Optimal control of state constrained integral equations

J. Frédéric Bonnans, INRIA Saclay et CMAP

JBHU Days, Château Neuf, IUT Bayonne

October 25-27, 2010

Référence

Common work with Constanza de la Vega, U. Buenos Aires

Published in Set-Valued and Variational Analysis

Online First (July 2010).

Example: the egg business I

Uncontrolled dynamics:

$$y_t = \int_{-1}^{t} c_{t-s} y_s \mathrm{d}s, \quad t \geq 0$$

where

- y_t represents the number of eggs produced at time t
- c_s fertility rate (zero if $s > 1$),
- y_t, $t < 0$ given

Example: the egg business II

Controlled dynamics: u_t proportion of unconsumed eggs

$$y_t = \int_{-1}^{t} c_{t-s} u_s y_s \mathrm{d}s, \quad t \geq 0$$

and given an utility function U, maximize

$$\int_0^T U((1-u_s)y_s)\mathrm{d}s + \phi(y_T)$$

with control constraints $u_t \in [0,1]$ (log penalty)
and state constraints $y_t \in [a,b]$.

A more serious application: control of cellular division by drugs

This is a simplified version of population dynamics

$y_{t,s}$ population of age s at time t

$$D_t y_{t,s} + D_s y_{t,s} = -(m_s + v_{t,s}) y_{t,s}$$

with

m_s mortality
$v_{t,s}$ proportion of harvesting
and again the birth law

$$y_{t,0} = \int_{t-1}^{t} c_s v_{t,s} \mathrm{d}s.$$

Setting

Optimal control problems of the following type:

$$\begin{cases} \mathrm{Min} \displaystyle\int_0^T \ell(u_t, y_t)\mathrm{d}t + \phi(y_0, y_T); \\ \quad \text{(i)} \qquad y_t = y_0 + \int_0^t f(t, s, u_s, y_s)\mathrm{d}s; \quad t \in (0, T); \\ \quad \text{(ii)} \qquad g(y_t) \le 0; \quad t \in [0, T], \\ \quad \text{(iii)} \quad \Phi(y_0, y_T) \in K, \end{cases} \tag{1}$$

$\ell : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$, $\phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $f : \mathbb{R}^2 \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$,
$g : \mathbb{R}^n \to \mathbb{R}^{n_g}$, $n_g \ge 1$, $c : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^{n_c}$, $\Phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n_\Phi}$,
K closed and non empty convex subset of $\mathbb{R}^{n_\Phi}$.

Well-posedness of state equation

All data f, g, c, ℓ, ϕ, Φ of class C^∞, f Lipschitz.
Set, for $q \in [1, \infty]$

$$\mathcal{U}_q := L^q(0, T, \mathbb{R}^m); \quad \mathcal{Y}_q := W^{1,q}(0, T, \mathbb{R}^n). \tag{2}$$

Control and state space

$$\mathcal{U} := \mathcal{U}_\infty \quad \mathcal{Y} := \mathcal{Y}_\infty$$

For given $y_0 \in \mathbb{R}^n$ and $u \in \mathcal{U}$, the *state equation* (1)(i) has a unique solution in $\mathcal{Y} := \mathcal{Y}_\infty$ denoted $y[u, y_0]$.

Lagrangian of the problem (unqualified form)

State derivative: here $D_\tau f$ partial derivative w.r.t. the second variable:

$$\dot{y}_t = f(t, t, u_t, y_t) + \int_0^t D_\tau f(t, s, u_s, y_s)\mathrm{d}s$$

Lagrangian function:

$$\begin{aligned} \mathcal{L} = \quad & \alpha \left(\int_0^T \ell(u_t, y_t)\mathrm{d}t + \phi(y_0, y_T) \right) \\ & + \int_0^T p_t \left(f(t, t, u_t, y_t) + \int_0^t D_\tau f(t, s, u_s, y_s)\mathrm{d}s - \dot{y}_t \right) \mathrm{d}t \\ & + \int_0^T g(y_t)\mathrm{d}\mu_t + \Psi\Phi(y_0, y_T) \end{aligned}$$

End-point Lagrangian function:

$$\Phi[\alpha, \Psi](y_0, y_T) := \alpha\phi(y_0, y_T) + \Psi\Phi(y_0, y_T).$$

Hamiltonian function (warning: the argument p is a function)

$$H[\alpha, p](t, u, y) := \alpha\ell(u, y) + p_t f(t, t, u, y) + \int_t^T p_s D_\tau f(s, t, u, y)\mathrm{d}s.$$

