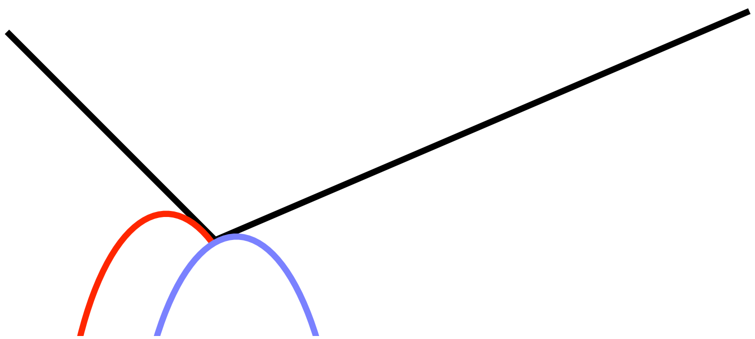
The set of all `contact slopes' of lower locally supporting parabolas is the **proximal subdifferential** $\partial_P f(\alpha)$



The set of all 'contact slopes' of lower locally supporting parabolas is the **proximal subdifferential** $\partial_P f(\alpha)$

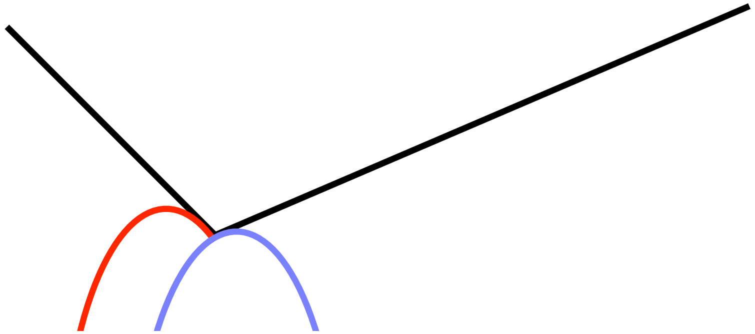


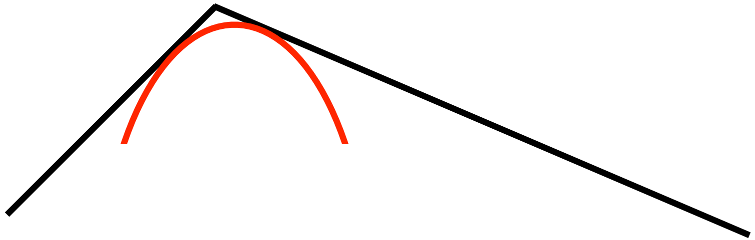
The set of all `contact slopes' of lower locally supporting parabolas is the **proximal subdifferential** $\partial_P f(\alpha)$



$$\partial_P f(\alpha) = [-2, 1]$$

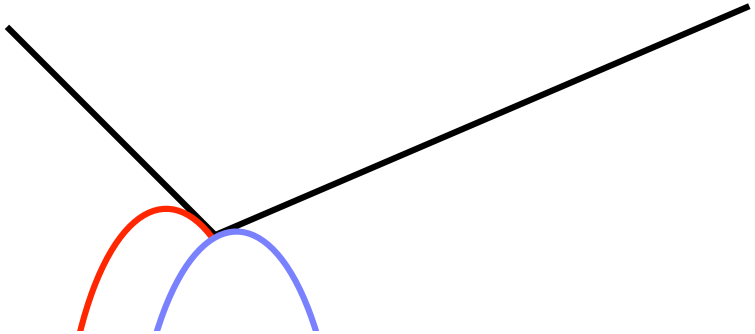
The set of all `contact slopes' of lower locally supporting parabolas is the **proximal subdifferential** $\partial_P f(\alpha)$



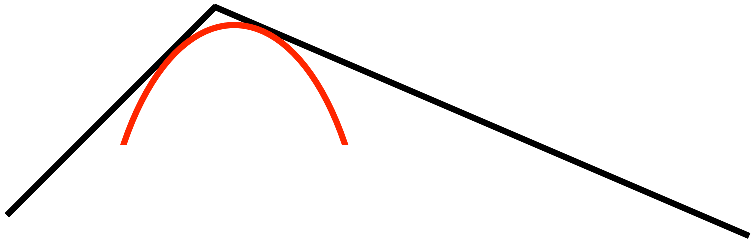


$$\partial_P f(\alpha) = [-2, 1]$$

The set of all `contact slopes' of lower locally supporting parabolas is the **proximal subdifferential** $\partial_P f(\alpha)$

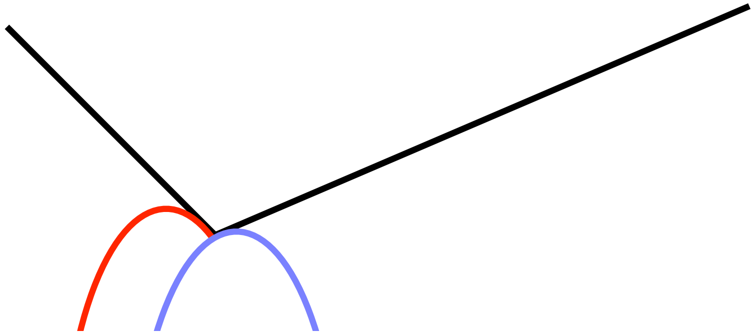


$$\partial_P f(\alpha) = [-2, 1]$$

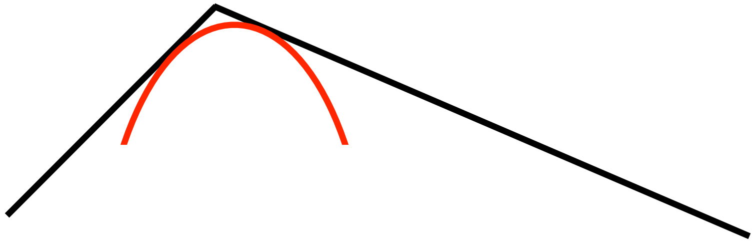


$$\partial_P f(\alpha) = \emptyset$$

The set of all 'contact slopes' of lower locally supporting parabolas is the **proximal subdifferential** $\partial_P f(\alpha)$



$$\partial_P f(\alpha) = [-2, 1]$$

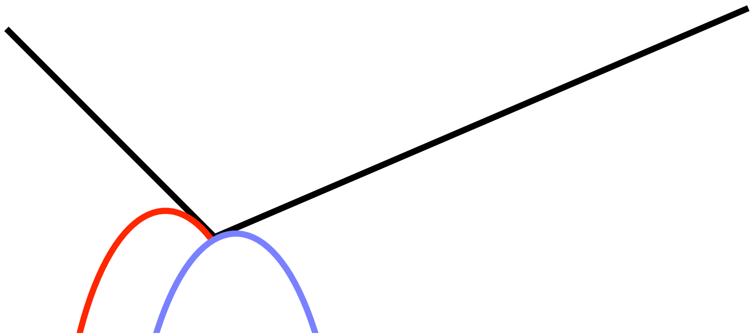


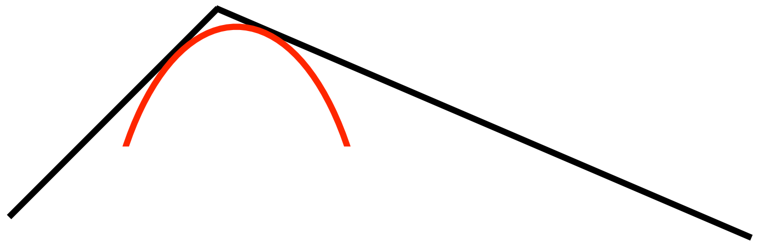
$$\partial_P f(\alpha) = \emptyset$$

$$\zeta \in \partial_P f(\alpha) \iff$$

$$f(x) \geq \langle \zeta, x - \alpha \rangle + f(\alpha) - \sigma |x - \alpha|^2 \text{ locally}$$

The set of all `contact slopes' of lower locally supporting parabolas is the **proximal subdifferential** $\partial_P f(\alpha)$





$$\partial_P f(\alpha) = [-2, 1]$$

$$\partial_P f(\alpha) = \emptyset$$

$$\zeta \in \partial_P f(\alpha) \iff$$

$$f(x) \geq \langle \zeta, x - \alpha \rangle + f(\alpha) - \sigma |x - \alpha|^2 \text{ locally}$$

$\partial_P f$ has a very complete (but fuzzy!) calculus...

Theorem

Let $\phi : R^n \to R_+$ **be a continuous positive definite function such that**

$$h(x,\zeta) + 1 = 0 \quad \forall \zeta \in \partial_P \phi(x), \; \forall x \neq 0.$$

Then

$$\phi(\cdot) = T(\cdot)$$

Theorem

Let $\phi : R^n \longrightarrow R_+$ **be a continuous positive definite function such that**

$$h(x,\zeta) + 1 = 0 \quad \forall\zeta \in \partial_P\phi(x), \ \forall x \neq 0.$$

Then

$$\phi(\cdot) = T(\cdot)$$

Remark **Large literature on H-J-B equation:**

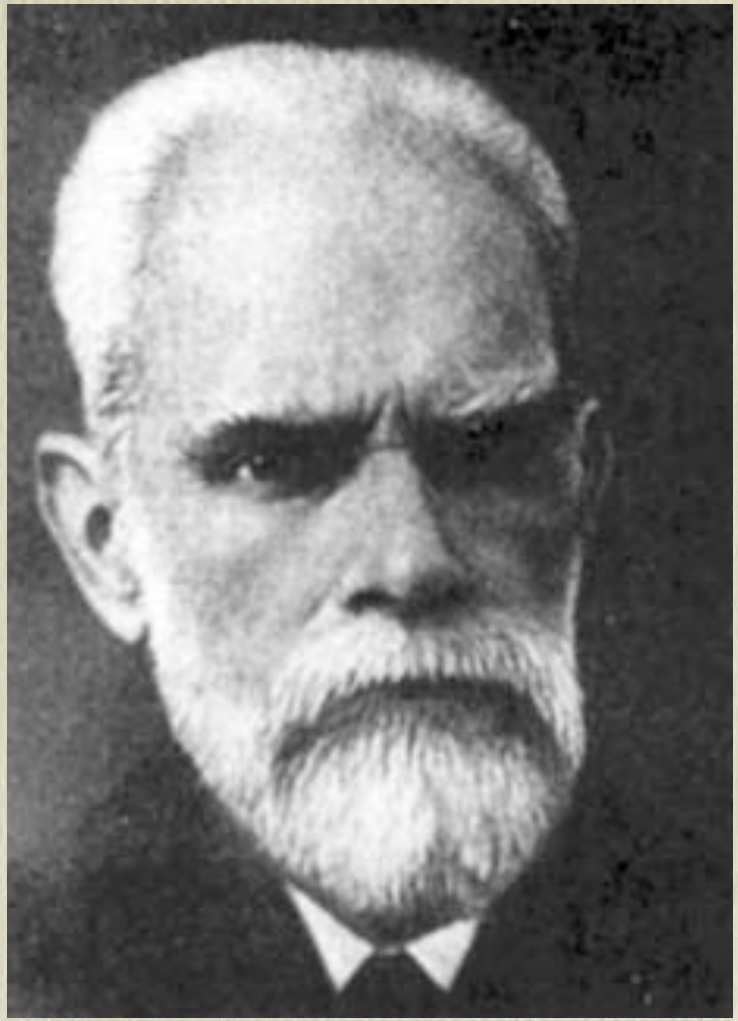
- **Clarke 1976 (Lipschitz, *generalized gradients*)**
- **Subbotin 1980 (invariance, Lipschitz, *minimax*)**
- **Crandall-Lions 1982 (comparison, continuous, *viscosity*)**
- **Clarke-Ledyaev 1994 (monotonicity, lsc, *proximal*)**
- **Fathi 1998 (*KAM solutions*)**
- **Dacorogna, DeVille... (*almost everywhere*)**

We now study the stability of the system trajectories: $x(t) \longrightarrow 0$ **, apparently very different**

We now study the **stability** of the system trajectories: $x(t) \longrightarrow 0$, apparently very different

Recall For the ordinary differential equation

$x'(t) = g\big(x(t)\big)$ we have:

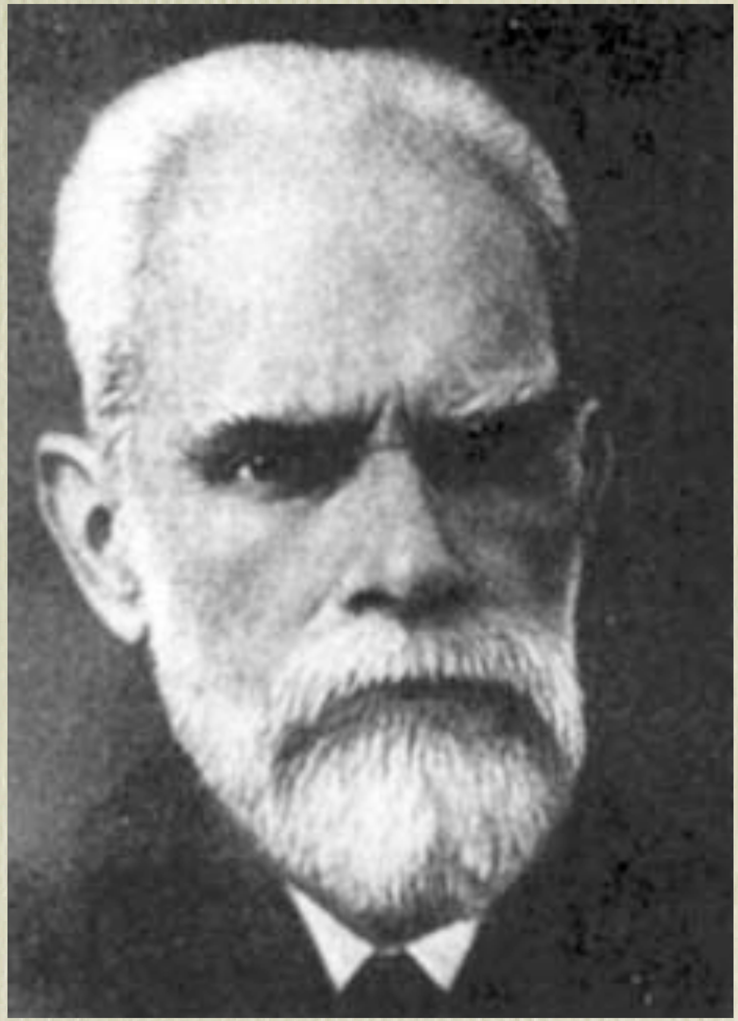


Theorem Let g be continuous. The differential equation is stable if and only if there is a Lyapunov function for g.

We now study the **stability** of the system trajectories: $x(t) \longrightarrow 0$, apparently very different

Recall For the ordinary differential equation $x'(t) = g\big(x(t)\big)$ we have:

Theorem Let g be continuous. The differential equation is stable if and only if there is a Lyapunov function for g.



Lyapunov for the sufficiency

Massera, Barbashin and Krasovskii, and Kurzweil for the necessity:
converse Lyapunov theorems

RECALL:

A Lyapunov function V for g is C^1 and satisfies:

RECALL:

A Lyapunov function V for g is C^1 and satisfies:

Positive definiteness:

$$V(0) = 0, \; V(x) > 0 \; \forall \; x \neq 0$$

RECALL:

A Lyapunov function V for g is C^1 and satisfies:

Positive definiteness:

$$V(0) = 0,\ V(x) > 0\ \forall\, x \neq 0$$

Properness:

The level sets $\{x : V(x) \leq c\}$ are compact for every c. Equivalently, V is radially unbounded: $V(x) \longrightarrow +\infty$ as $|x| \longrightarrow +\infty$

RECALL:

A Lyapunov function V for g is C^1 and satisfies:

Positive definiteness:

$$V(0) = 0, \ V(x) > 0 \,\forall\, x \neq 0$$

Properness:

The level sets $\{x : V(x) \leq c\}$ are compact for every c. Equivalently, V is radially unbounded: $V(x) \longrightarrow +\infty$ as $|x| \longrightarrow +\infty$

Infinitesimal decrease:

$$\langle \nabla V(x), g(x) \rangle < 0 \,\forall\, x \neq 0.$$

For the <u>controlled</u> differential equation

$$(*) \begin{cases} x'(t) = f\big(x(t), u(t)\big) \\ u(t) \in U \end{cases}$$

there are two principal scenarios:

For the <u>controlled</u> differential equation

$$(*) \begin{cases} x'(t) = f\big(x(t), u(t)\big) \\ u(t) \in U \end{cases}$$

there are two principal scenarios:

- **all trajectories go to 0, from any initial condition**

For the controlled differential equation

$$(*) \begin{cases} x'(t) = f\big(x(t), u(t)\big) \\ u(t) \in U \end{cases}$$

there are two principal scenarios:

- **all trajectories go to 0, from any initial condition**

Strong stability

For the <u>controlled</u> differential equation

$$(*) \begin{cases} x'(t) = f\big(x(t), u(t)\big) \\ u(t) \in U \end{cases}$$

there are two principal scenarios:

- **all trajectories go to 0, from any initial condition**

Strong stability

- **from any initial condition, some trajectory goes to 0**

For the controlled differential equation

$$(*) \begin{cases} x'(t) = f\big(x(t), u(t)\big) \\ u(t) \in U \end{cases}$$

there are two principal scenarios:

- **all** trajectories go to 0, from any initial condition

  Strong stability

- from any initial condition, **some** trajectory goes to 0

  Weak stability?

Strong stability:

$$(*) \begin{cases} x'(t) = f(x(t), u(t)) \\ u(t) \in U \end{cases}$$

all trajectories go to 0, from any initial condition

Strong stability:

$$(*) \begin{cases} x'(t) = f(x(t), u(t)) \\ u(t) \in U \end{cases}$$

all trajectories go to 0, from any initial condition

Theorem [Clarke, Ledyaev, Stern 1998]

The system is strongly stable if and only if there exists a strong Lyapunov function for it:
V smooth, positive definite, proper, such that

$$\max_{u \in U} \langle \nabla V(x), f(x,u) \rangle < 0 \; \forall \, x \neq 0.$$

Strong stability:

$$(*)\begin{cases} x'(t) = f(x(t),u(t)) \\ u(t) \in U \end{cases}$$

all trajectories go to 0, from any initial condition

Theorem [Clarke, Ledyaev, Stern 1998]

The system is strongly stable if and only if there exists a strong Lyapunov function for it:

V smooth, positive definite, proper, such that

$$\max_{u\in U}\langle \nabla V(x), f(x,u)\rangle < 0 \; \forall\, x \neq 0.$$

Weak stability

$$(*) \begin{cases} x'(t) = f(x(t), u(t)) \\ u(t) \in U \end{cases}$$

from any initial condition, some trajectory goes to 0

Weak stability

$$(*) \begin{cases} x'(t) = f(x(t), u(t)) \\ u(t) \in U \end{cases}$$

from any initial condition, some trajectory goes to 0

The system is said to be open-loop Globally Asymptotically Controllable to the origin if:

For every α , there exists a control $u_\alpha(t)$ and a state trajectory $x(t)$ such that

$$x'(t) = f(x(t), u_\alpha(t)),\ x(0) = \alpha \text{ and } x(t) \to 0$$

(plus a technical condition at 0)

Weak stability

$$(*)\begin{cases} x'(t) = f(x(t), u(t)) \\ u(t) \in U \end{cases}$$

from any initial condition, some trajectory goes to 0

GAC

The system is said to be open-loop Globally Asymptotically Controllable to the origin if:

For every α , there exists a control $u_\alpha(t)$ and a state trajectory $x(t)$ such that

$$x'(t) = f(x(t), u_\alpha(t)),\ x(0) = \alpha \text{ and } x(t) \to 0$$

(plus a technical condition at 0)

In view of the above (and linear systems) we surmise:

Theorem ??

The system is GAC if and only if there exists a weak Lyapunov function for it:

V smooth, positive definite, proper, such that

$$h\big(x, \nabla V(x)\big) := \min_{u \in U} \big\langle \nabla V(x), f(x,u) \big\rangle < 0 \,\forall\, x \neq 0.$$

In view of the above (and linear systems) we surmise:

Theorem ??

The system is GAC if and only if there exists a weak Lyapunov function for it:

V smooth, positive definite, proper, such that

$$h\big(x, \nabla V(x)\big) := \min_{u \in U} \langle \nabla V(x), f(x,u) \rangle < 0 \; \forall\, x \neq 0.$$

In view of the above (and linear systems) we surmise:

Theorem ??

The system is GAC if and only if there exists a weak Lyapunov function for it:

V smooth, positive definite, proper, such that

$$h(x, \nabla V(x)) := \min_{u \in U} \langle \nabla V(x), f(x,u) \rangle < 0 \; \forall \, x \neq 0.$$

Definition: **Such a function is called a (smooth) CLF**

In view of the above (and linear systems) we surmise:

Theorem ??

The system is GAC if and only if there exists a weak Lyapunov function for it:

V smooth, positive definite, proper, such that

$$h(x, \nabla V(x)) := \min_{u \in U} \langle \nabla V(x), f(x,u) \rangle < 0 \,\forall\, x \neq 0.$$

Definition: **Such a function is called a (smooth) CLF**

In view of the above (and linear systems) we surmise:

Theorem ??

The system is GAC if and only if there exists a weak Lyapunov function for it:

V smooth, positive definite, proper, such that

$$h(x, \nabla V(x)) := \min_{u \in U} \langle \nabla V(x), f(x,u) \rangle < 0 \; \forall x \neq 0.$$

Definition: Such a function is called a (smooth) CLF

THEOREM [Clarke, Ledyaev, Stern 1998]

Suppose that the system $(*)$ admits a smooth CLF. Then for every $\delta > 0$, the following set is a neighborhood of 0:

$$\{v \in \text{ co } f(x, U) : x \in B(0, \delta)\}.$$

In view of the above (and linear systems) we surmise:

Theorem ??

The system is GAC if and only if there exists a weak Lyapunov function for it:

V smooth, positive definite, proper, such that

$$h(x, \nabla V(x)) := \min_{u \in U} \langle \nabla V(x), f(x,u) \rangle < 0 \; \forall \, x \neq 0.$$

Definition: Such a function is called a (smooth) CLF

THEOREM [Clarke, Ledyaev, Stern 1998]

Suppose that the system $(*)$ admits a smooth CLF. Then for every $\delta > 0$, the following set is a neighborhood of 0:

$$\{v \in \text{ co } f(x,U) : x \in B(0,\delta)\}.$$

So any system which fails to satisfy this covering condition cannot admit a smooth CLF

Example: nonholonomic integrator (NHI)

$$x_1'(t) = u_1(t)$$
$$x_2'(t) = u_2(t)$$
$$x_3'(t) = x_1(t)u_2(t) - x_2(t)u_1(t)$$

$$U = \{(u_1, u_2) : u_1^2 + u_2^2 \leq 1\}$$

Example: nonholonomic integrator (NHI)

$$x_1'(t) = u_1(t)$$
$$x_2'(t) = u_2(t)$$
$$x_3'(t) = x_1(t)u_2(t) - x_2(t)u_1(t)$$

$$U = \{(u_1, u_2) : u_1^2 + u_2^2 \leq 1\}$$

This is a nonlinear system (of real interest) that is close to being a classical linear system: it is linear in u, linear in x (separately), with an ample control set.

Example: nonholonomic integrator (NHI)

$$x_1'(t) = u_1(t)$$
$$x_2'(t) = u_2(t)$$
$$x_3'(t) = x_1(t)u_2(t) - x_2(t)u_1(t)$$

$$U = \{(u_1, u_2) : u_1^2 + u_2^2 \leq 1\}$$

This is a nonlinear system (of real interest) that is close to being a classical linear system: it is linear in u, linear in x (separately), with an ample control set.

It is easy to verify directly that the system is GAC.

RECALL: THEOREM

Suppose that the system $(*)$ admits a smooth CLF. Then for every $\delta > 0$, the following set is a neighborhood of 0:

$$\{v \in \text{co}\, f(x, U) : x \in B(0, \delta)\}.$$

RECALL: THEOREM

Suppose that the system $(*)$ admits a smooth CLF. Then for every $\delta > 0$, the following set is a neighborhood of 0:

$$\{v \in \text{ co } f(x, U) : x \in B(0, \delta)\}.$$

For NHI :

$$x_1'(t) = u_1(t) \qquad U = \{(u_1, u_2) : u_1^2 + u_2^2 \leq 1\}$$

$$x_2'(t) = u_2(t)$$

$$x_3'(t) = x_1(t)u_2(t) - x_2(t)u_1(t)$$

$$\begin{bmatrix} 0 \\ 0 \\ \gamma \end{bmatrix} \notin \text{ co } f\big(B(0, r), U\big) = f\big(B(0, r), U\big) \text{ if } \gamma \neq 0$$

RECALL: THEOREM

Suppose that the system $(*)$ admits a smooth CLF. Then for every $\delta > 0$, the following set is a neighborhood of 0:

$$\{v \in \text{ co } f(x, U) : x \in B(0, \delta)\}.$$

For NHI :

$$x_1'(t) = u_1(t)$$
$$x_2'(t) = u_2(t)$$
$$x_3'(t) = x_1(t)u_2(t) - x_2(t)u_1(t)$$

$$U = \{(u_1, u_2) : u_1^2 + u_2^2 \leq 1\}$$

$$\begin{bmatrix} 0 \\ 0 \\ \gamma \end{bmatrix} \notin \text{ co } f\big(B(0,r), U\big) = f\big(B(0,r), U\big) \text{ if } \gamma \neq 0$$

So: no smooth CLF.

RECALL: THEOREM

Suppose that the system $(*)$ admits a smooth CLF. Then for every $\delta > 0$, the following set is a neighborhood of 0:

$$\{v \in \text{ co } f(x, U) : x \in B(0, \delta)\}.$$

For NHI :

$$x_1'(t) = u_1(t) \qquad U = \{(u_1, u_2) : u_1^2 + u_2^2 \le 1\}$$

$$x_2'(t) = u_2(t)$$

$$x_3'(t) = x_1(t)u_2(t) - x_2(t)u_1(t)$$

$$\begin{bmatrix} 0 \\ 0 \\ \gamma \end{bmatrix} \notin \text{ co } f\big(B(0,r), U\big) = f\big(B(0,r), U\big) \text{ if } \gamma \neq 0$$

So: no smooth CLF.

The property GAC is <u>not</u> characterized by the existence of a smooth CLF

Definition: **V is a Dini CLF if it is continuous, proper, positive definite, and satisfies infinitesimal decrease in the Dini sense:**

$$\min_{u \in U} dV\big(x; f(x,u)\big) \;<\; -W(x)\;\forall\, x \neq 0.$$

Definition: **V is a Dini CLF if it is continuous, proper, positive definite, and satisfies infinitesimal decrease in the Dini sense:**

$$\min_{u \in U} \underbrace{dV\big(x; f(x,u)\big)}_{\text{Dini derivate}} < -W(x) \; \forall\, x \neq 0.$$

Definition: **V is a Dini CLF if it is continuous, proper, positive definite, and satisfies infinitesimal decrease in the Dini sense:**

$$\min_{u \in U} dV\left(x; f(x,u)\right) \;<\; -W(x) \;\forall\, x \neq 0.$$

Dini derivate

rate function

Definition: **V is a Dini CLF if it is continuous, proper, positive definite, and satisfies infinitesimal decrease in the Dini sense:**

$$\underbrace{\min_{u\in U} dV(x; f(x,u))}_{\text{Dini derivate}} \; < \; -\underbrace{W(x)}_{\text{rate function}} \;\forall\, x \neq 0.$$

THEOREM [Sontag 1983]

The system $(*)$ is GAC if and only if there exists a Dini CLF.

Definition: **V is a Dini CLF if it is continuous, proper, positive definite, and satisfies infinitesimal decrease in the Dini sense:**

$$\underbrace{\min_{u\in U} dV(x; f(x,u))}_{\text{Dini derivate}} < -\underbrace{W(x)}_{\text{rate function}} \;\forall\, x \neq 0.$$

THEOREM [Sontag 1983]

The system $(*)$ is GAC if and only if there exists a Dini CLF.

THEOREM

The system $(*)$ is GAC if and only if there exists a proximal CLF:

$$\max_{\zeta\in\partial_P V(x)} \min_{u\in U} \langle \zeta, f(x,u)\rangle < -W(x) \;\forall\, x \neq 0.$$

How are CLF's found?

The value function technique

The value function technique

Let (*) be GAC. Fix r>0, and, for a given rate function W, define

$$\phi(\alpha) \;:=\; \min \int_0^T W\big(x(t)\big)\, dt,$$

where the minimum is taken over all trajectories x such that

$$x(0) = \alpha,\; x(T) \in B(0,r),\; T \text{ free}$$

The function ϕ is an example of a value function, in which α is the parameter. Such functions play a central role in pde's, optimization, and differential games.

