SECOND-ORDER VARIATIONAL MODELS FOR IMAGE DENOISING AND TEXTURE EXTRACTION

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COLLOQUE JBHU 2010
JBHU RI OMENALDIA
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MAPMO Orléans
Fédération Denis Poisson
Orléans - Tours

OUTLINE

1. Functional framework
2. Variational model for denoising and texture extraction
3. Numerical methods (discretization and algorithms)
4. Examples
5. Anisotropic strategy

1. Functional framework

1.1 Functions of bounded variation

Ω open subset of $\mathbb{R}^2$.

$$|f|_{BV(\Omega)} := \sup \left\{ \int_\Omega f \,\mathrm{div}(\phi) \mid \phi \in \mathcal{C}_c^1(\Omega, \mathbb{R}^2),\ \|\phi\|_\infty \leq 1 \right\}$$

$$BV(\Omega) = \left\{ f \in L^1(\Omega) \mid |f|_{BV(\Omega)} < +\infty \right\}$$

1.2. Functions of second order bounded variation

$$BV^2(\Omega) = \left\{ f \in W^{1,1}(\Omega) \mid |f|_{BV^2(\Omega)} < +\infty \right\}$$

$$|f|_{BV^2(\Omega)} := \sup \left\{ \int_\Omega \langle \nabla f, \operatorname{div}(\phi) \rangle_{\mathbb{R}^2} \mid \phi \in \mathcal{C}_c^2(\Omega, \mathbb{R}^{2\times 2}),\ \|\phi\|_\infty \leq 1 \right\}$$

$$\operatorname{div}(\phi) = (\operatorname{div}(\phi_1), \operatorname{div}(\phi_2)), \quad \text{with} \ \ \forall i,\ \phi_i = (\phi_i^1, \phi_i^2) : \Omega \to \mathbb{R}^2$$

$$W^{1,1}(\Omega) \subset BV(\Omega) \qquad\qquad H^1(\Omega) \subset BV^2(\Omega)$$

$f \in BV^2(\Omega)$ if and only if

$$f \in W^{1,1}(\Omega) \text{ and } \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\right) \in (BV(\Omega))^2.$$

$$|f|_{BV^2(\Omega)} \leq \sum_{i=1}^{n} \left|\frac{\partial f}{\partial x_i}\right|_{BV(\Omega)} \leq n\,|f|_{BV^2(\Omega)}$$

2. VARIATIONAL MODELS

2.1 Rudin-Osher-Fatemi (ROF) model for denoising

$$\mathcal{P}_{\mathcal{ROF}} : \qquad \min_{u \in BV(\Omega)} \mathcal{F}_R(u) := \lambda |u|_{BV(\Omega)} + \frac{1}{2} \|u_d - u\|^2_{L^2(\Omega)}$$

Problem $\mathcal{P}_{\mathcal{ROF}}$ has a unique solution

2.1 Second order model
(for denoising and texture extraction)

$$\min_{v \in BV^2(\Omega)} \mathcal{F}_{R2}(v) := \lambda |v|_{BV^2(\Omega)} + \frac{1}{2}\|u_d - v\|^2_{L^2(\Omega)} + \delta \|v\|_{W^{1,1}(\Omega)}$$

Problem $\mathcal{P}_2$ has a unique solution

3.1 ROF finite difference discretization

$$(\nabla u)_{i,j} = \left((\nabla u)^1_{i,j}, (\nabla u)^2_{i,j} \right),$$

$$(\nabla u)^1_{i,j} = \begin{cases} u_{i+1,j} - u_{i,j} & \text{if } i < N \\ 0 & \text{if } i = N, \end{cases}$$

$$(\nabla u)^2_{i,j} = \begin{cases} u_{i,j+1} - u_{i,j} & \text{if } j < N \\ 0 & \text{if } j = N. \end{cases}$$

$$(\text{div } p)_{i,j} = \begin{cases} p^1_{i,j} - p^1_{i-1,j} & \text{if } 1 < i < N \\ p^1_{i,j} & \text{if } i = 1 \\ -p^1_{i-1,j} & \text{if } i = N \end{cases}$$

$$+ \begin{cases} p^1_{i,j} - p^2_{i,j-1} & \text{if } 1 < j < N \\ p^2_{i,j} & \text{if } j = 1 \\ -p^1_{i,j-1} & \text{if } i = N. \end{cases}$$

$$\inf_{u \in X} \left(J_1(u) + \frac{1}{2\lambda} \|u_d - u\|_X^2 \right)$$

$$J_1(u) := \sum_{1 \le i,j \le N} \| (\nabla u)_{i,j} \|_{\mathbb{R}^2},$$

$$X = \mathbb{R}^{N \times N} \text{ and } Y = X \times X$$

Chambolle projection algorithm

The solution to the discretized problem is given by

$$u = u_d - P_{\lambda K_1}(u_d)$$

$$K_1 = \{\mathrm{div}\ g \mid g \in Y,\ \|g_{i,j}\|_{\mathbb{R}^2} \leq 1\ \forall i, j\}$$

$$\min\{\|\lambda \mathrm{div}\, p - f\|_X^2 \mid \|p_{i,j}\|_{\mathbb{R}^2} \leq 1 \ \forall i, j = 1, \ldots, N\}$$

$$p^0 = 0$$

$$p_{i,j}^{n+1} = \frac{p_{i,j}^n + \tau \left(\nabla \left(\mathrm{div}\, p^n - f/\lambda\right)\right)_{i,j}}{1 + \tau \| \left(\nabla \left(\mathrm{div}\, p^n - f/\lambda\right)\right)_{i,j} \|_{\mathbb{R}^2}}$$

$$\tau \leq 1/8 \Longrightarrow \lambda \mathrm{div}\, p^n \to P_{\lambda K_1}(f)$$

3.2 ROF2 discretization

$$
(Hv)^{11}_{i,j} = \begin{cases} v_{i+1,j} - 2v_{i,j} + v_{i-1,j} & \text{for } 1 < i < N, \\ v_{i+1,j} - v_{i,j} & \text{for } i = 1, \\ v_{i-1,j} - v_{i,j} & \text{for } i = N, \end{cases}
$$

$$
(Hv)^{12}_{i,j} = \begin{cases} v_{i,j+1} - v_{i,j} - v_{i-1,j+1} + v_{i-1,j} & \text{for } 1 < i \leq N,\ 1 \leq j < N, \\ 0 & \text{for } i = 1, \\ 0 & \text{for } i = N, \end{cases}
$$

$$
(Hv)^{21}_{i,j} = \begin{cases} v_{i+1,j} - v_{i,j} - v_{i+1,j-1} + v_{i,j-1} & \text{for } 1 \leq i < N,\ 1 < j \leq N, \\ 0 & \text{for } i = 1, \\ 0 & \text{for } i = N, \end{cases}
$$

$$
(Hv)^{22}_{i,j} = \begin{cases} v_{i,j+1} - 2v_{i,j} + v_{i,j-1} & \text{for } 1 < j < N, \\ v_{i,j+1} - v_{i,j} & \text{for } j = 1, \\ v_{i,j-1} - v_{i,j} & \text{for } j = N. \end{cases}
$$

3.2 ROF2 discretization

$$\min_{v \in X} \left(J_2(v) + \frac{\|u_d - v\|_X^2}{2\lambda} \right).$$

$$J_2(v) := \sum_{1 \le i,j \le N} \|(Hv)_{i,j}\|_{\mathbb{R}^4}$$

$$v = u_d - P_{\lambda K_2}(u_d)$$

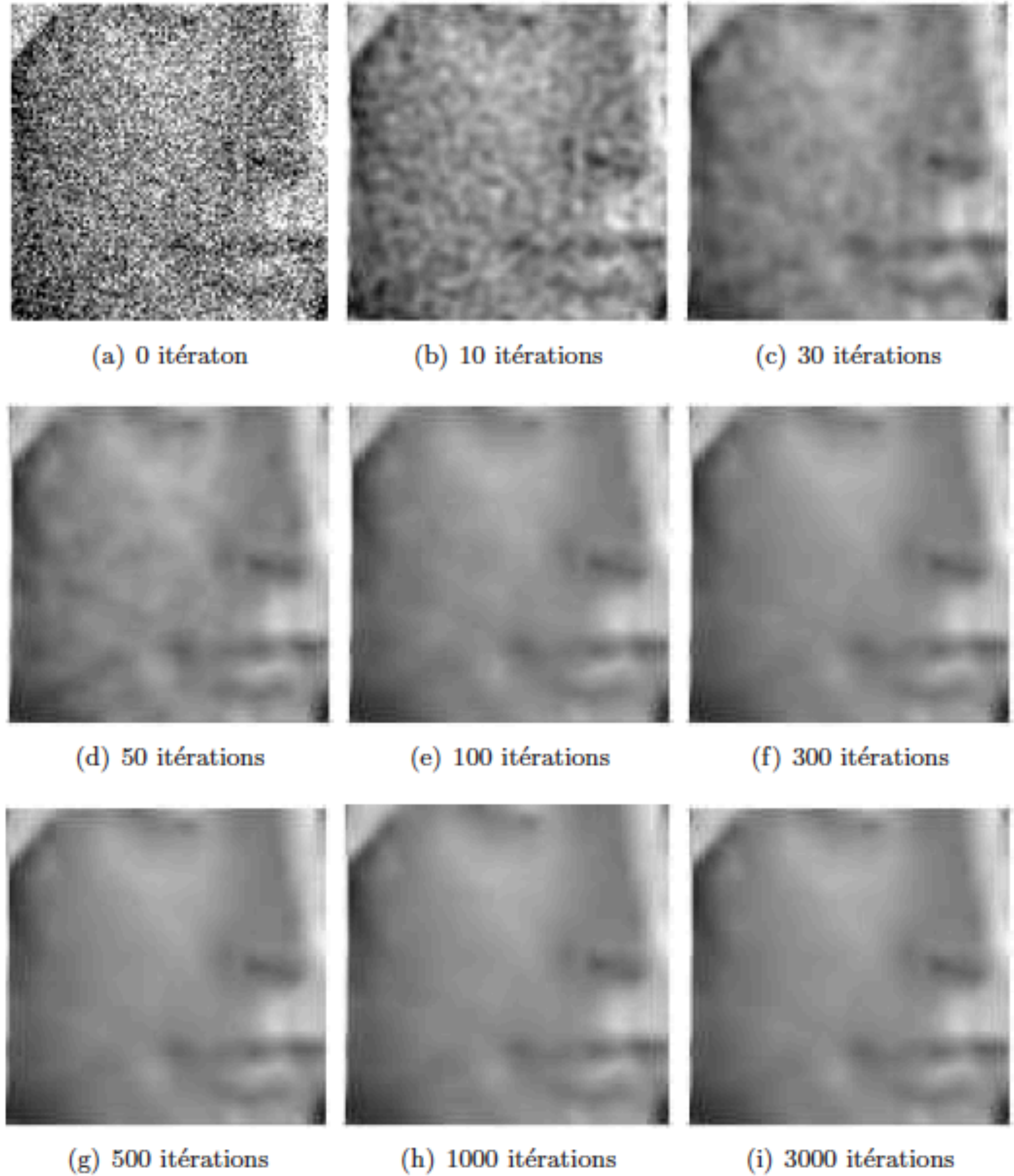
$$K_2 := \{H^* p \mid \|p_{i,j}\|_{\mathbb{R}^4} \le 1, \ \forall i,j = 1, \ldots, N\}$$

$$p^0 = 0$$

$$p^{n+1}_{i,j} = \frac{p^n_{i,j} - \tau \left(H \left[H^* p^n - u_d/\lambda \right] \right)_{i,j}}{1 + \tau \| \left(H \left[H^* p^n - u_d/\lambda \right] \right)_{i,j} \|_{\mathbb{R}^4}}$$