Lagrangian function again:

$$\mathcal{L} = \int_0^T (H[\alpha, p](t, u_t, y_t) - \dot{y}_t)\,\mathrm{d}t + \int_0^T g(y_t)\mathrm{d}\mu_t + \Phi[\alpha, \Psi](y_0, y_T)$$

Costate equation

From the above equations, it must be "as usual"

$$-\mathrm{d}p_t = D_y H[\alpha, p](t, u_t, y_t)\mathrm{d}t + \mathrm{d}\mu_t g'(y_t), \quad t \in (0, T),$$

plus usual end-point conditions, which means in fact

$$\begin{aligned}
-\mathrm{d}\bar{p}_t \quad &= \quad \bar{\alpha} D_y \ell(\bar{u}_t, \bar{y}_t)\mathrm{d}t + \bar{p}_t D_y f(t, t, \bar{u}_t, \bar{y}_t)\mathrm{d}t + \sum_{i=1}^{n_g} g_i'(\bar{y}_t)\mathrm{d}\bar{\eta}_{i,t} \\
&\quad + \int_t^T \bar{p}_s D^2_{\tau,y} f(s, t, \bar{u}_t, \bar{y}_t)\mathrm{d}s, \\
(-\bar{p}_{0-}, \bar{p}_{T+}) \quad &= \quad \Phi'[\bar{\alpha}, \bar{\Psi}](\bar{y}_0, \bar{y}_T).
\end{aligned}$$

Pontryagin multipliers

Let $(\bar{u}, \bar{y}) \in F(P)$. We say that $(\bar{\alpha}, \bar{\eta}, \bar{\Psi}, \bar{p})$ in $\mathbb{R}_+ \times \mathcal{M} \times \mathbb{R}^{n_\Phi *} \times \mathcal{P}$, is a *Pontryagin multiplier* associated with $(\bar{u}, \bar{y}) \in F(P)$ if the costate equation is satisfied, as well as:

$$\bar{\alpha} + \|\bar{\eta}\| + |\bar{\Psi}| > 0, \quad \textit{nontriviality} \tag{3}$$

$$\mathrm{d}\bar{\eta} \geq 0; \quad \sum_{i=1}^{n_g} \int_0^T g_i(y_t) \mathrm{d}\bar{\eta}_{i,t} = 0, \quad \textit{complementarity}, \tag{4}$$

$$\bar{\Psi} \in N_K(\Phi(\bar{y}_0, \bar{y}_T)), \quad \textit{transversality condition} \tag{5}$$

and *Hamiltonian inequality*

$$H[\bar{\alpha}, \bar{p}](t, \bar{u}_t, \bar{y}_t) \leq H[\bar{\alpha}, \bar{p}](t, u, \bar{y}_t), \quad \text{for all } u \in \mathbb{R}^m, \text{ a.a. } t \in (0, T). \tag{6}$$

Pontryagin principle

We say that $(\bar{u}, \bar{y}) \in F(P)$ is a *Pontryagin extremal*, or that it satisfies *Pontryagin's principle*, if the set of associated Pontryagin multipliers is not empty.

We say that $(\bar{u}, \bar{y}) \in F(P)$, is a local solution of (P) *in the* L^1 *norm* if the following holds:

$$\begin{cases} \int_0^T \ell(\bar{u}_t, \bar{y}_t)\mathrm{d}t + \phi(\bar{y}_0, \bar{y}_T) \leq \int_0^T \ell(u_t, y_t)\mathrm{d}t + \phi(y_0, y_T), \\ \text{for all } (u, y) \in F(P) \quad \text{such that } \|u - \bar{u}\|_1 + |y_0 - \bar{y}_0| \text{ is small enough.} \end{cases}$$

Theorem 1. *(PMP) Any local solution of problem (P), in the L^1 norm, is a Pontryagin extremal.*

Proof of the PMP

Based on

- Needle variations and "Pontryagin linearization"

$$z_t = y_0 - \bar{y}_0 + \int_0^t \left[D_y f(t,s,\bar{u}_s,\bar{y}_s) z_s + f(t,s,u_s,\bar{y}_s) - f(t,s,\bar{u}_s,\bar{y}_s) \right] \mathrm{d}s,$$

that satisfies

$$\|\bar{y} + z - y\|_\infty \leq C_1 \left(\|u - \bar{u}\|_1^2 + |y_0 - \bar{y}_0|^2 \right).$$

- Ekeland's principle of a penalized problem.