The value function technique

Let (*) be GAC. Fix r>0, and, for a given rate function W, define

$$\phi(\alpha) \ := \ \min \int_0^T W\big(x(t)\big)\, dt,$$

where the minimum is taken over all trajectories x such that

$$x(0) = \alpha,\ x(T) \in B(0,r),\ T \text{ free}$$

The function ϕ is an example of a value function, in which α is the parameter. Such functions play a central role in pde's, optimization, and differential games.

Fact: ϕ satisfies infinitesimal decrease except on $B(0,r)$

The value function technique

Let (*) be GAC. Fix r>0, and, for a given rate function W, define

$$\phi(\alpha) \;:=\; \min \int_0^T W\big(x(t)\big)\, dt,$$

where the minimum is taken over all trajectories x such that

$$x(0) = \alpha,\; x(T) \in B(0,r),\; T \text{ free}$$

The function ϕ is an example of a value function, in which α is the parameter. Such functions play a central role in pde's, optimization, and differential games.

Fact: ϕ satisfies infinitesimal decrease except on $B(0,r)$

ϕ is rather close to being a CLF for the system. But in which sense? Certainly not the smooth sense, for value functions are notoriously nonsmooth.

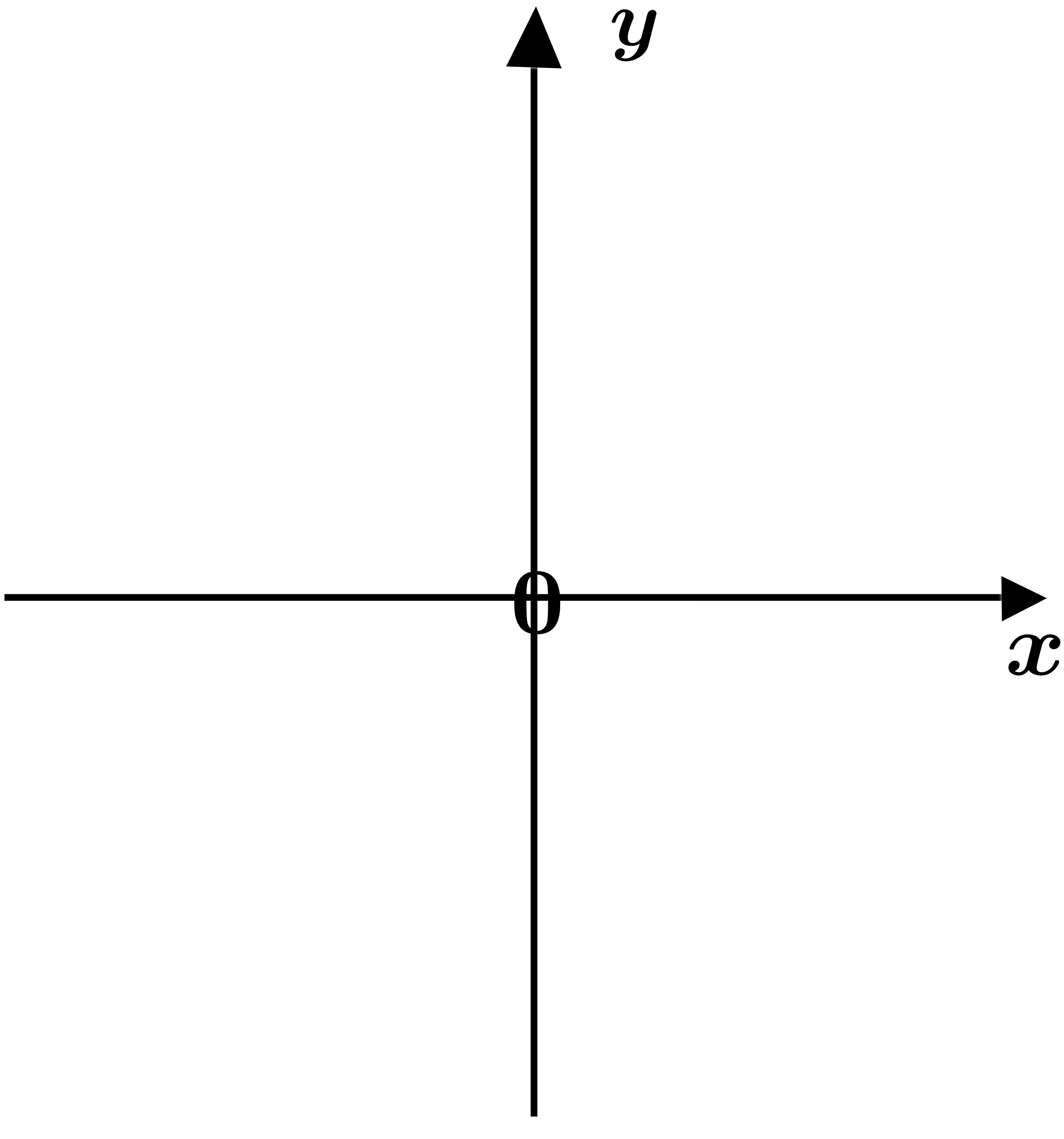
How are CLF's found?

The “field of trajectories” approach

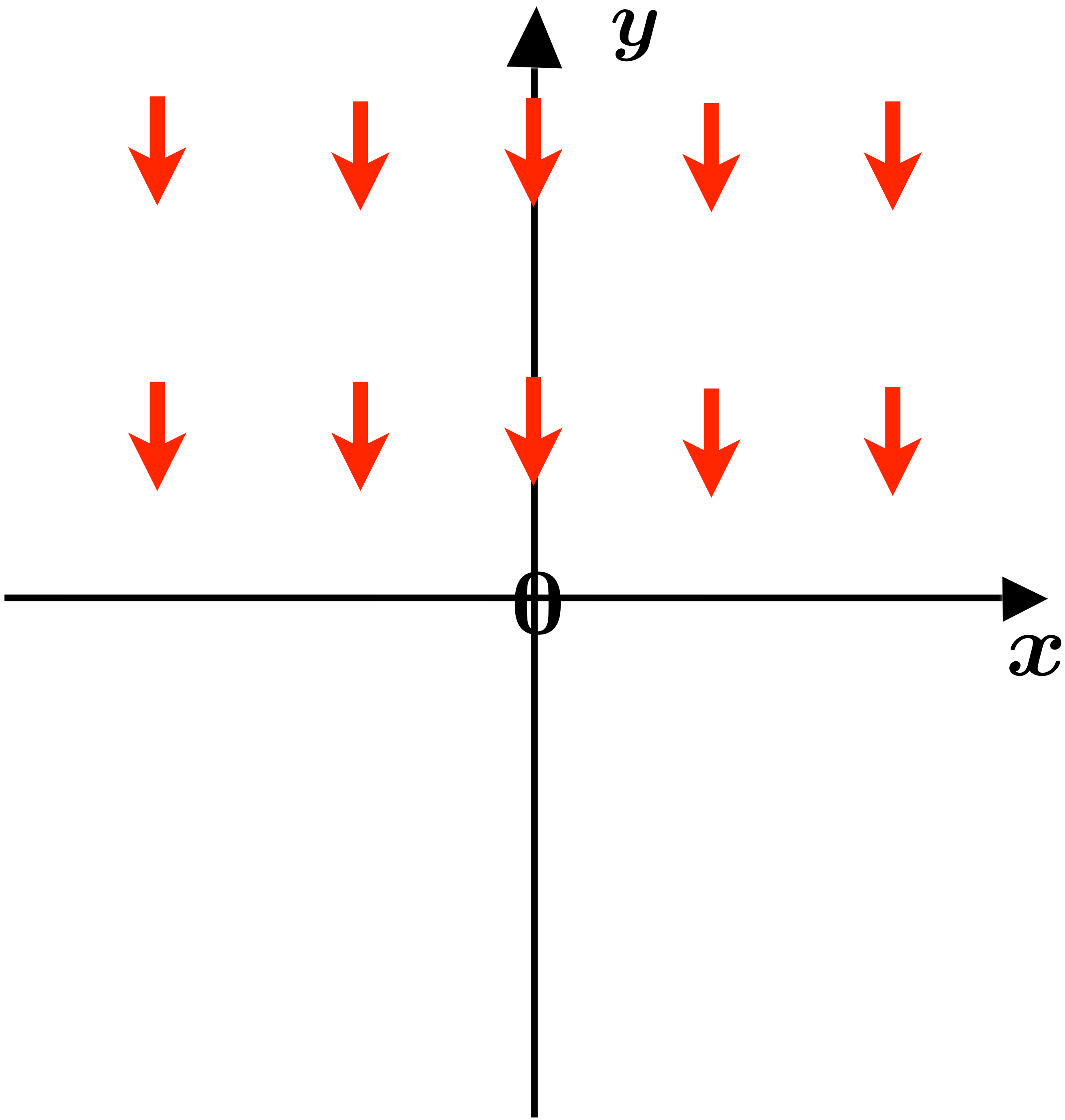
The “field of trajectories” approach

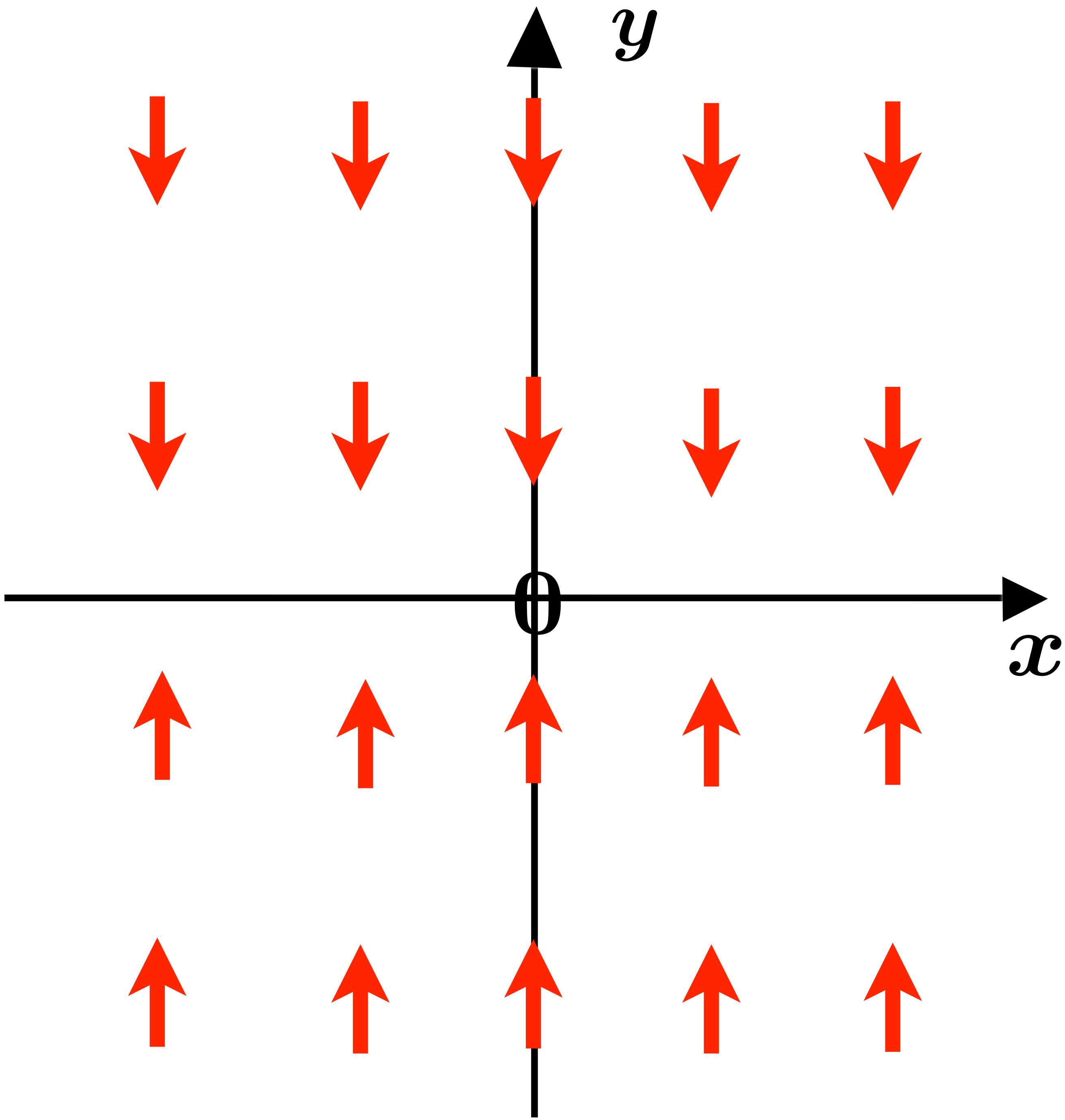
Exhibit a “reasonable, consistent” scheme for attaining a target S. Let V(α) be the time to the target, starting at α , and according to the scheme.

Then V is a Dini (and hence proximal) CLF (relative to the target S).

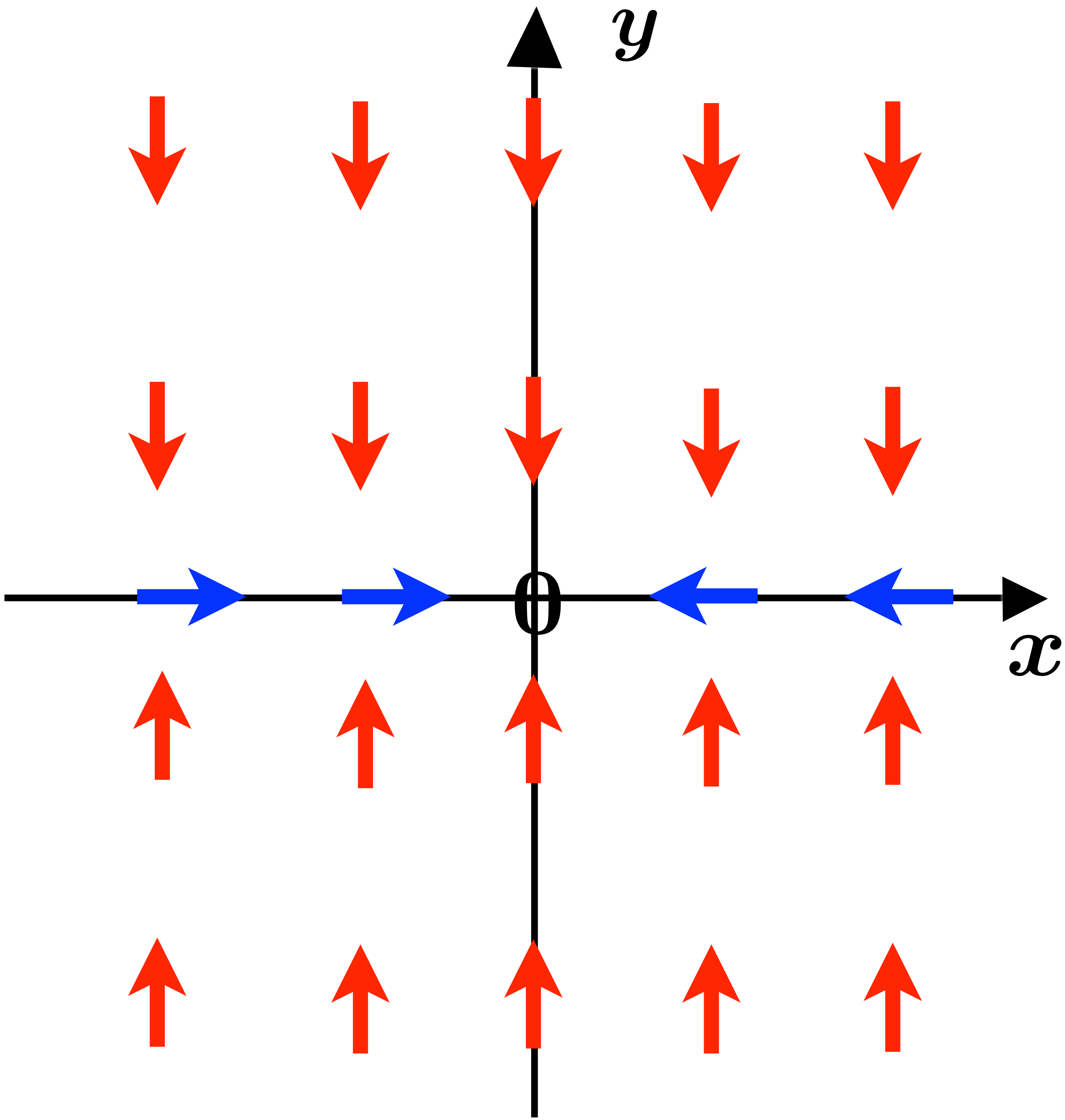


y
0
x

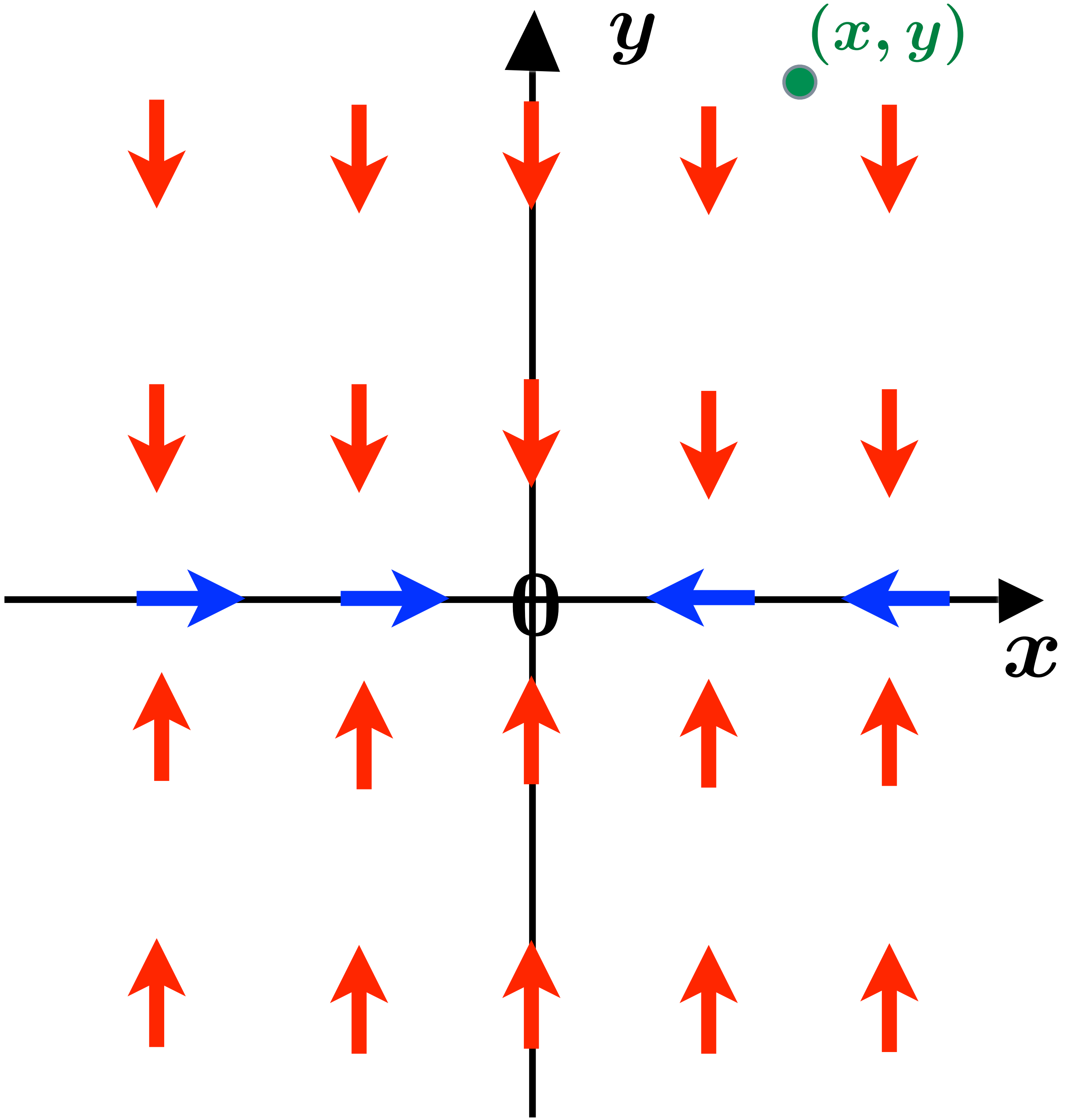

y
0
x



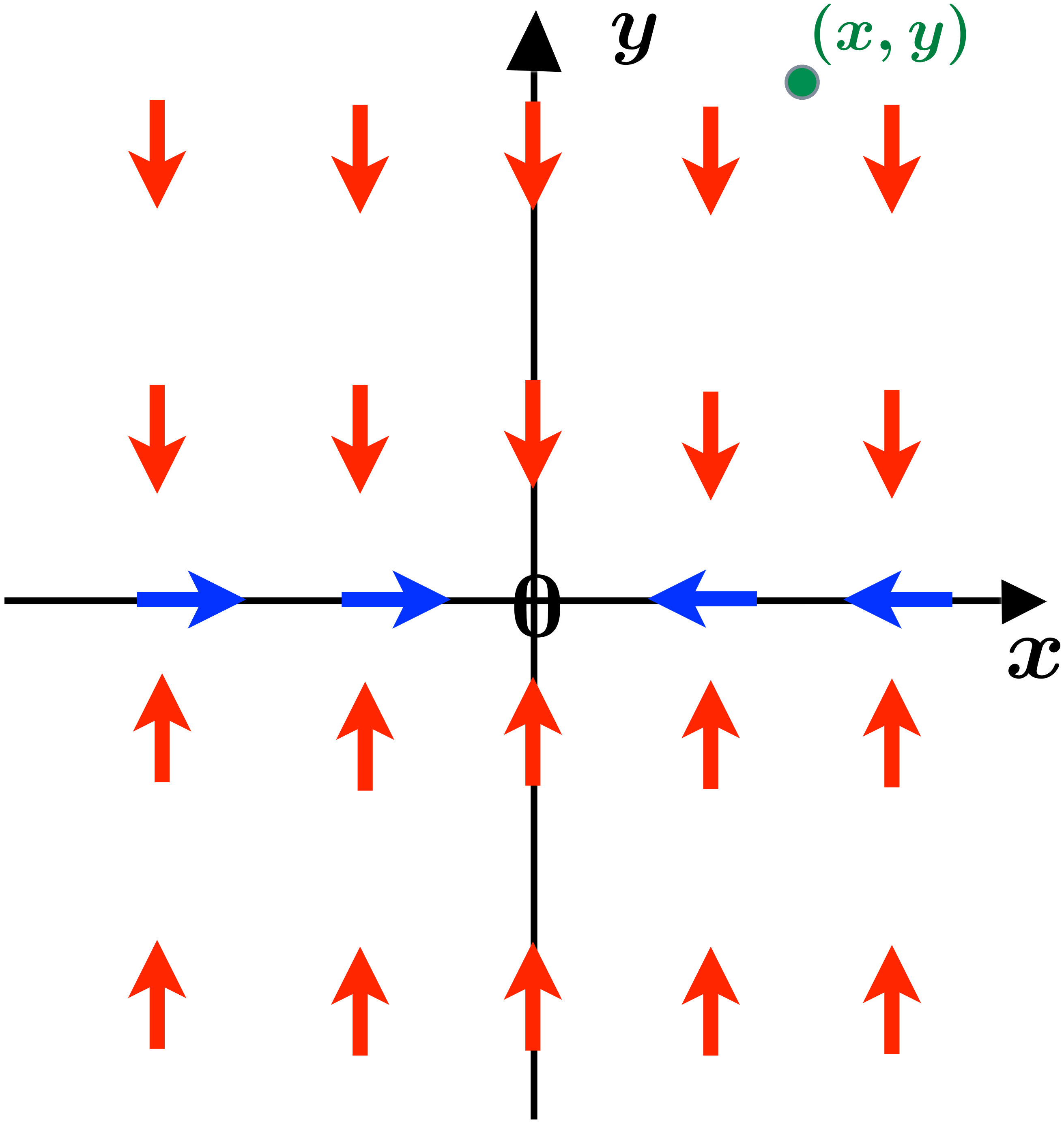
y
0
x



y
0
x

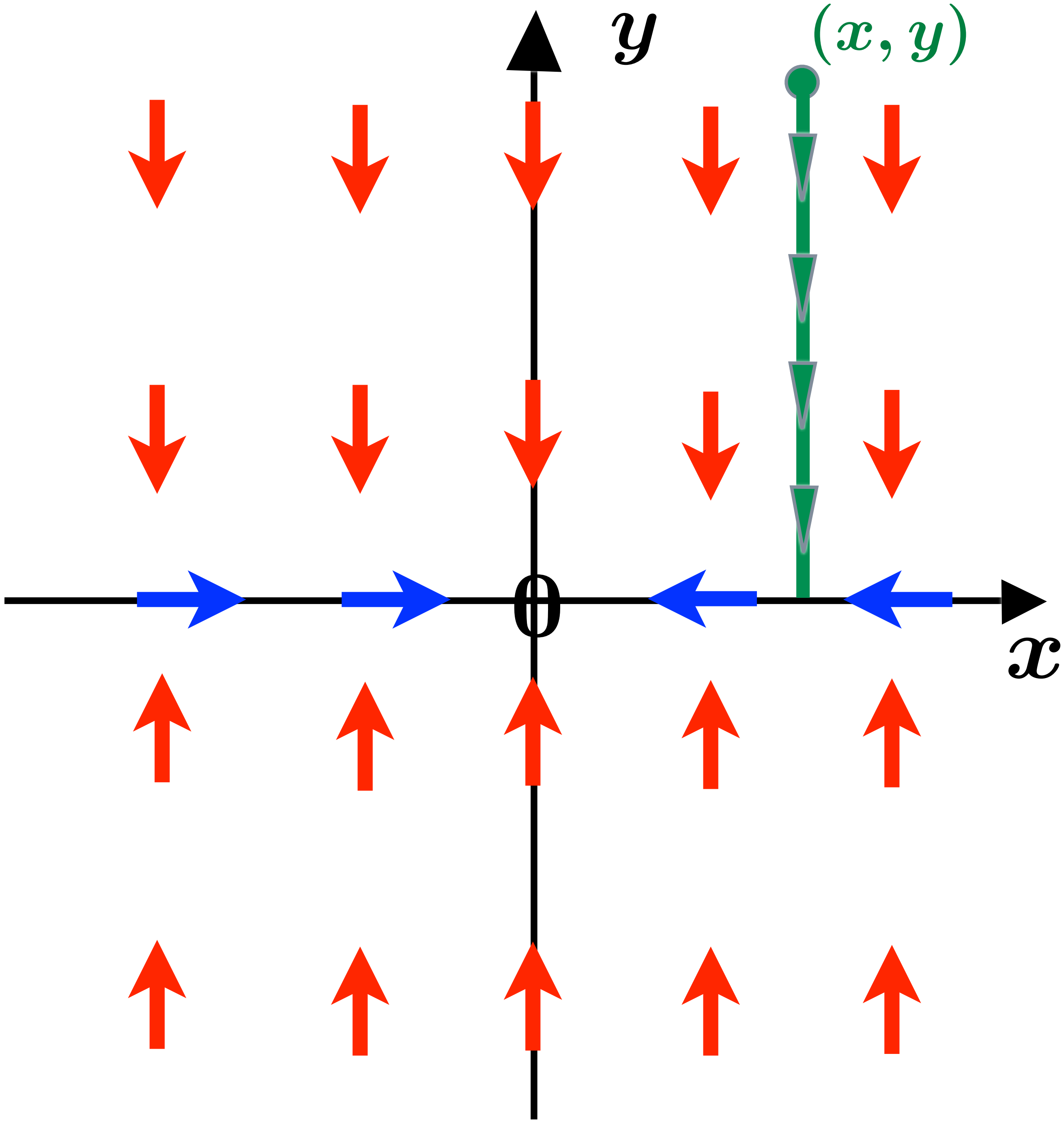


y
(x, y)
0
x



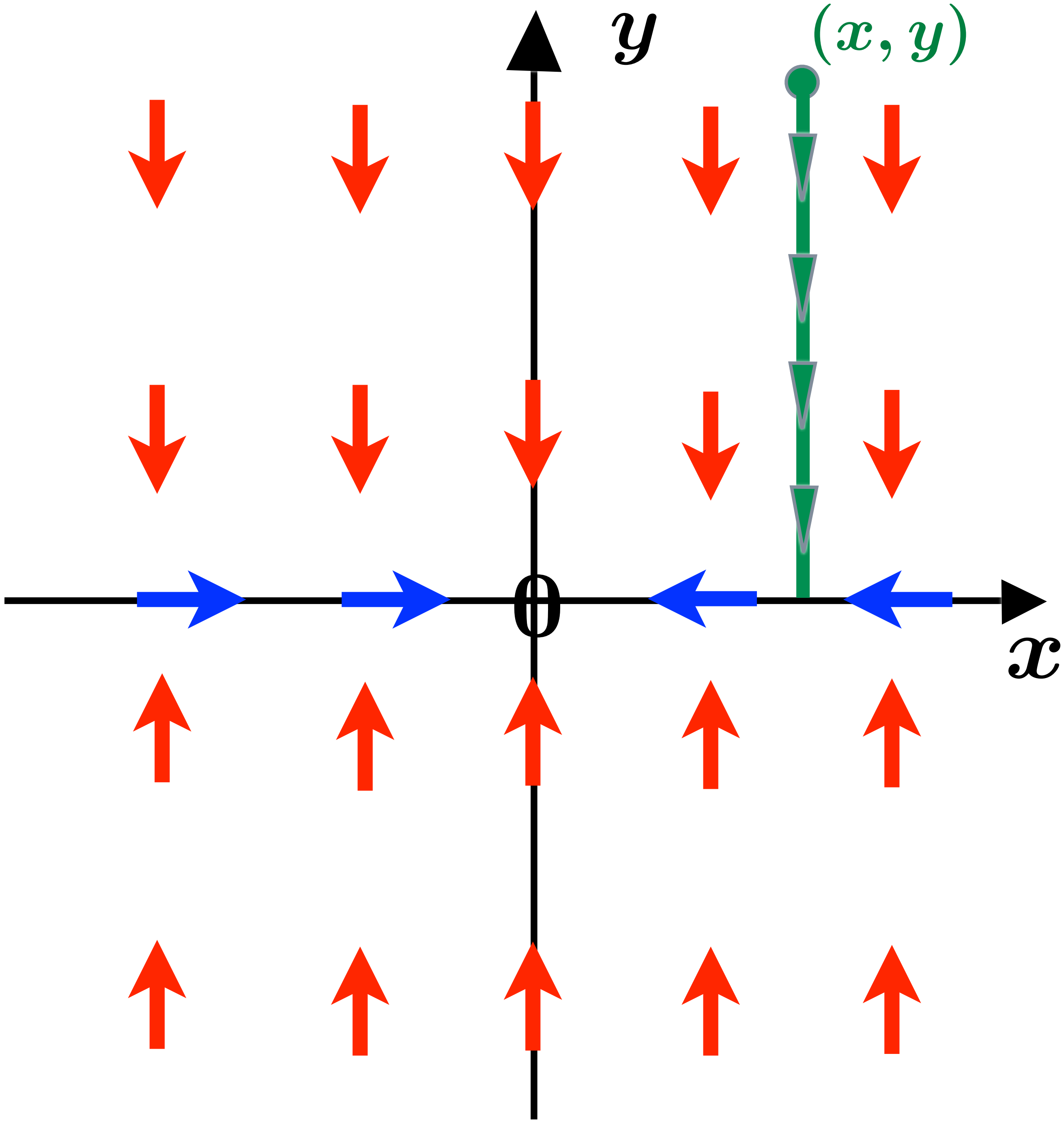
From (x,y):