$$\tau \leq 1/64 \Longrightarrow \lambda H^* p^n \to P_{\lambda K_2}(f)$$

Number of iterations

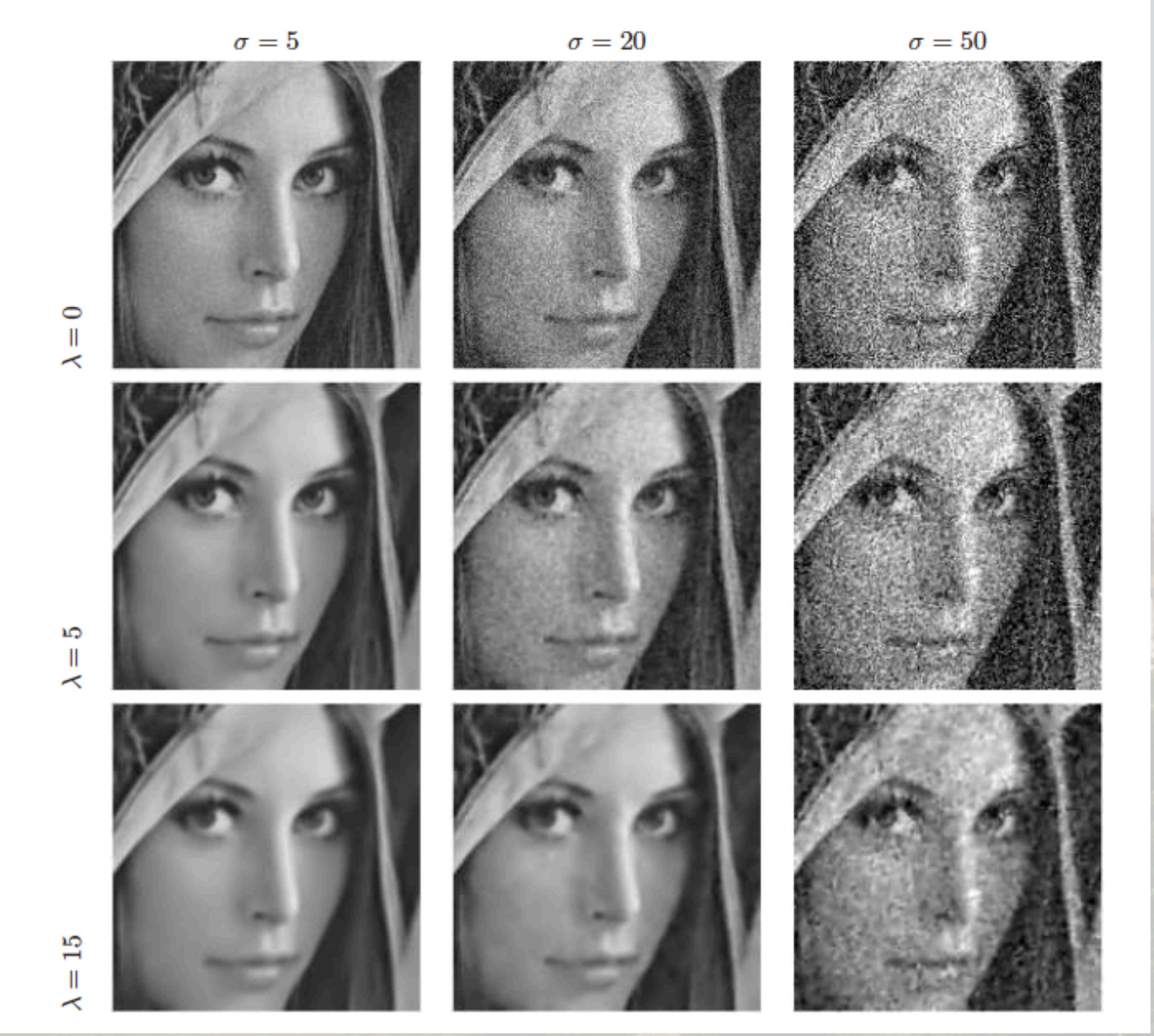


(a) 0 itératon (b) 10 itérations (c) 30 itérations

(d) 50 itérations (e) 100 itérations (f) 300 itérations

(g) 500 itérations (h) 1000 itérations (i) 3000 itérations

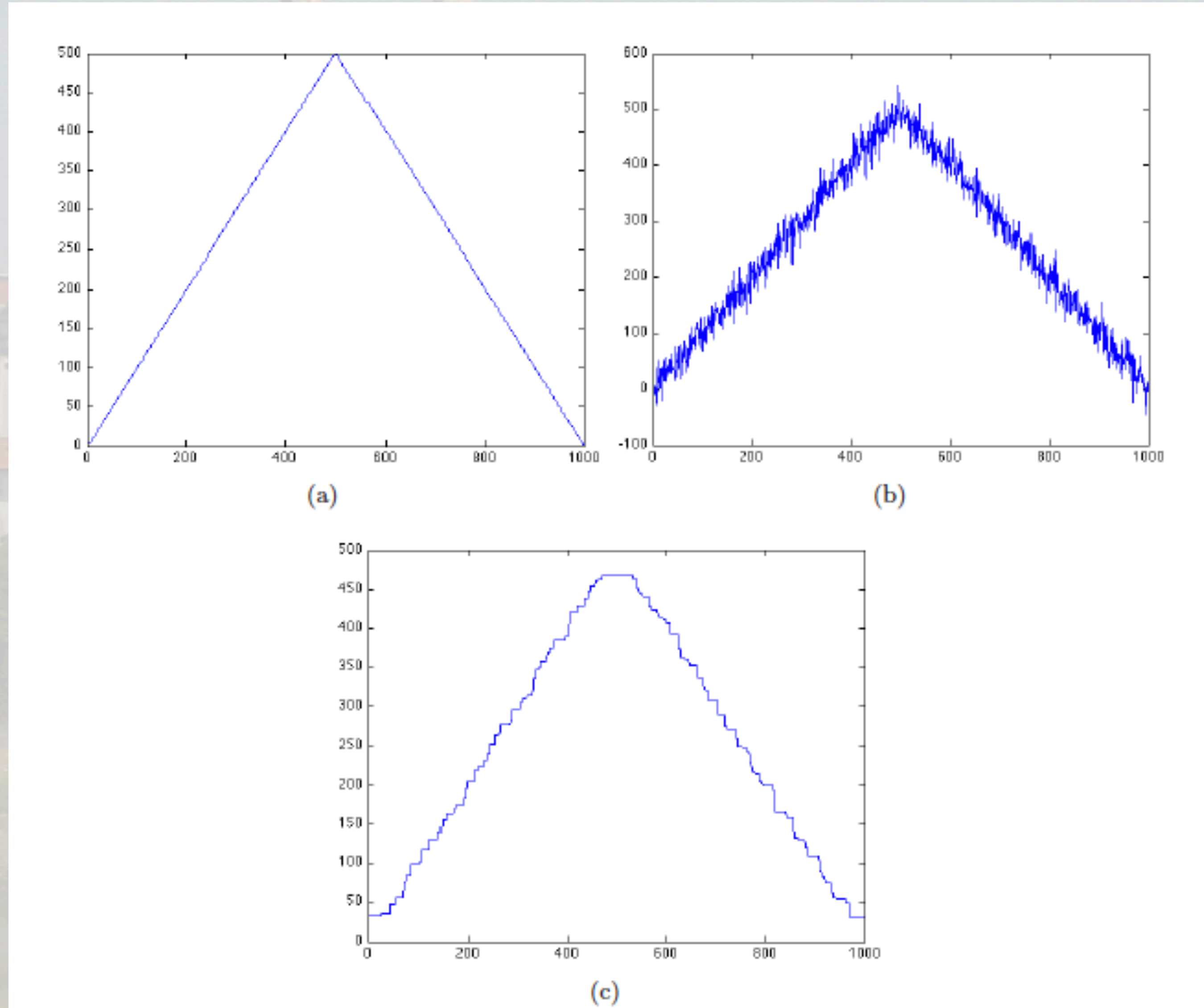
Influence of λ



4. NUMERICAL EXAMPLES

4.1 Denoising and staircasing effect

ROF



(a) (b) (c)

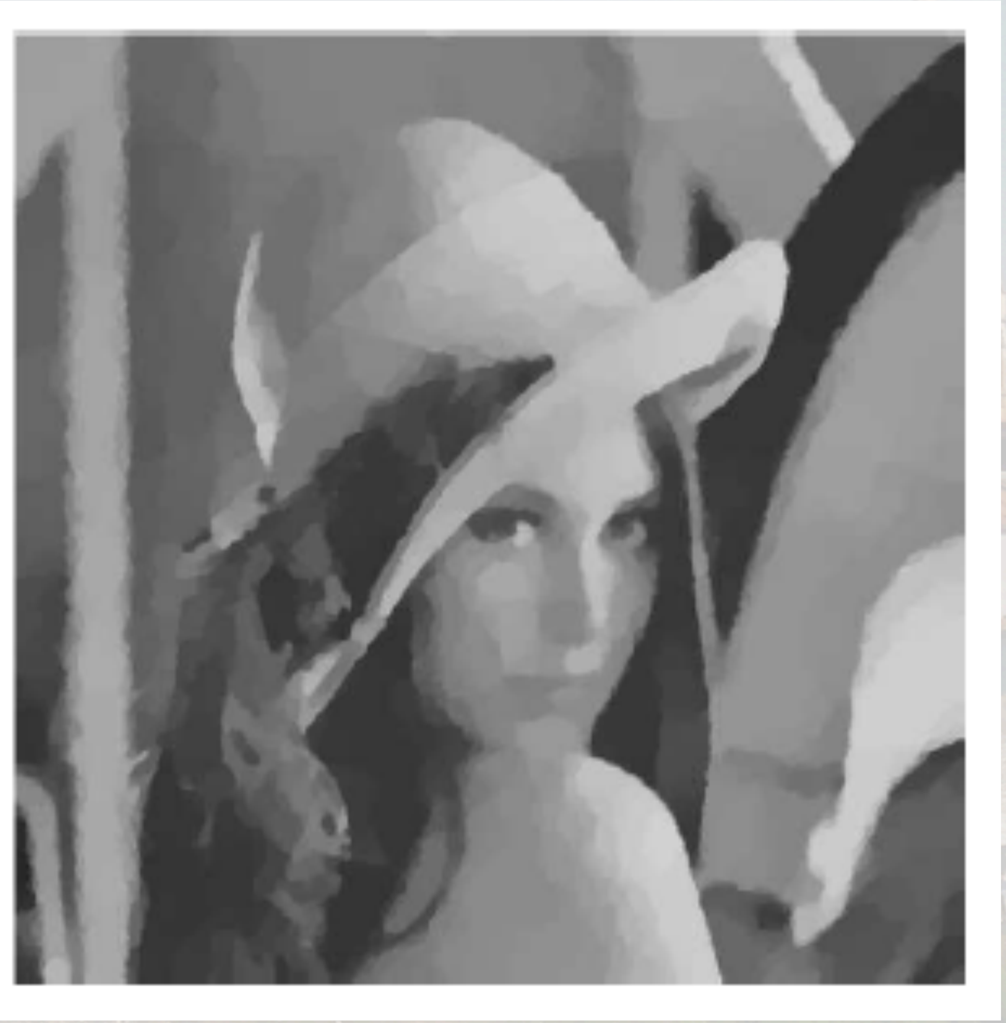
ROF
ROF2

ROF

ROF2

ROF

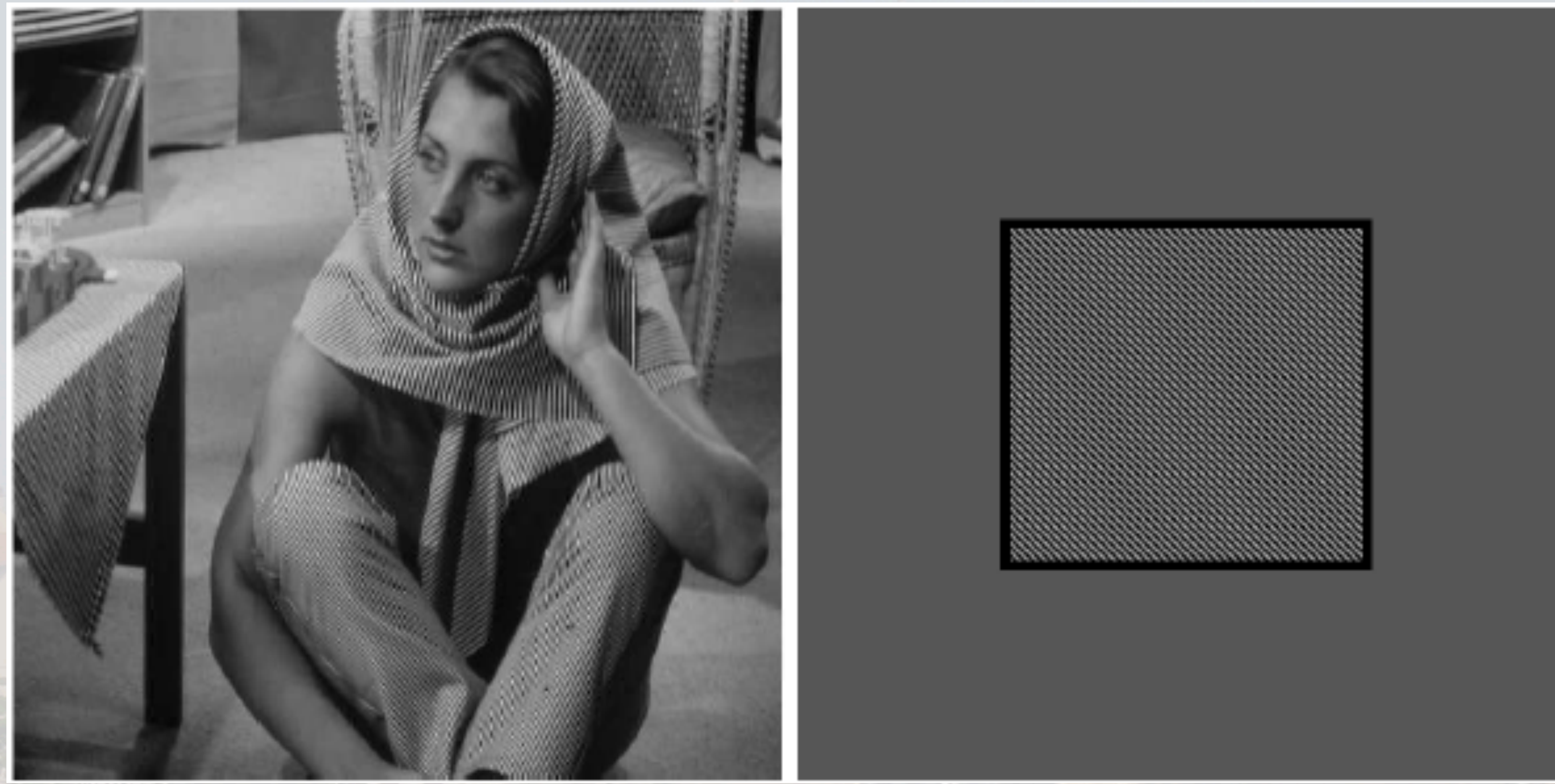
ROF2

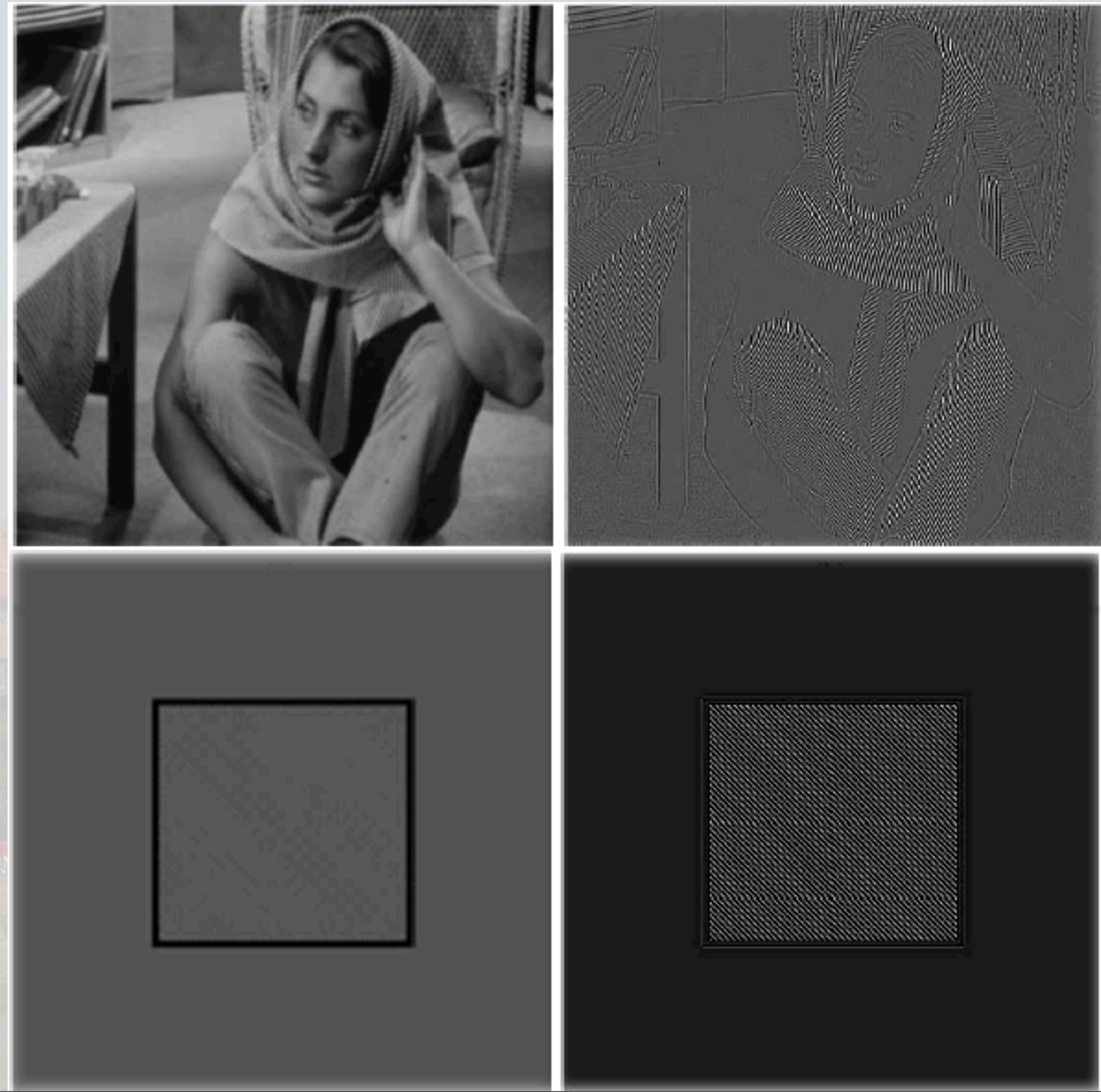


ROF

ROF2

4.2 Texture extraction





5. ANISOTROPIC STRATEGY

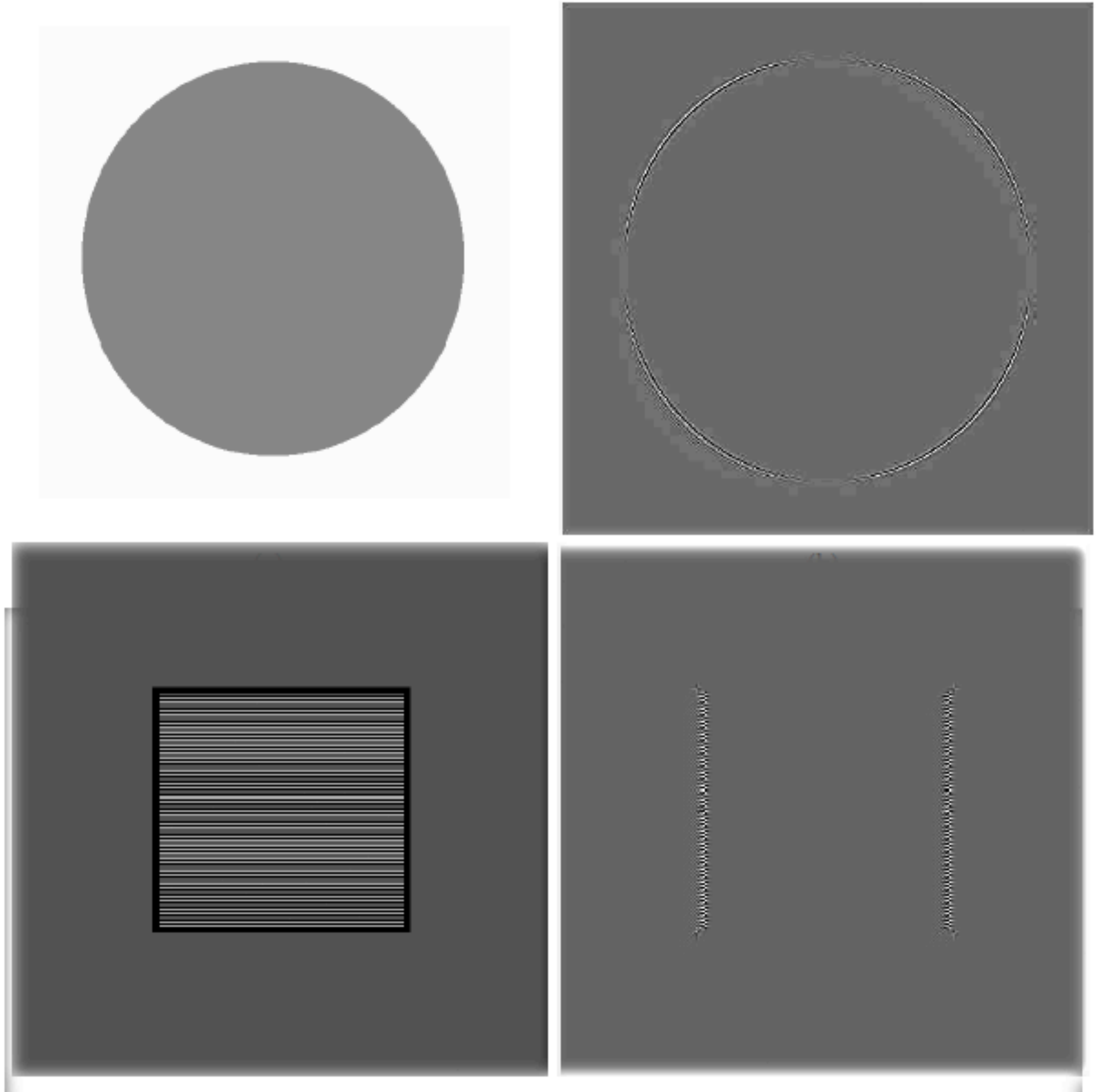
5.1 Global strategy

Pixel (i,j)