Total derivative of a function of the state

Remember that

$$\dot{y}_t = f(t,t,u_t,y_t) + \int_0^t D_\tau f(t,s,u_s,y_s)\mathrm{d}s.$$

Total derivative of $G(t,y_t)$, along the trajectory (y,u):

$$\begin{aligned} G^{(1)}(t,\tilde{u},\tilde{y},u,y) := \quad & D_t G(t,\tilde{y}) + D_{\tilde{y}} G(t,\tilde{y}) f(t,t,\tilde{u},\tilde{y}) \\ & + D_{\tilde{y}} G(t,\tilde{y}) \int_0^t D_\tau f(t,s,u_s,y_s)\mathrm{d}s. \end{aligned}$$

First order state constraint

The total derivative of the ith state constraint is $g_i^{(1)}(t, u_t, y_t, u, y)$, where

$$g_i^{(1)}(t, \tilde{u}, \tilde{y}, u, y) = g_i'(\tilde{y}) f(t, t, \tilde{u}, \tilde{y}) + g_i'(\tilde{y}) \int_0^t D_\tau f(t, s, u_s, y_s) \mathrm{d}s.$$

We say that the ith state constraint is of *first order* if the dependence w.r.t. $\tilde{u}$ of the above expression is non trivial, i.e., if

$$g_i'(\tilde{y}) D_u f(t, t, \tilde{u}, \tilde{y}) \neq 0, \quad \text{for some } (t, \tilde{u}, \tilde{y}) \in \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^n.$$

High order state constraint

If the state constraint is not of first order we can compute the second total derivative of the state constraint:

$$g_i^{(2)}(t, u_t, y_t, u, y) = D_t g_i^{(1)}(t, y_t, u, y) + D_{\tilde{y}} g_i^{(1)}(t, y_t, u, y)\dot{y}_t.$$

and we have

$$D_{\tilde{u}} g_i^{(2)}(t, u_t, y_t, u, y) = D_{\tilde{y}} g_i^{(1)}(t, y_t, u, y) D_u f(t, t, u_t, y_t).$$

and so on for higher orders ...

Continuity of the control

Let $(\bar{u}, \bar{y})$ be a Pontryagin extremal. We say that $\bar{u}$ has *side limits* on $[0, T]$ if it has left limits on $(0, T]$ and right limits on $[0, T)$. When $t \in (0, T)$ is such that $\bar{u}_t$ has left and right limits at time t, denoted by $\bar{u}_{t\pm}$, with jump $[\bar{u}_t] := \bar{u}_{t+} - \bar{u}_{t-}$, define

$$\bar{u}_t^{\sigma} := \bar{u}_{t-} + \sigma[\bar{u}_t], \quad \sigma \in [0, 1],$$

so that $\bar{u}_t^0 = \bar{u}_{t-}$ and $\bar{u}_t^1 = \bar{u}_{t+}$; we use the same convention for other functions. Set, for $\sigma \in [0, 1]$:

$$H^{\sigma}[\bar{\alpha}, \bar{p}](t, u, y) := \bar{\alpha}\ell(u, y) + \bar{p}_t^{\sigma} f(t, t, u, y) + \int_t^T \bar{p}_s D_{\tau} f(s, t, u, y) \mathrm{d}s.$$

Hamiltonian coercivity hypothesis

$$\left\{ \begin{array}{r} \text{For some } \alpha_H > 0,\ \alpha_H |[\bar{u}_t]|^2 \leq D^2_{uu} H^\sigma[\bar{\alpha}, \bar{p}](t, \bar{u}^\sigma_t, \bar{y}_t)([\bar{u}_t], [\bar{u}_t]), \\ \text{for all } \sigma \in [0,1],\ t \in [0,T]. \end{array} \right. \tag{7}$$

I_1 (resp. $I_1(t)$): the set of (resp. of active at time t) first order state constraints,

Positive linear independence w.r.t. control of *first-order* active state constraints

$$\sum_{i \in I_1(t)} \beta_i D_{\tilde{u}} g_i^{(1)}(t, \bar{u}_t, \bar{y}_t, \bar{u}, \bar{y}) = 0 \text{ and } \beta \geq 0 \text{ implies } \beta = 0, \text{ for all } t \in [0,T]. \tag{8}$$

$$H[\bar{\alpha}, p](t_\pm, u, y) := \bar{\alpha}\ell(u, y) + p_{t\pm} f(t, t, u, y) + \int_t^T p_s D_\tau f(s, t, u, y) \mathrm{d}s.$$