1. Go directly to the x-axis



From (x,y):

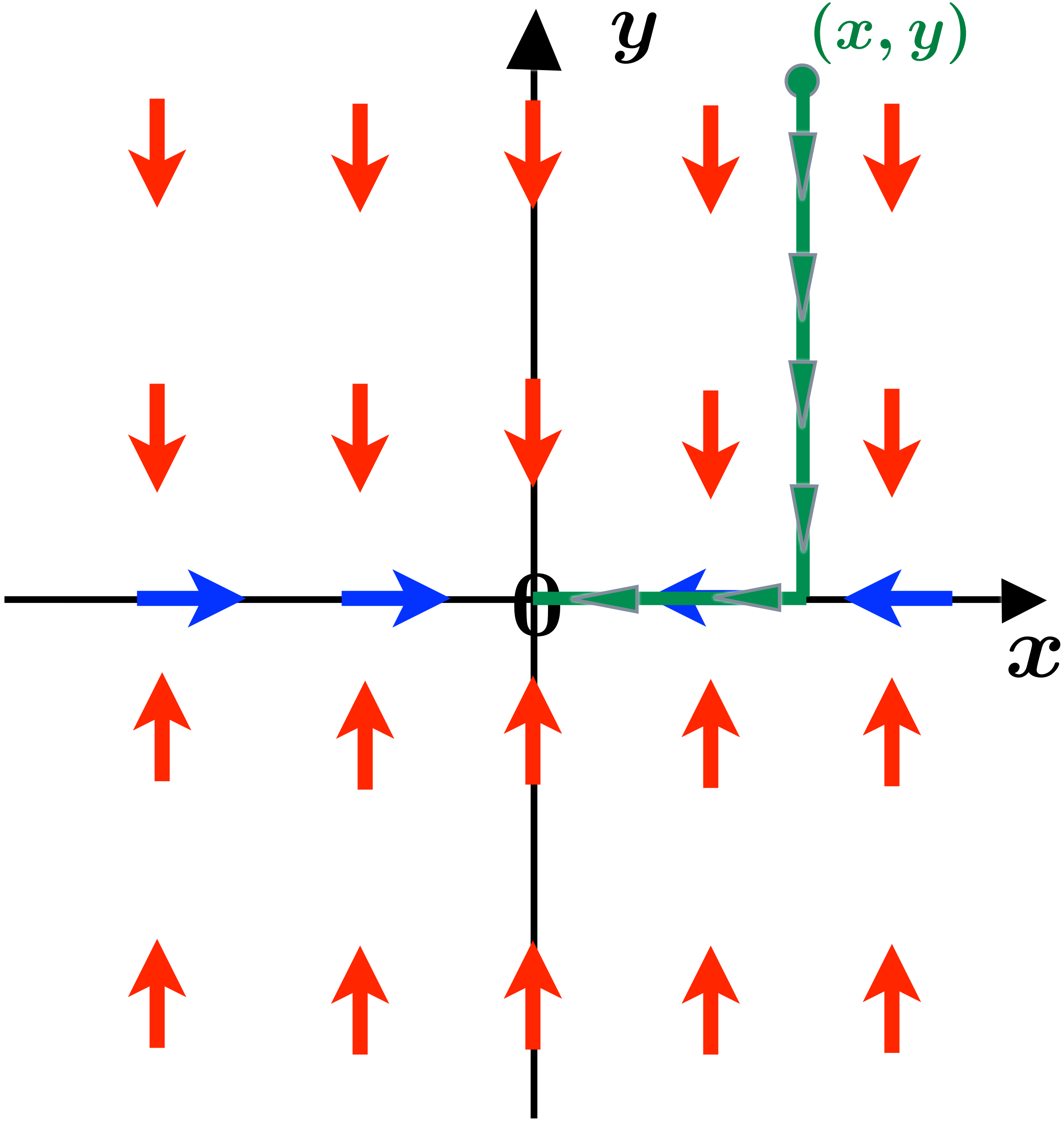
1. Go directly to the x-axis



From (x,y):

1. Go directly to the x-axis

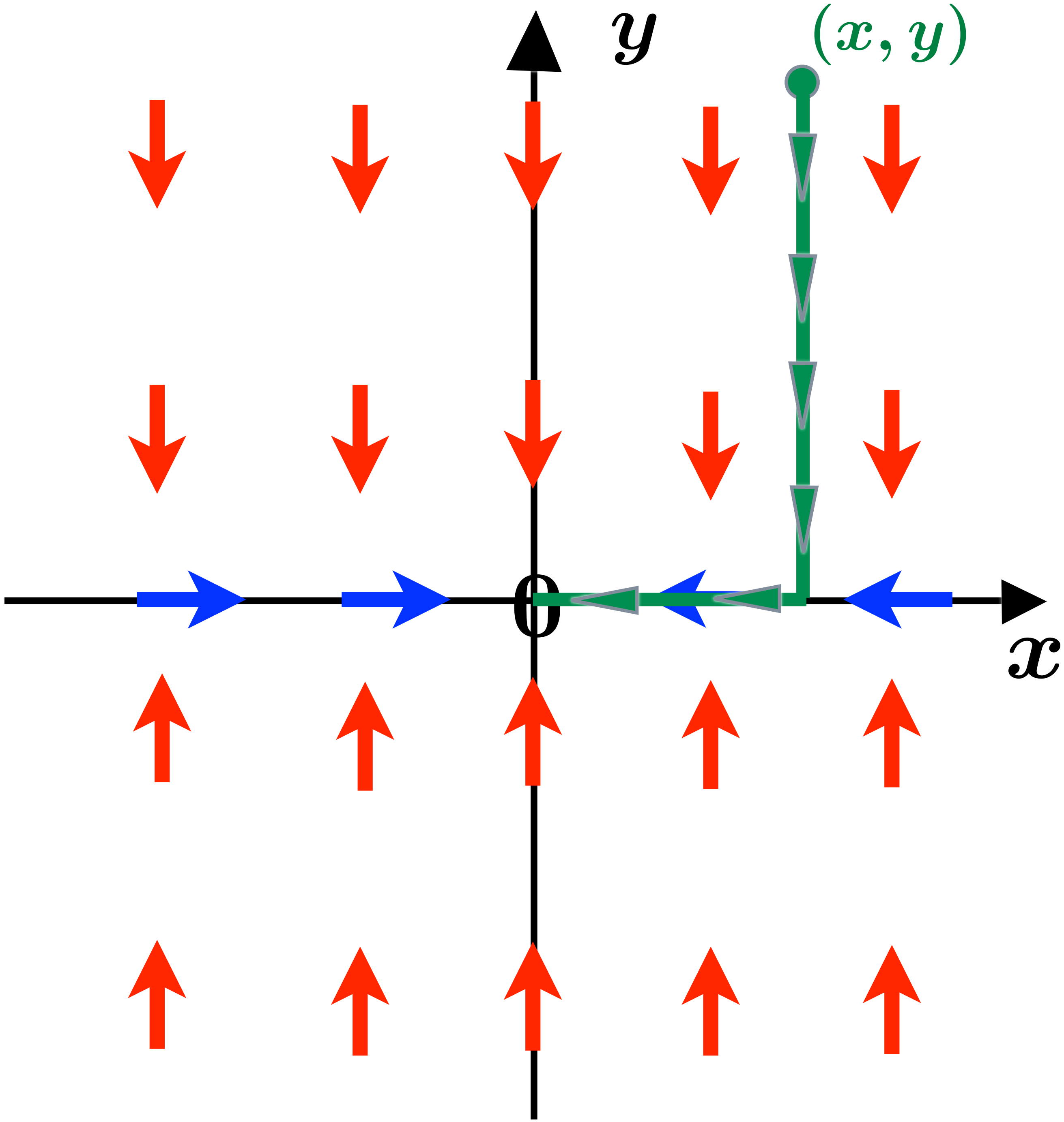
2. Then go directly to the origin



From (x,y):

1. Go directly to the x-axis

2. Then go directly to the origin



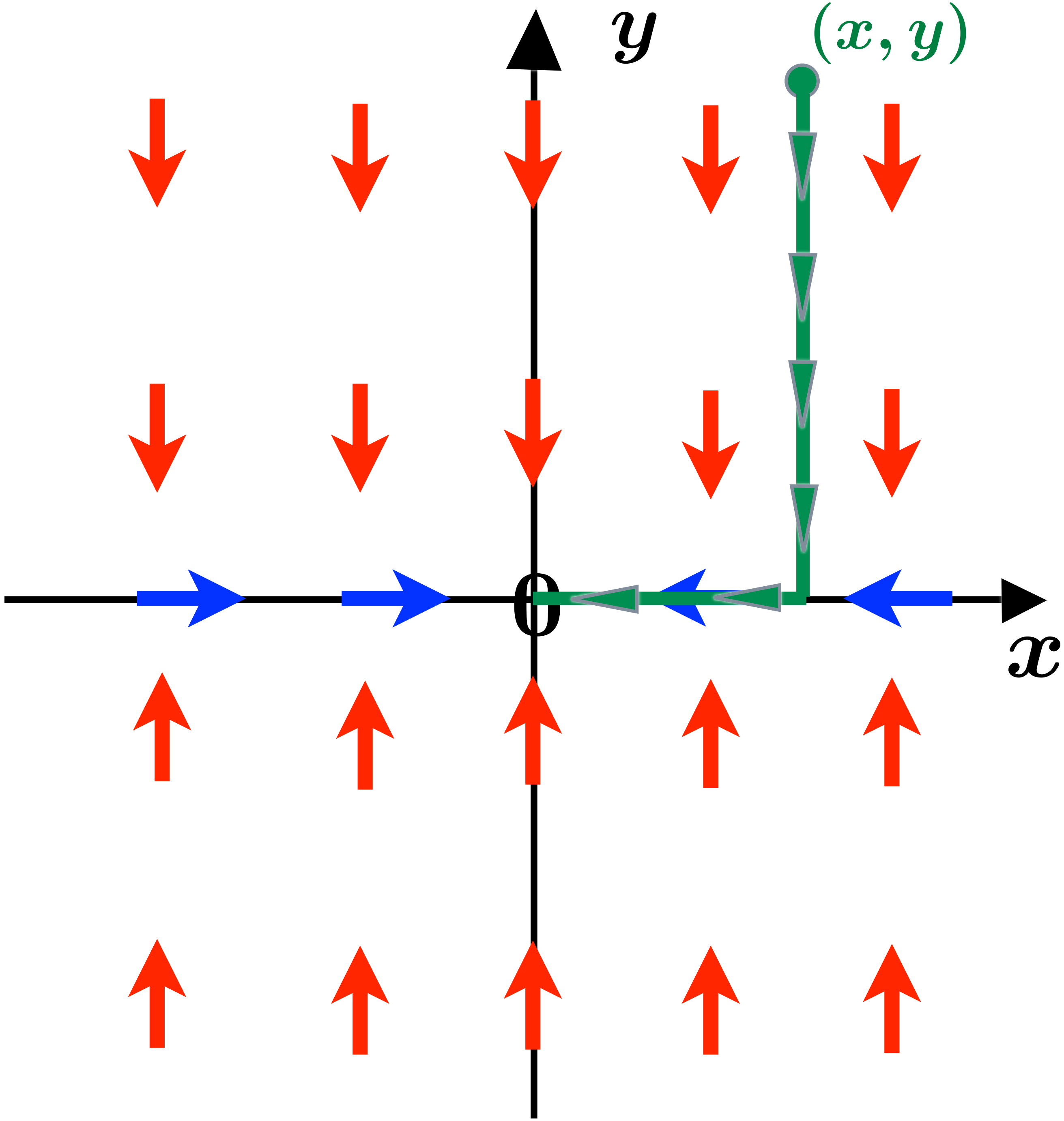
From (x,y):

1. Go directly to the x-axis

   Time: |y|

2. Then go directly to the origin

   Time: |x|



From (x,y):

1. Go directly to the x-axis

   Time: |y|

2. Then go directly to the origin

   Time: |x|

$\Longrightarrow$

V(x,y) = |x|+|y|

Stabilizing Feedback

$$(*) \quad \begin{cases} x'(t) = f\big(x(t), u(t)\big) \\ u(t) \in U \end{cases}$$

Goal: steer the state $x(t)$ to 0 .

Stabilizing Feedback

Goal: steer the state $x(t)$ to 0 .

$$(*) \quad \begin{cases} x'(t) = f\big(x(t), u(t)\big) \\ u(t) \in U \end{cases}$$

More explicitly, find a feedback $u = k(x)$ so that the differential equation

$$x'(t) = g\big(x(t)\big) := f\big(x(t), k(x(t))\big)$$

is stable:

$$\begin{cases} x'(t) = g\big(x(t)\big) \\ x(0) = \alpha \end{cases} \Longrightarrow x(t) \to 0$$

Stabilizing Feedback

Goal: steer the state $x(t)$ to 0 .

$$(*) \begin{cases} x'(t) = f\big(x(t), u(t)\big) \\ u(t) \in U \end{cases}$$

More explicitly, find a feedback $\boxed{u = k(x)}$ so that the differential equation

closed-loop control

$$x'(t) = g\big(x(t)\big) := f\big(x(t), k(x(t))\big)$$

is stable:

$$\begin{cases} x'(t) = g\big(x(t)\big) \\ x(0) = \alpha \end{cases} \Longrightarrow x(t) \to 0$$

Stabilizing Feedback

Goal: steer the state $x(t)$ to 0 .

$$(*) \begin{cases} x'(t) = f\big(x(t), u(t)\big) \\ u(t) \in U \end{cases}$$

More explicitly, find a feedback $u = k(x)$ so that the differential equation

closed-loop control

$$x'(t) = g\big(x(t)\big) := f\big(x(t), k(x(t))\big)$$

is stable:

$$\begin{cases} x'(t) = g\big(x(t)\big) \\ x(0) = \alpha \end{cases} \Longrightarrow x(t) \to 0$$

For $g(x)$ continuous, we require $k(x)$ continuous.

Stabilizing Feedback

Goal: steer the state $x(t)$ to 0 .

$$(*)\quad \begin{cases} x'(t) = f(x(t), u(t)) \\ u(t) \in U \end{cases}$$

More explicitly, find a feedback $\boxed{u = k(x)}$ **so that the differential equation**

closed-loop control

$$x'(t) = g(x(t)) := f(x(t), k(x(t)))$$

is stable:

$$\begin{cases} x'(t) = g(x(t)) \\ x(0) = \alpha \end{cases} \Longrightarrow x(t) \to 0$$

For $g(x)$ continuous, we require $k(x)$ continuous.

Q: Does k exist? (The system is then said to be *stabilizable*.) How to construct such a feedback?

A necessary condition for this is GAC

A necessary condition for this is GAC

Q: Is every (reasonable) GAC system stabilizable by feedback ?

A necessary condition for this is GAC

Q: Is every (reasonable) GAC system stabilizable by feedback ?

That is, can we synthesize the various open-loop controls $u_\alpha(t)$ into one coherent (continuous) feedback law $\boxed{k(x)}$?

This question has played a central role for decades.

A necessary condition for this is GAC

Q: Is every (reasonable) GAC system stabilizable by feedback ?

That is, can we synthesize the various open-loop controls $u_\alpha(t)$ into one coherent (continuous) feedback law $\boxed{k(x)}$?

This question has played a central role for decades.

The classic linear case:

$$(*) \qquad x'(t) = Ax(t) + Bu(t),\ U = \mathbb{R}^m$$

linear systems theory

A necessary condition for this is GAC

Q: Is every (reasonable) GAC system stabilizable by feedback ?

That is, can we synthesize the various open-loop controls $u_\alpha(t)$ into one coherent (continuous) feedback law $\boxed{k(x)}$?

This question has played a central role for decades.

The classic linear case:

linear systems theory

$$(*) \qquad x'(t) = Ax(t) + Bu(t), \; U = \mathbb{R}^m$$

Then: $(*)$ is GAC $\iff$ $(*)$ is stabilizable (by a linear feedback $k(x) = Kx$)

In applications, systems are rarely linear, yet linear systems theory is applied:

In applications, systems are rarely linear, yet linear systems theory is applied: **linearization**

$$x'(t) = f\big(x(t), u(t)\big) \Rightarrow \quad x'(t) = A\, x(t) + B\, u(t)$$

$$A = D_x f(0,0) \qquad B = D_u f(0,0)$$

In applications, systems are rarely linear, yet linear systems theory is applied: **linearization**

$$x'(t) = f\big(x(t), u(t)\big) \Longrightarrow \quad x'(t) = A\,x(t) + B\,u(t)$$

$$A \uparrow D_x f(0,0) \qquad B \uparrow D_u f(0,0)$$

This requires:

In applications, systems are rarely linear, yet linear systems theory is applied: **linearization**

$$x'(t) = f\big(x(t), u(t)\big) \Rightarrow \quad x'(t) = A\,x(t) + B\,u(t)$$

$$A \uparrow D_x f(0,0) \qquad B \uparrow D_u f(0,0)$$

This requires:

- **smooth data**

In applications, systems are rarely linear, yet linear systems theory is applied: **linearization**

$$x'(t) = f\big(x(t), u(t)\big) \Rightarrow \quad x'(t) = A\,x(t) + B\,u(t)$$

$$A \leftarrow D_x f(0,0) \qquad B \leftarrow D_u f(0,0)$$

This requires:

- **smooth data**
- **nondegenerate linearization**

In applications, systems are rarely linear, yet linear systems theory is applied: **linearization**

$$x'(t) = f\big(x(t), u(t)\big) \Rightarrow \quad x'(t) = A\,x(t) + B\,u(t)$$

$$A \leftarrow D_x f(0,0) \qquad B \leftarrow D_u f(0,0)$$

This requires:

- **smooth data**
- **nondegenerate linearization**
- **small *(x,u)***

In applications, systems are rarely linear, yet linear systems theory is applied: **linearization**

$$x'(t) = f\big(x(t), u(t)\big) \Longrightarrow \quad x'(t) = A\,x(t) + B\,u(t)$$

$$A \leftarrow D_x f(0,0) \qquad B \leftarrow D_u f(0,0)$$

This requires:

- **smooth data**
- **nondegenerate linearization**
- **small *(x,u)***
- **all *(x,u)* near *(0,0)* available**

In applications, systems are rarely linear, yet linear systems theory is applied: **linearization**

$$x'(t) = f\big(x(t), u(t)\big) \Longrightarrow \quad x'(t) = A\,x(t) + B\,u(t)$$

$A \uparrow D_x f(0,0)$ $\quad$ $B \uparrow D_u f(0,0)$

This requires:

- **smooth data** false for some problems
- **nondegenerate linearization**
- **small *(x,u)***
- **all *(x,u)* near *(0,0)* available**

In applications, systems are rarely linear, yet linear systems theory is applied: **linearization**

$$x'(t) = f\big(x(t), u(t)\big) \Longrightarrow \quad x'(t) = A\,x(t) + B\,u(t)$$

$A \uparrow D_x f(0,0)$ $\quad B \uparrow D_u f(0,0)$

This requires:

- **smooth data** false for some problems
- **nondegenerate linearization** fails for NHI
- **small *(x,u)***
- **all *(x,u)* near *(0,0)* available**

In applications, systems are rarely linear, yet linear systems theory is applied: **linearization**

$$x'(t) = f\big(x(t), u(t)\big) \Longrightarrow \quad x'(t) = A\,x(t) + B\,u(t)$$

$$A \leftarrow D_x f(0,0) \qquad B \leftarrow D_u f(0,0)$$

This requires:

- **smooth data** false for some problems
- **nondegenerate linearization** fails for NHI
- **small *(x,u)*** not the case in pursuit-evasion
- **all *(x,u)* near *(0,0)* available**

In applications, systems are rarely linear, yet linear systems theory is applied: **linearization**

$$x'(t) = f\big(x(t), u(t)\big) \implies x'(t) = A\,x(t) + B\,u(t)$$

$A \leftarrow D_x f(0,0)$, $B \leftarrow D_u f(0,0)$

This requires:

- **smooth data** — false for some problems
- **nondegenerate linearization** — fails for NHI
- **small *(x,u)*** — not the case in pursuit-evasion
- **all *(x,u)* near *(0,0)* available** — locally unrestricted state, control

A famous diagnostic tool for the feedback issue:

Theorem (Brockett 1983)

If $(*)$ is stabilizable by a continuous feedback k, then, for every $r > 0$, the set $f(B(0,r),U)$ contains a neighborhood of 0.

A famous diagnostic tool for the feedback issue:

Theorem (Brockett 1983)

If $(*)$ is stabilizable by a continuous feedback k, then, for every $r > 0$, the set $f(B(0,r),U)$ contains a neighborhood of 0.

For NHI (GAC and reasonable!) :

$$x_1'(t) = u_1(t)$$
$$x_2'(t) = u_2(t)$$
$$x_3'(t) = x_1(t)u_2(t) - x_2(t)u_1(t)$$

$$U = \{(u_1, u_2) : u_1^2 + u_2^2 \leq 1\}$$

$$\begin{bmatrix} 0 \\ 0 \\ \gamma \end{bmatrix} \notin f(B(0,r),U) \text{ if } \gamma \neq 0$$

so: no continuous stabilizer

A famous diagnostic tool for the feedback issue:

Theorem (Brockett 1983)

> **If $(*)$ is stabilizable by a continuous feedback k, then, for every $r > 0$, the set $f(B(0,r),U)$ contains a neighborhood of 0.**

For NHI (GAC and reasonable!) :

$$x_1'(t) = u_1(t)$$
$$x_2'(t) = u_2(t)$$
$$x_3'(t) = x_1(t)u_2(t) - x_2(t)u_1(t)$$

$$U = \{(u_1, u_2) : u_1^2 + u_2^2 \leq 1\}$$

$$\begin{bmatrix} 0 \\ 0 \\ \gamma \end{bmatrix} \notin f(B(0,r),U) \text{ if } \gamma \neq 0$$

so: no continuous stabilizer

Note: The problem cannot be "approximated away"