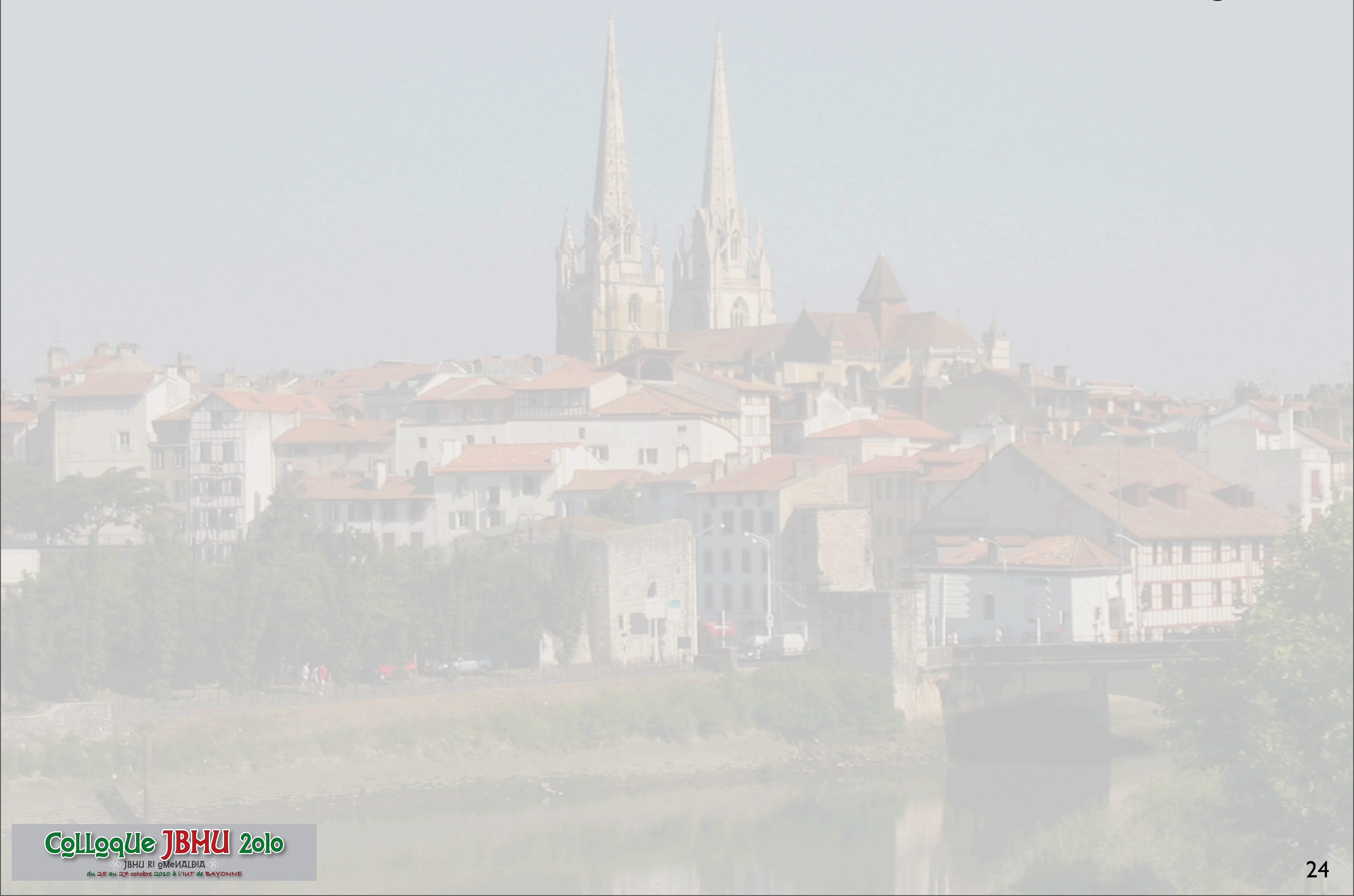
$$H(i,j) = \begin{bmatrix} H^{11} & H^{12} \\ H^{21} & H^{22} \end{bmatrix}$$

$$\Longrightarrow \qquad \tilde{H}(i,j) = \begin{bmatrix} 0 & H^{12} \\ H^{21} & H^{22} \end{bmatrix}$$

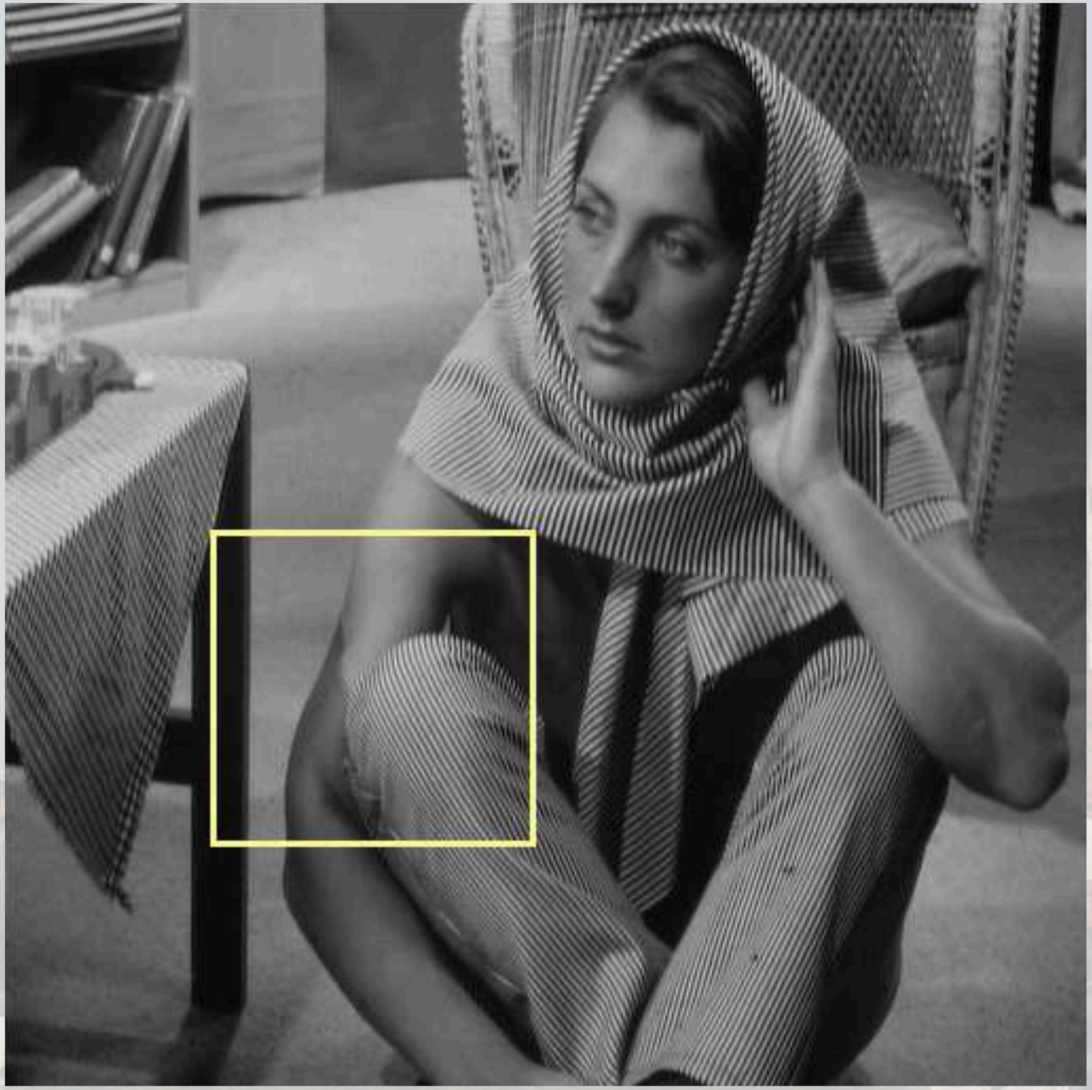
$\Longrightarrow$ No vertical contour any longer



5.2 Local strategy



5.2 Local strategy



5.2 Local strategy

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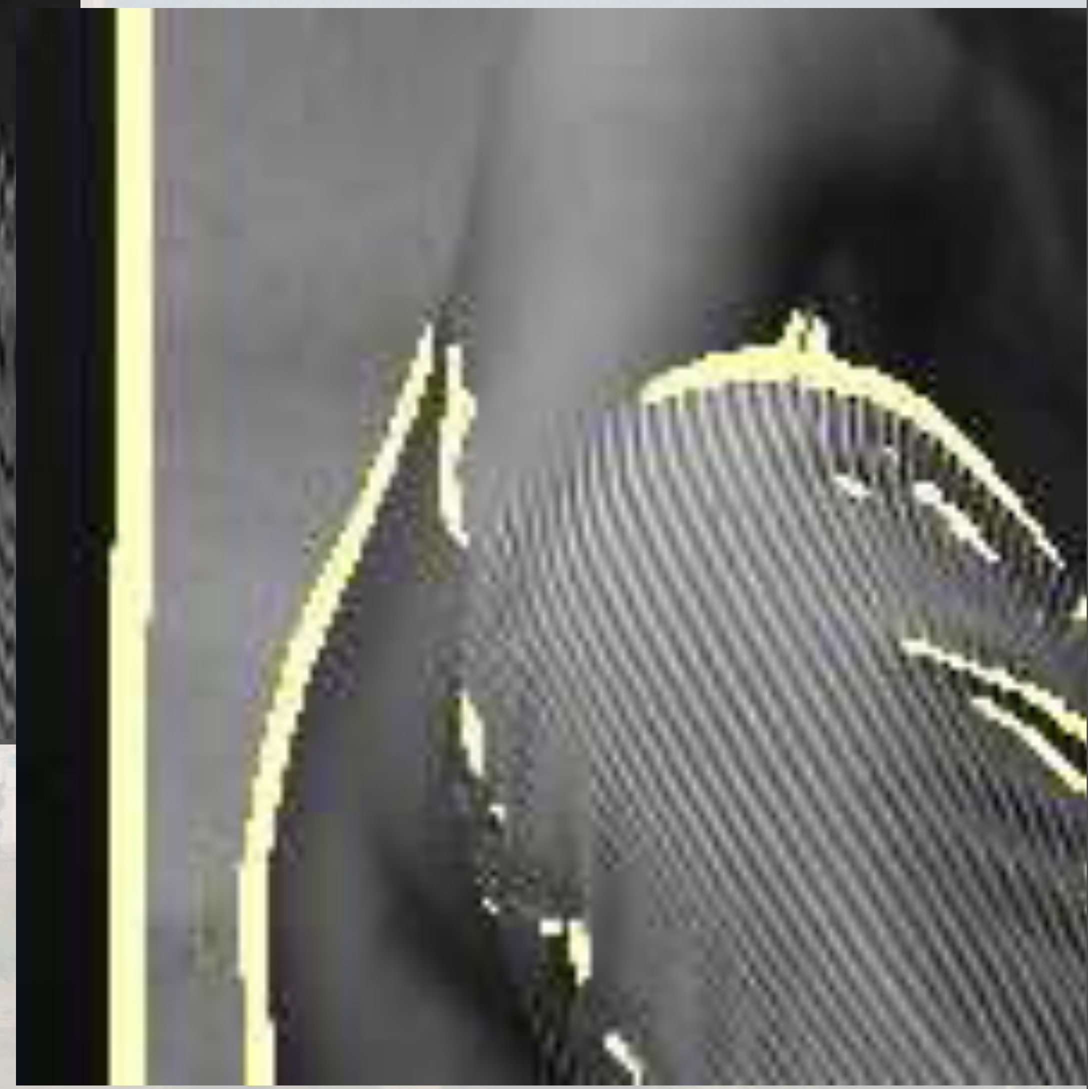