Theorem 2 (Continuity of the control)**.** *Let $(\bar{u}, \bar{y})$ be a Pontryagin extremal for (P) with associated Pontryagin multiplier $(\bar{\alpha}, \bar{\eta}, \bar{\Psi}, \bar{p})$.*
(i) *Assume that, for some $R > \|\bar{u}\|_\infty$, $H[\bar{\alpha}, \bar{p}](t_\pm, \cdot, \bar{y}_t)$ has, for all $t \in (0, T)$, a unique minimum w.r.t. the control over $B(0, R)$, denoted $\hat{u}_{t\pm}$. Then (a representative of) $\bar{u}$ has side limits on $[0, T]$, equal to $\hat{u}_{t\pm}$.*
(ii) *Assume that $\bar{u}$ has side limits on $[0, T]$ and that* (7) *holds. Then $\bar{u}$ is continuous.*
(iii) *Assume that the control is continuous and that* (8) *holds. Then the multipliers η_i associated with components g_i of the state constraint of* first order *($q_i = 1$) are continuous on $[0, T]$.*

First-order alternative system

Similarly to Hager [79] define the *alternative multiplier and costate*:

$$\eta^1_t := -\bar{\eta}_t; \quad p^1_t := \bar{p}_t - \eta^1_t g'(\bar{y}_t), \quad t \in [0,T].$$

Alternative Hamiltonian:

$$\begin{aligned} H^1[\alpha, p^1, \eta^1](t, \tilde{u}, \tilde{y}, u, y) := \quad & H[\alpha, p^1](t, \tilde{u}, \tilde{y}) + \eta^1 g^{(1)}(t, \tilde{u}, \tilde{y}, u, y) + \\ & \int_t^T \eta^1_s g'(\bar{y}_s) D_\tau f(s, t, \tilde{u}, \tilde{y}) \mathrm{d}s. \end{aligned}$$

Then

$$-\dot{p}^1_t = D_{\tilde{y}} H^1[\bar{\alpha}, p^1, \eta^1](t, \bar{u}_t, \bar{y}_t, \bar{u}, \bar{y}), \quad t \in (0,T).$$

Dependence w.r.t. the control:

$$H^1[\alpha, p^1, \eta^1](t, \tilde{u}, \bar{y}_t, \bar{u}, \bar{y}) = H[\alpha, \bar{p}](t, \tilde{u}, \bar{y}_t) + \eta^1_t g'(\bar{y}_t) \int_0^t D_\tau f(t, s, \bar{u}_s, \bar{y}_s) \mathrm{d}s.$$

It follows that stationarity or minimality of H w.r.t. u holds iff H^1 has the same property w.r.t. $\tilde{u}$.

"A trajectory is a Pontryagin extremal iff it is a Pontryagin extremal for the alternative system"

Lipschitz regularity results

- Feasibility of realization of the control taking into account technological constraints

- Estimates of variation of solution for perturbed problems (Dontchev and Hager 1993, 98)

- Numerical analysis and error estimates: open problem
  First order state constraints: see Dontchev and Hager, 2001

Literature on Lipschitz regularity

- Hager (1979)
- Shvartsman and Vinter: first order state constraints (2006)
- Do Rosario de Pinho and Shvartsman: first order state constraints (to appear)
- Hermant: second order state constraints (2009)

Hager's lemma (1979): preparation

Let X be a Banach space, and x be a continuous function $[0,T] \to X$. Let $I : [0,T] \to \{1,\dots,n\}$ be upper continuous, i.e.,

If $t_n \to t \in [0,T]$, and $i \in I(t_n)$, then $i \in I(t)$.

In our application $I(t)$ is the set of active constraints.
We say that the pair (a,b) in $[0,T]^2$ is *compatible* if

$$a < b; \quad I(a) = I(b); \quad I(t) \subset I(a), \quad \text{for all } \ t \in (a,b),$$

We say that $L > 0$ is a Lipschitz constant for x over $E \subset [0,T]^2$ if

$$\|x(a) - x(b)\| \leq L|b-a| \quad \text{whenever } (a,b) \in E.$$

Hager's lemma (1979): statement

Lemma 1. *Assume that* $x \in C([0,T], X)$ *and that* I *is upper continuous. Let* $L > 0$ *be a Lipschitz constant for* x *over the set of compatible pairs. Then* L *is a Lipschitz constant for* x *i.e., we have that*

$$\|x(a) - x(b)\| \leq L|b - a|, \quad \textit{for all } (a,b) \in [0,T]^2.$$

Lipschitz continuity of the control: hypotheses

Hypthesis of continuous control
All state constraints of first order: $I(t) = I_1(t)$.
Constraint qualification:

$$\sum_{i \in I(t)} \beta_i D_{\tilde{u}} g_i^{(1)}(t, \bar{u}_t, \bar{y}_t, \bar{u}, \bar{y}) = 0 \quad \text{implies } \beta = 0, \quad \text{for all } t \in [0, T]. \quad (9)$$