So we reluctantly consider discontinuous feedbacks

$$x'(t) = f\big(x(t), k(x(t))\big)$$

So we reluctantly consider discontinuous feedbacks

$$x'(t) = f\big(x(t), k(x(t))\big)$$

A natural solution concept: *sample-and-hold*

So we reluctantly consider discontinuous feedbacks

$$x'(t) = f\big(x(t), k(x(t))\big)$$

A natural solution concept: *sample-and-hold*

This means that $k(x)$ is applied on a piecewise-constant basis between sampling moments:

$$x'(t) \;=\; f(x(t), k(x_i)), \quad \text{for } t_i \le t \le t_{i+1}$$

relative to a given partition $\{t_i\}$ of the underlying interval

So we reluctantly consider discontinuous feedbacks

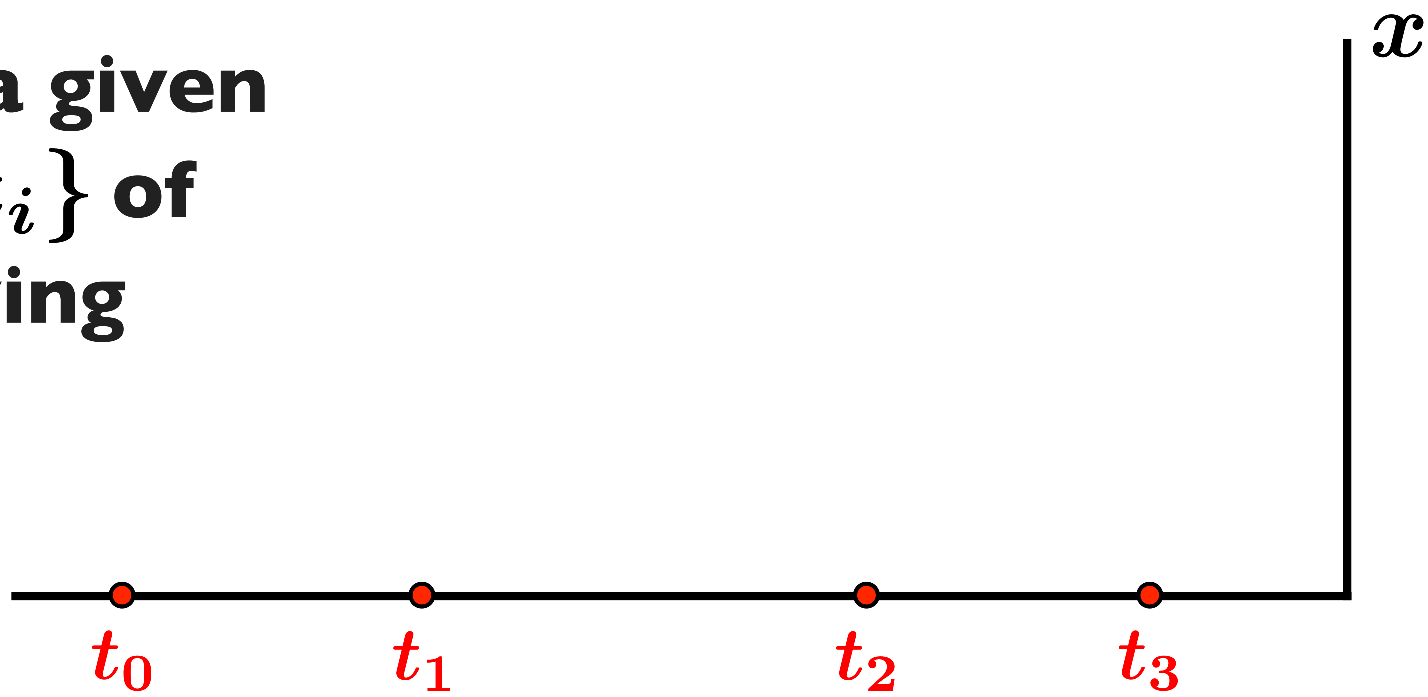
$$x'(t) = f\big(x(t), k(x(t))\big)$$

A natural solution concept: *sample-and-hold*

This means that $k(x)$ is applied on a piecewise-constant basis between sampling moments:

$$x'(t) \;=\; f(x(t), k(x_i)), \quad \text{for } t_i \le t \le t_{i+1}$$

relative to a given partition $\{t_i\}$ of the underlying interval



So we reluctantly consider discontinuous feedbacks

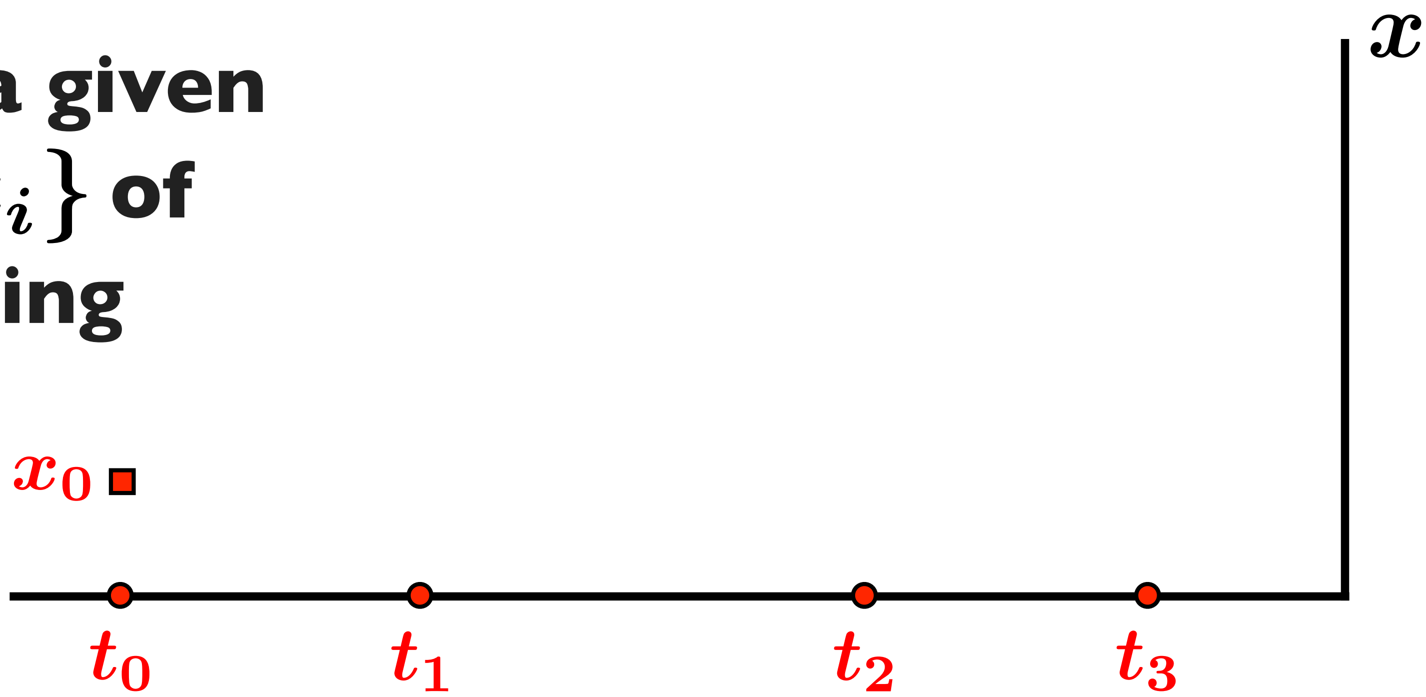
$$x'(t) = f\big(x(t), k(x(t))\big)$$

A natural solution concept: *sample-and-hold*

This means that $k(x)$ is applied on a piecewise-constant basis between sampling moments:

$$x'(t) \;=\; f(x(t), k(x_i)), \quad \text{for } t_i \le t \le t_{i+1}$$

relative to a given partition $\{t_i\}$ of the underlying interval



So we reluctantly consider discontinuous feedbacks

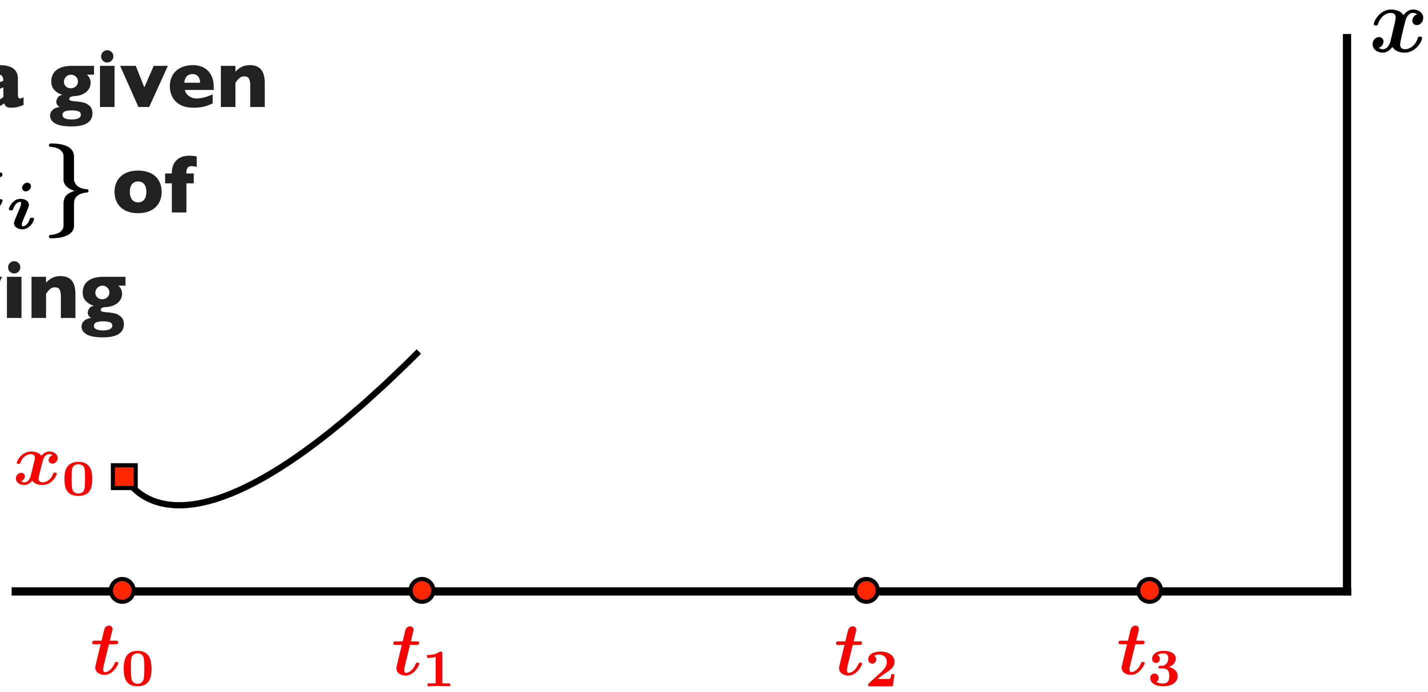
$$x'(t) = f(x(t), k(x(t)))$$

A natural solution concept: *sample-and-hold*

This means that $k(x)$ is applied on a piecewise-constant basis between sampling moments:

$$x'(t) \;=\; f(x(t), k(x_i)), \quad \text{for } t_i \le t \le t_{i+1}$$

relative to a given partition $\{t_i\}$ of the underlying interval



So we reluctantly consider discontinuous feedbacks

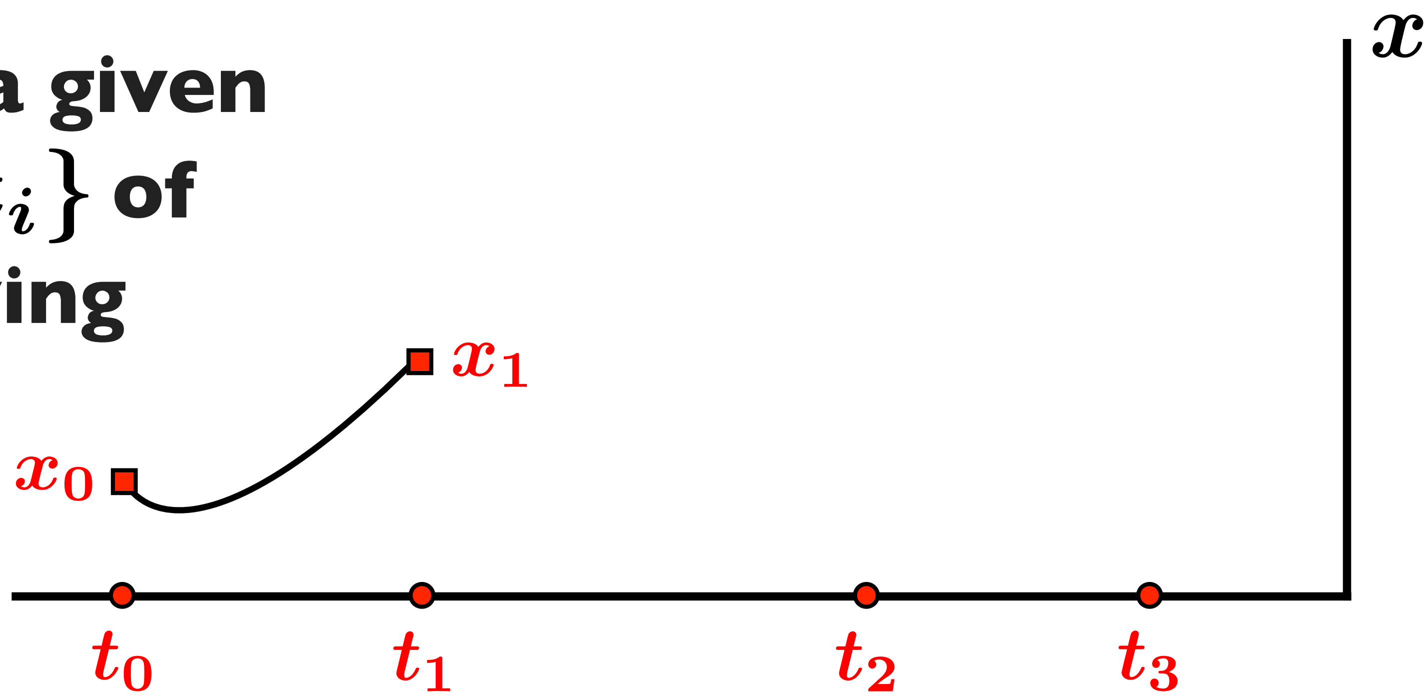
$$x'(t) = f\big(x(t), k(x(t))\big)$$

A natural solution concept: *sample-and-hold*

This means that $k(x)$ is applied on a piecewise-constant basis between sampling moments:

$$x'(t) \;=\; f(x(t), k(x_i)), \quad \text{for } t_i \le t \le t_{i+1}$$

relative to a given partition $\{t_i\}$ of the underlying interval



So we reluctantly consider discontinuous feedbacks

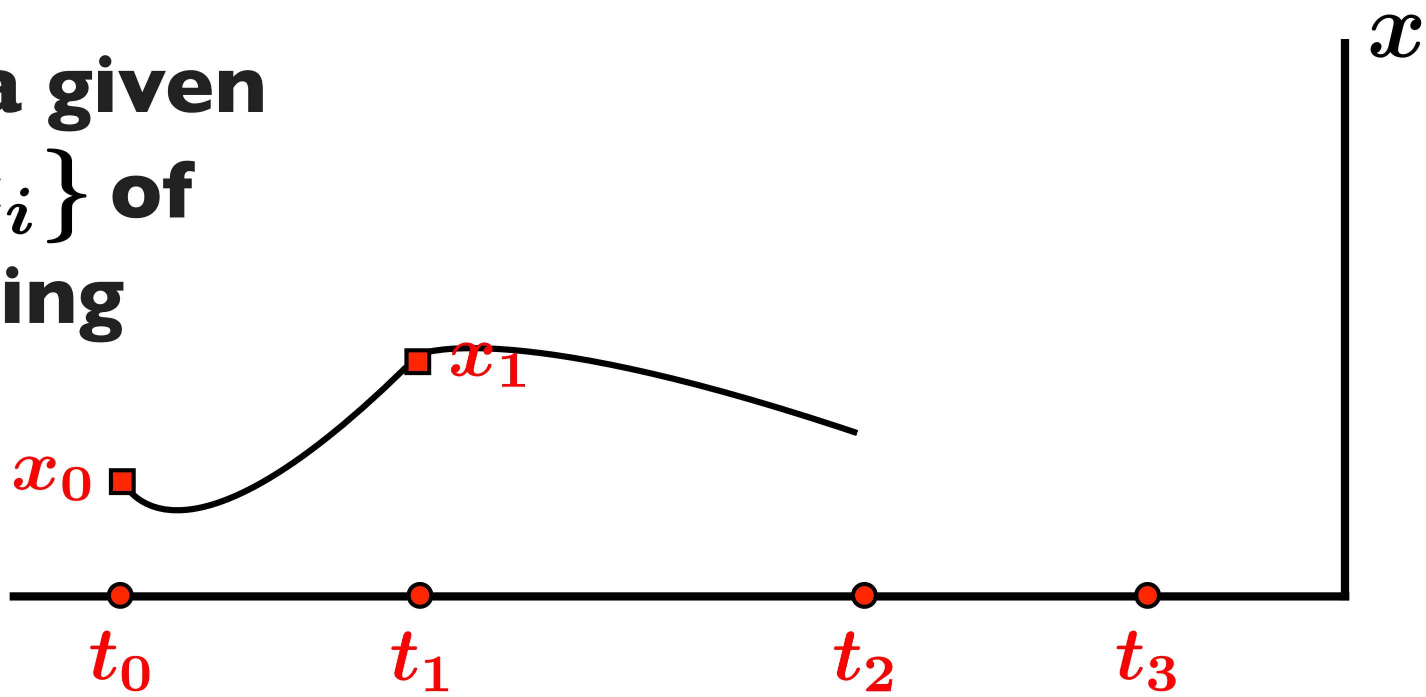
$$x'(t) = f\big(x(t), k(x(t))\big)$$

A natural solution concept: *sample-and-hold*

This means that $k(x)$ is applied on a piecewise-constant basis between sampling moments:

$$x'(t) \;=\; f(x(t), k(x_i)), \quad \text{for } t_i \leq t \leq t_{i+1}$$

relative to a given partition $\{t_i\}$ of the underlying interval



So we reluctantly consider discontinuous feedbacks

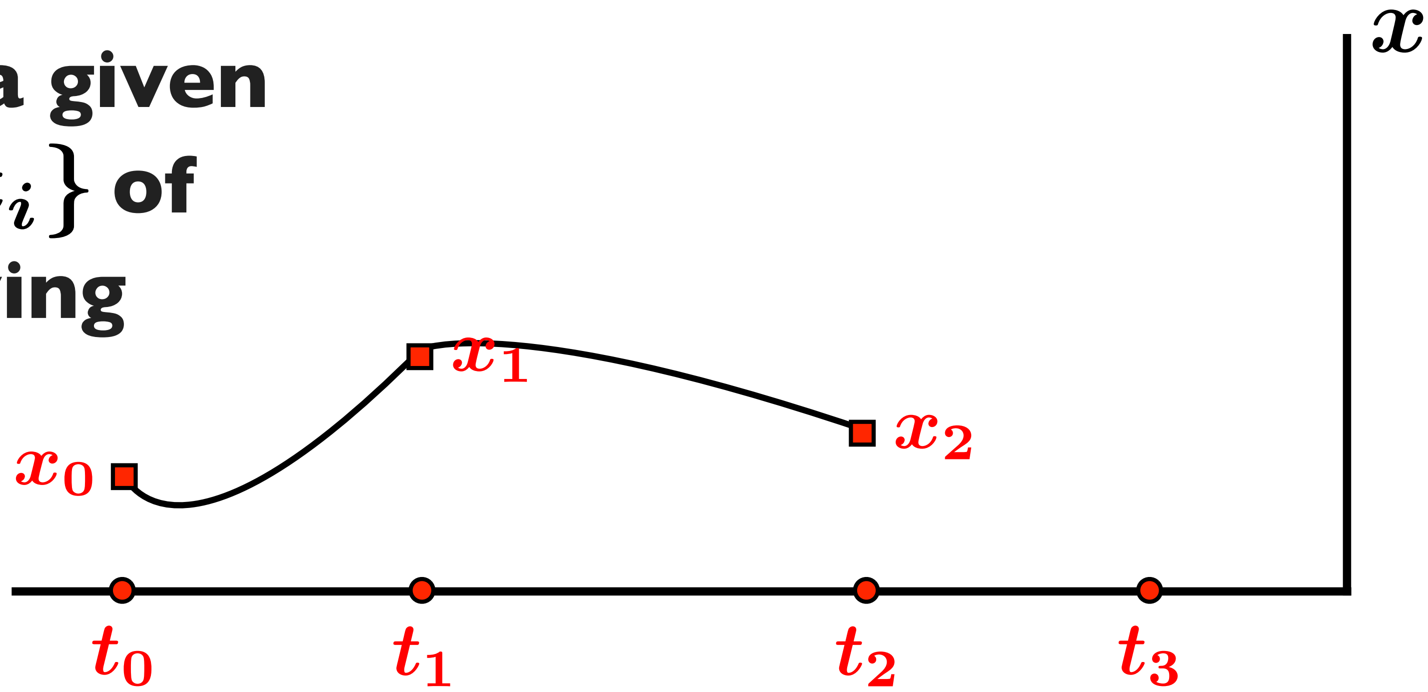
$$x'(t) = f\big(x(t), k(x(t))\big)$$

A natural solution concept: *sample-and-hold*

This means that $k(x)$ is applied on a piecewise-constant basis between sampling moments:

$$x'(t) \;=\; f(x(t), k(x_i)), \quad \text{for } t_i \le t \le t_{i+1}$$

relative to a given partition $\{t_i\}$ of the underlying interval



So we reluctantly consider discontinuous feedbacks

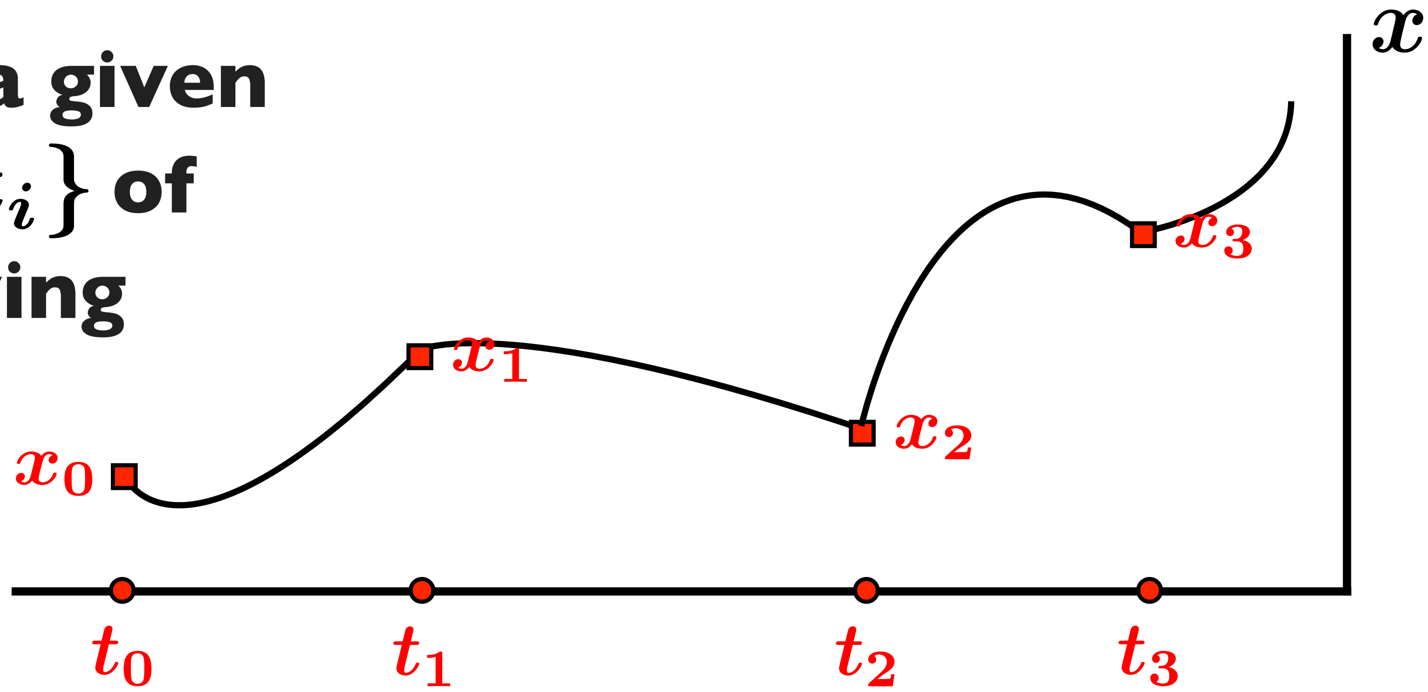
$$x'(t) = f\big(x(t), k(x(t))\big)$$

A natural solution concept: *sample-and-hold*

This means that $k(x)$ is applied on a piecewise-constant basis between sampling moments:

$$x'(t) \;=\; f(x(t), k(x_i)), \quad \text{for } t_i \le t \le t_{i+1}$$

relative to a given partition $\{t_i\}$ of the underlying interval



We say that k stabilizes (in the s&h sense) if:

Given B(0,R) and B(0,r), then with sufficiently fine partitions, k drives all points in B(0,R) to B(0,r)

We say that k stabilizes (in the s&h sense) if:

Given B(0,R) and B(0,r), then with sufficiently fine partitions, k drives all points in B(0,R) to B(0,r)

Theorem (Clarke, Ledyaev, Sontag, Subbotin 1997)

Any GAC system is stabilizable, with possibly discontinuous feedback, implemented in the sample-and-hold sense. (The converse is evident)

We say that k stabilizes (in the s&h sense) if:

Given B(0,R) and B(0,r), then with sufficiently fine partitions, k drives all points in B(0,R) to B(0,r)

Theorem (Clarke, Ledyaev, Sontag, Subbotin 1997)

Any GAC system is stabilizable, with possibly discontinuous feedback, implemented in the sample-and-hold sense. (The converse is evident)

Remarks:

-
-
-

We say that k stabilizes (in the s&h sense) if:

Given B(0,R) and B(0,r), then with sufficiently fine partitions, k drives all points in B(0,R) to B(0,r)

Theorem (Clarke, Ledyaev, Sontag, Subbotin 1997)

> **Any GAC system is stabilizable, with possibly discontinuous feedback, implemented in the sample-and-hold sense.** (The converse is evident)

Remarks:

- The s&h stabilization is "meaningful"...
-
-

We say that k stabilizes (in the s&h sense) if:

Given B(0,R) and B(0,r), then with sufficiently fine partitions, k drives all points in B(0,R) to B(0,r)

Theorem (Clarke, Ledyaev, Sontag, Subbotin 1997)

> **Any GAC system is stabilizable, with possibly discontinuous feedback, implemented in the sample-and-hold sense.** (The converse is evident)

Remarks:

- The s&h stabilization is "meaningful"...
- There is robustness with respect to implementation, as well as small error; such analysis becomes possible
-

We say that k stabilizes (in the s&h sense) if:

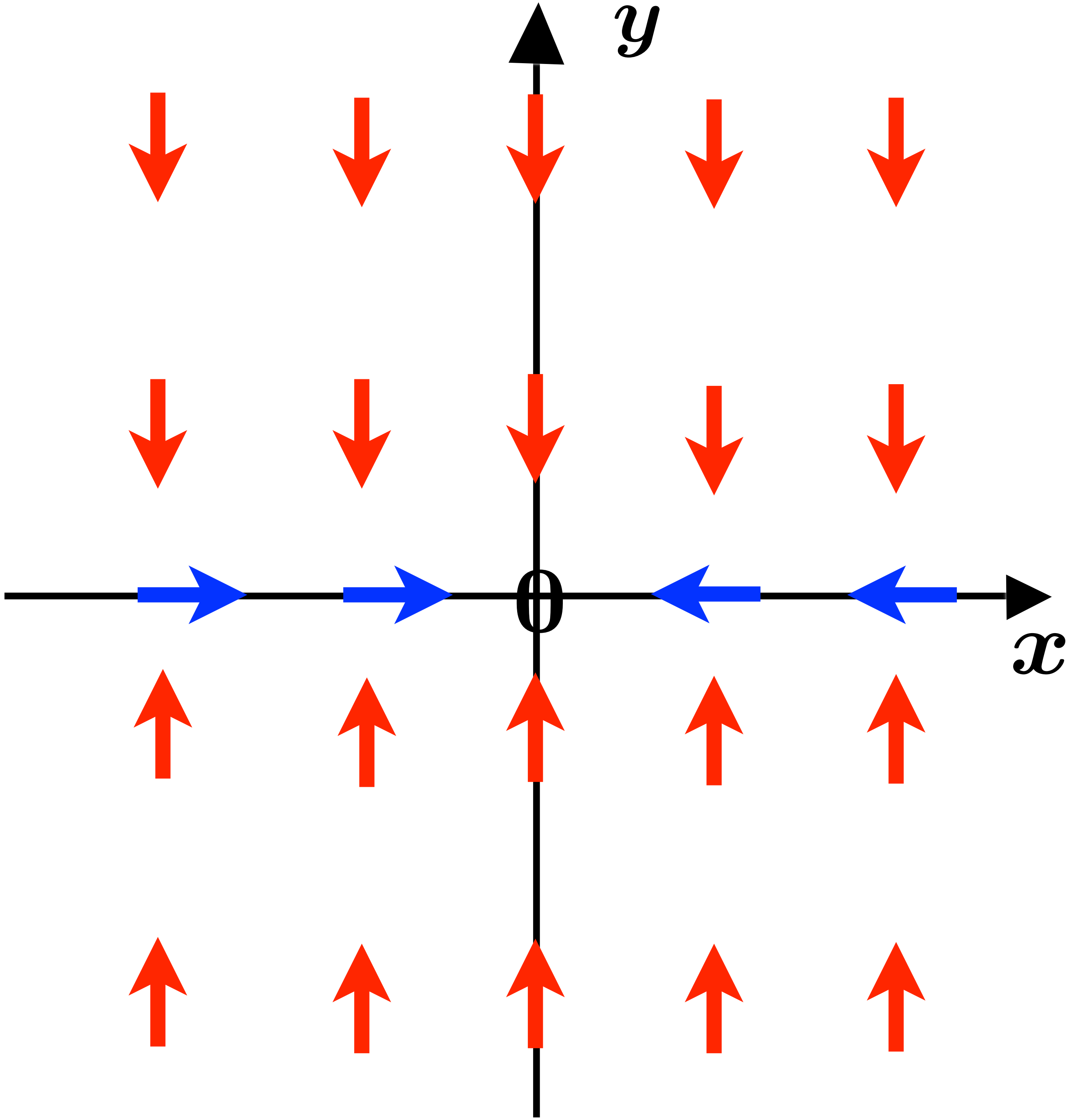
Given B(0,R) and B(0,r), then with sufficiently fine partitions, k drives all points in B(0,R) to B(0,r)

Theorem (Clarke, Ledyaev, Sontag, Subbotin 1997)

Any GAC system is stabilizable, with possibly discontinuous feedback, implemented in the sample-and-hold sense. (The converse is evident)

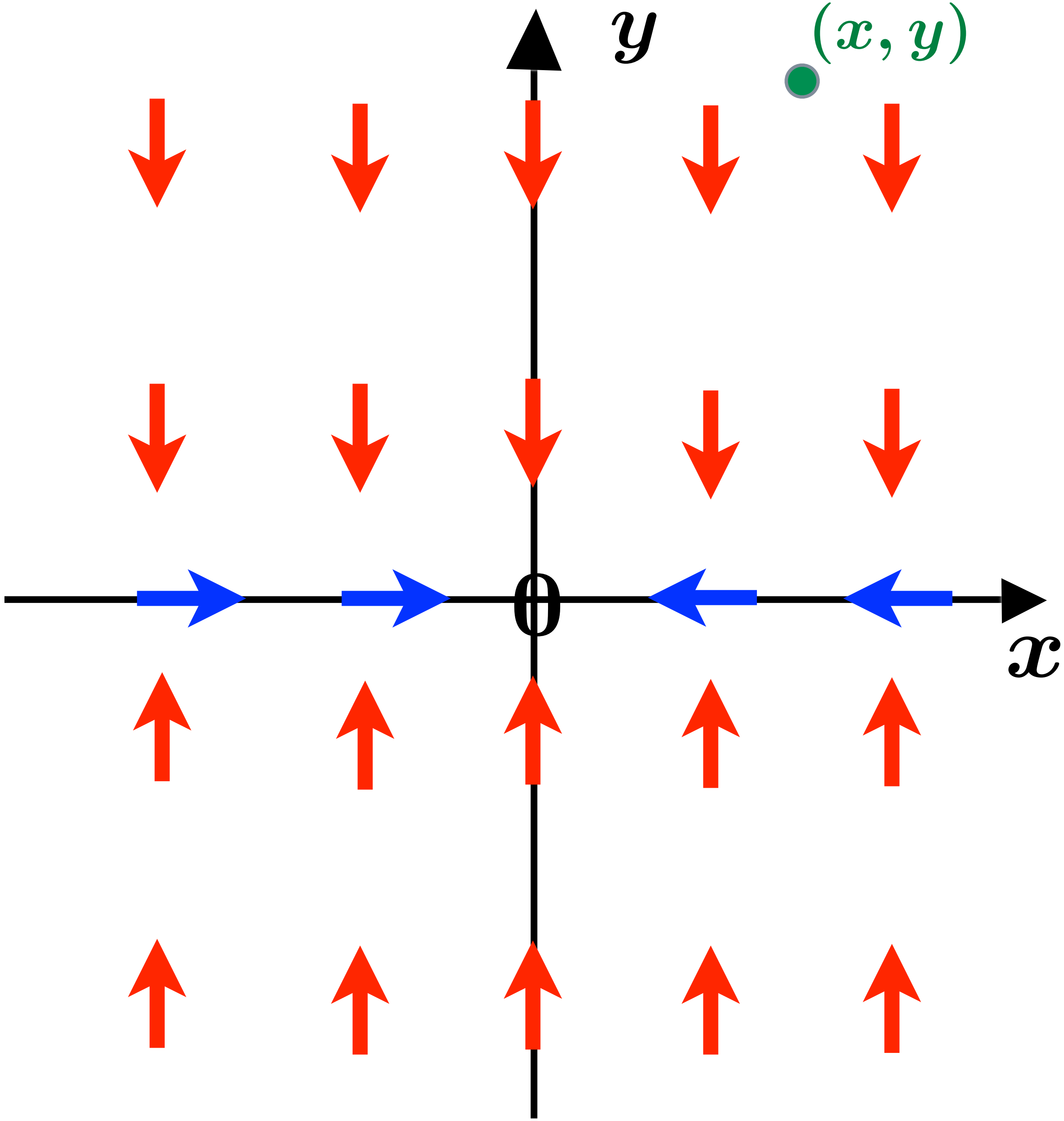
Remarks:

- The s&h stabilization is "meaningful"...
- There is robustness with respect to implementation, as well as small error; such analysis becomes possible
- Filippov solutions don't work.