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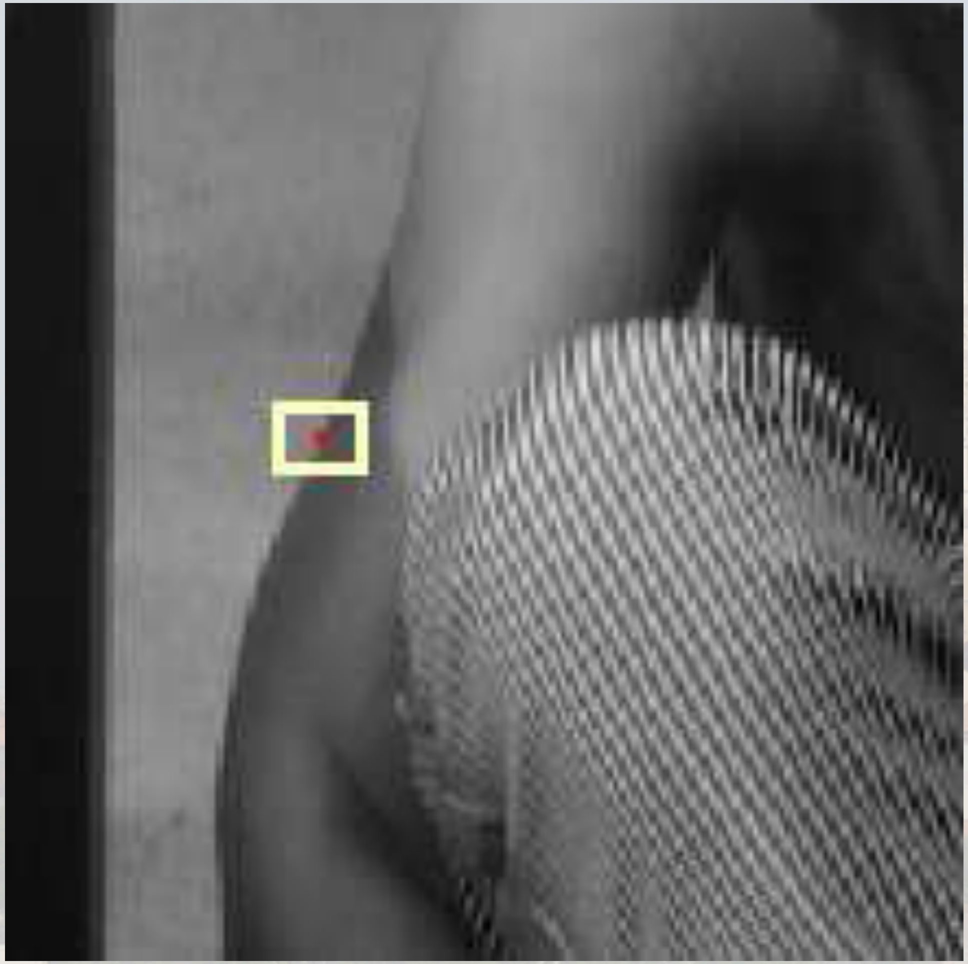
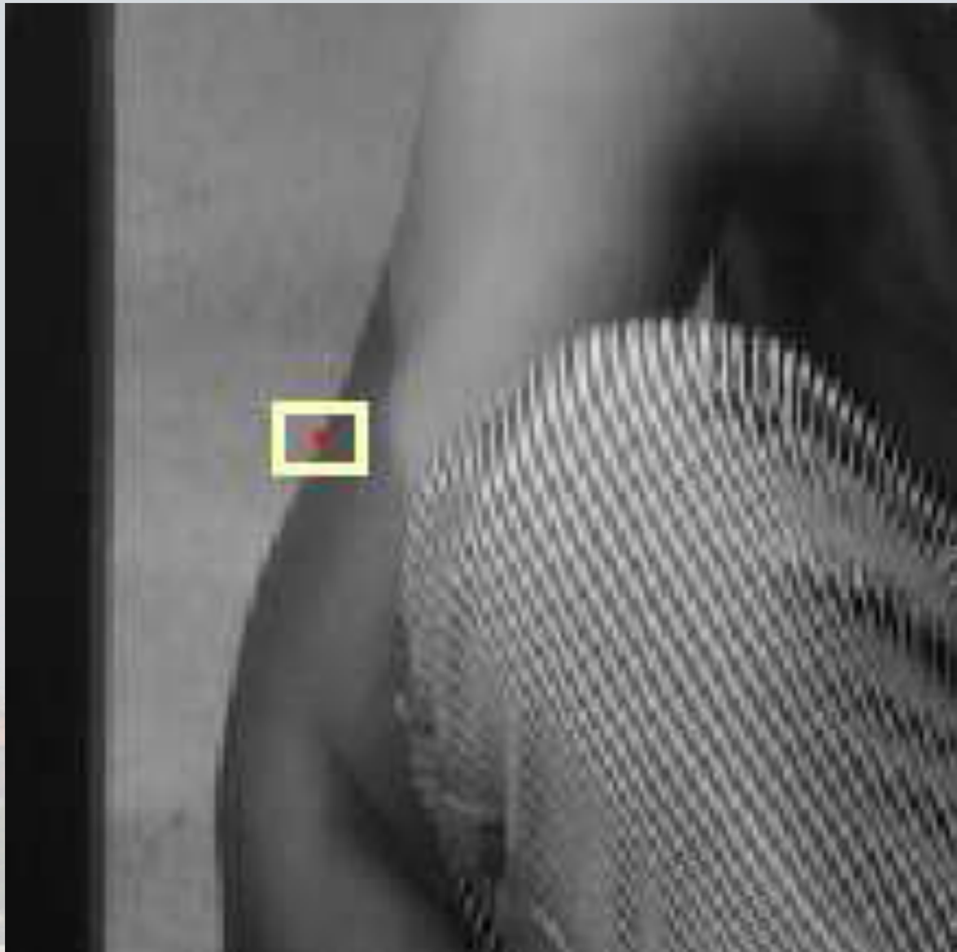
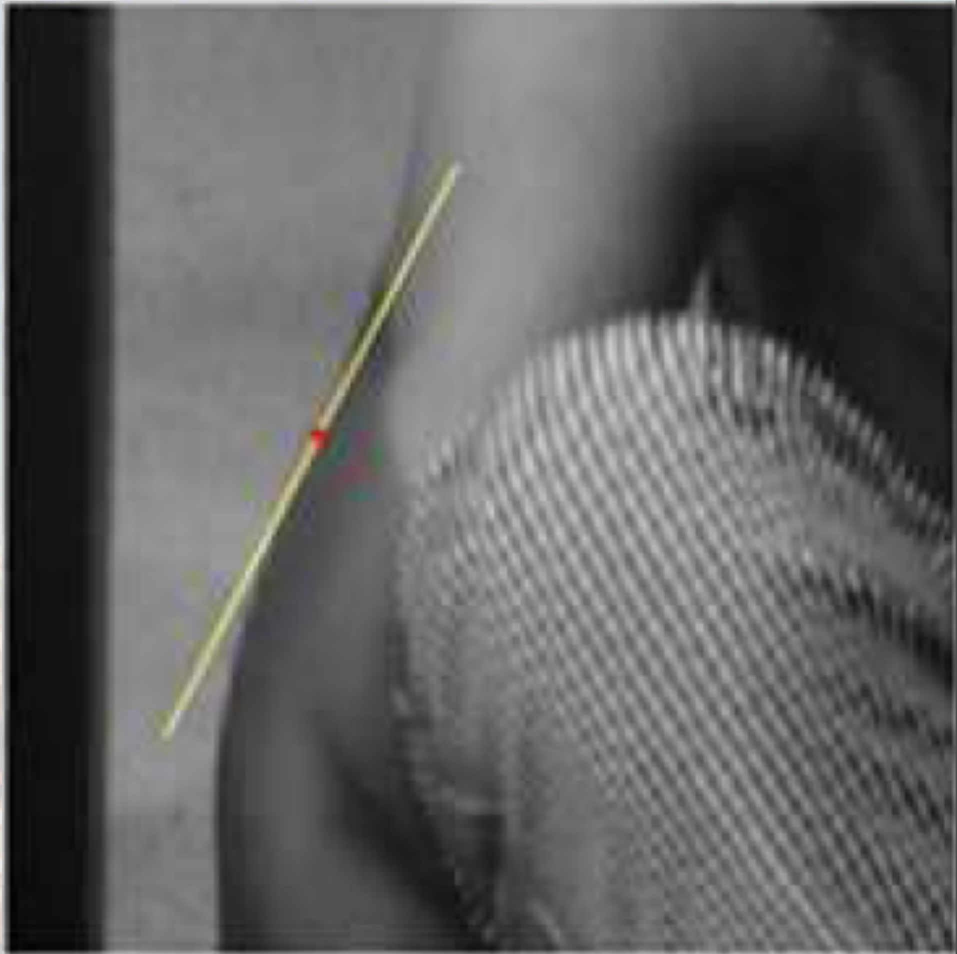
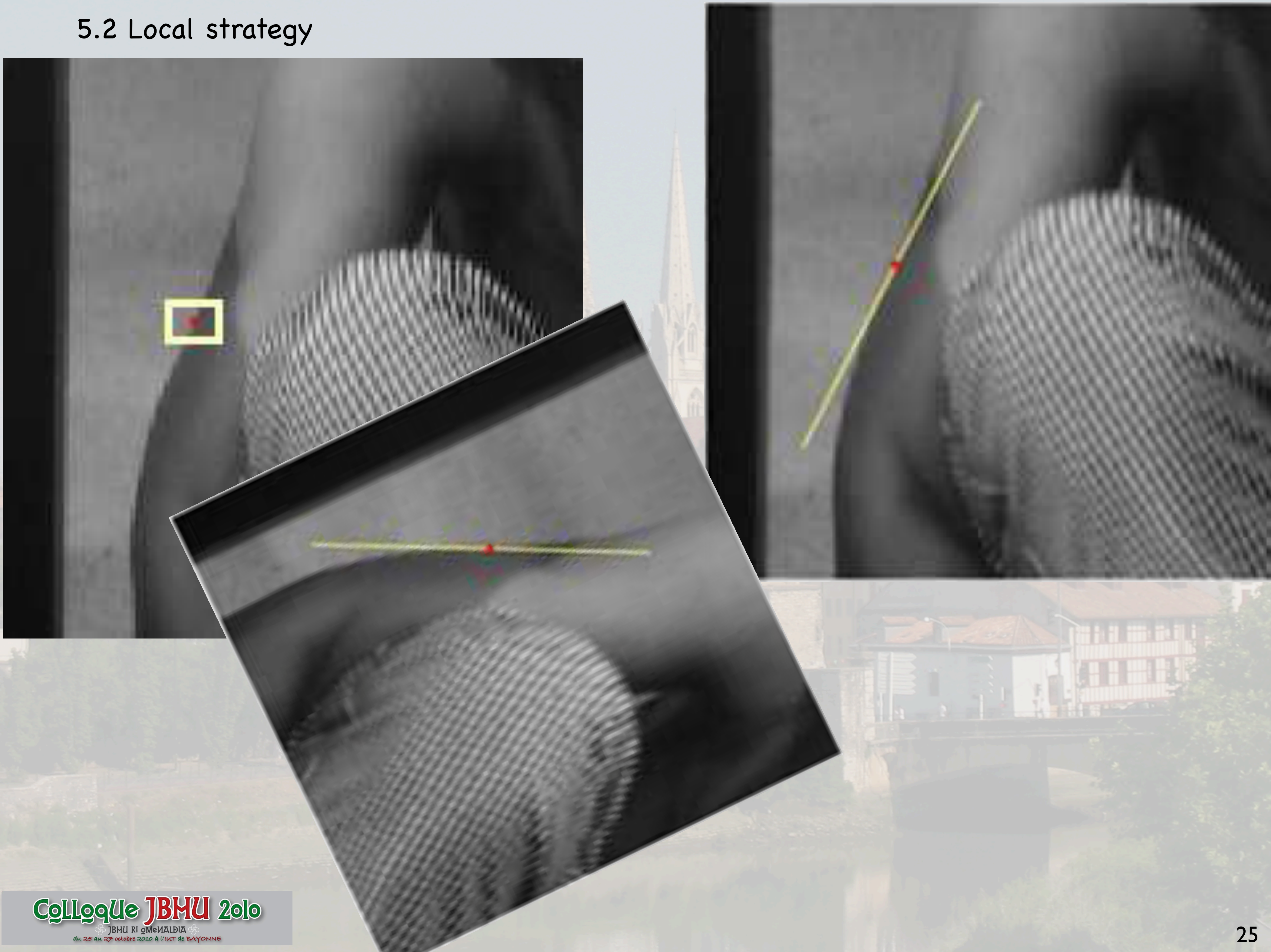


5.2 Local strategy

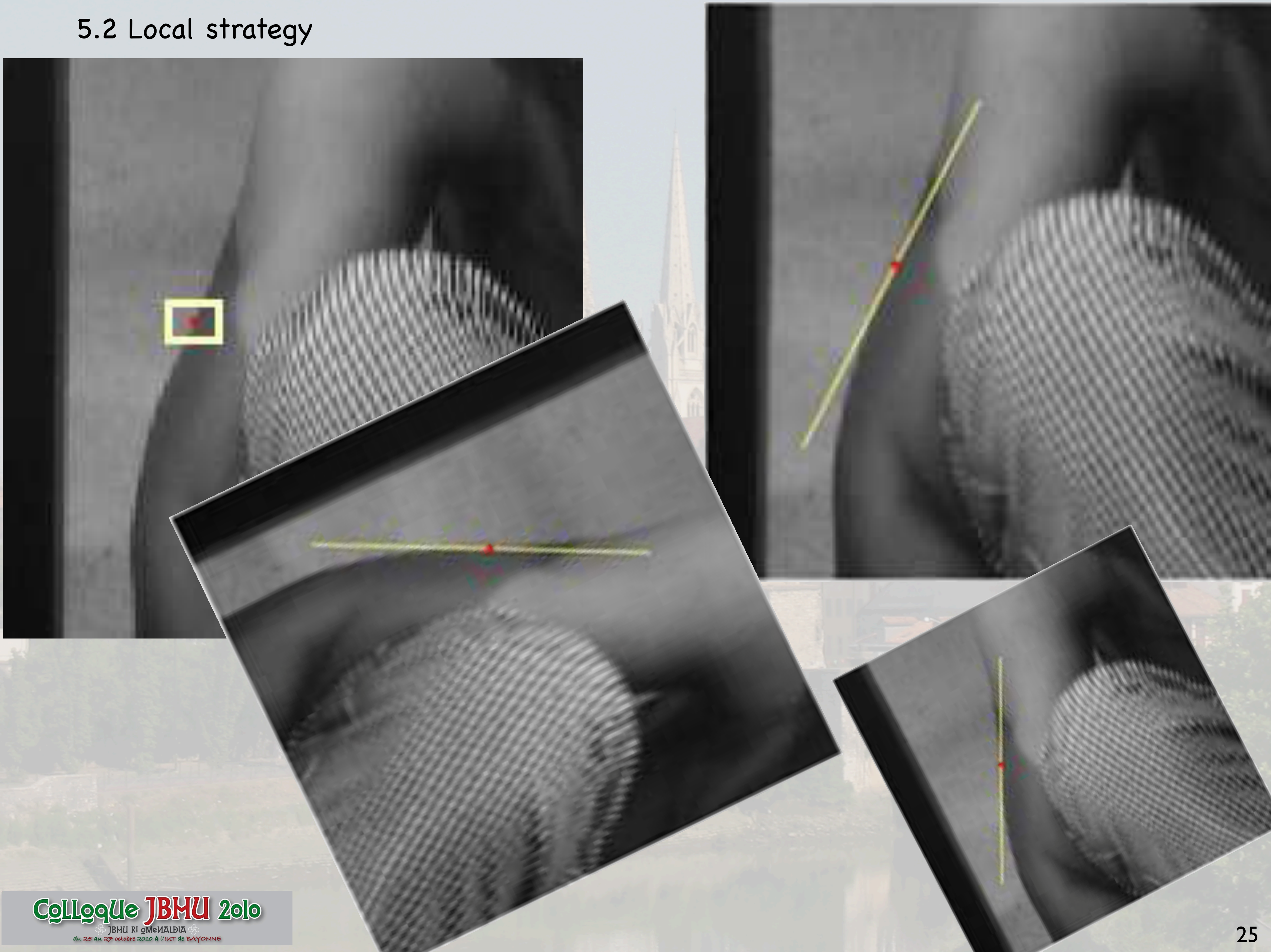




5.2 Local strategy



5.2 Local strategy



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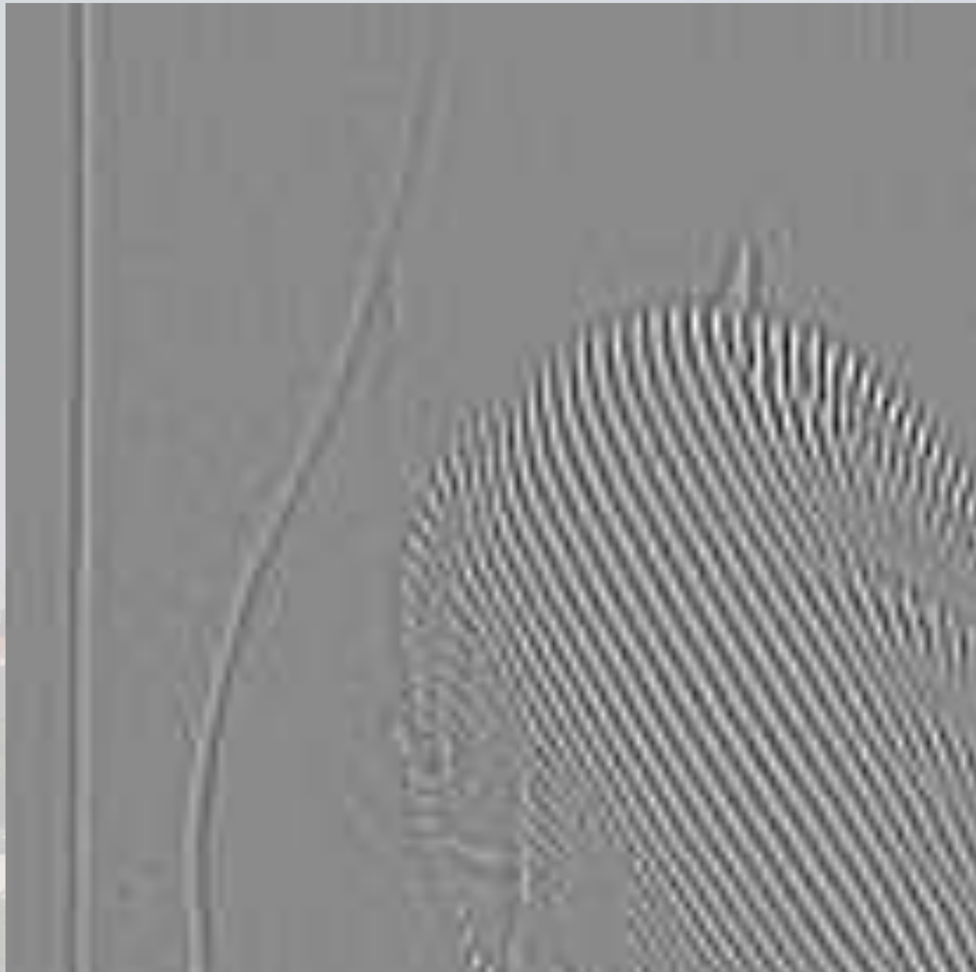