Strong Legendre-Clebsch condition, reduced to a subspace:

$$\begin{array}{l} \text{For some } \alpha_H > 0: \;\; \alpha_H |v|^2 \leq D^2_{uu} H[\bar{\alpha}, \bar{p}](t, \bar{u}_t, \bar{y}_t)(v, v), \\ \text{whenever } D_{\tilde{u}} g_i^{(1)}(t, \bar{u}_t, \bar{y}_t, \bar{u}, \bar{y}) v = 0, \text{ pour tout } \; i \in I(t), \; t \in [0, T]. \end{array} \quad (10)$$

Lipschitz continuity of the control: hypotheses II

Definition 1. *Let* $(\bar u, \bar y) \in F(P)$. *We say that* $(\bar\alpha, \bar\eta, \bar\Psi, \bar p)$ *in* $\mathbb{R}_+ \times \mathcal{M} \times \mathbb{R}^{n_\Phi *} \times \mathcal{P}$, *is a* first order multiplier *if the PMP conditions holds except for the Hamiltonian minimality replaced by the* Hamiltonian stationarity *condition*

$$D_u H[\bar\alpha, \bar p](t, \bar u_t, \bar y_t) = 0, \qquad \text{for a.a. } t \in (0, T). \tag{11}$$

We say that $(\bar u, \bar y) \in F(P)$ *is a* first-order extremal *if the set of associated first order multipliers is not empty.*

The theory of alternative optimality system has a straighforward extension to first order extremals, replacing the Hamiltonian minimality condition by

$$D_u H^1[\bar\alpha, p^1, \eta^1](t, \bar u_t, \bar y_t, \bar u, \bar y) = 0, \text{for a.a. } t \in (0, T). \tag{12}$$

Lipschitz continuity of the control: main result

Theorem 3. *Let* $(\bar{u}, \bar{y}, \bar{p}, \bar{\eta})$ *be a first-order extremal and associated multipliers, with* $\bar{u}$ *continuous. If hypotheses* (9) *and* (10) *hold, then* $\bar{u}$ *and* $\bar{\eta}$ *are Lipschitz functions of time.*

Proof. Define $\tilde{\eta} \in \mathcal{M}$

$$\tilde{\eta}_{i,t} = 0 \ \text{ if } i \in I(t), \text{ and } \tilde{\eta}_{i,t} = \eta^1_{i,t} \text{ otherwise.} \tag{13}$$

Consider the function, where $\eta^\sharp \in \mathbb{R}^{n_g *}$:

$$\begin{aligned} F[t, \bar{u}, \bar{y}, \alpha, p^1, \eta^1, \eta^\sharp](u) := \quad & H[\alpha, p^1](t, u, \bar{y}_t) + \eta^\sharp g^{(1)}(t, u, \bar{y}_t, \bar{u}, \bar{y}) \\ & + \int_t^T \eta^1_s g'(\bar{y}_s) D_\tau f(s, t, u, \bar{y}_t) \mathrm{d}s, \end{aligned} \tag{14}$$

whose expression is similar to the one of the alternative Hamiltonian, but with $\eta^\sharp$ instead of η^1_t in the second term of the sum in the r.h.s. Consider, fot $t \in (0,T)$, the nonlinear programming problem

$$\operatorname*{Min}_{u \in \mathbb{R}^m} \; F[t, \bar{u}, \bar{y}, \alpha, p^1, \eta^1, \tilde{\eta}_t](u) \text{ subject to } g_i^{(1)}(t, u, \bar{y}_t, \bar{u}, \bar{y}) = 0, \; i \in I(t). \tag{15}$$

We claim that $\bar{u}_t$ is a local solution of this problem. Indeed, let $i \in I(t)$, for some $t \in [0,T]$. Then $g_i(\bar{y}_t)$ reaches a local maximum, and hence its total derivative $g_i^{(1)}(t, u, \bar{y}_t, \bar{u}, \bar{y})$ is equal to zero. Therefore, $\bar{u}_t$ is a feasible point of problem (15).