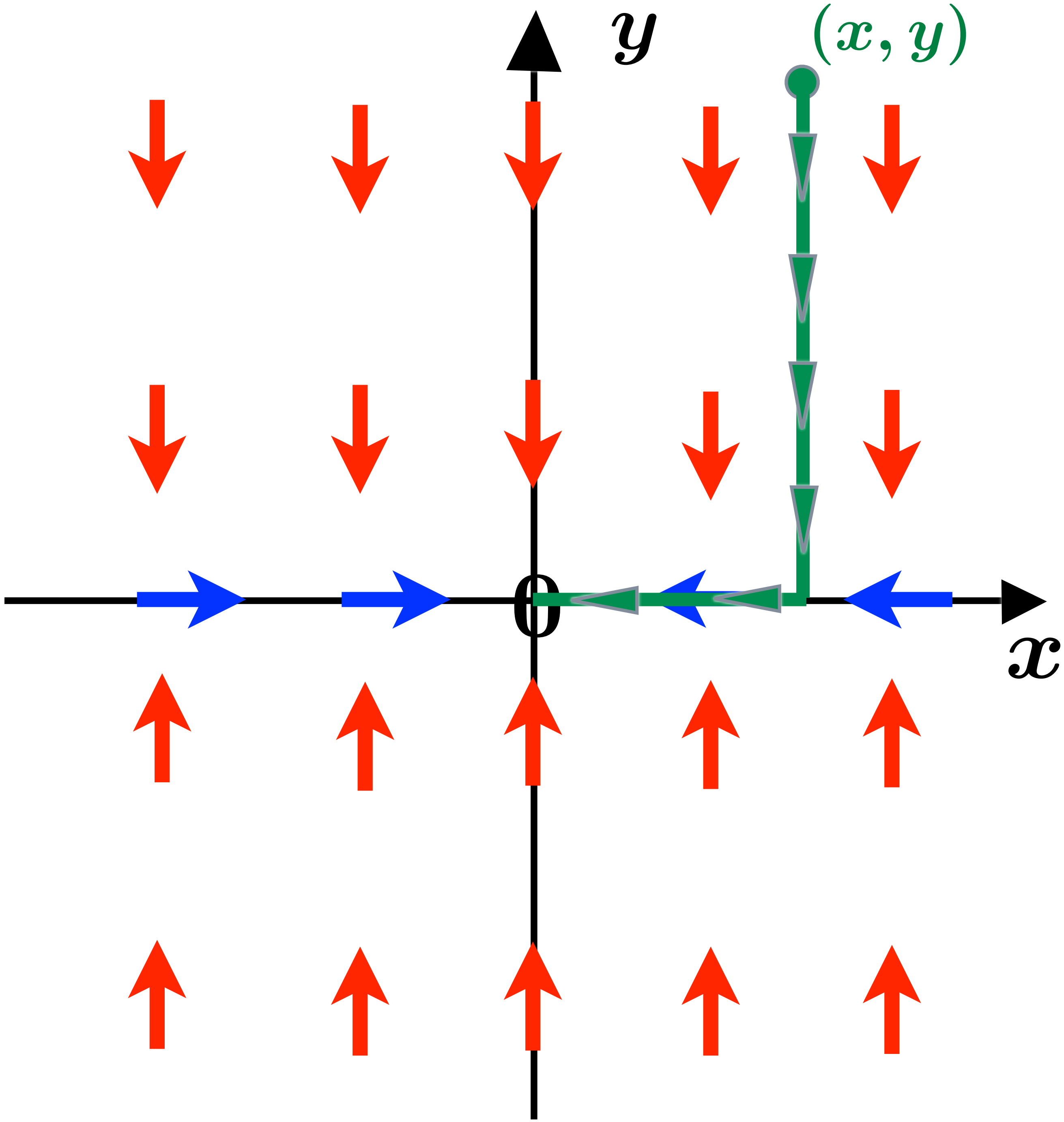
From (x,y):

1. Go directly to the x-axis
2. Then go directly to the origin



From (x,y):

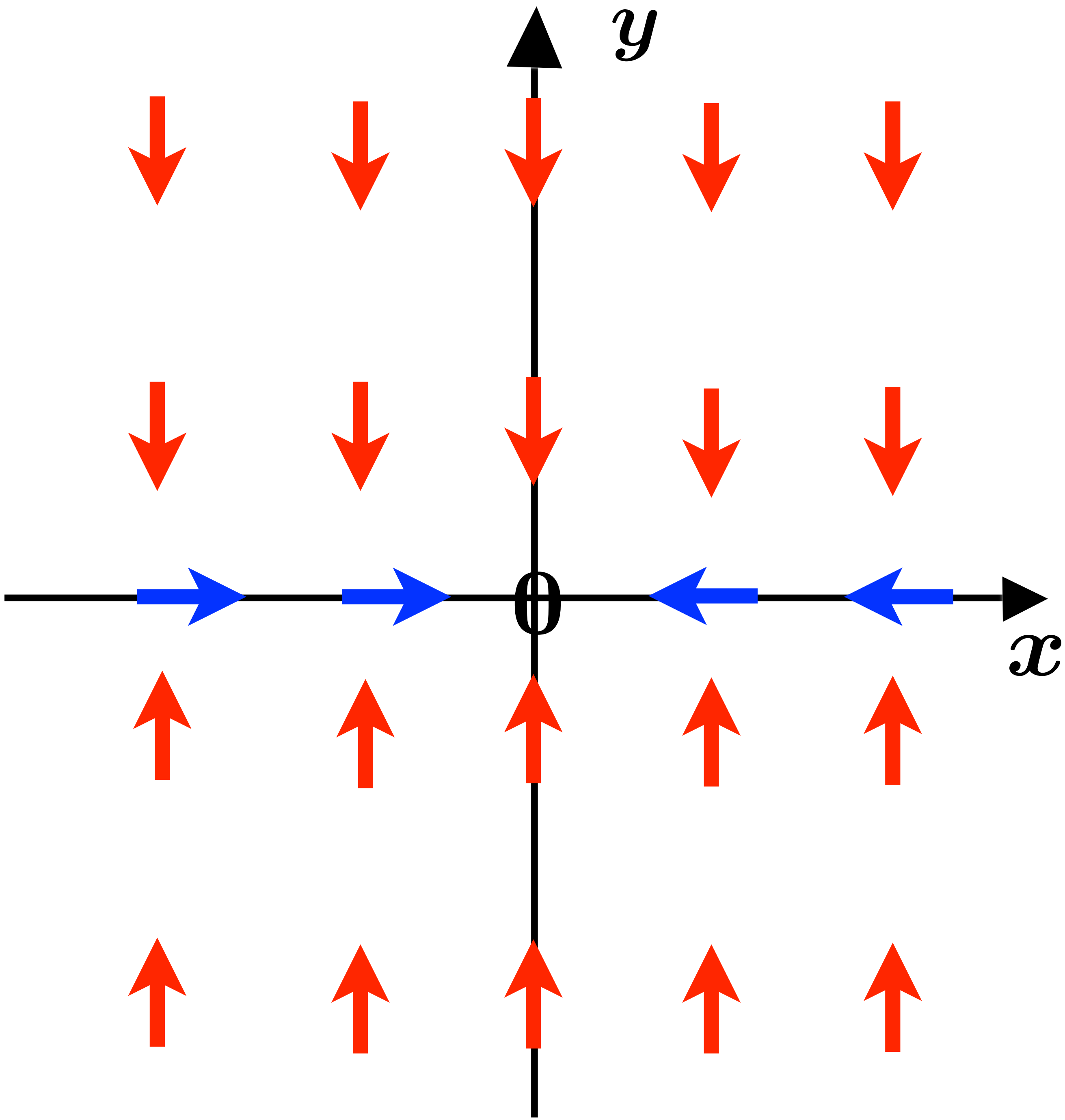
1. Go directly to the x-axis
2. Then go directly to the origin



From (x,y):

1. Go directly to the x-axis
2. Then go directly to the origin

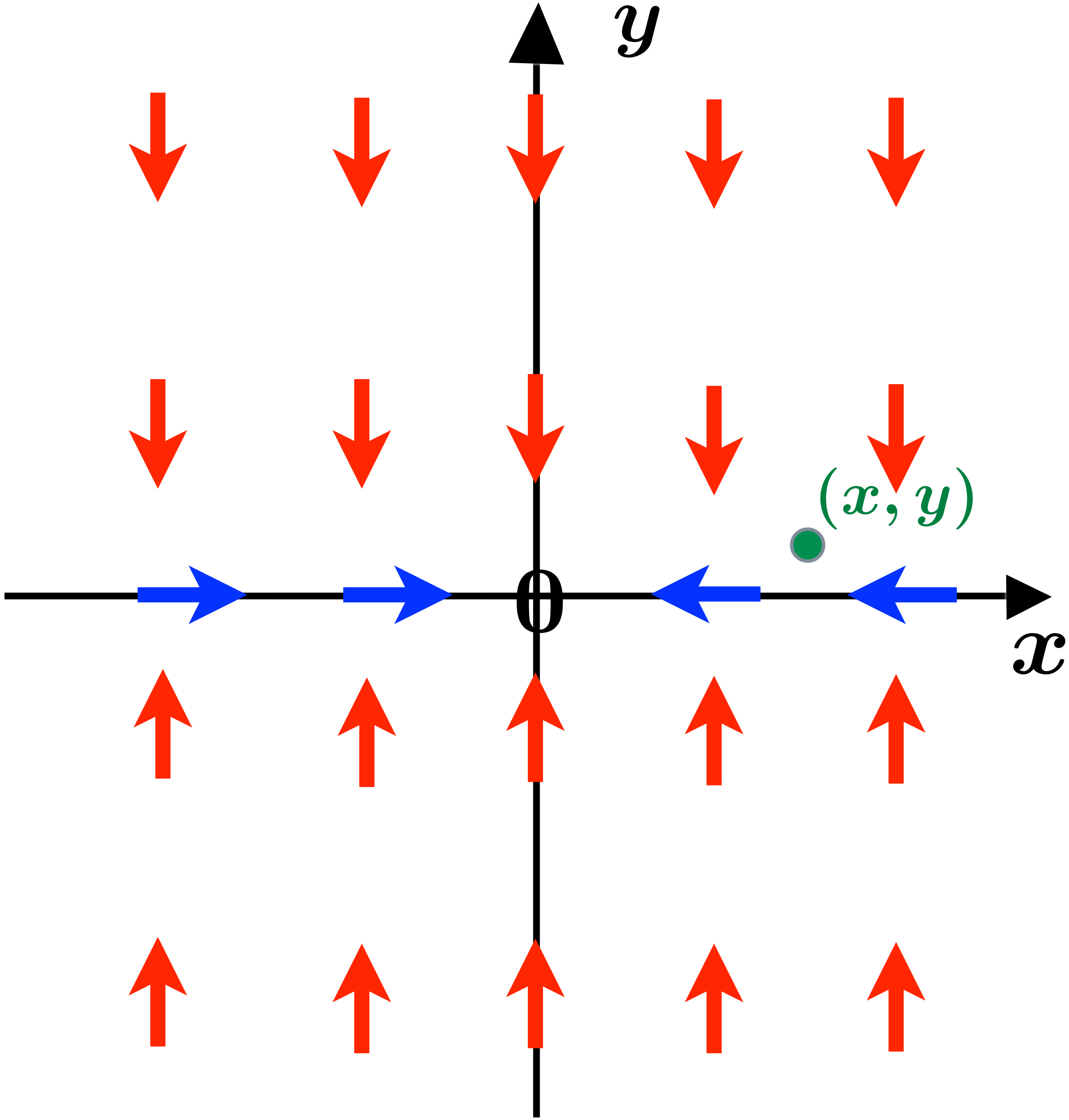
This defines a feedback, but a meaningless one



From (x,y):

1. Go directly to the x-axis
2. Then go directly to the origin

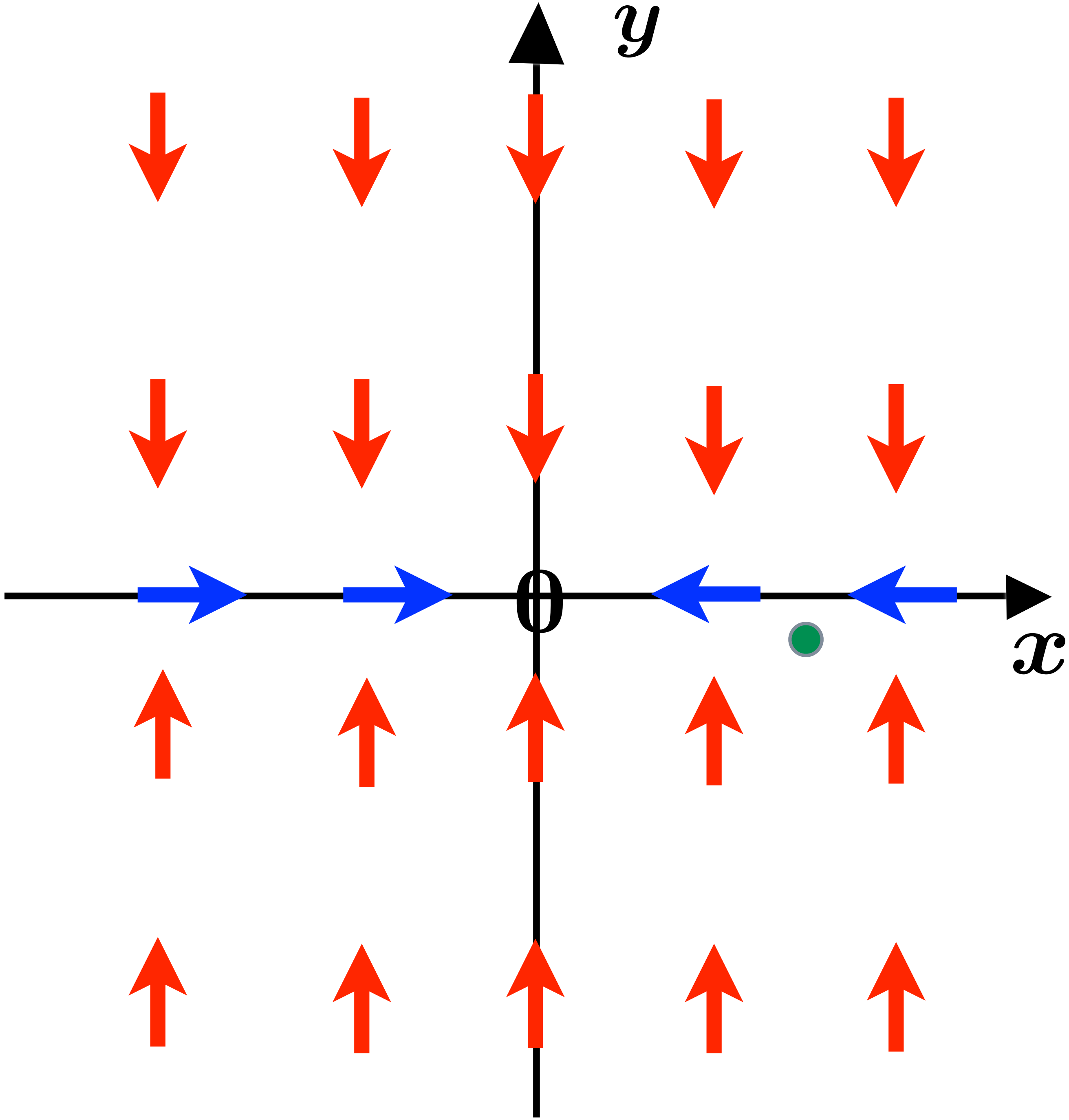
This defines a feedback, but a meaningless one



From (x,y):

1. Go directly to the x-axis
2. Then go directly to the origin

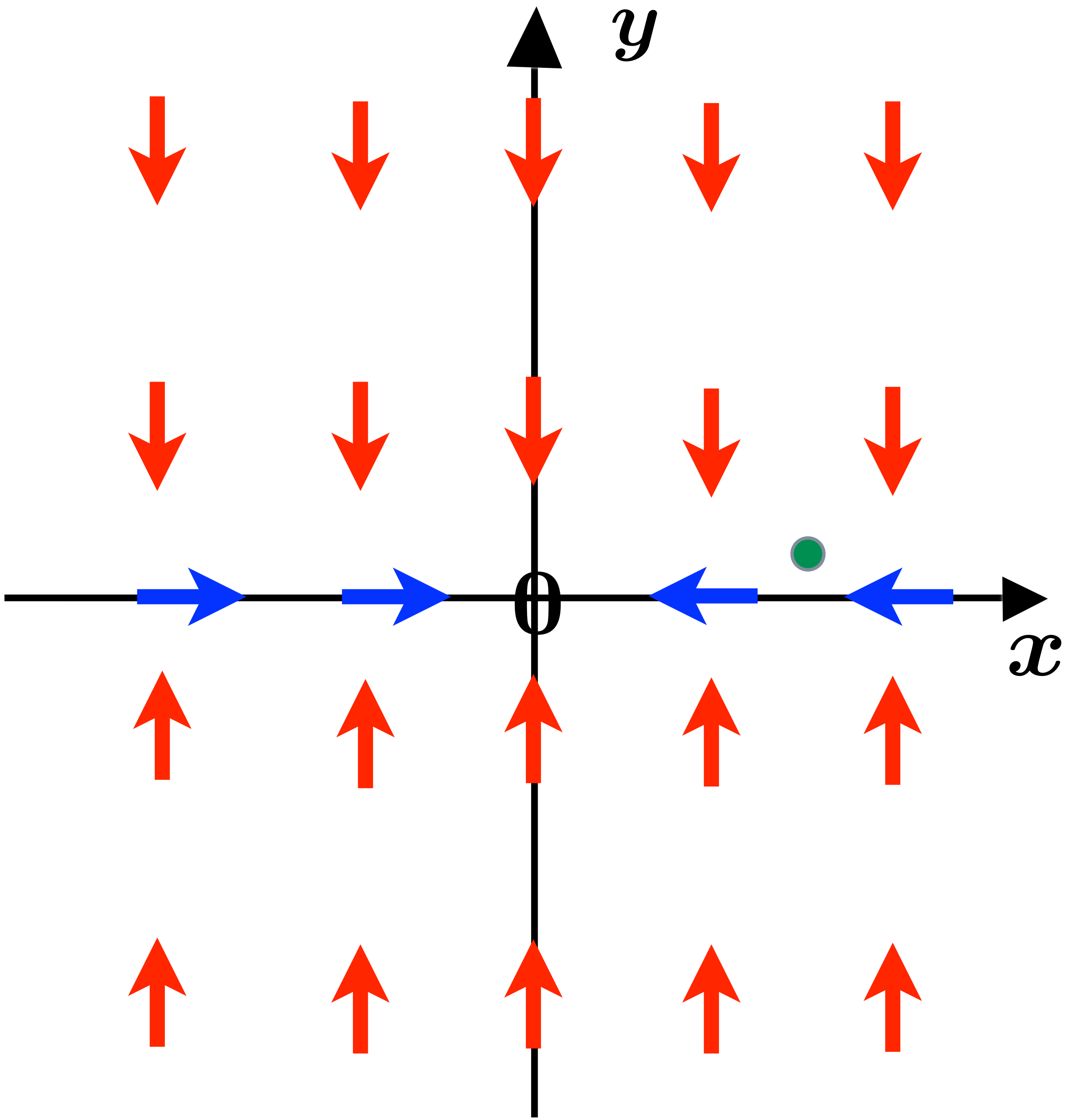
This defines a feedback, but a meaningless one



From (x,y):

1. Go directly to the x-axis
2. Then go directly to the origin

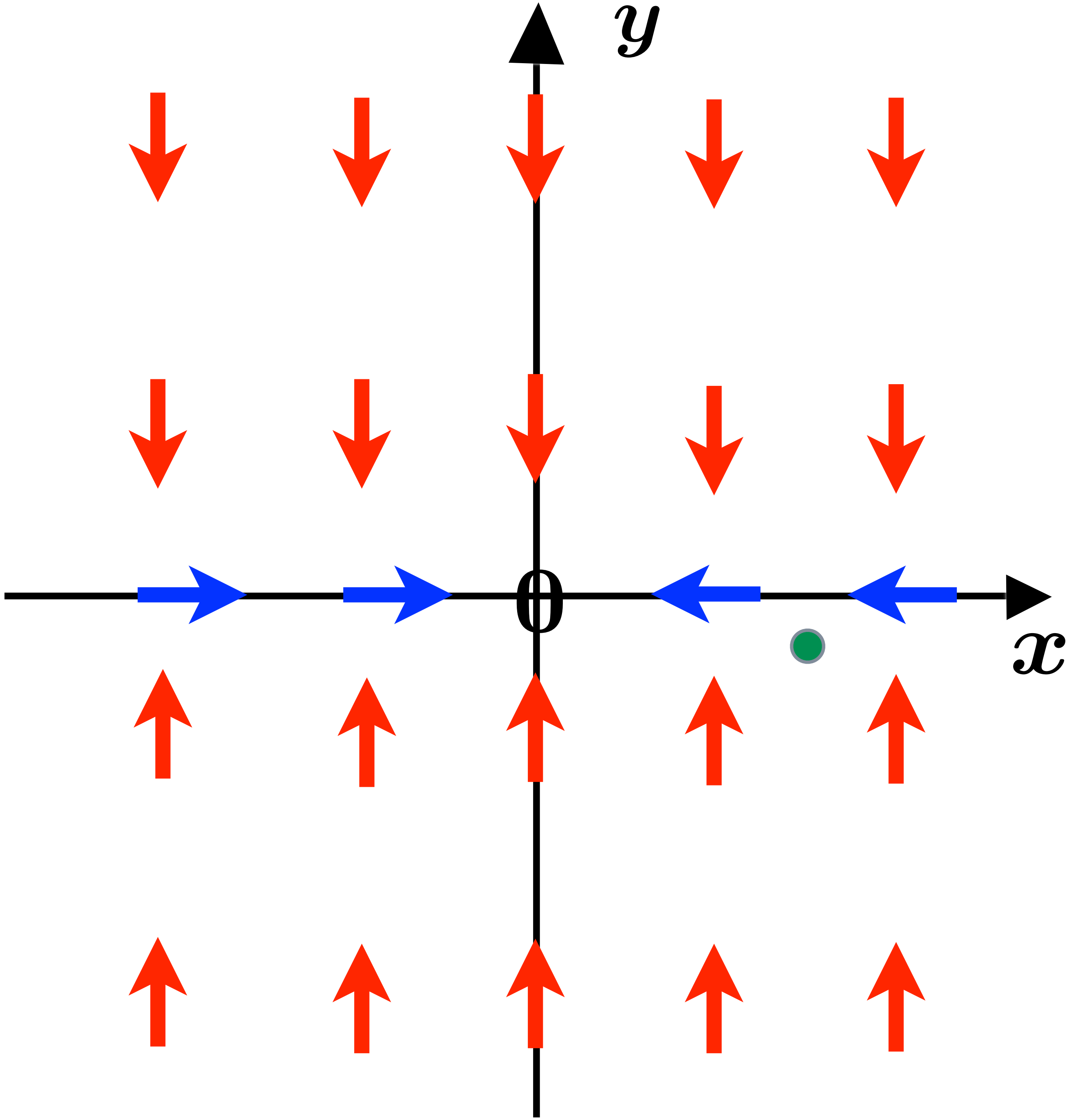
This defines a feedback, but a meaningless one



From (x,y):

1. Go directly to the x-axis
2. Then go directly to the origin

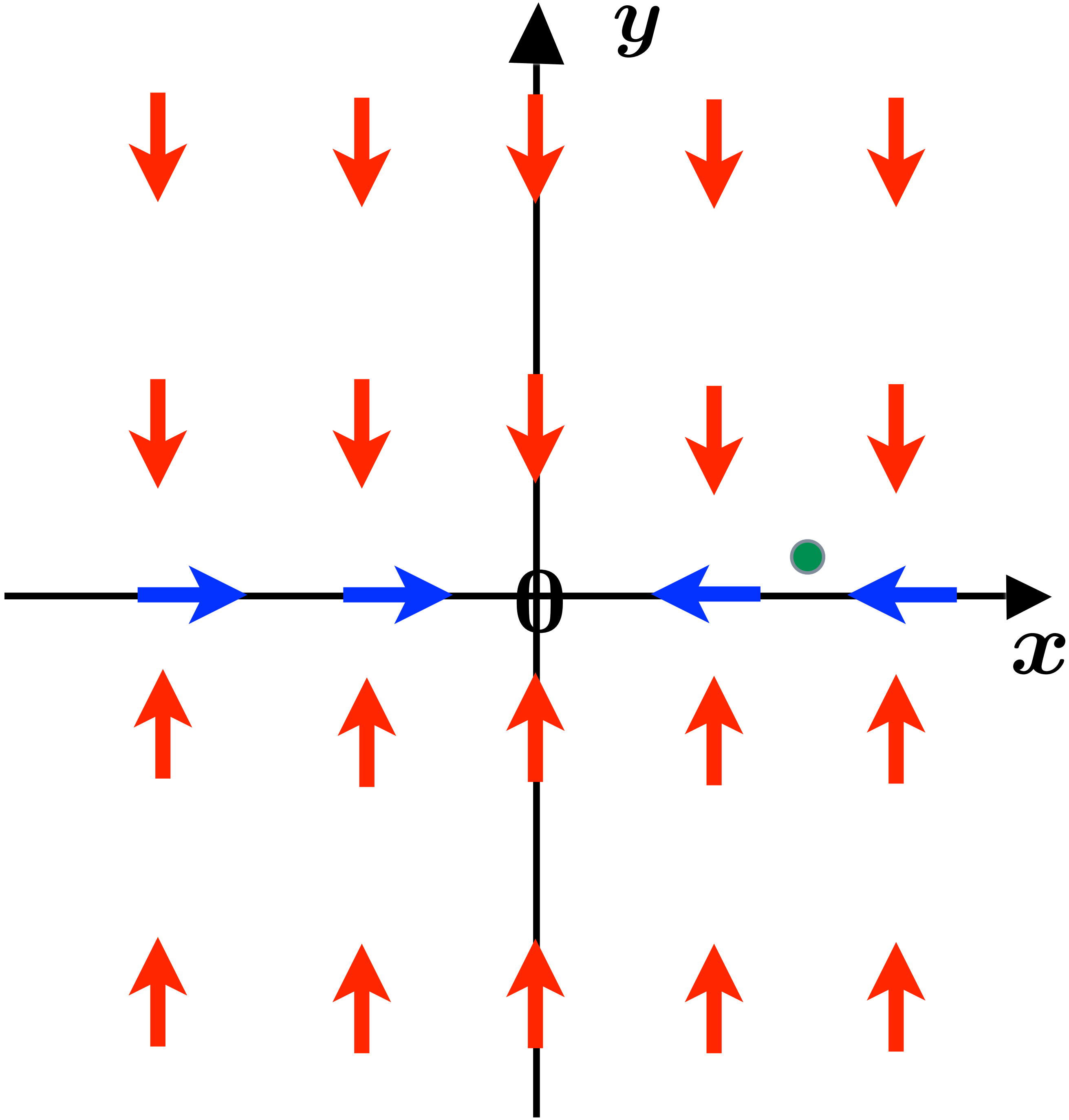
This defines a feedback, but a meaningless one