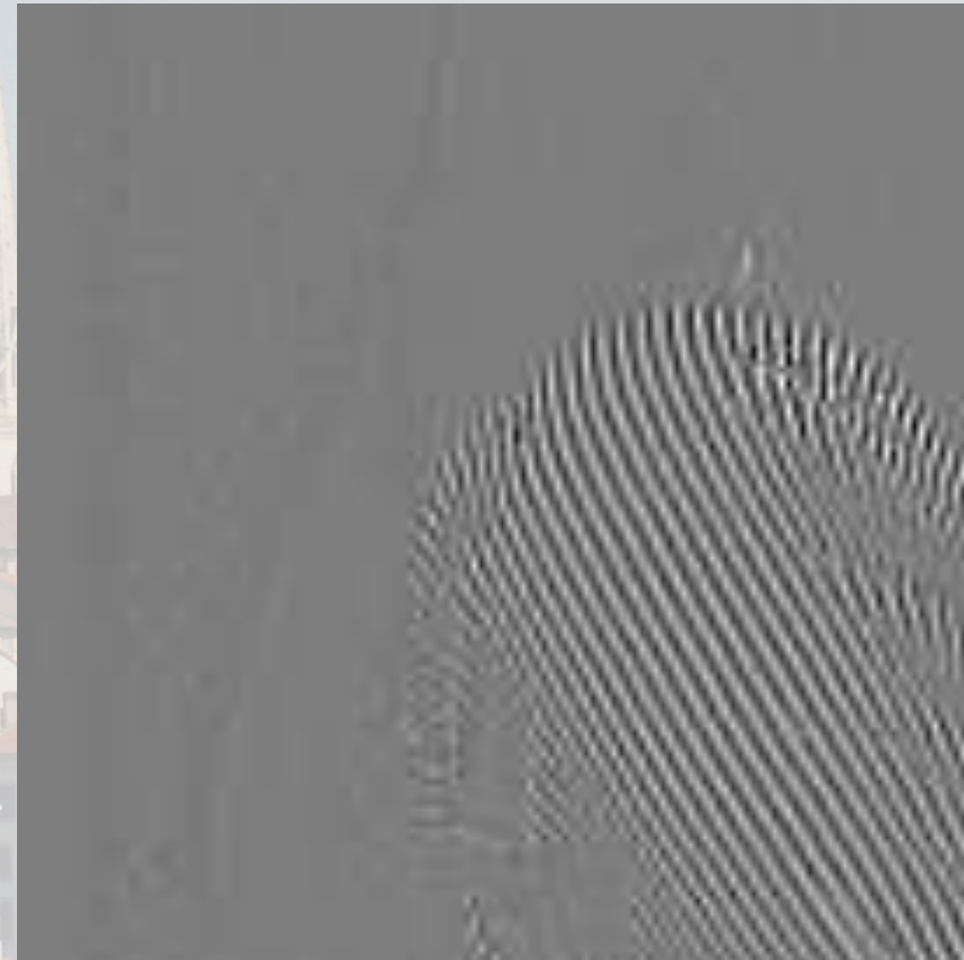

4. EXAMPLES

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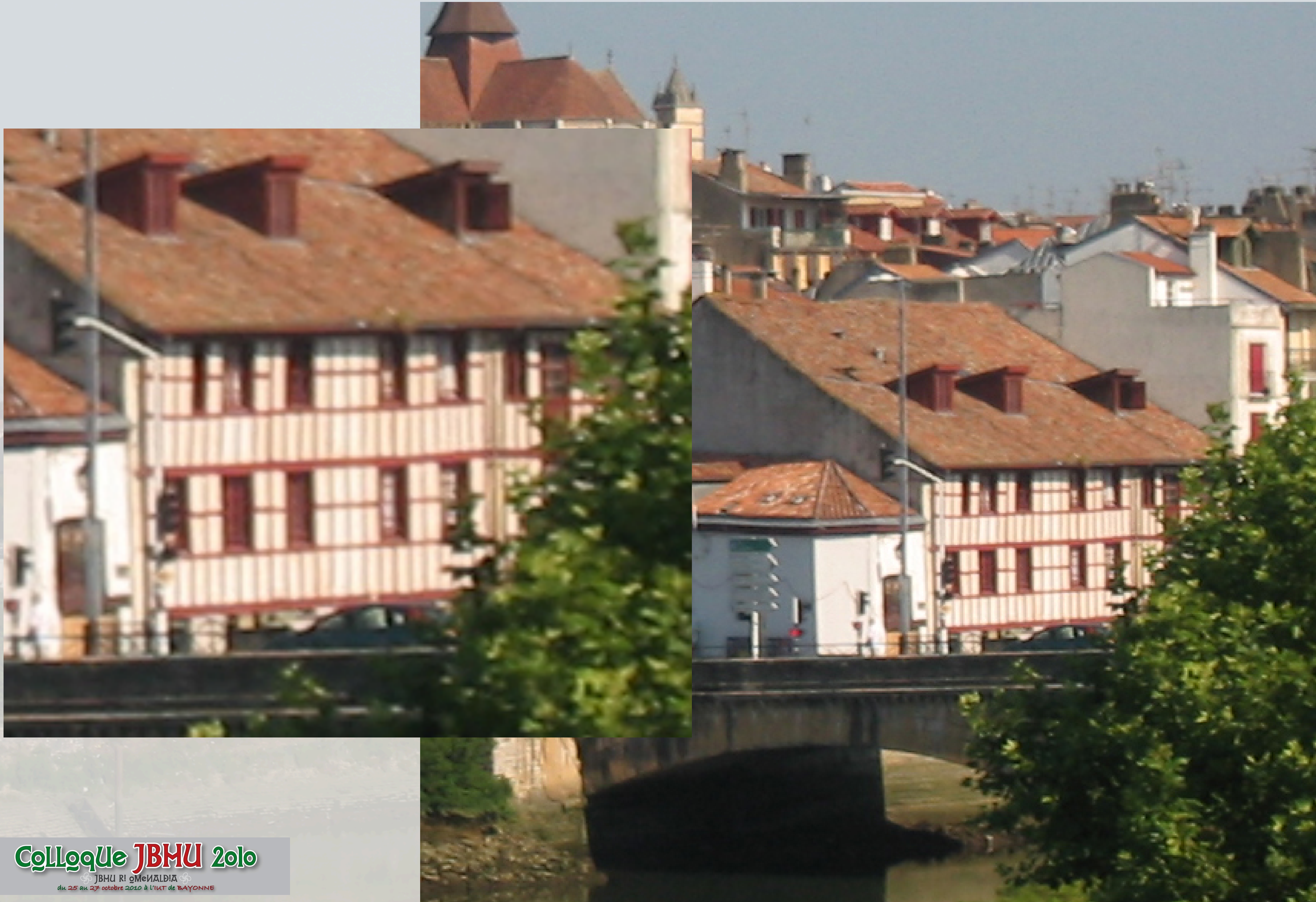

Cartoon 100 iterations Texture

Without anisotropy strategy

$$\lambda = 10$$

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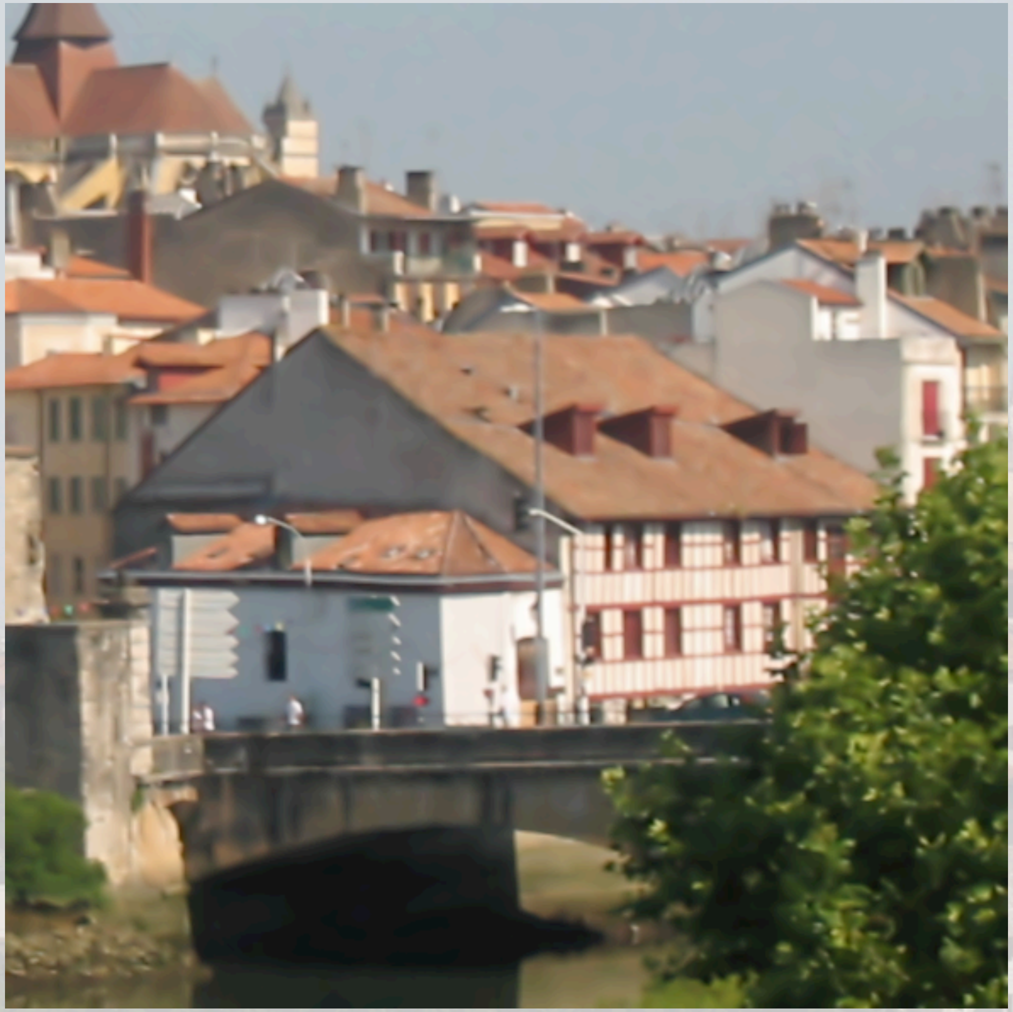
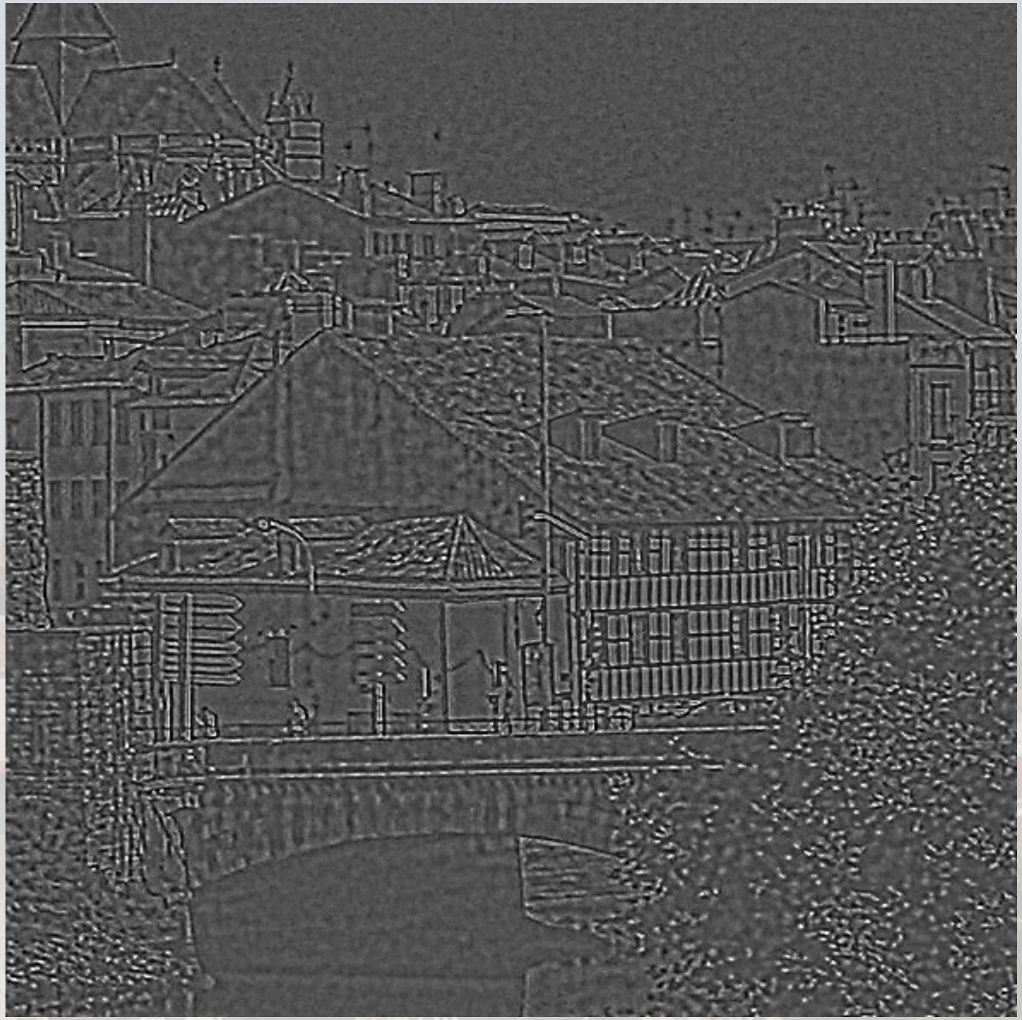
Cartoon 100 iterations Texture





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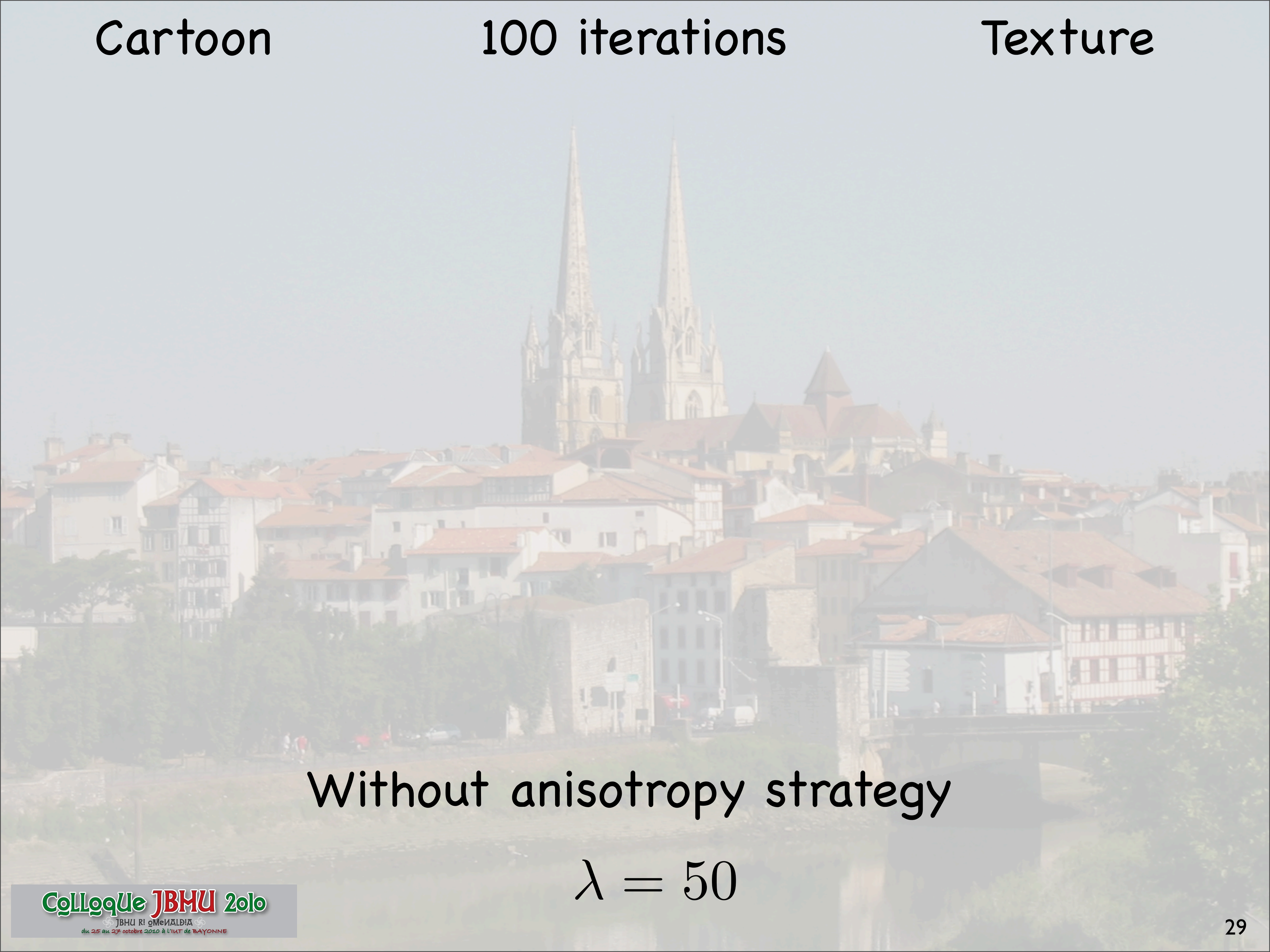
Cartoon 100 iterations Texture