In view of the qualification hypothesis (9), there exists at most one Lagrange multiplier associated with $\bar{u}_t$, and this multiplier is characterized by the condition of stationarity of the Lagrangian of the problem w.r.t. the control. We may consider the Lagrange multiplier denoted $\eta^\flat$ as an element of $\mathbb{R}^{n_g *}$ whose all components in $\bar{I}(t)$ are set to zero. With that convention, the first-order optimality conditions of problem (15) may be expressed as

$$D_u F[t, \bar{u}, \bar{y}, \alpha, p^1, \eta^1, \tilde{\eta}_t + \eta^\flat](u) = 0; \quad g_i^{(1)}(t, u, \bar{y}_t, \bar{u}, \bar{y}) = 0, \; i \in I(t). \tag{16}$$

In view of the Hamiltonian stationarity condition (12), we see that the multiplier is nothing but $\hat{\eta}_t$, defined by (compare to (13)):

$$\hat{\eta}_{i,t} = \eta_{i,t} \text{ if } i \in I(t), \text{ and } \hat{\eta}_{i,t} = 0 \text{ otherwise.} \tag{17}$$

In view of (**??**), we see that

$$F[t, \bar{u}, \bar{y}, \alpha, p^1, \eta^1, \tilde{\eta}_t + \hat{\eta}_t](u) = F[t, \bar{u}, \bar{y}, \alpha, p^1, \eta^1, \eta_t^1](u) = H[\bar{\alpha}, \bar{p}](t, \bar{u}_t, \bar{y}_t). \tag{18}$$

Therefore hypothesis (10) can be interpreted as the condition of positive curvature of the Lagrangian of problem (15) over the set of critical directions (which are identical to the set of tangent directions since there are only equality constraints). This is a well-known sufficient condition for local optimality for nonlinear programming problems, see e.g. "BGLS". It follows that $\bar{u}_t$ is a local solution of (15), as was claimed.

In addition, by the previous discussion, the Jacobian of optimality conditions (16) w.r.t. unknowns $(u, \hat{\eta})$ is

$$\begin{pmatrix} D^2_{uu} H[\bar{\alpha}, \bar{p}](t, \bar{u}_t, \bar{y}_t) & D_{\tilde{u}} g^{(1)}_{I(t)}(t, \bar{u}_t, \bar{y}_t, \bar{u}, \bar{y})^\top \\ D_{\tilde{u}} g^{(1)}_{I(t)}(t, \bar{u}_t, \bar{y}_t, \bar{u}, \bar{y}) & 0 \end{pmatrix}, \tag{19}$$

and is, in view of hypotheses (9)-(10), invertible at the point $(\bar{u}_t, \hat{\eta}_t)$.

Let $0 < T_1 < T_2 < T$, and (a, b) be a compatible pair, in the sense of section **??**, for the set $I(t)$. For $t \in [0, T]$, denote the set of non active first-order constraints by $\bar{I}(t) := \{1, \ldots, n_g\} \setminus I(t)$. Then $\bar{I}(a) = \bar{I}(b)$. The data of problem (15) satisfy a Lipschitz condition, with a constant not depending on the particular (a,b), since either they are indeed Lipschitz functions of time, or, in the case of $\tilde{\eta}$, it has the same value at time a and b. By the implicit function theorem, applied to (16), and standard compactness arguments, there exists $\varepsilon > 0$ and $c > 0$ such that, if $b < a+\varepsilon$, then

$$|\bar{u}_b - \bar{u}_a| + |\eta^1_b - \eta^1_a| \leq c(b-a), \quad \text{for all compatible pairs } (a,b) \text{ such that } b < a+\varepsilon.$$

By lemma 1, $(\bar{u}, \eta^1)$ is Lipschitz over (a,b) whenever $b < a+\varepsilon$. It follows that $(\bar{u}, \eta^1)$ is Lipschitz over $[T_1, T_2]$. But since the Lipschitz constant (which is the one for compatible pairs) may be taken uniform over $(0,T)$, and since $\bar{u}$ is continuous, the conclusion follows. $\square$

Extensions ?

- High order state constraints: Lipschitz regularity of the control
- Second order necessary and sufficient conditions
- Sensitivity analysis, directional derivatives of solutions
- Numerical analysis
  In the ODE framework, for control constrained problems:
  consequence of theory of partitioned Runge-Kutta methods.
  See Hager (2000), Bonnans and Laurent-Varin (2006).

FIN de l'exposé !