From (x,y):

1. Go directly to the x-axis
2. Then go directly to the origin

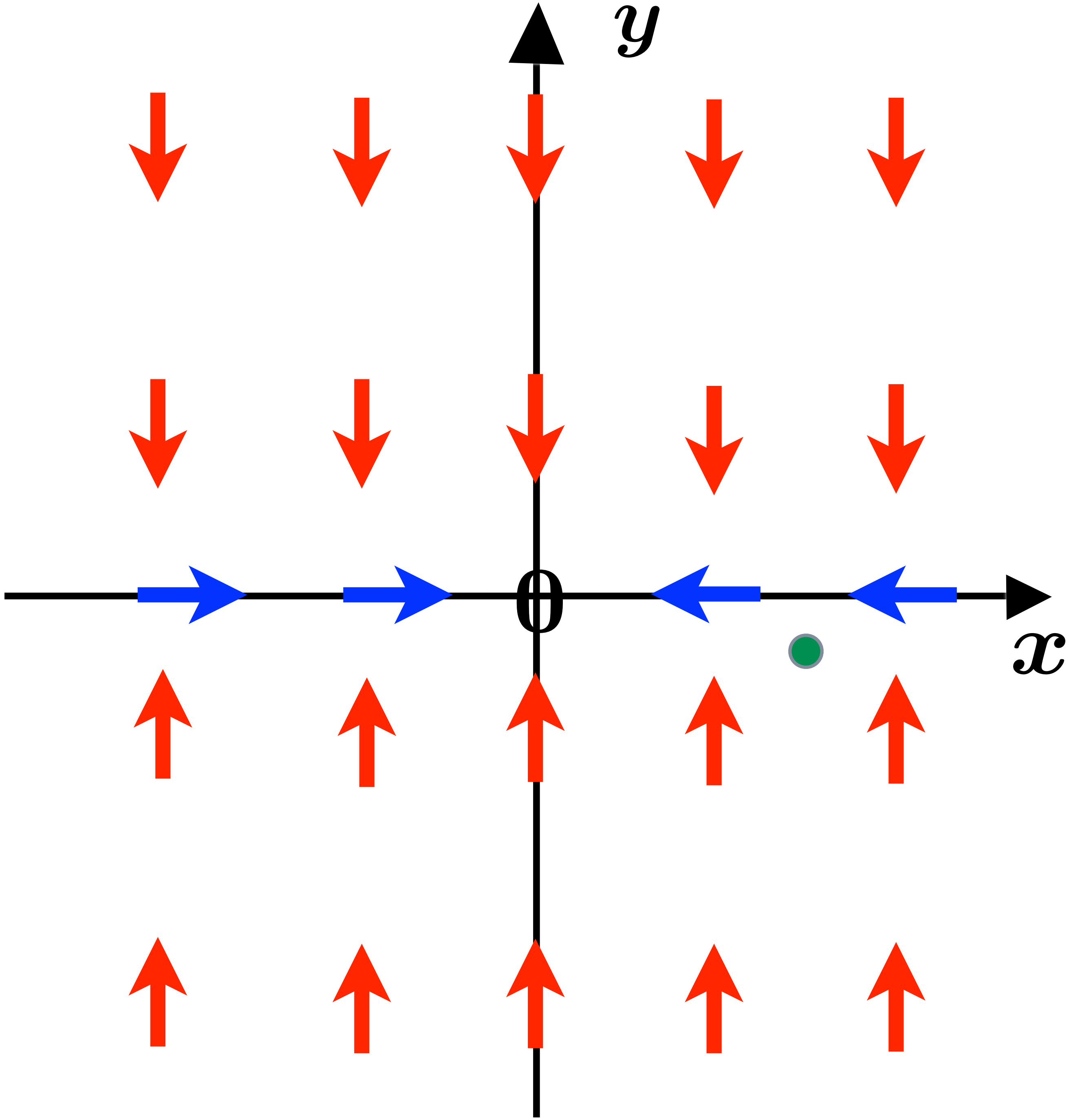
This defines a feedback, but a meaningless one



From (x,y):

1. Go directly to the x-axis
2. Then go directly to the origin

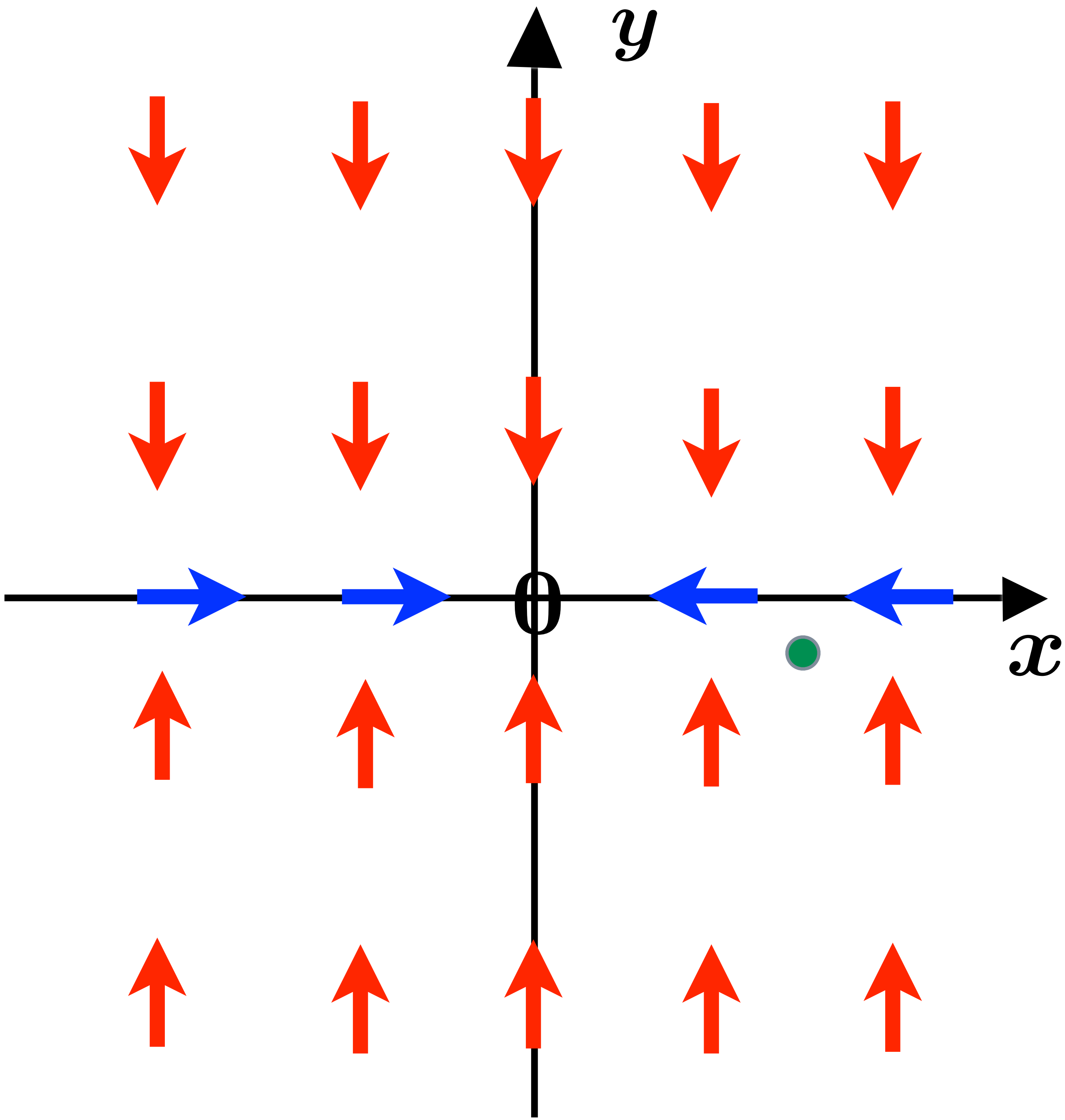
This defines a feedback, but a meaningless one



From (x,y):

1. Go directly to the x-axis
2. Then go directly to the origin

This defines a feedback, but a meaningless one



From (x,y):

1. Go directly to the x-axis

2. Then go directly to the origin

DITHER

This defines a feedback, but a meaningless one

This is an example of the thin set fallacy:

a feedback whose effect depends on its values on a set of measure zero

This is an example of the thin set fallacy:

a feedback whose effect depends on its values on a set of measure zero

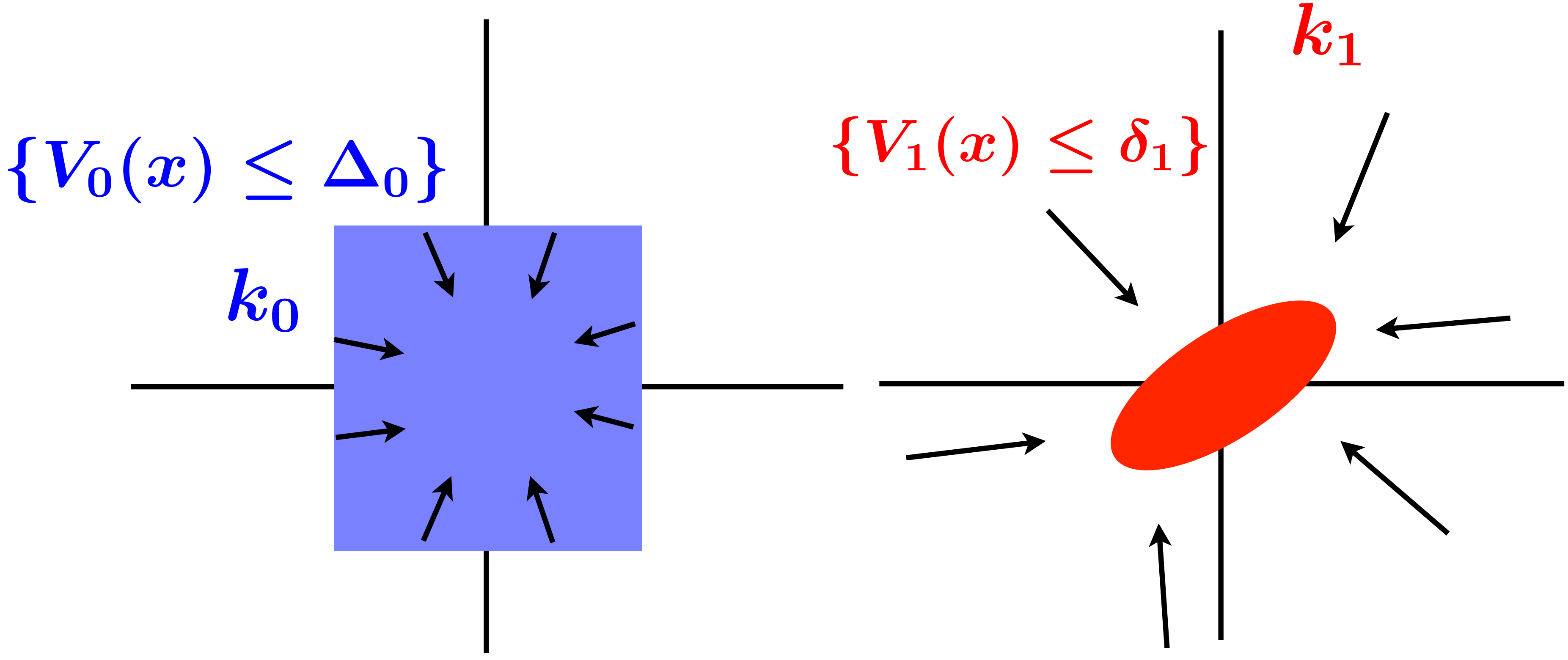
Lesson: In using discontinuous feedback, take account from the beginning of the implementation procedure.
Sample-and-hold forces one to do so.

This is an example of the **thin set fallacy:**

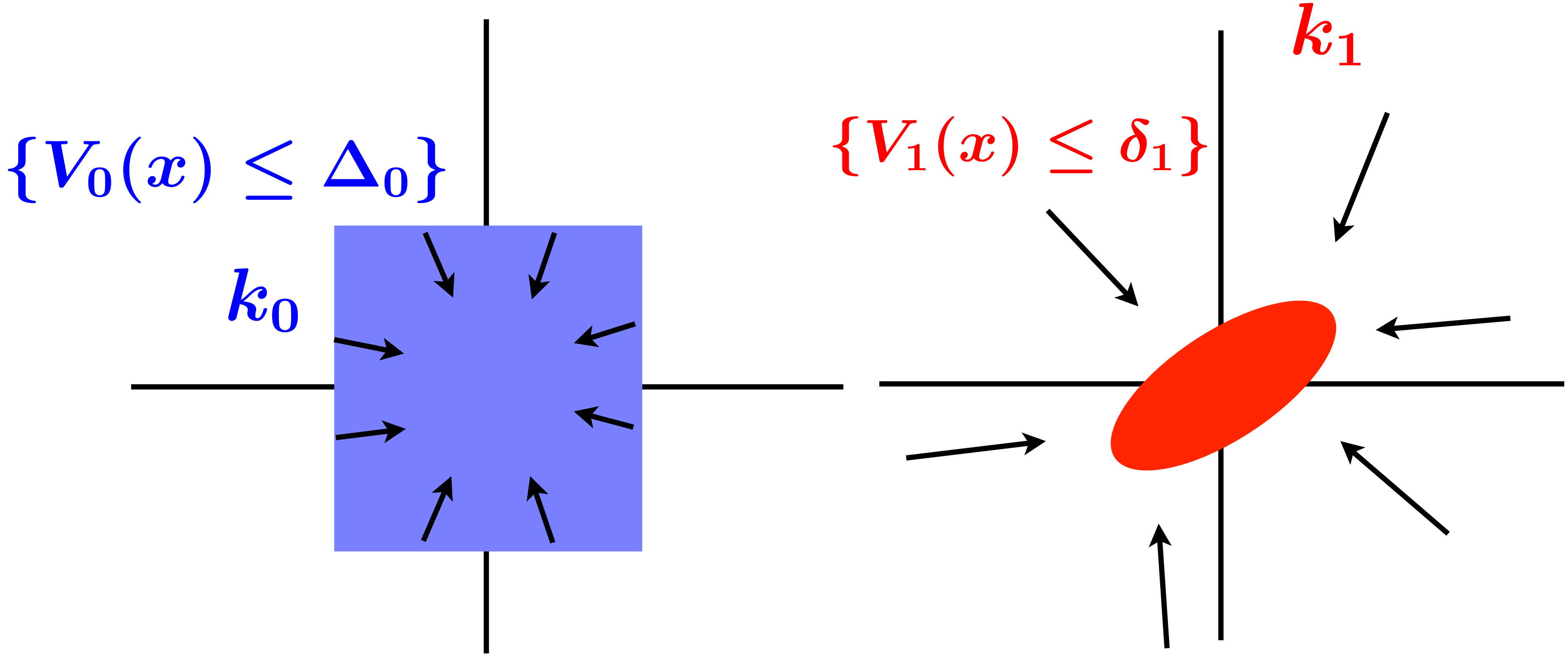
a feedback whose effect depends on its values on a set of measure zero

Lesson: **In using discontinuous feedback, take account from the beginning of the implementation procedure. Sample-and-hold forces one to do so.**

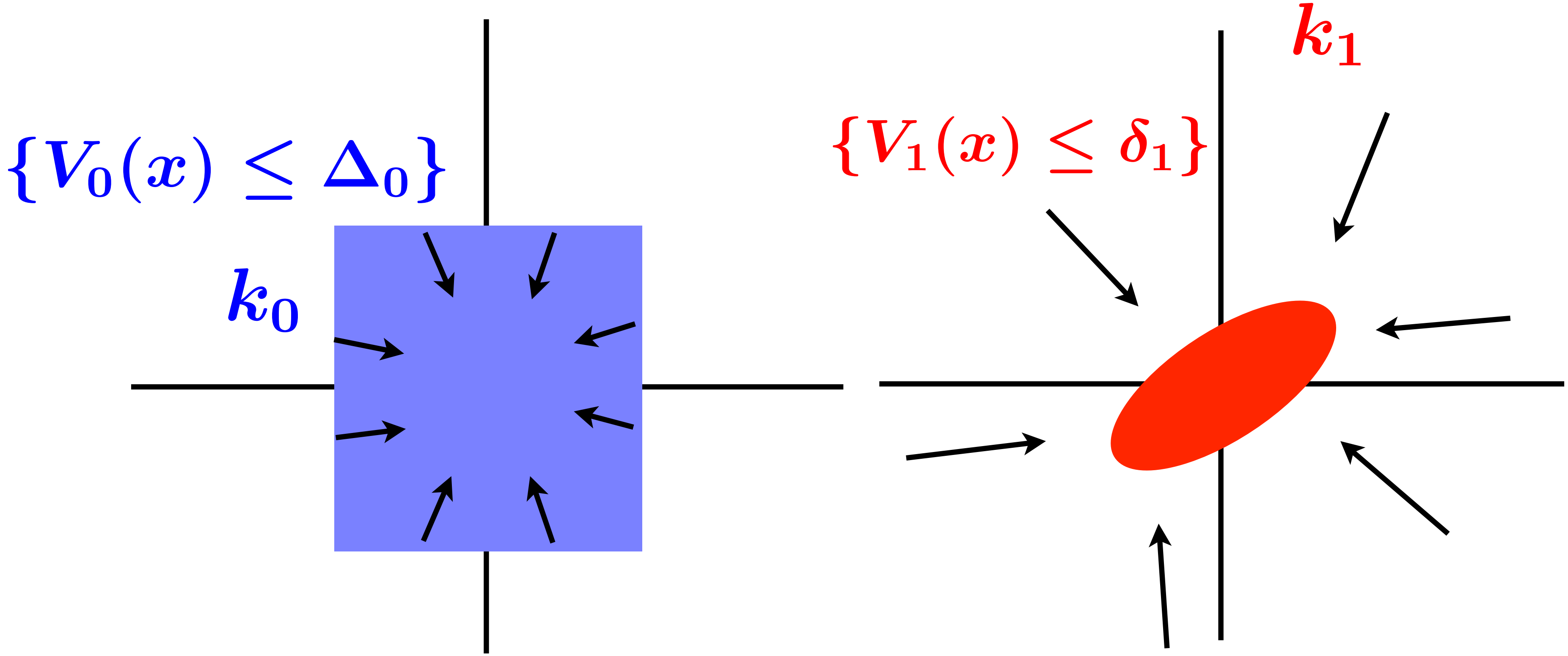
This issue does not arise with continuous feedbacks. So discontinuous feedbacks must be designed with extra care. But they also have some advantages (such as blending, sliding).



$\{V_0(x) \leq \Delta_0\}$
k_0
k_1
$\{V_1(x) \leq \delta_1\}$

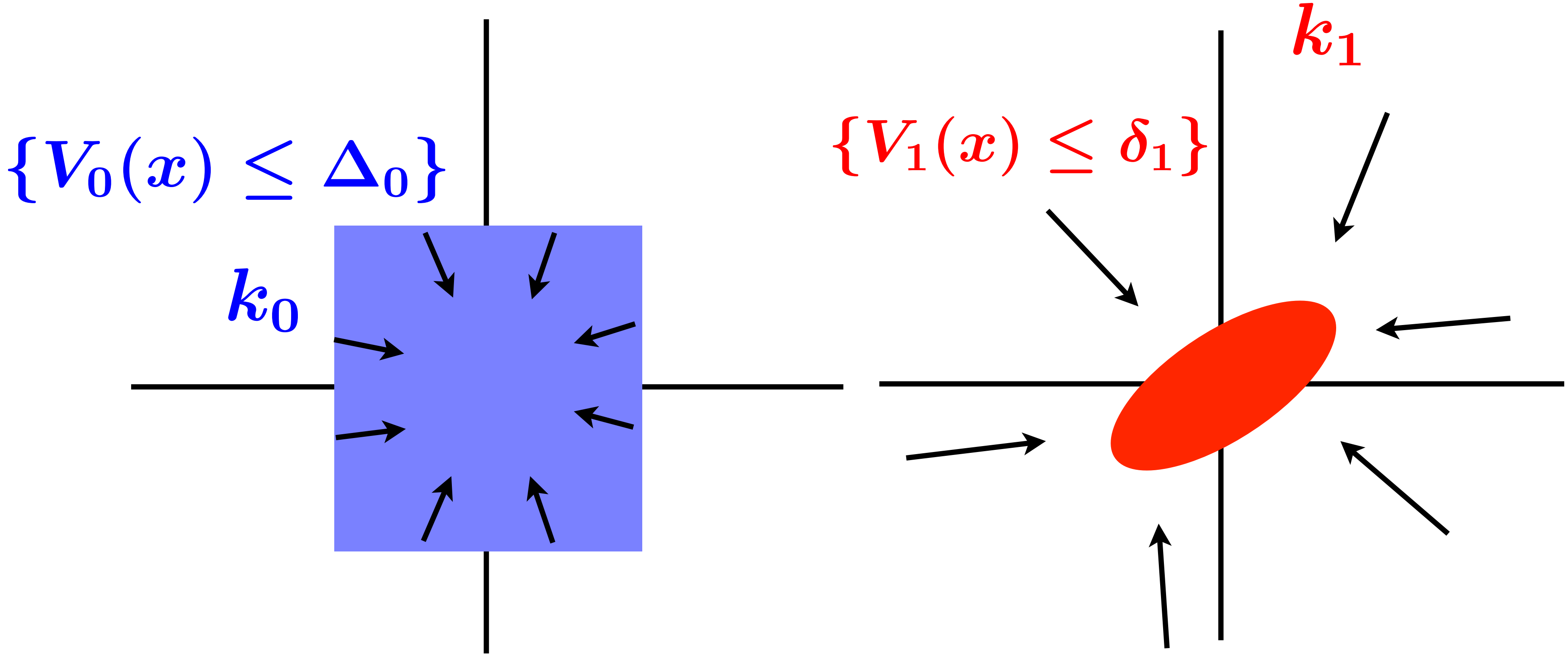


Goal: combine, or "blend", the two feedbacks



Goal: combine, or "blend", the two feedbacks

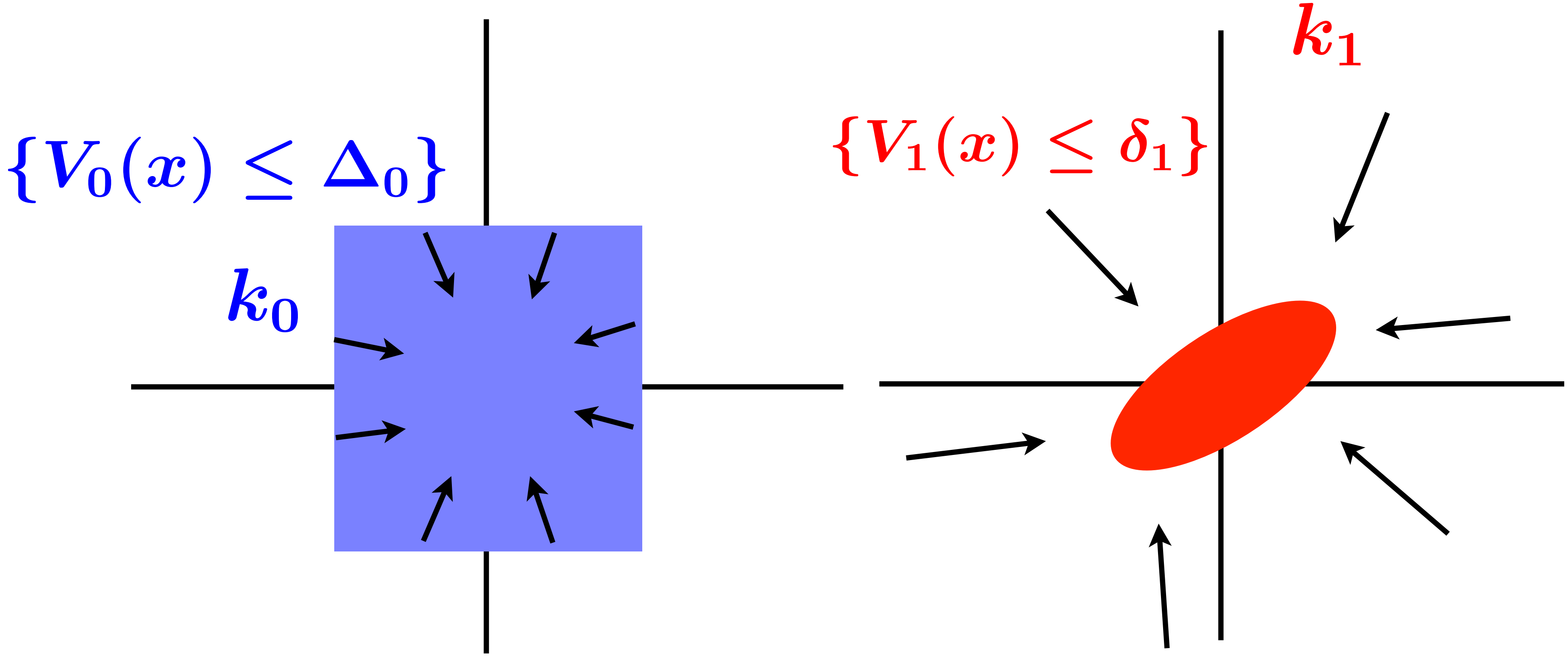
(We require overlap: $\{V_1(x) \leq \delta_1\} \subset \{V_0(x) \leq \Delta_0\}$)



Goal: combine, or "blend", the two feedbacks

(We require overlap: $\{V_1(x) \leq \delta_1\} \subset \{V_0(x) \leq \Delta_0\}$)

If continuity is **not** an issue, then we can switch, in sample-and-hold, from k_0 to k_1.



Goal: combine, or "blend", the two feedbacks

(We require overlap: $\{V_1(x) \leq \delta_1\} \subset \{V_0(x) \leq \Delta_0\}$)

If continuity is not an issue, then we can switch, in sample-and-hold, from k_0 to k_1.
Or, we can first blend V_0 and V_1 by taking "lower envelopes"...

Q: How to construct meaningful stabilizing feedbacks?

Q: How to construct meaningful stabilizing feedbacks?

The case of a smooth CLF

$$\inf_{u \in U} \langle \nabla V(x), f(x,u) \rangle \; < \; -W(x) \quad x \neq 0.$$

Q: How to construct meaningful stabilizing feedbacks?

The case of a smooth CLF

$$\inf_{u\in U} \langle \nabla V(x), f(x,u)\rangle \ < \ -W(x) \quad x \neq 0.$$

Natural approach: choose k(x) in U so that

$$\Big\langle \nabla V(x), f\big(x, k(x)\big)\Big\rangle < -W(x) \ \forall\, x \neq 0.$$

(A "steepest-descent" feedback induced by V)

Q: How to construct meaningful stabilizing feedbacks?

The case of a smooth CLF

$$\inf_{u \in U} \langle \nabla V(x), f(x,u) \rangle \; < \; -W(x) \quad x \neq 0.$$

Natural approach: choose k(x) in U so that

$$\left\langle \nabla V(x), f\left(x, k(x)\right)\right\rangle < -W(x) \;\; \forall\, x \neq 0.$$

(A "steepest-descent" feedback induced by V)

Theorem

k stabilizes the system in the s & h sense

THEOREM A steepest descent feedback k stabilizes the system in the sample-and-hold sense.

PROOF. For ease of exposition, we shall suppose that V (on $\mathbb{R}^n$) and ∇V (on $\mathbb{R}^n\backslash\{0\}$) are locally Lipschitz rather than merely continuous (otherwise, the argument is carried out with moduli of continuity). We also restrict attention to uniform partitions.