Without anisotropy strategy

$$\lambda = 50$$

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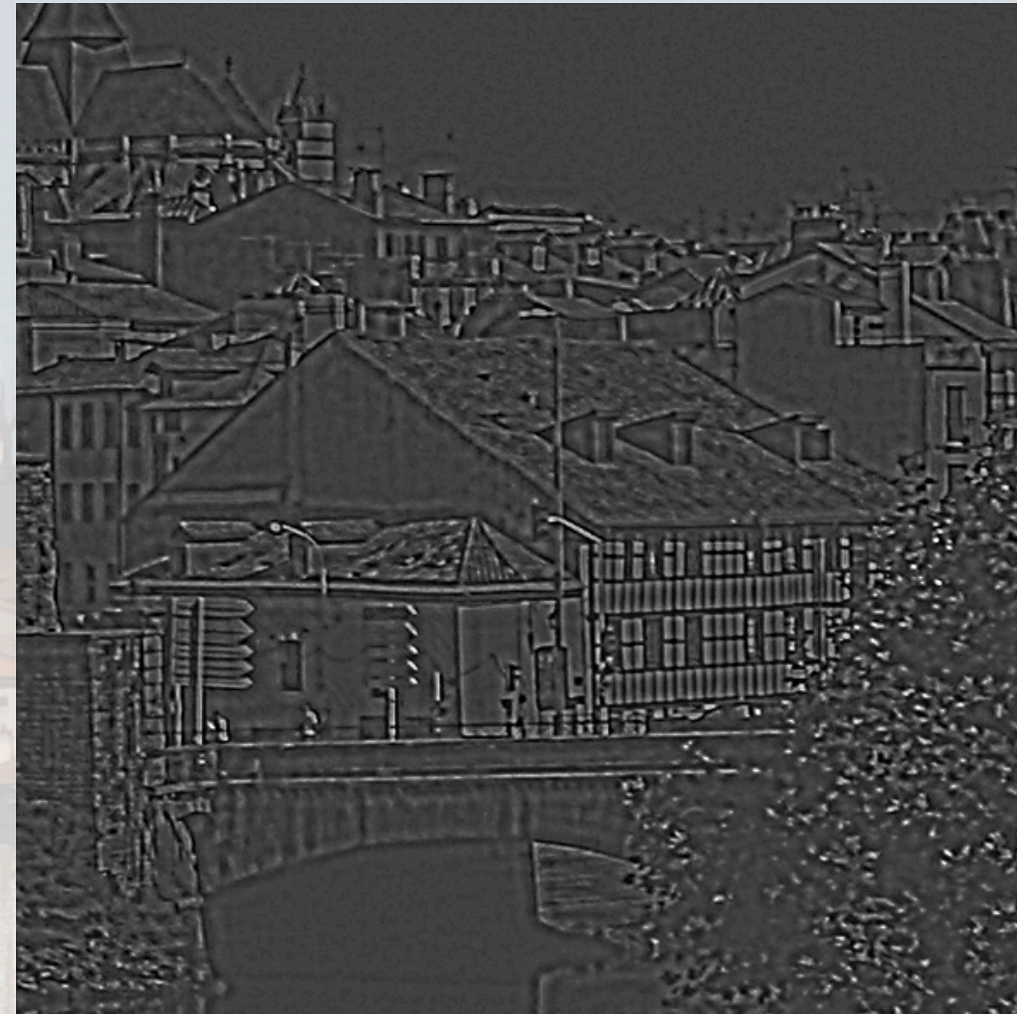
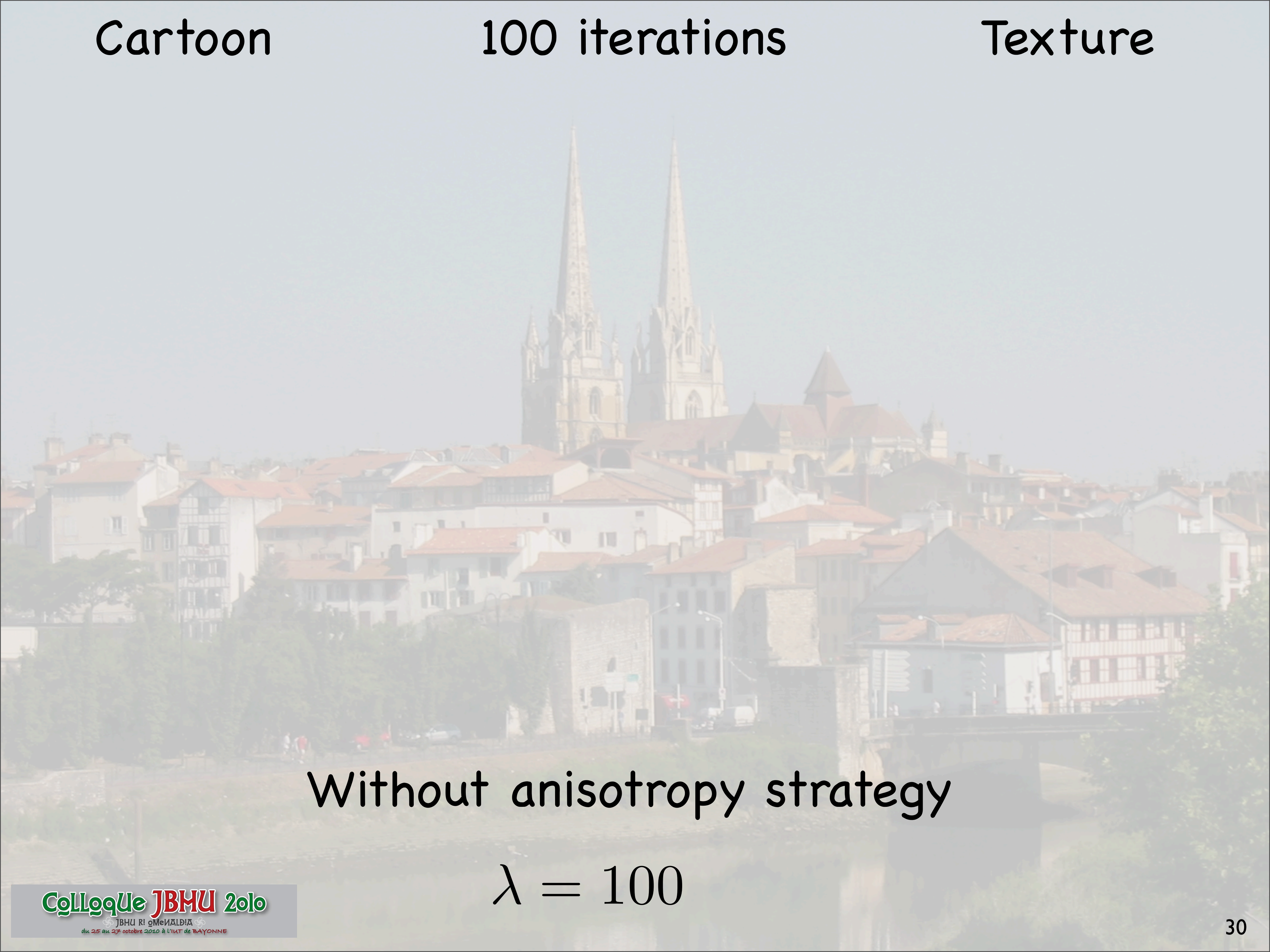
Cartoon 100 iterations Texture



Without anisotropy strategy

$$\lambda = 50$$

Cartoon 100 iterations Texture



Without anisotropy strategy

$$\lambda = 100$$

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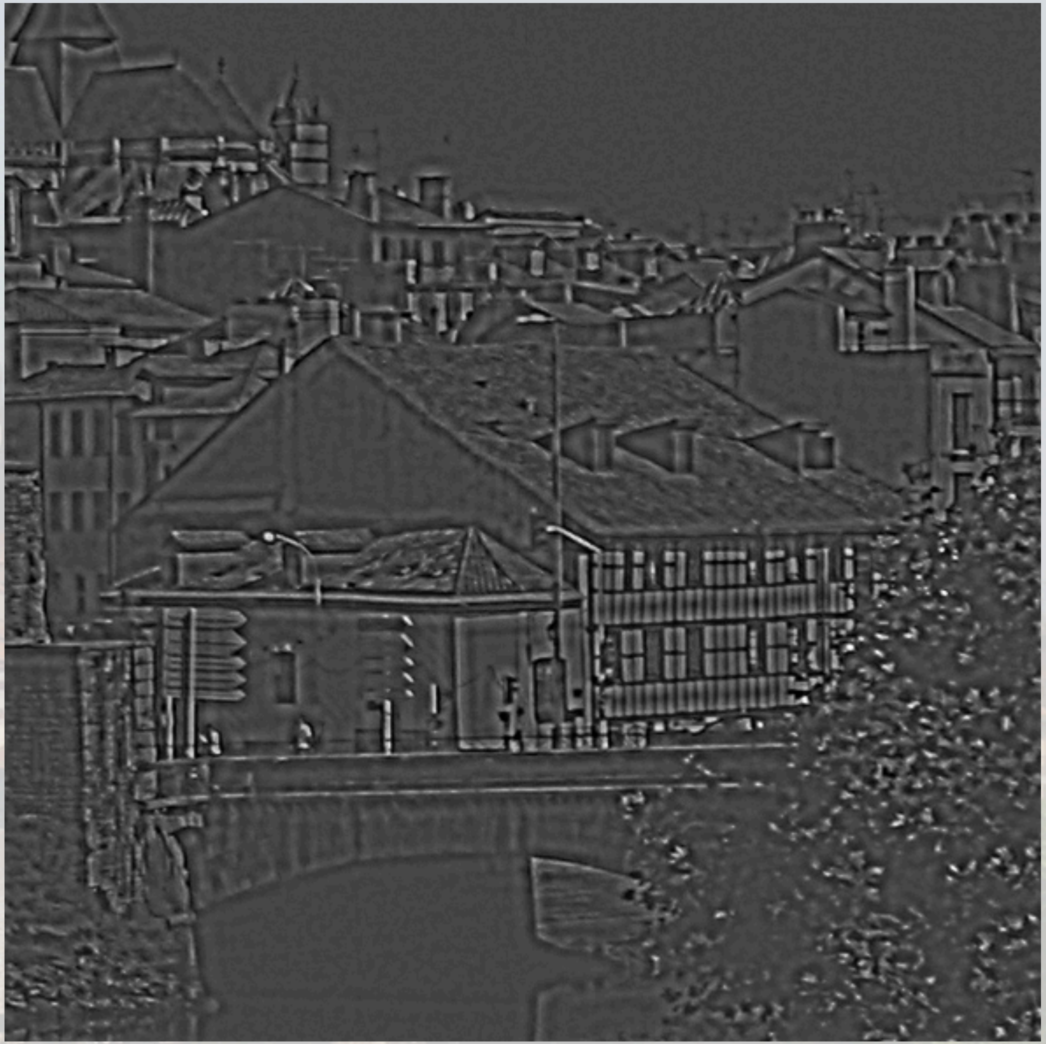
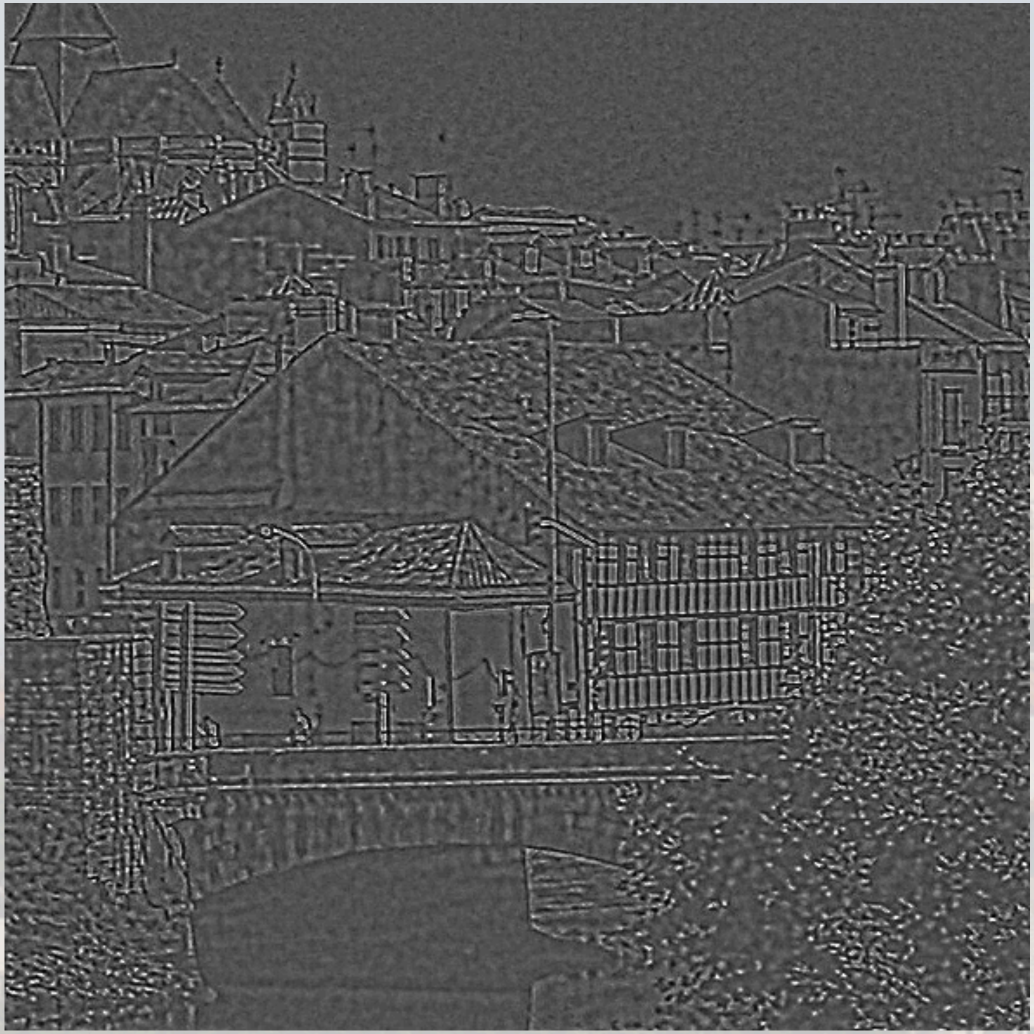
Cartoon 100 iterations Texture