Let $B(0,R)$ and $B(0,r)$ be the initial values and target set under consideration. The properties of V imply the existence of positive numbers $e < E$ such that

$$\{x : V(x) \le e\} \subset B(0,r), \ \ \{x : V(x) \le E\} \supset B(0,R).$$

Fix $E' > E$. There exist positive constants K, L,M such that, for all x, y in the compact set $\{x : V(x) \le E'\}$ and $u \in U$, we have

$$|V(x) - V(y)| \ \le \ L|x-y|, \ |f(x,u)| \ \le \ M, \quad |f(x,u) - f(y,u)| \ \le \ K|x-y|. \quad (1)$$

Now pick e' and e'' so that $0 < e'' < e' < e$, and set

$$X := \{x : e'' \le V(x) \le E'\}.$$

Then there exist constants N and $\omega > 0$ such that

$$|\nabla V(x) - \nabla V(y)| \ \le \ N|x-y|, \ \ W(x) \ \ge \ \omega \ \forall x,y \in X. \quad (2)$$

Let $B(0,R)$ and $B(0,r)$ be the initial values and target set under consideration. The properties of V imply the existence of positive numbers $e < E$ such that

$$\{x : V(x) \le e\} \subset B(0,r), \ \ \{x : V(x) \le E\} \supset B(0,R).$$

Fix $E' > E$. There exist positive constants K, L,M such that, for all x, y in the compact set $\{x : V(x) \le E'\}$ and $u \in U$, we have

$$|V(x) - V(y)| \ \le \ L|x-y|, \ |f(x,u)| \ \le \ M, \quad |f(x,u) - f(y,u)| \ \le \ K|x-y|. \quad (1)$$

Now pick e' and e'' so that $0 < e'' < e' < e$, and set

$$X := \{x : e'' \le V(x) \le E'\}.$$

Then there exist constants N and $\omega > 0$ such that

$$|\nabla V(x) - \nabla V(y)| \ \le \ N|x-y|, \ \ W(x) \ \ge \ \omega \ \forall x,y \in X. \quad (2)$$

Let π be a uniform partition of diameter $\delta \in (0,1)$ such that

$$\delta LM \ < \ \min\{e-e', e'-e'', E'-E\}, \quad \delta(LK+MN)M \ < \ \omega/2. \quad (3)$$

Now let x_0 be any point in $B(0,R)$, and proceed to implement the feedback k via the partition π. On the first time interval $[t_0,t_1]$ the trajectory x corresponding to k is generated by the differential equation

$$x'(t) = f(x(t), k(x_0)), \ x(t_0) = x_0, \ t_0 \le t \le t_1.$$

The solution to this differential equation exists on some interval of positive length, and is unique because f is locally Lipschitz in the state variable. If the solution fails to exist on the entire interval, it is because blow-up has occurred. Then there exists a first $\tau \in (t_0,t_1]$ for which $V(x(\tau)) = E'$. On the interval $[t_0,\tau)$, the Lipschitz constant L of (1) is valid, as well as the bound M, whence

$$V(x(t)) \le V(x_0) + L|x(t) - x_0| \le E + \delta LM \ \forall t \in [t_0,\tau).$$

But then $V(x(\tau)) \le E + \delta LM < E'$ by (3), a contradiction. It follows that blow-up cannot occur, and that the solution of the differential equation exists on the entire interval $[t_0,t_1]$ and satisfies $V(x(t)) < E'$ there.

Case 1 $\quad V(x_0) \le e'$.

It follows then from $\delta LM < e - e'$ (see (3)) that we have

$$V(x(t)) < e \ \forall \ t \in [t_0,t_1].$$

Case 2 $\quad e' < V(x_0)$.

Now we have $x_0 \in X$ and

$$\langle \nabla V(x_0), f(x_0, k(x_0))\rangle \ < \ -\omega$$

from the way $k(x_0)$ is defined, and since $W(x_0) > \omega$.

Let $t \in (t_0,t_1]$; then, at least while $x(t)$ remains in the set X, we can argue as follows:

$$V(x(t)) - V(x(t_0)) = \langle \nabla V(x(t^*)), x'(t^*)\rangle\,(t-t_0)$$

(by the Mean Value Theorem, for some $t^* \in (0,t)$)

$$\begin{aligned}
&= \langle \nabla V(x(t^*)), f(x(t^*), k(t_0))\rangle\,(t-t_0)\\
&= \langle \nabla V(x(t_0)), f(x(t_0), k(t_0))\rangle\,(t-t_0)\\
&\quad + \langle \nabla V(x(t_0)), f(x(t^*), k(t_0)) - f(x(t_0), k(t_0))\rangle (t-t_0)\\
&\quad + \langle \nabla V(x(t^*)) - \nabla V(x(t_0)), f(x(t^*), k(t_0))\rangle\,(t-t_0)\\
&\le \langle \nabla V(x(t_0)), f(x(t_0), k(t_0))\rangle\,(t-t_0)\\
&\quad + LK|x(t^*) - x_0|(t-t_0) + NM|x(t^*) - x_0|(t-t_0) \quad \text{(by (1) and (2))}\\
&\le -\omega(t-t_0) + LKM\delta(t-t_0) + M^2N\delta(t-t_0) \quad \text{(by definition of } k)\\
&= \{-\omega + \delta\,(LK+MN)\,M\}\,(t-t_0)\\
&\le -(\omega/2)(t-t_0), \text{ by (3).}
\end{aligned}$$

Thus the value of V has decreased. It follows from this, together with the inequality $\delta LM < e' - e''$ provided by (3), that $x(t)$ remains in X throughout $[t_0,t_1]$, so that the estimates above apply.
To summarize, we have in Case 2 the following decrease property:

$$V(x(t)) - V(x(t_0)) \ \le \ -(\omega/2)(t-t_0) \, \forall \, t \in [t_0,t_1].$$

It follows that, in either case, we have $V(x(t)) \le E$ for $t \in [t_0,t_1]$, and in particular $V(x_1) \le E$, where $x_1 := x(t_1)$ is the next node in the implementation scheme.

We now repeat the procedure on the next interval $[t_1,t_2]$, but using the constant control value $k(x_1)$. Precisely the same arguments as above apply to this and to all subsequent steps: either we are at a node x_i for which $V(x_i) \le e'$ (Case 1), or else $V(x(t))$ continues to decrease at a rate of at least $\omega/2$ (Case 2).
Since V is nonnegative, the case of continued decrease cannot persist indefinitely. Let x_J ($J \ge 0$) be the first node satisfying $V(x_J) \le e'$. If $J > 0$, then

$$\begin{aligned} e' < V(x_{J-1}) \ &\le \ V(x_0) - (\omega/2)(t_{J-1} - t_0)\\ &= \ V(x_0) - (\omega/2)(J-1)\delta, \end{aligned}$$

whence

$$(\omega/2)(J-1)\delta \ < \ V(x_0) - e' \ \le \ E - e',$$

and so

$$J\delta \ < \ 2(E-e')/\omega + \delta \ \le \ 2(E-e')/\omega + 1 =: T,$$

which provides a uniform upper bound T independent of δ for the time $J\delta$ required to attain the condition $V(x_J) \le e'$. Once this condition is satisfied, the above analysis shows that for all $t \ge t_J$, we have $V(x(t) < e$, which implies $x(t) \in B(0,r)$.

Since for all $t \ge 0$ the trajectory x satisfies $V(x(t)) \le E$, and since $\{x : V(x) \le E\} \subset B(0,R)$, there exists C depending only on R such that $|x(t)| \le C \, \forall \, t \ge 0$. This completes the proof that the required stabilization takes place.

THEOREM A steepest descent feedback k stabilizes the system in the sample-and-hold sense.

PROOF. For ease of exposition, we shall suppose that V (on $\mathbb{R}^n$) and ∇V (on $\mathbb{R}^n \backslash \{0\}$) are locally Lipschitz rather than merely continuous (otherwise, the argument is carried out with moduli of continuity). We also restrict attention to uniform partitions.

Let $B(0,R)$ and $B(0,r)$ be the initial values and target set under consideration. The properties of V imply the existence of positive numbers $e < E$ such that

$$\{x : V(x) \le e\} \subset B(0,r), \quad \{x : V(x) \le E\} \supset B(0,R).$$

Fix $E' > E$. There exist positive constants K, L,M such that, for all x, y in the compact set $\{x : V(x) \le E'\}$ and $u \in U$, we have

$$|V(x) - V(y)| \ \le \ L|x-y|, \ |f(x,u)| \ \le \ M, \quad |f(x,u) - f(y,u)| \ \le \ K|x-y|. \quad (1)$$

Now pick e' and e'' so that $0 < e'' < e' < e$, and set

$$X := \{x : e'' \le V(x) \le E'\}.$$

Then there exist constants N and $\omega > 0$ such that

$$|\nabla V(x) - \nabla V(y)| \ \le \ N|x-y|, \quad W(x) \ \ge \ \omega \ \forall x, y \in X. \quad (2)$$

Let $B(0,R)$ and $B(0,r)$ be the initial values and target set under consideration. The properties of V imply the existence of positive numbers $e < E$ such that

$$\{x : V(x) \le e\} \subset B(0,r), \quad \{x : V(x) \le E\} \supset B(0,R).$$

Fix $E' > E$. There exist positive constants K, L,M such that, for all x, y in the compact set $\{x : V(x) \le E'\}$ and $u \in U$, we have

$$|V(x) - V(y)| \ \le \ L|x-y|, \ |f(x,u)| \ \le \ M, \quad |f(x,u) - f(y,u)| \ \le \ K|x-y|. \quad (1)$$

Now pick e' and e'' so that $0 < e'' < e' < e$, and set

$$X := \{x : e'' \le V(x) \le E'\}.$$

Then there exist constants N and $\omega > 0$ such that

$$|\nabla V(x) - \nabla V(y)| \ \le \ N|x-y|, \quad W(x) \ \ge \ \omega \ \forall x, y \in X. \quad (2)$$

Let π be a uniform partition of diameter $\delta \in (0,1)$ such that

$$\delta LM \ < \ \min\{e-e', e'-e'', E'-E\}, \quad \delta(LK+MN)M \ < \ \omega/2. \quad (3)$$

Now let x_0 be any point in $B(0,R)$, and proceed to implement the feedback k via the partition π. On the first time interval $[t_0, t_1]$ the trajectory x corresponding to k is generated by the differential equation

$$x'(t) = f(x(t), k(x_0)), \ x(t_0) = x_0, \ t_0 \le t \le t_1.$$

The solution to this differential equation exists on some interval of positive length, and is unique because f is locally Lipschitz in the state variable. If the solution fails to exist on the entire interval, it is because blow-up has occurred. Then there exists a first $\tau \in (t_0, t_1]$ for which $V(x(\tau)) = E'$. On the interval $[t_0, \tau)$, the Lipschitz constant L of (1) is valid, as well as the bound M, whence

$$V(x(t)) \le V(x_0) + L|x(t) - x_0| \le E + \delta LM \ \forall t \in [t_0, \tau).$$

But then $V(x(\tau)) \le E + \delta LM < E'$ by (3), a contradiction. It follows that blow-up cannot occur, and that the solution of the differential equation exists on the entire interval $[t_0, t_1]$ and satisfies $V(x(t)) < E'$ there.

Case 1 $\quad V(x_0) \le e'$.

It follows then from $\delta LM < e - e'$ (see (3)) that we have

$$V(x(t)) < e \ \forall \ t \in [t_0, t_1].$$

Case 2 $\quad e' < V(x_0)$.

Now we have $x_0 \in X$ and

$$\langle \nabla V(x_0), f(x_0, k(x_0)) \rangle \ < \ -\omega$$

from the way $k(x_0)$ is defined, and since $W(x_0) > \omega$.

Let $t \in (t_0, t_1]$; then, at least while $x(t)$ remains in the set X, we can argue as follows:

$$V(x(t)) - V(x(t_0)) = \langle \nabla V(x(t^*)), x'(t^*) \rangle \ (t - t_0)$$

(by the Mean Value Theorem, for some $t^* \in (0,t)$)

$$\begin{aligned}
&= \langle \nabla V(x(t^*)), f(x(t^*), k(t_0)) \rangle \ (t-t_0) \\
&= \langle \nabla V(x(t_0)), f(x(t_0), k(t_0)) \rangle \ (t-t_0) \\
&\quad + \langle \nabla V(x(t_0)), f(x(t^*), k(t_0)) - f(x(t_0), k(t_0)) \rangle (t-t_0) \\
&\quad + \langle \nabla V(x(t^*)) - \nabla V(x(t_0)), f(x(t^*), k(t_0)) \rangle \ (t-t_0) \\
&\le \langle \nabla V(x(t_0)), f(x(t_0), k(t_0)) \rangle \ (t-t_0) \\
&\quad + LK|x(t^*) - x_0|(t-t_0) + NM|x(t^*) - x_0|(t-t_0) \quad \text{(by (1) and (2))} \\
&\le -\omega(t-t_0) + LKM\delta(t-t_0) + M^2N\delta(t-t_0) \quad \text{(by definition of } k) \\
&= \{-\omega + \delta\,(LK+MN)\,M\}\,(t-t_0) \\
&\le -(\omega/2)(t-t_0), \ \text{by (3)}.
\end{aligned}$$

Thus the value of V has decreased. It follows from this, together with the inequality $\delta LM < e' - e''$ provided by (3), that $x(t)$ remains in X throughout $[t_0, t_1]$, so that the estimates above apply.

To summarize, we have in Case 2 the following decrease property:

$$V(x(t)) - V(x(t_0)) \ \le \ -(\omega/2)(t-t_0) \ \forall \, t \in [t_0, t_1].$$

It follows that, in either case, we have $V(x(t)) \le E$ for $t \in [t_0, t_1]$, and in particular $V(x_1) \le E$, where $x_1 := x(t_1)$ is the next node in the implementation scheme.

We now repeat the procedure on the next interval $[t_1, t_2]$, but using the constant control value $k(x_1)$. Precisely the same arguments as above apply to this and to all subsequent steps: either we are at a node x_i for which $V(x_i) \le e'$ (Case 1), or else $V(x(t))$ continues to decrease at a rate of at least $\omega/2$ (Case 2).

Since V is nonnegative, the case of continued decrease cannot persist indefinitely. Let x_J ($J \ge 0$) be the first node satisfying $V(x_J) \le e'$. If $J > 0$, then

$$\begin{aligned}
e' \ < V(x_{J-1}) \ &\le \ V(x_0) - (\omega/2)(t_{J-1} - t_0) \\
&= \ V(x_0) - (\omega/2)(J-1)\delta,
\end{aligned}$$

whence

$$(\omega/2)(J-1)\delta \ < \ V(x_0) - e' \ \le \ E - e',$$

and so

$$J\delta \ < \ 2(E-e')/\omega + \delta \ \le \ 2(E-e')/\omega + 1 =: T,$$

which provides a uniform upper bound T independent of δ for the time $J\delta$ required to attain the condition $V(x_J) \le e'$. Once this condition is satisfied, the above analysis shows that for all $t \ge t_J$, we have $V(x(t) < e$, which implies $x(t) \in B(0,r)$.

Since for all $t \ge 0$ the trajectory x satisfies $V(x(t)) \le E$, and since $\{x : V(x) \le E\} \subset B(0,R)$, there exists C depending only on R such that $|x(t)| \le C \, \forall \, t \ge 0$. This completes the proof that the required stabilization takes place.

QED

Consider now a Dini CLF:

$$\inf_{u \in U} dV\big(x; f(x,u)\big) \;<\; -W(x) \quad x \neq 0.$$

Consider now a Dini CLF:

$$\inf_{u\in U} dV\big(x; f(x,u)\big) \;<\; -W(x) \quad x \neq 0.$$

Natural approach: choose k(x) in U so that

$$dV\big(x; f(x,k(x))\big) \;<\; -W(x) \quad x \neq 0.$$

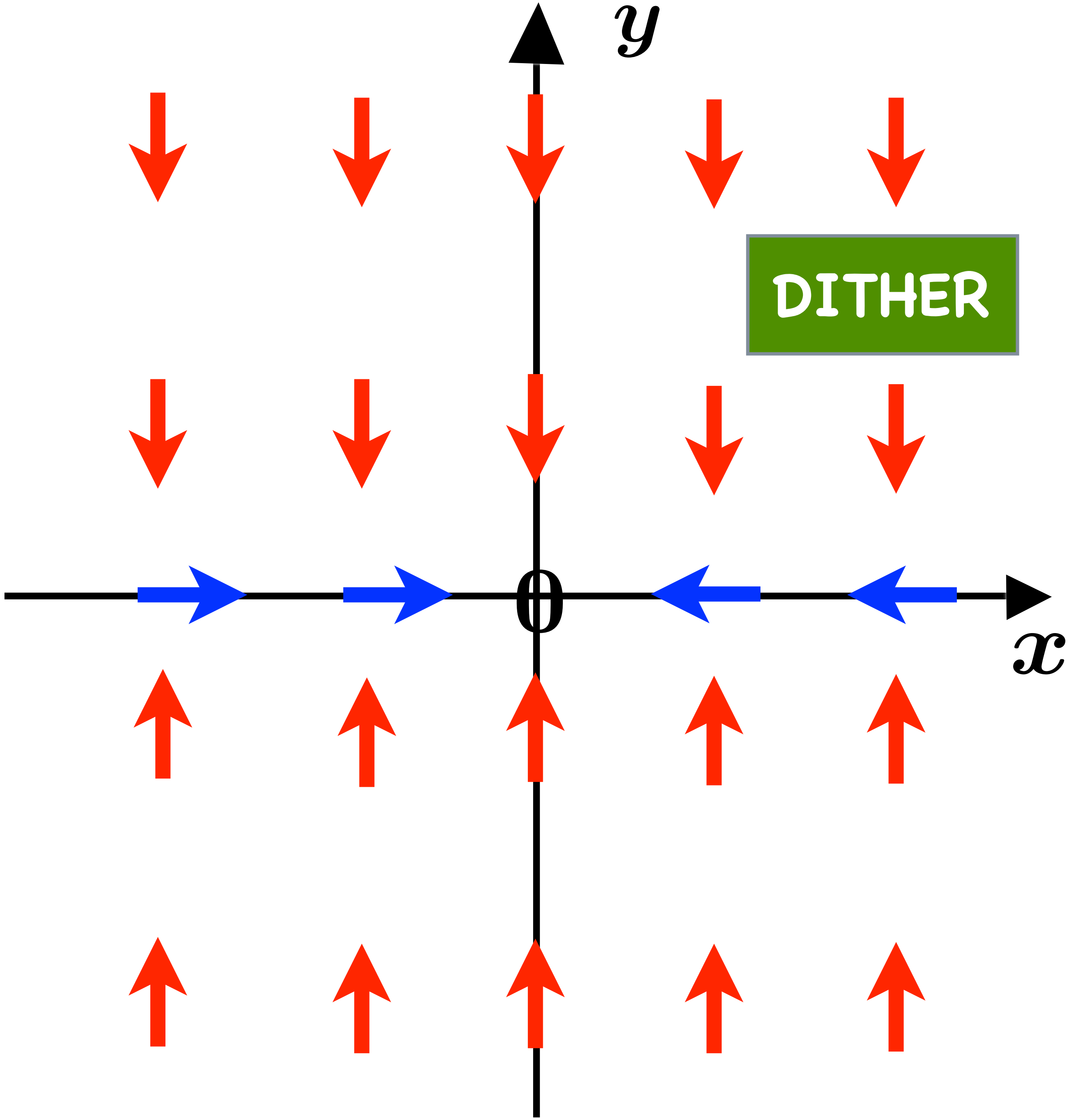
Consider now a Dini CLF:

$$\inf_{u \in U} dV\big(x; f(x,u)\big) \; < \; -W(x) \quad x \neq 0.$$

Natural approach: choose k(x) in U so that

$$dV\big(x; f(x,k(x))\big) \; < \; -W(x) \quad x \neq 0.$$

But this can give a meaningless, non-stabilizing feedback



From (x,y):

1. Go directly to the x-axis
2. Then go directly to the origin

By construction, the feedback is of steepest descent type for the CLF it induces:

$$V(x,y) = |x|+|y|$$

SO: for a Dini CLF:

$$\inf_{u \in U} dV\big(x; f(x,u)\big) \;<\; -W(x) \quad x \neq 0.$$

the natural approach: choose k(x) in U so that

$$dV\big(x; f(x,k(x))\big) \;<\; -W(x) \quad x \neq 0.$$

can fail.

SO: for a Dini CLF:

$$\inf_{u \in U} dV\big(x; f(x,u)\big) \; < \; -W(x) \quad x \neq 0.$$

the natural approach: choose k(x) in U so that

$$dV\big(x; f(x,k(x))\big) \; < \; -W(x) \quad x \neq 0.$$

can fail.

However, the natural steepest descent approach DOES work if V is semiconcave:

SO: for a Dini CLF:

$$\inf_{u \in U} dV\big(x; f(x,u)\big) \; < \; -W(x) \quad x \neq 0.$$

the natural approach: choose k(x) in U so that

$$dV\big(x; f(x,k(x))\big) \; < \; -W(x) \quad x \neq 0.$$

can fail.

However, the natural steepest descent approach DOES work if V is semiconcave:

V is locally Lipschitz and (locally)

$$V(y) - V(z) - \langle \zeta, y - z \rangle \; \leq \; \sigma |y - z|^{1+\eta}$$

$$\forall \, \zeta \in \partial_C V(z)$$

SO: for a Dini CLF:

$$\inf_{u \in U} dV\big(x; f(x,u)\big) \ < \ -W(x) \quad x \neq 0.$$

the natural approach: choose k(x) in U so that

$$dV\big(x; f(x,k(x))\big) \ < \ -W(x) \quad x \neq 0.$$

can fail.

However, the natural steepest descent approach DOES work if V is semiconcave:

V is locally Lipschitz and (locally)

$$V(y) - V(z) - \langle \zeta, y - z \rangle \ \leq \ \sigma |y - z|^{1+\eta}$$

$$\forall \, \zeta \in \partial_C V(z)$$

true for smooth and concave V

SO: for a Dini CLF:

$$\inf_{u \in U} dV\big(x; f(x,u)\big) \;<\; -W(x) \quad x \neq 0.$$

the natural approach: choose k(x) in U so that

$$dV\big(x; f(x,k(x))\big) \;<\; -W(x) \quad x \neq 0.$$

can fail.

However, the natural steepest descent approach DOES work if V is semiconcave:

V is locally Lipschitz and (locally)

true for smooth and concave V

$$V(y) - V(z) - \langle \zeta, y - z \rangle \;\leq\; \sigma |y - z|^{1+\eta}$$

$$\forall\, \zeta \in \partial_C V(z)$$

Theorem

k stabilizes the system in the s & h sense

4. If ϕ is concave or $C^{1,\eta}$ near x, then ϕ satisfies SC at x.

5. The positive linear combination (and in particular, the sum) of a finite number of functions each of which satisfies SC at x also satisfies SC at x.

6. If $\phi = g \circ h$, where $h : \mathbb{R}^n \to \mathbb{R}^m$ is $C^{1,\eta}$ near x, and where $g : \mathbb{R}^m \to \mathbb{R}$ is concave, then ϕ satisfies SC at x.

7. If $\phi = g \circ h$, where $h : \mathbb{R}^n \to \mathbb{R}$ is concave, and where $g : \mathbb{R} \to \mathbb{R}$ is $C^{1,\eta}$ near $h(x)$, then ϕ satisfies SC at x.

8. If $\phi = gh$, where h is convex, and where $g : \mathbb{R}^n \to (-\infty, 0\,]$ is $C^{1,\eta}$ near x, then ϕ satisfies SC at x.

9. If $\phi = gh$, where g is $C^{1,\eta}$ near x, with $g(x) > 0$, and where h is concave, then ϕ satisfies SC at x.

10. If $\phi = \min \phi_i$, where $\{\phi_i\}$ is a finite family of functions each of which satisfies SC at x, then ϕ satisfies SC at x.

11. If ϕ satisfies SC at x, then the directional derivative $\phi'(x; v)$ exists for each v, and one has

$$d\phi(x; v) = \phi'(x; v) = \min_{\zeta \in \partial_C \phi(x)} \langle \zeta, v \rangle \;\; \forall\, v \in \mathbb{R}^n.$$

Two Dini CLF's for NHI:

$$V_1(x) := x_1^2 + x_2^2 + 2x_3^2 - 2|x_3|\sqrt{x_1^2 + x_2^2}.$$

$$V_2(x) := \max\left\{\sqrt{x_1^2 + x_2^2},\ |x_3| - \sqrt{x_1^2 + x_2^2}\right\}.$$

Two Dini CLF's for NHI:

$$V_1(x) := x_1^2 + x_2^2 + 2x_3^2 - 2|x_3|\sqrt{x_1^2 + x_2^2}.$$

$$V_2(x) := \max\left\{\sqrt{x_1^2 + x_2^2},\ |x_3| - \sqrt{x_1^2 + x_2^2}\right\}.$$

Only one of these is semiconcave.

Two Dini CLF's for NHI:

$$V_1(x) := x_1^2 + x_2^2 + 2x_3^2 - 2|x_3|\sqrt{x_1^2 + x_2^2}.$$

$$V_2(x) := \max\left\{\sqrt{x_1^2 + x_2^2},\ |x_3| - \sqrt{x_1^2 + x_2^2}\right\}.$$

Only one of these is semiconcave.

The collection of facts about operations that preserve that property (positive linear combinations, certain products and compositions, lower envelopes) allows us to see easily that V_1 is semiconcave.

The corresponding steepest-descent feedback induced by V_1 is given by

For $x \neq 0$:

When $\sigma \neq 0$ and $x_3 \neq 0$, set

$$k(x) = \begin{cases} (x_1, x_2)/\rho & \text{if } |x_3| - \rho \geq \rho|\rho\,\mathrm{sgn}\,(x_3) - 2x_3| \\ -(x_1, x_2)/\rho & \text{if } \rho - |x_3| \geq \rho|\rho\,\mathrm{sgn}\,(x_3) - 2x_3| \\ (x_2, -x_1)/\rho & \text{if } \rho\big(2x_3 - \rho\,\mathrm{sgn}\,(x_3)\big) > |\rho - |x_3|| \\ -(x_2, -x_1)/\rho & \text{if } \rho\big(\rho\,\mathrm{sgn}\,(x_3) - 2x_3\big) > |\rho - |x_3|| \end{cases}$$

where $\rho := \sqrt{x_1^2 + x_2^2}$.

When $\sigma = 0$ (then $x_3 \neq 0$), set $k(x) = (1,1)/\sqrt{2}$.

When $x_3 = 0$ (then $\sigma \neq 0$), set $k(x) = -(x_1, x_2)/\sigma$

(Set $k(0)$ equal to any point in U)

Four types of regularity for Dini or proximal CLF's:

-
-
-
-

Four types of regularity for Dini or proximal CLF's:

- Continuous
-
-
-

Four types of regularity for Dini or proximal CLF's:

- Continuous

  $\cup$

- Locally Lipschitz

Four types of regularity for Dini or proximal CLF's:

- Continuous

$\cup$

- Locally Lipschitz

$\cup$

- Semiconcave

-

Four types of regularity for Dini or proximal CLF's:

- Continuous

$\cup$

- Locally Lipschitz

$\cup$

- Semiconcave

$\cup$

- Smooth (C^1)

Four types of regularity for Dini or proximal CLF's:

- Continuous

$$\cup$$

- Locally Lipschitz

$$\cup$$

- Semiconcave

$$\cup$$

- Smooth (C^1)

Theorem [Rifford 2000] The system is GAC if and only if it admits a semiconcave CLF.

What if V is merely locally Lipschitz, not smooth or semiconcave?

What if V is merely locally Lipschitz, not smooth or semiconcave?

Fact: given r and R, then, for λ sufficiently large, the steepest descent feedback generated by

$$V_\lambda(x) := \min_{z \in \mathbb{R}^n} \{V(z) + (\lambda/2)|x - z|^2\}.$$

stabilizes B(0,R) to B(0,r).

What if V is merely locally Lipschitz, not smooth or semiconcave?

Fact: given r and R, then, for λ sufficiently large, the steepest descent feedback generated by

inf-convolution

$$V_\lambda(x) := \min_{z \in \mathbb{R}^n} \{ V(z) + (\lambda/2) |x - z|^2 \}.$$

stabilizes B(0,R) to B(0,r).

What if V is merely locally Lipschitz, not smooth or semiconcave?

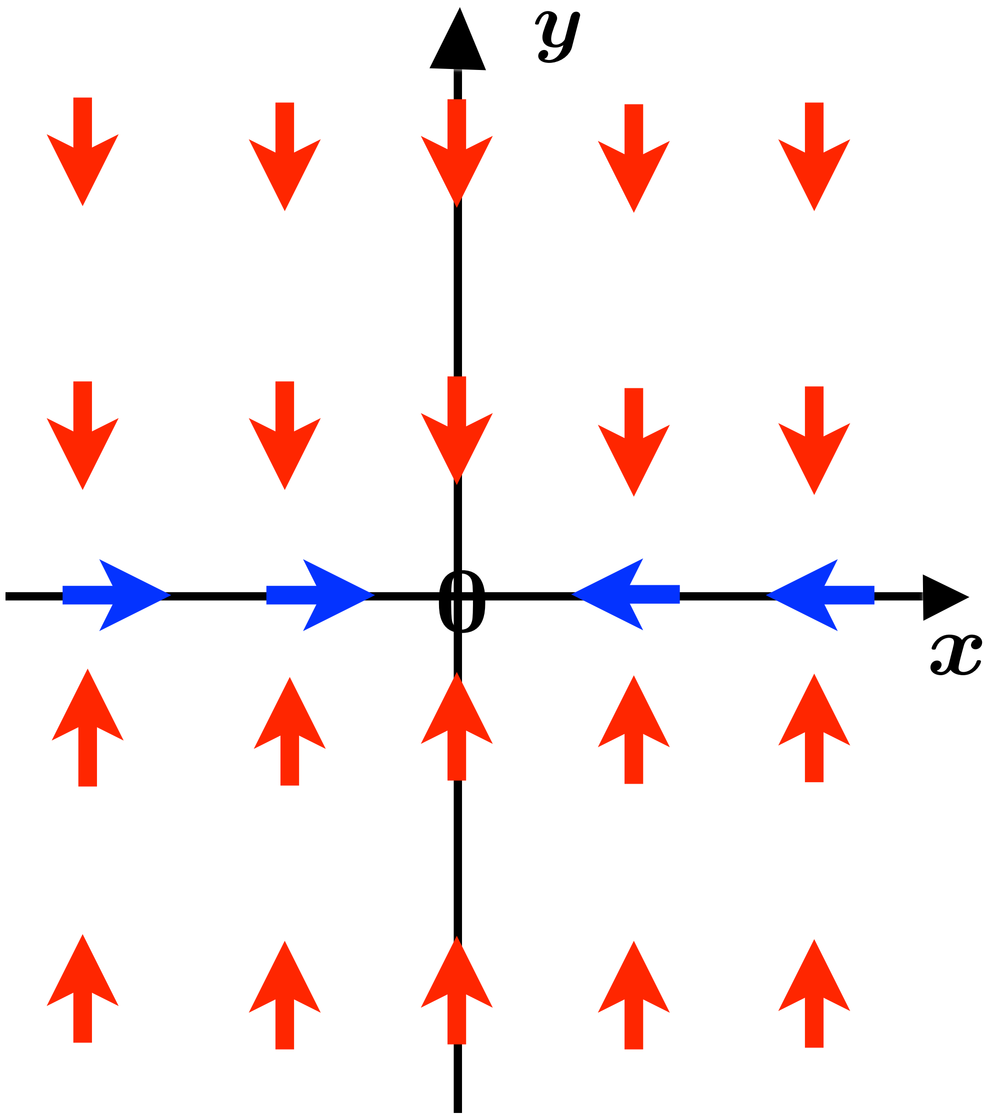
Fact: given r and R, then, for λ sufficiently large, the steepest descent feedback generated by

inf-convolution

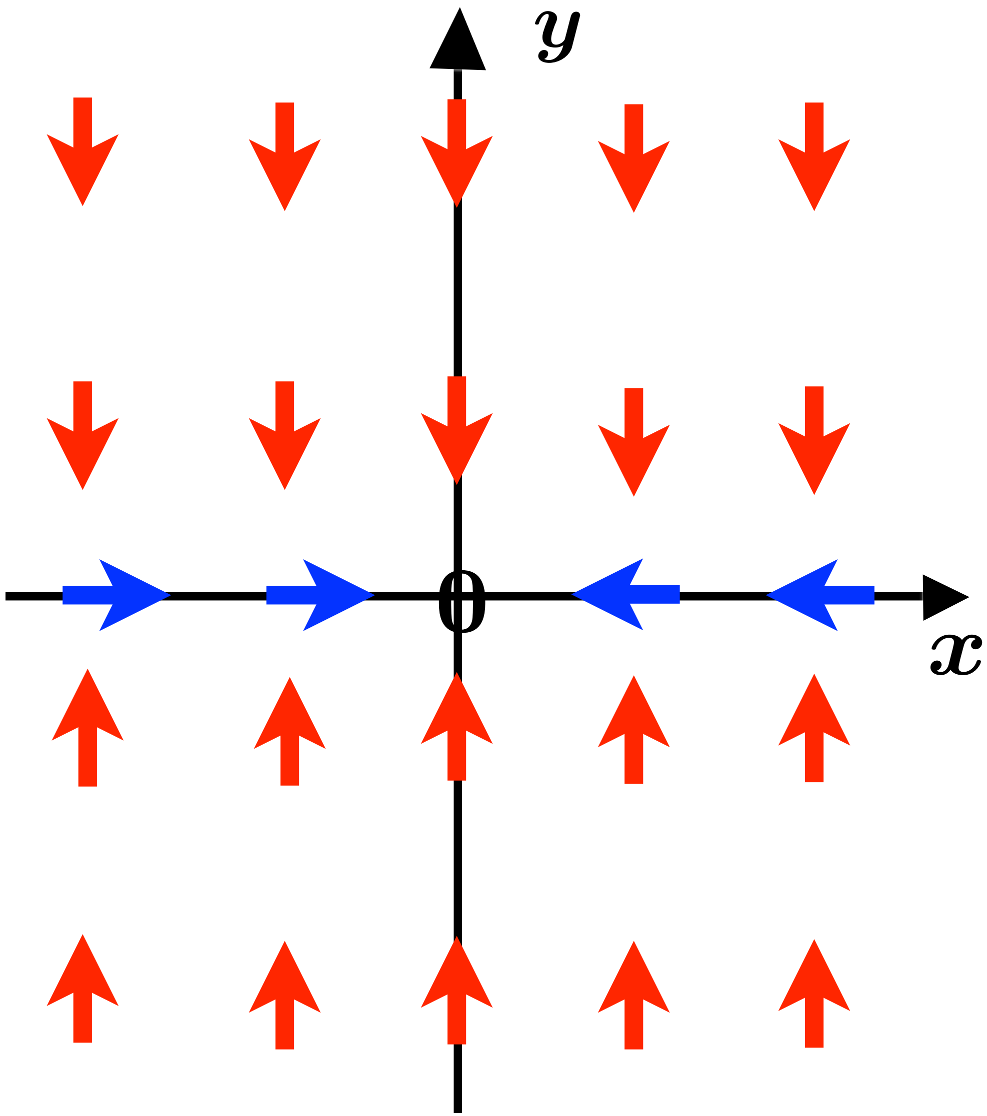
$$V_\lambda(x) := \min_{z \in \mathbb{R}^n} \{V(z) + (\lambda/2)|x - z|^2\}.$$

stabilizes B(0,R) to B(0,r).

So we get feedbacks for "practical semiglobal stabilization"

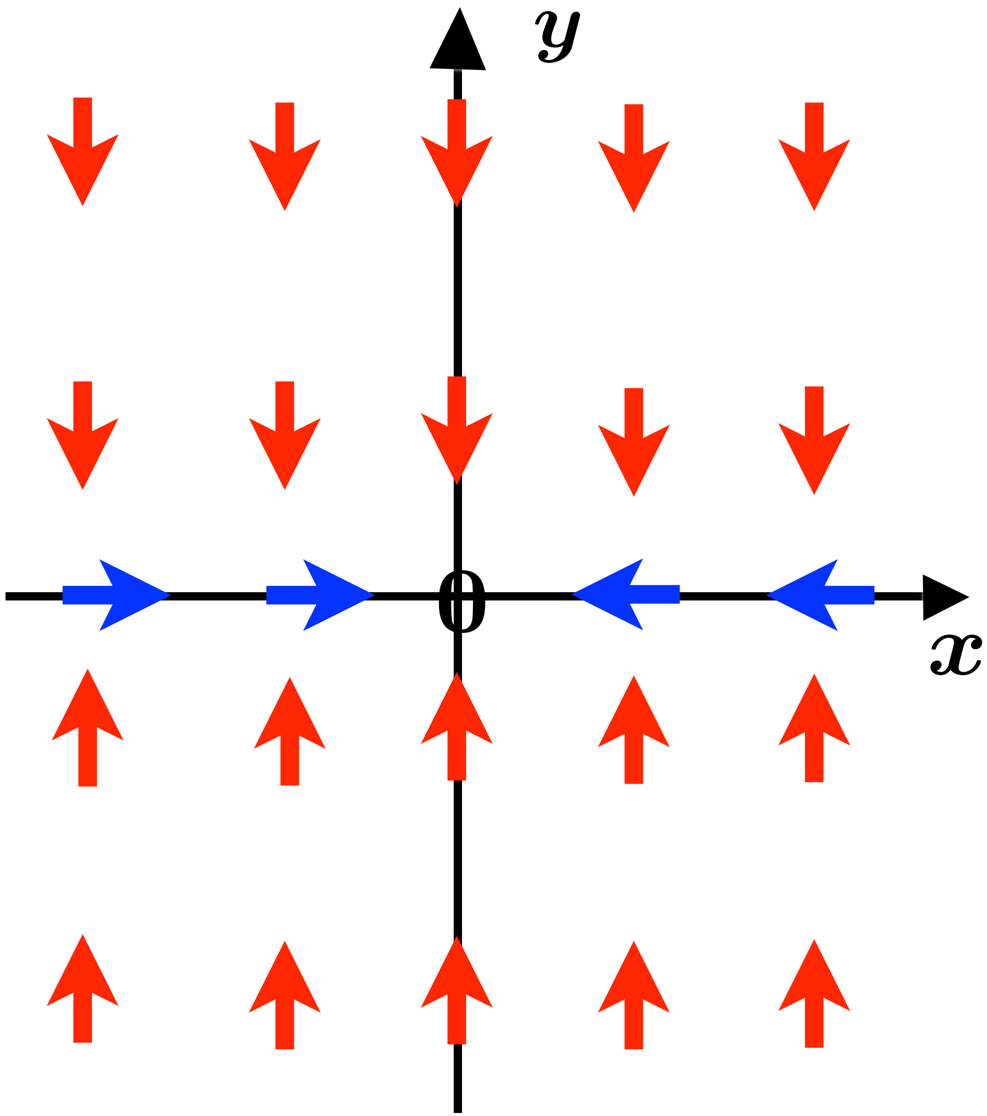


Steepest descent for
$V(x,y) = |x|+|y|$
(dither)



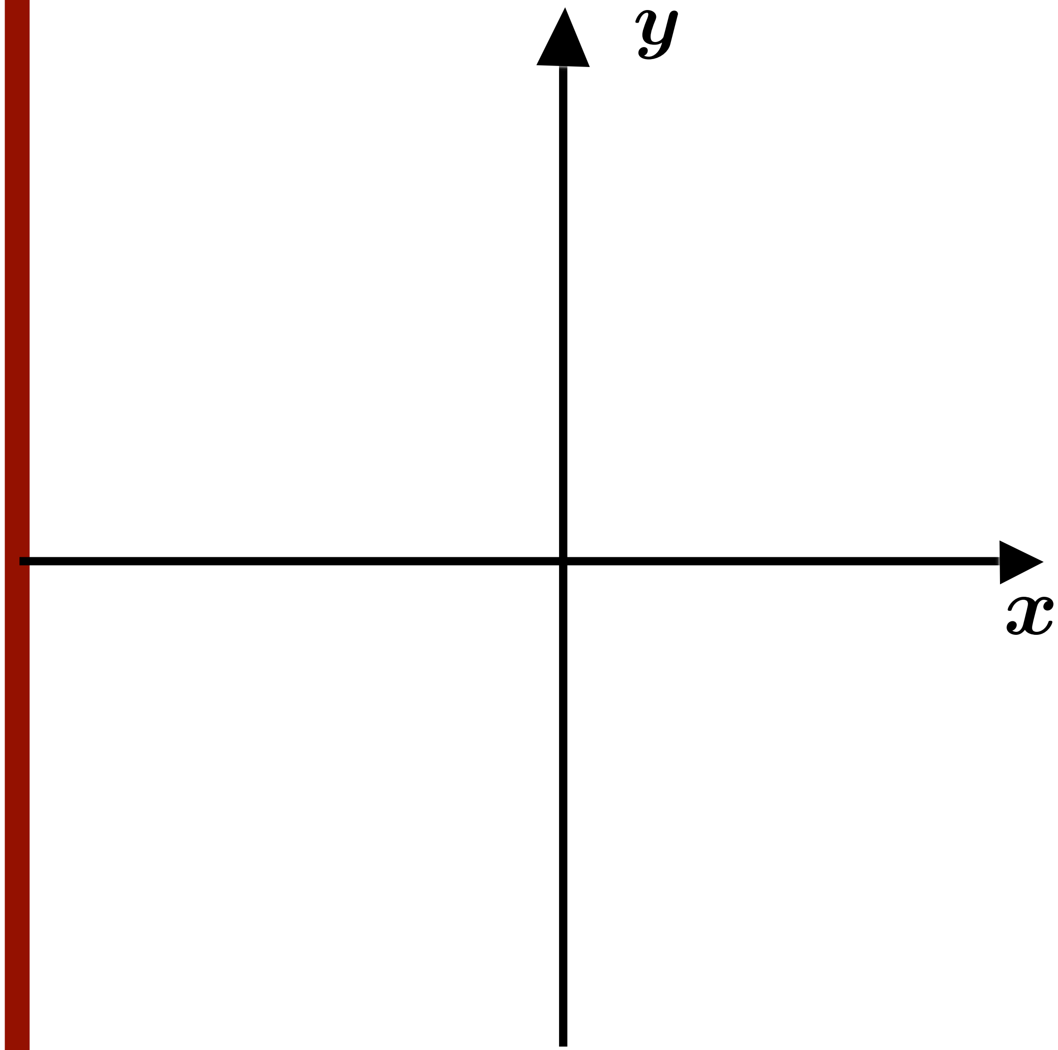
Steepest descent for
$V(x,y) = |x|+|y|$
(dither)

not semiconcave!

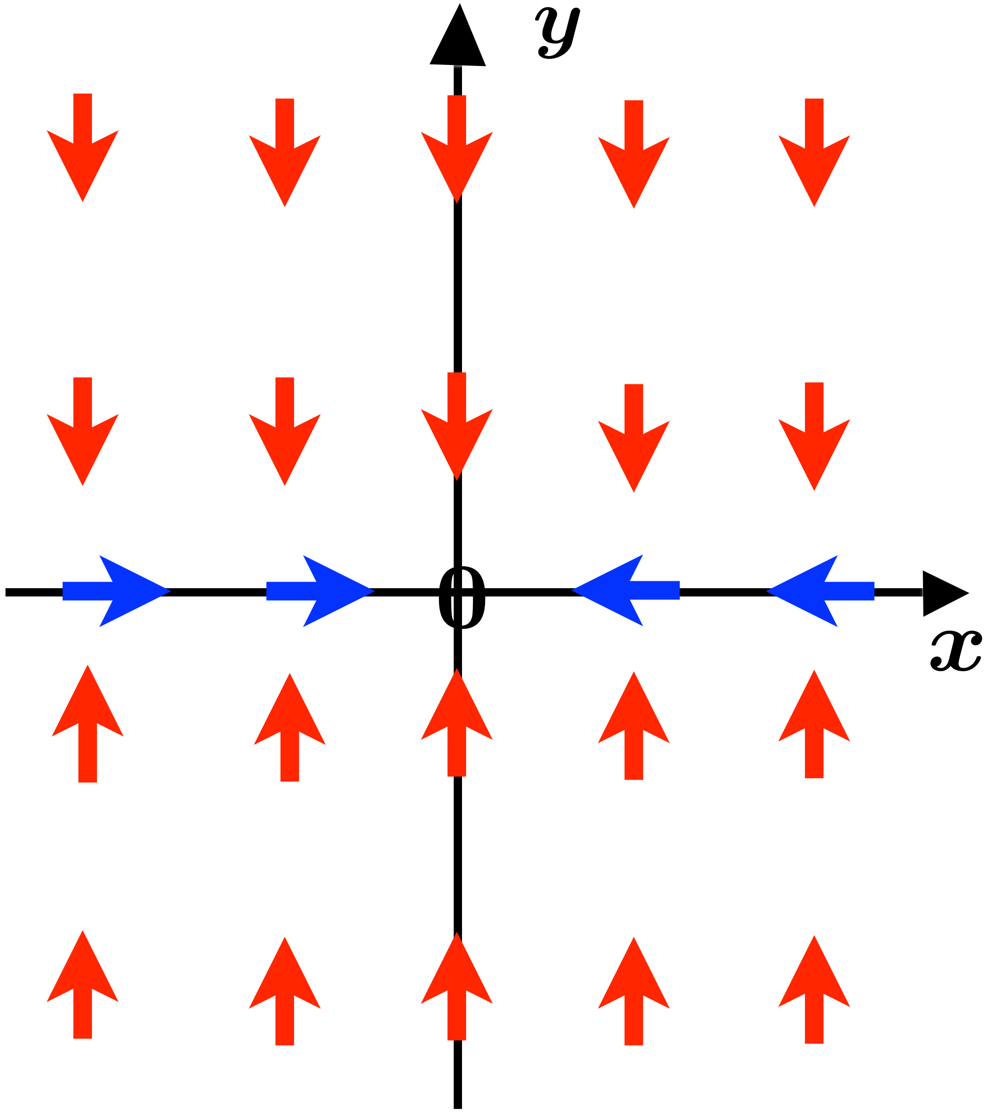


Steepest descent for $V(x,y) = |x|+|y|$ (dither)

not semiconcave!

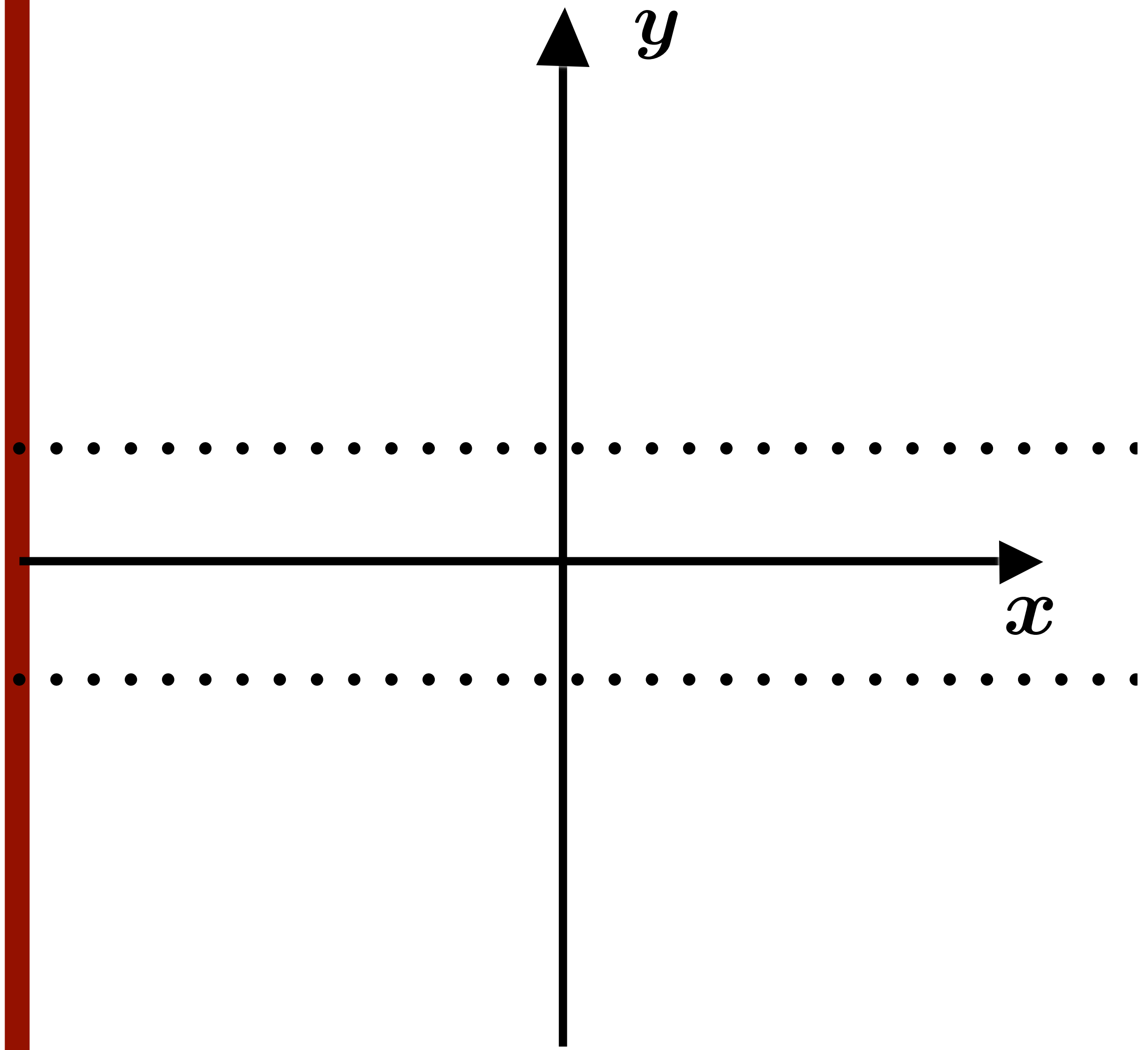


Steepest descent for $V_\lambda(x,y)$ (s & h stabilization)

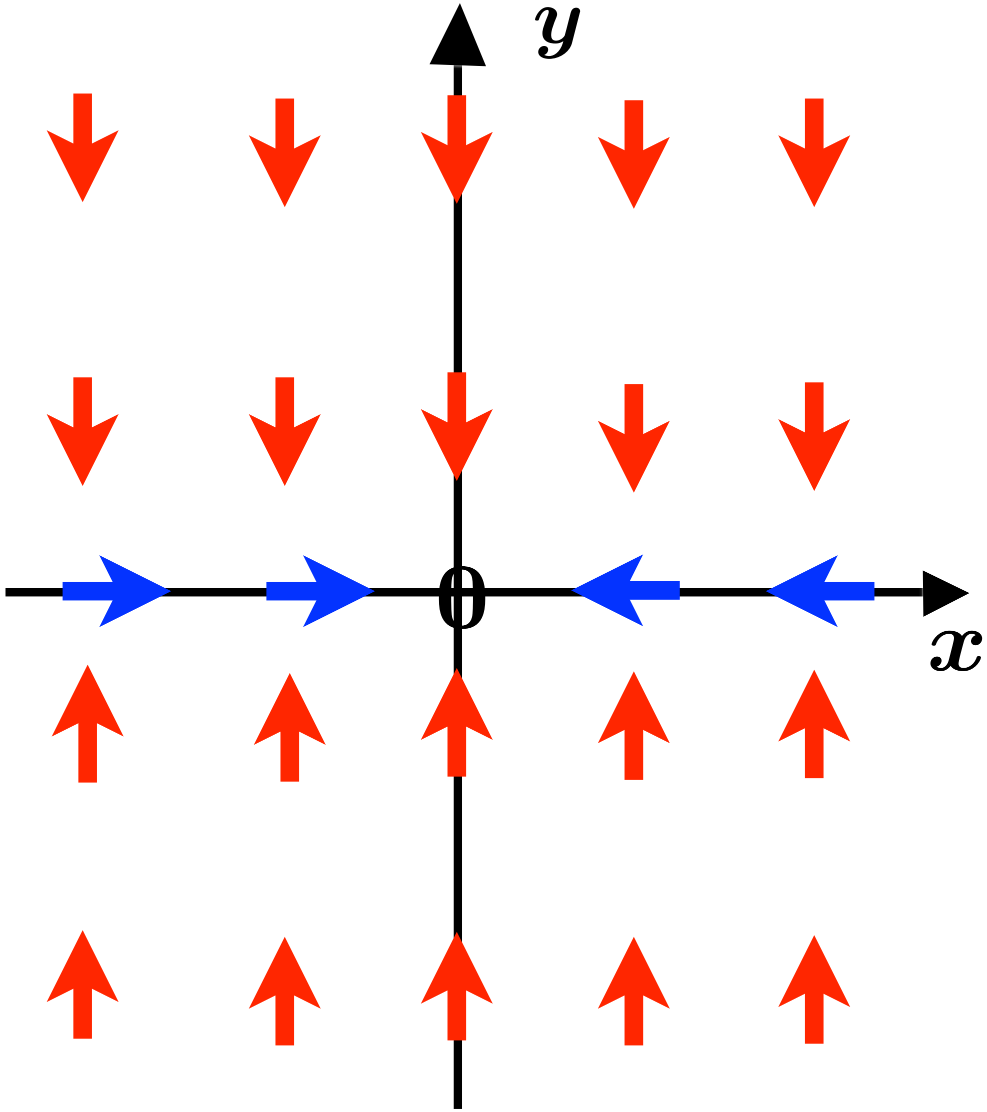


Steepest descent for
V(x,y) = |x|+|y|
(dither)

not semiconcave!

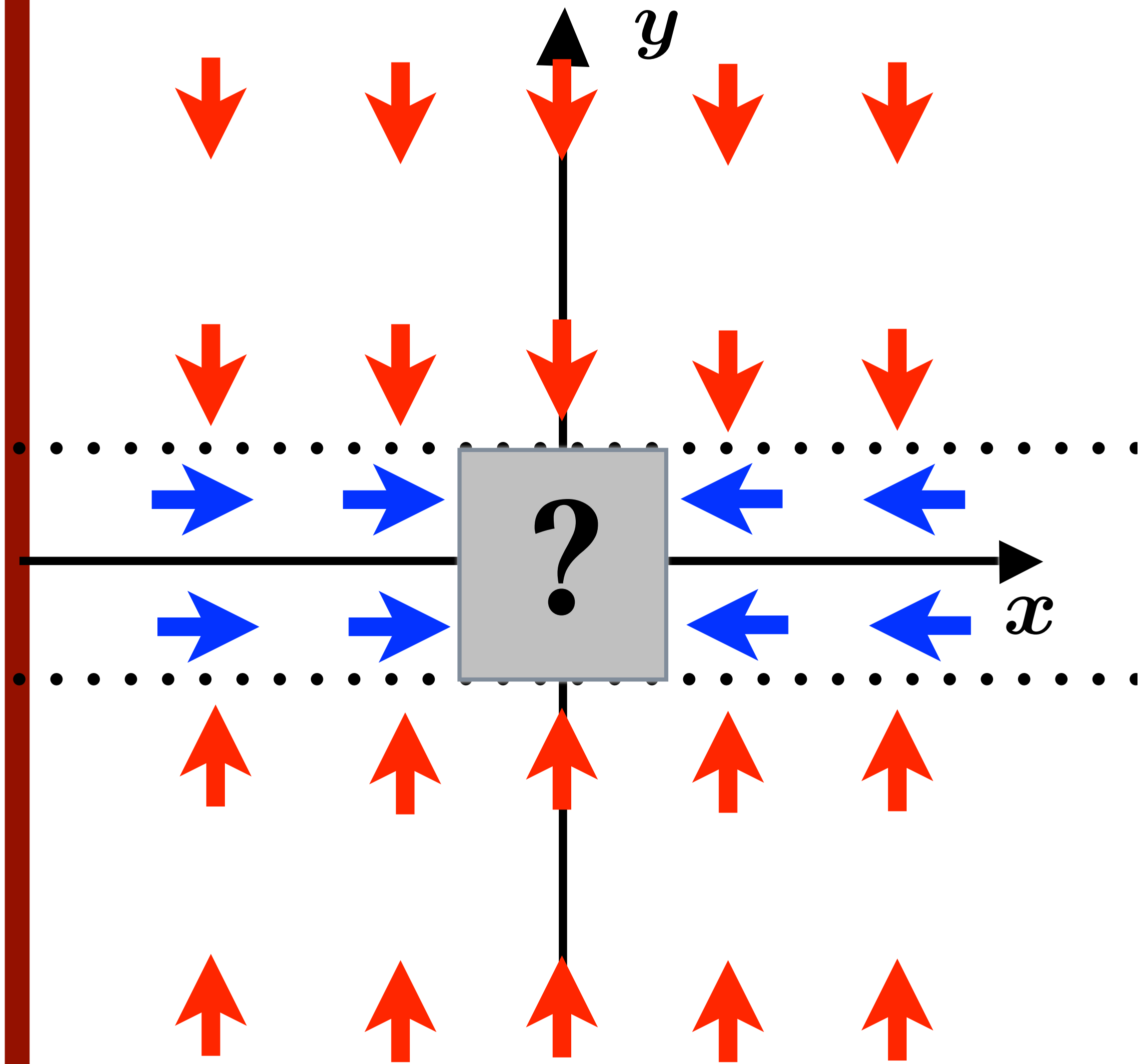


Steepest descent
for $V_\lambda(x,y)$
(s & h stabilization)



Steepest descent for
V(x,y) = |x|+|y|
(dither)

not semiconcave!



Steepest descent
for $V_\lambda(x,y)$
(s & h stabilization)

Conclusions

Discontinuous feedbacks appear to be essential in nonlinear control settings

They must be handled with more care than continuous ones, and require more effort, but they offer certain advantages

There is a growing body of theory and techniques on the subject, based on sample-and-hold analysis

THE END