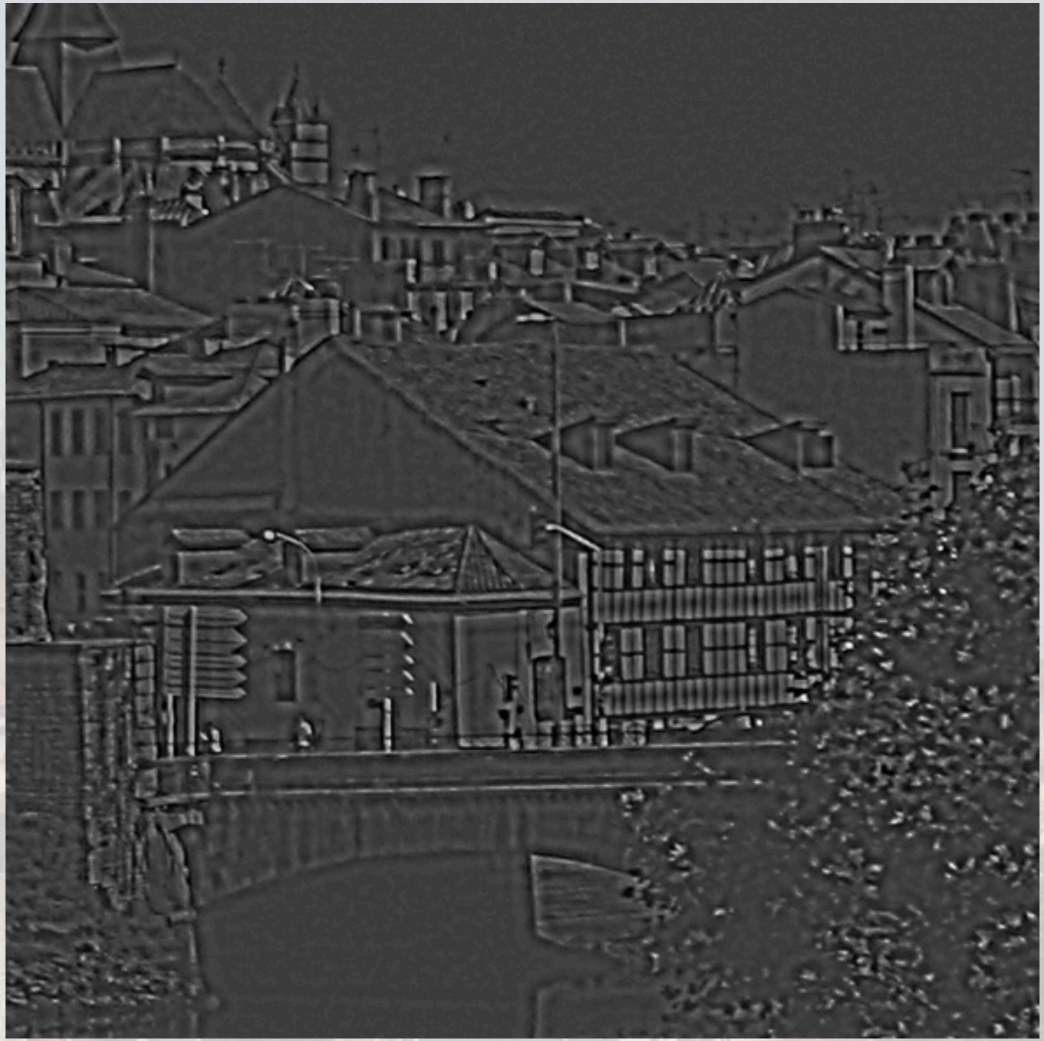
Without anisotropy strategy

$$\lambda = 100$$

Texture



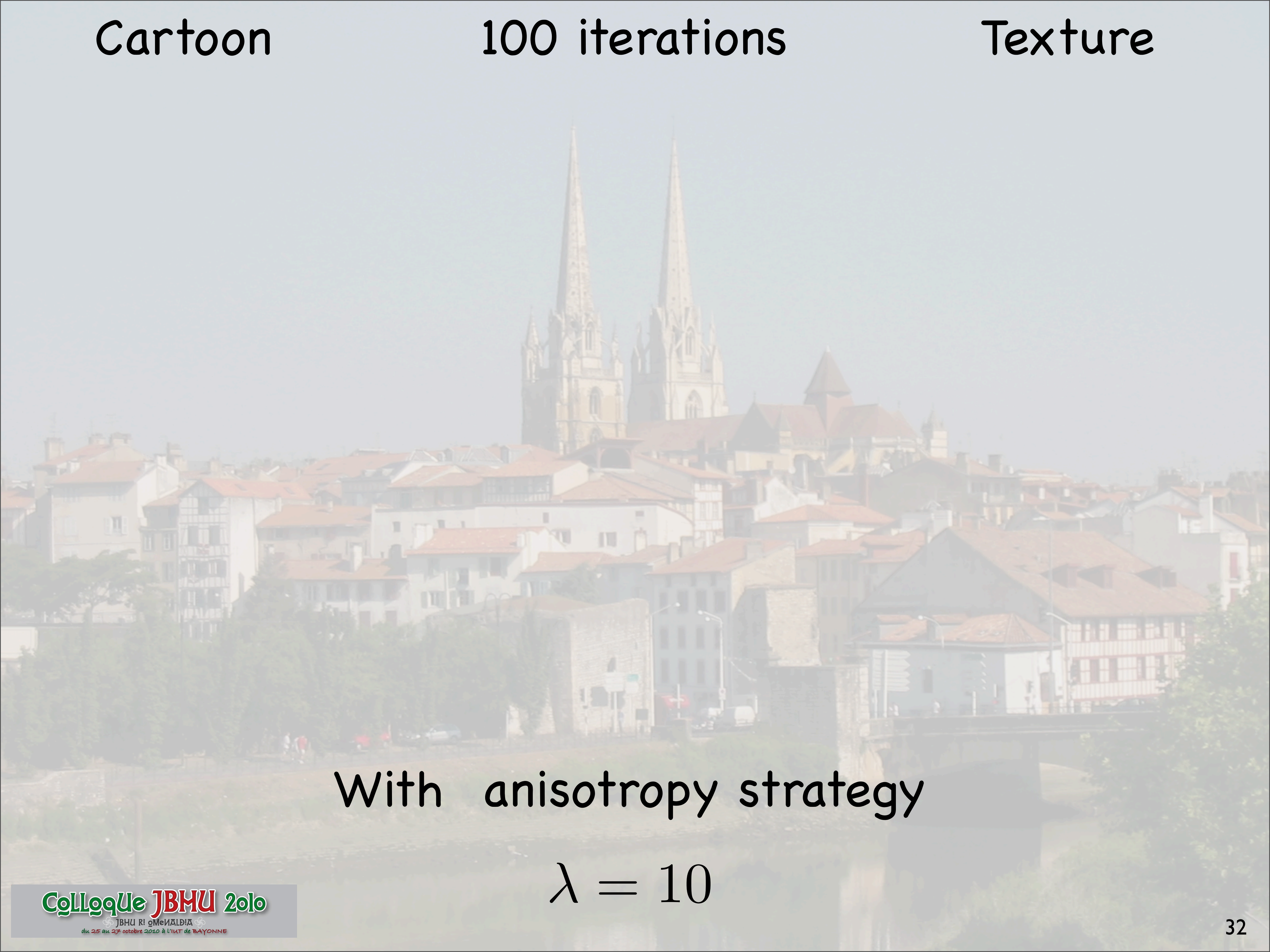
$\lambda = 10$



$\lambda = 100$

Without anisotropy strategy

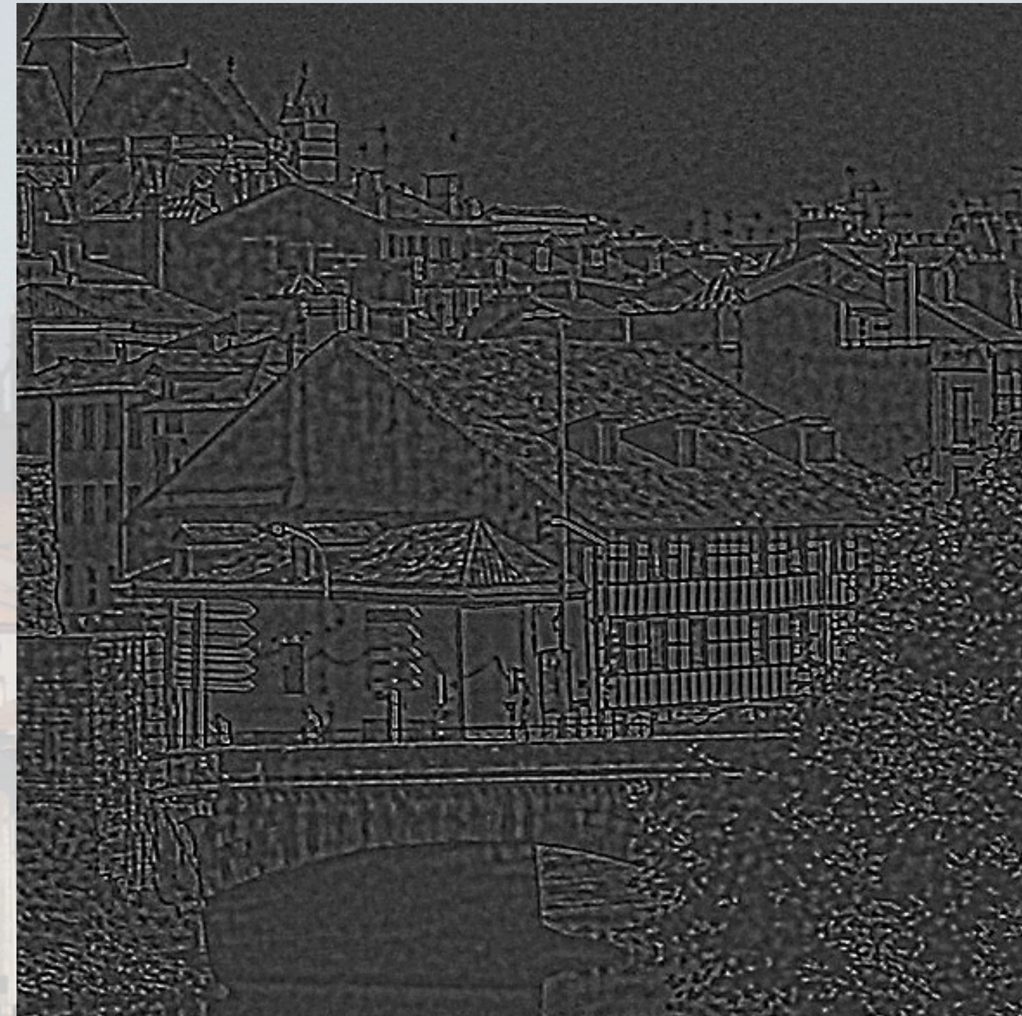
Cartoon 100 iterations Texture



With anisotropy strategy

$$\lambda = 10$$

Cartoon 100 iterations Texture



With anisotropy strategy

$$\lambda = 10$$

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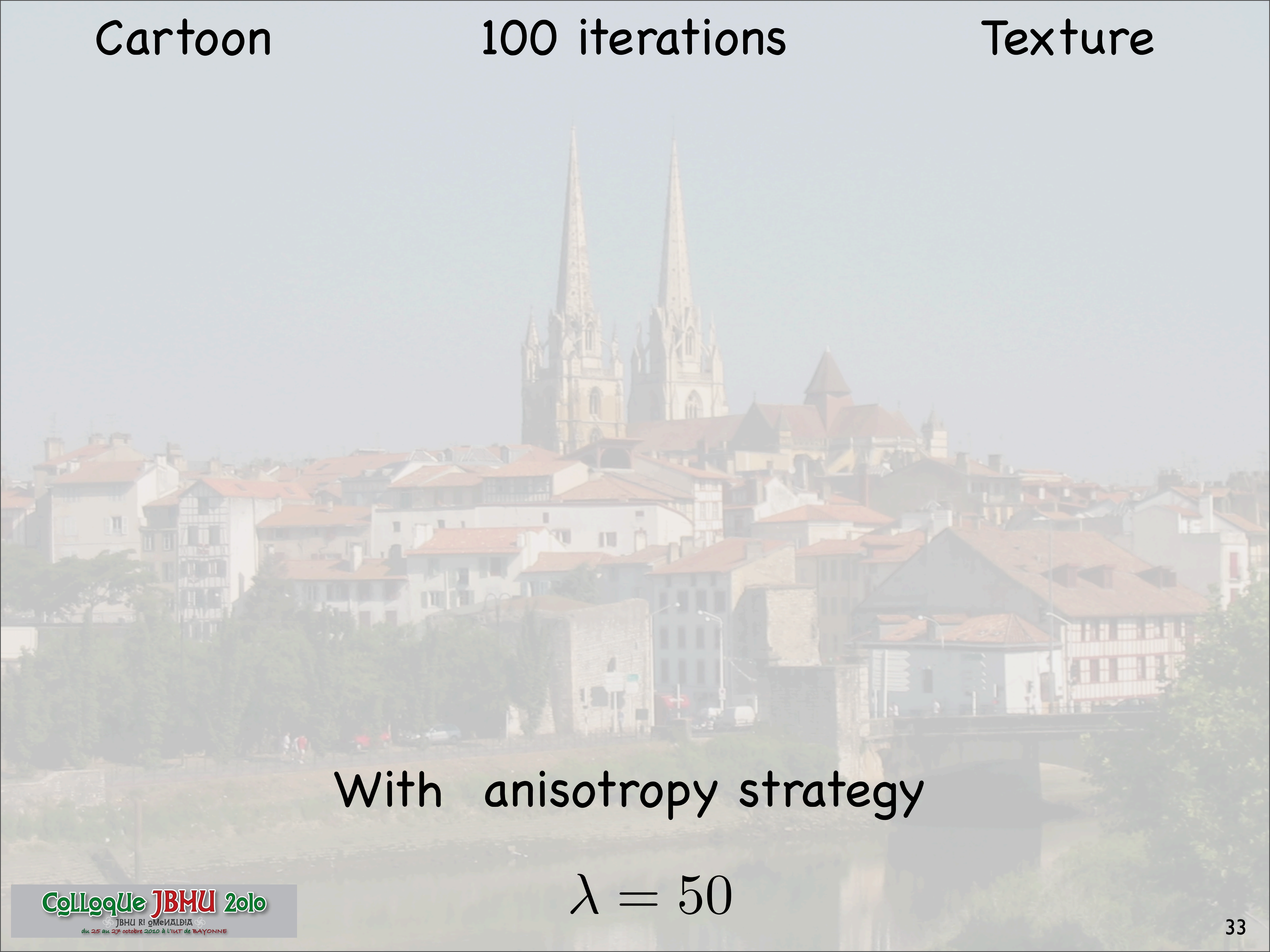
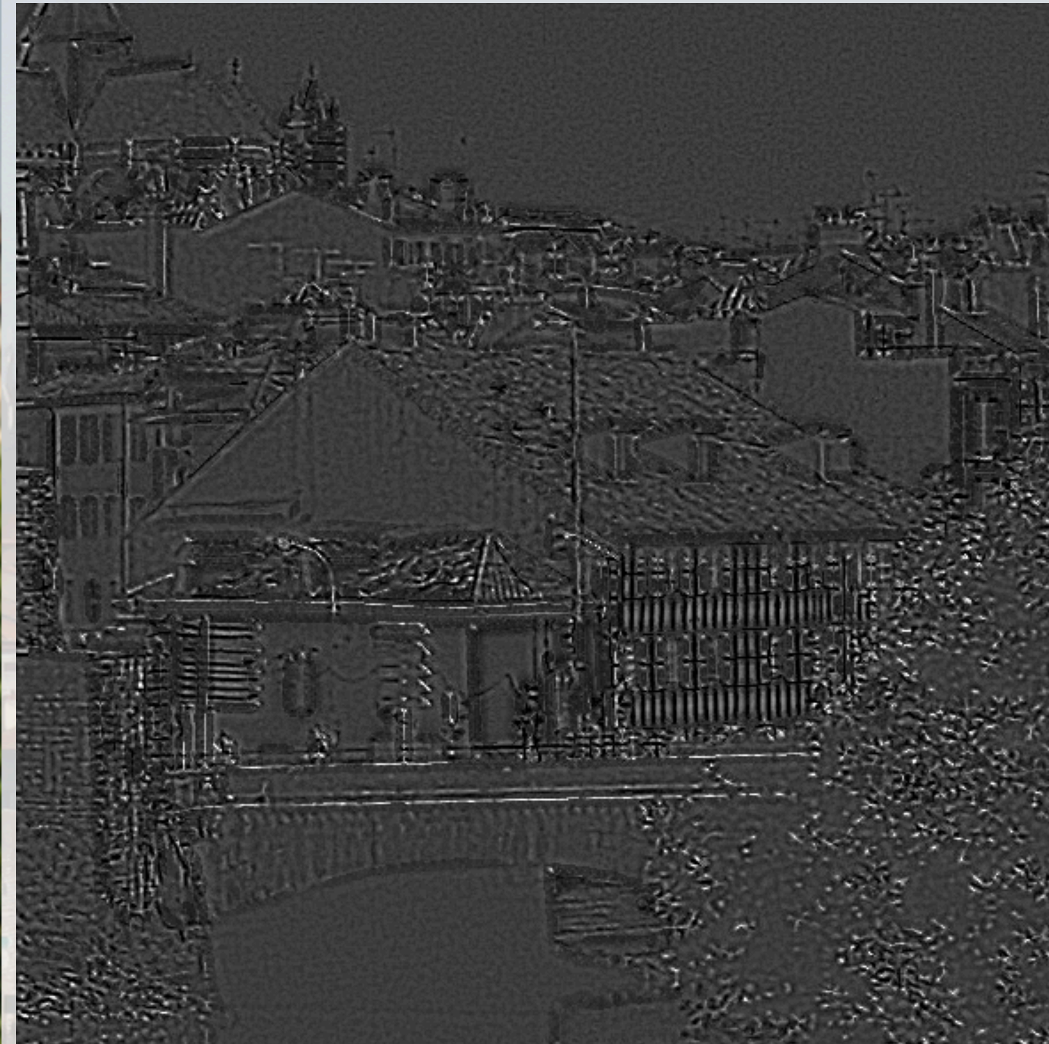


Cartoon 100 iterations Texture

With anisotropy strategy

$$\lambda = 50$$

Cartoon 100 iterations Texture



With anisotropy strategy

$$\lambda = 50$$

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Cartoon 100 iterations Texture

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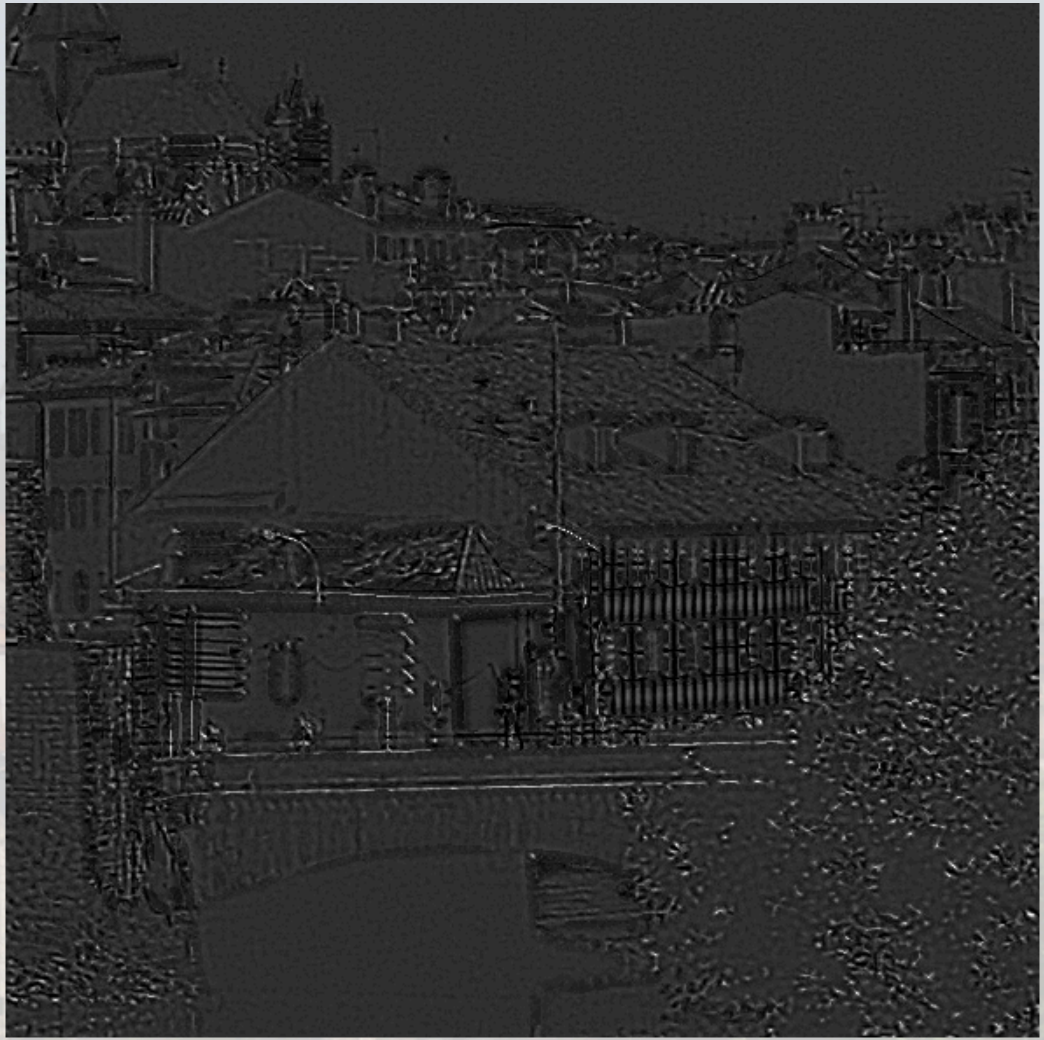


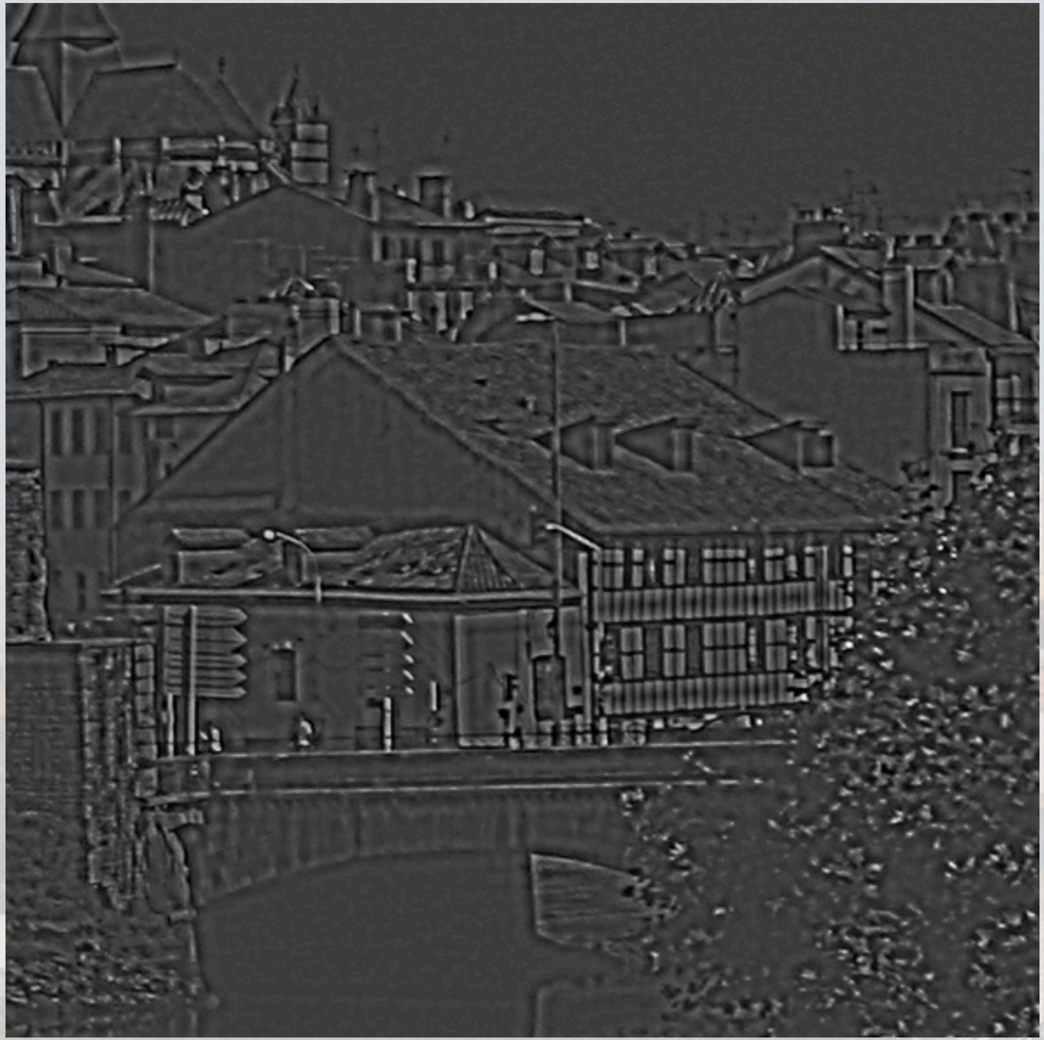
With anisotropy strategy

$$\lambda = 100$$

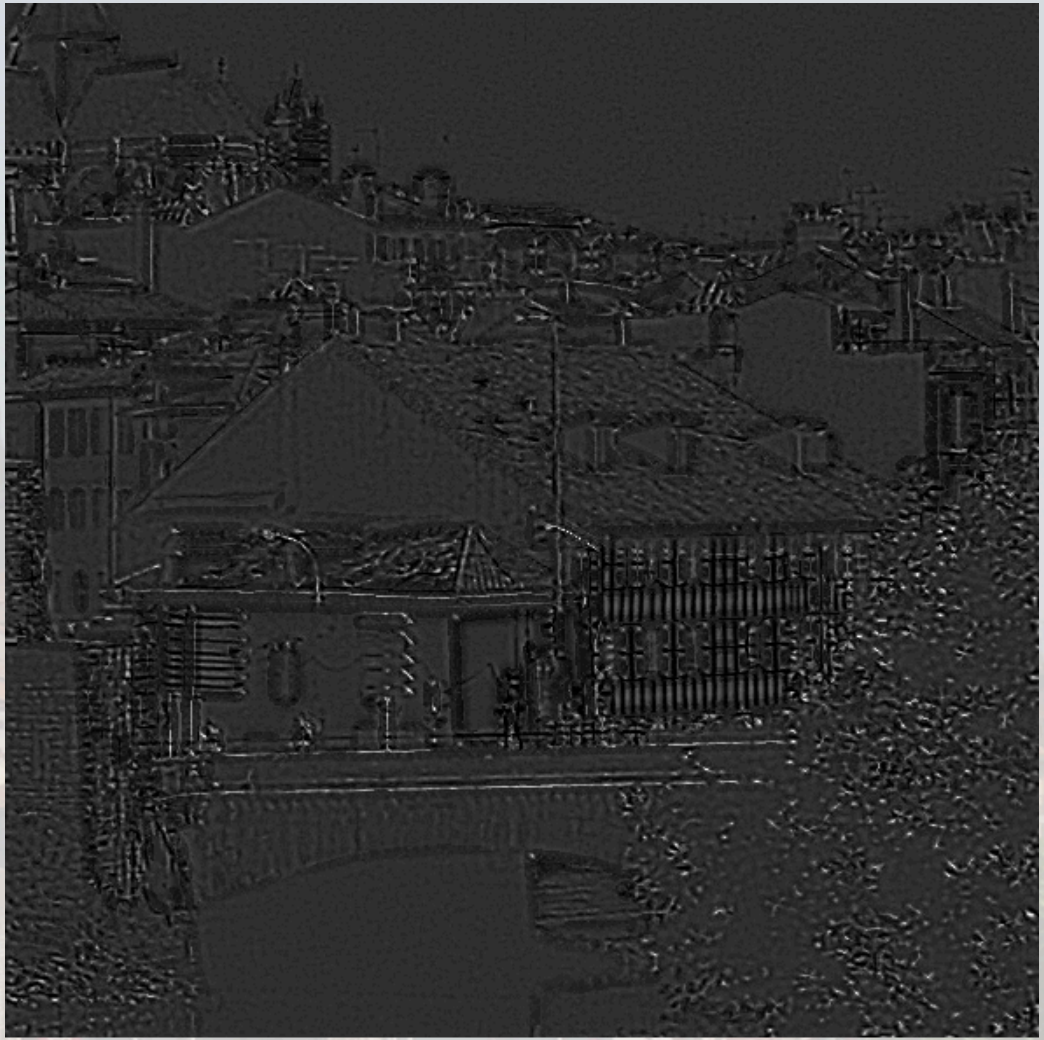
Without

With

$$\lambda = 100$$

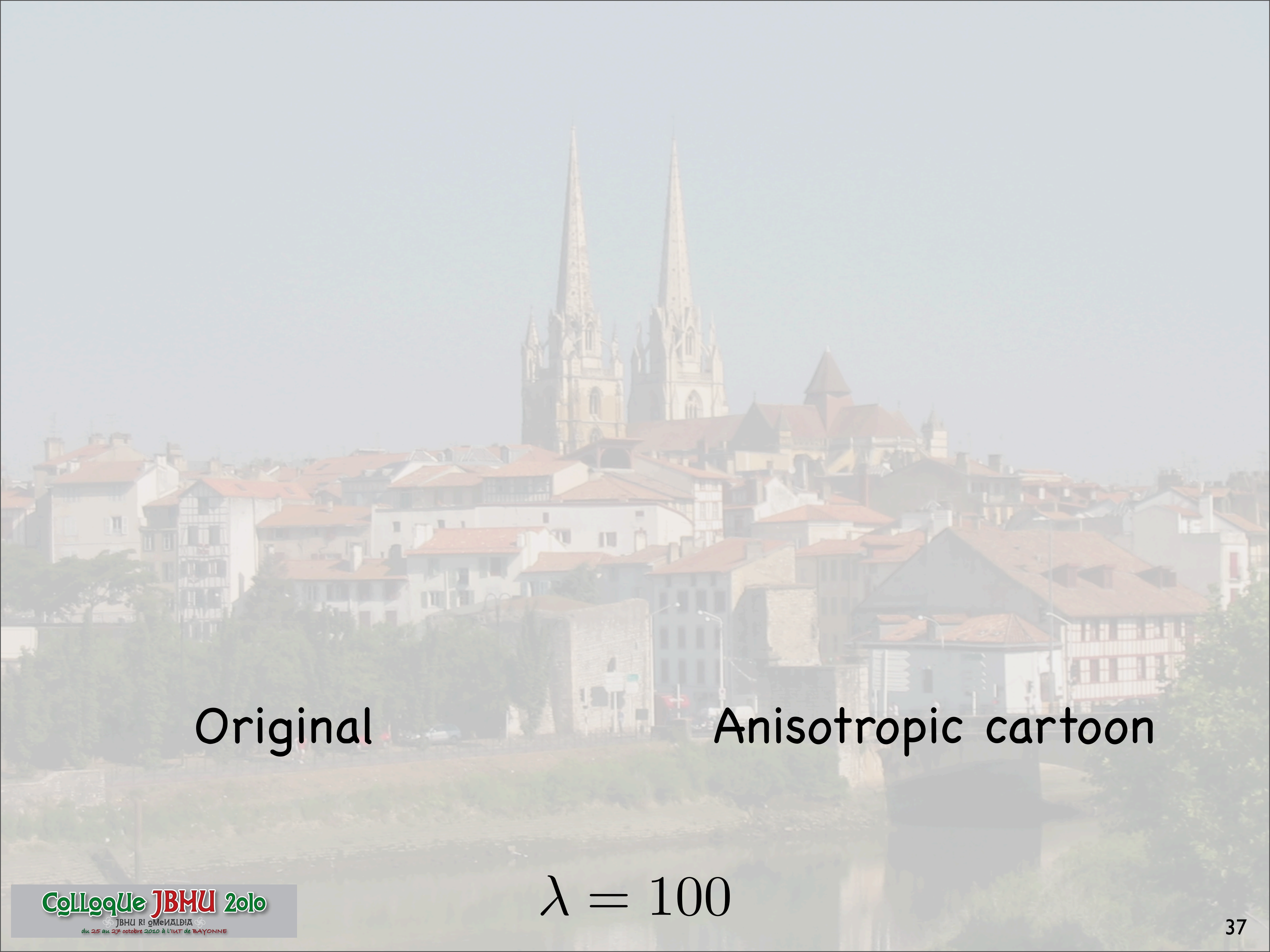


Without



With

$$\lambda = 100$$



Original

Anisotropic cartoon

$$\lambda = 100$$

Original

Anisotropic cartoon

$$\lambda = 100$$

Original

Anisotropic cartoon

$$\lambda = 100$$

Original

Anisotropic cartoon

$$\lambda = 100$$

Original

Anisotropic cartoon

$$\lambda = 100$$

JBHU (and Fermat) lifting...

$$\lambda = 20$$

50 iterations
without anisotropy

JBHU (and Fermat) lifting...

$$\lambda = 20$$

50 iterations
without anisotropy

100 iterations
without anisotropy

$$\lambda = 50$$

100 iterations
without anisotropy

$$\lambda = 50$$

100 iterations
without anisotropy

$$\lambda = 50$$

100 iterations
with anisotropy

$$\lambda = 50$$

100 iterations
with anisotropy

$$\lambda = 50$$

100 iterations
with anisotropy

$\lambda = 50$

Without

With

$$\lambda = 50$$

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