When is $L^r(\mathbb{R})$ contained in $L^p(\mathbb{R}) + L^q(\mathbb{R})$?

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Abstract

We prove a necessary and sufficient condition on the exponents $p, q, r \geq 1$ such that $L^r(\mathbb{R}) \subset L^p(\mathbb{R}) + L^q(\mathbb{R})$. In doing so, we explore the structure of $L^p(\mathbb{R}) + L^q(\mathbb{R})$ as a normed vector space.

1 Introduction.

In a recent mathematical note aimed at undergraduate students and their teachers ([3]), J.-B. Hiriart-Urruty and M. Pradel proposed a way to extend the Fourier transformation to all the spaces $L^r(\mathbb{R})$ with $1 \leq r \leq 2$ in the following manner.

– First, they classically define the Fourier transformation on $L^1(\mathbb{R})$. Then they define it on $L^2(\mathbb{R})$ using in that case the much less known Wiener’s approach which relies on a specific Hilbertian basis made of the so-called Bernstein functions.

– After having checked the coherence of both definitions on $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, they extend the definition of the Fourier transformation to the space $L^1(\mathbb{R}) + L^2(\mathbb{R})$.

– Finally, and this is the key-point, they prove the inclusion $L^r(\mathbb{R}) \subset L^1(\mathbb{R}) + L^2(\mathbb{R})$ for all $1 \leq r \leq 2$, so that the Fourier transformation can be extended to all the Lebesgue spaces $L^r(\mathbb{R})$.

An obvious question which arises is, what happens if $r \notin [1, 2]$? For example, is the inclusion $L^3(\mathbb{R}) \subset L^1(\mathbb{R}) + L^2(\mathbb{R})$ true or not? The objective of the present mathematical note is to answer this question. We provide a necessary and sufficient condition for the inclusion $L^r(\mathbb{R}) \subset L^p(\mathbb{R}) + L^q(\mathbb{R})$ to hold true. For that purpose, concentrate on the sum $L^p(\mathbb{R}) + L^q(\mathbb{R})$ and see how it is structured as a normed vector space.

2 How to norm a sum of normed vector spaces?

Let $V$ and $W$ be two vector subspaces of a "holdall" vector space, $E$. If $V$ is equipped with a norm $\| \cdot \|_1$ and $W$ with a norm $\| \cdot \|_2$, is there a natural way to define a norm on the vector subspace $V + W$? Of course, we do not assume
that \( V \cap W = \{O_E\} \). If this was the case, a natural way to define a norm \( N \) on \( V + W \) would be

\[
N(u) = \|v\|_1 + \|w\|_2,
\]

whenever \( u \in V + W \) is (uniquely) decomposed as \( u = v + w \), with \( v \in V \) and \( w \in W \).

What we have in mind is indeed \( V = L^p(\mathbb{R}) \), \( W = L^q(\mathbb{R}) \) and \( E = L(\mathbb{R}) \) as the “holdall” vector space (\( L(\mathbb{R}) \) stands for the set of all Lebesgue classes of measurable functions on \( \mathbb{R} \)).

The theorem below answers the question posed above. It does not seem to be well-known, except by people who have to deal with the interpolation of functional spaces (like in [1]).

**Theorem 1** Let \( N \) be defined on \( V + W \) as follows

\[
N(u) := \inf \{ \|v\|_1 + \|w\|_2 : u = v + w \text{ with } v \in V \text{ and } w \in W \}. \tag{1}
\]

Then, \( N \) is a semi-norm on \( V + W \). It is a norm under the following “compatibility” assumption:

\[
(T) \begin{cases} z_k \in V \cap W, \\
z_k \rightarrow a \text{ in } (V, ||.||_1) \\
z_k \rightarrow b \text{ in } (W, ||.||_2) \end{cases} \implies a = b.
\]

**Proof:** To check that \( N(O_E) = 0 \), \( N(\lambda u) = |\lambda| N(u) \) for all \( \lambda \in \mathbb{R} \), \( u \in V + W \), and \( N(u_1 + u_2) \leq N(u_1) + N(u_2) \) for all \( u_1 \) and \( u_2 \) in \( V + W \), does not raise any difficulty. It suffices to use the definition of the lower bound (or infimum) of a set of real numbers.  

To prove that \( N(u) = 0 \) implies that \( u = O_E \) is a bit more tricky. Our experience with that question with undergraduate students shows that they usually fail to answer it correctly. Their common mistake is to deduce that a sequence \( \langle v_k + w_k \rangle_k \) converges to \( O_E \) using the fact that \( \langle v_k \rangle_k \) converges to \( O_E \) in \( V \) and \( \langle w_k \rangle_k \) converges to \( O_E \) in \( W \). We therefore provide a proof here.

We first begin by observing that

\[
\nu : V \cap W \ni u \mapsto \nu(u) := \max(\|u\|_1, \|u\|_2) \tag{2}
\]

is a norm on \( V \cap W \); this is an easy result to prove.

Consider therefore, \( u \in V + W \) such that \( N(u) = 0 \). We take for example

\[
u = v + w, \text{ with } v \in V \text{ and } w \in W. \tag{3}
\]

Due to the definition (1) of \( N(u) \), for all positive integers \( k \), there exists \( v_k \in V \) and \( w_k \in W \) such that

\[
u = v_k + w_k, \text{ and } \|v_k\|_1 + \|w_k\|_2 \leq N(u) + \frac{1}{k} = \frac{1}{k}. \tag{4}
\]
Thus, \((v_k)_k\) converges to \(O_E\) in \(V\) and \((w_k)_k\) converges to \(O_E\) in \(w\). But what about \((v_k + w_k)_k\)? Recall that no norm, hence, no topology, has yet been defined on \(V + W\). We infer from (3) and (4) that

\[ v + w = v_k + w_k, \]

and thus

\[ v - v_k = w_k - w. \]  \(6\)

As a consequence, this common vector \(z_k := v - v_k = w_k - w\) lies in \(V \cap W\) and, since \(\nu(z_k) = \|v - v_k\|_1\) or \(\|w_k - w\|_2\),

\[ z_k \to v \text{ in } (V, \|\cdot\|_1) \text{ and } z_k \to -w \text{ in } (W, \|\cdot\|_2). \]

The assumption \((T)\) then ensures that

\[ v = -w, \text{ that is } u = v + w = O_E. \]

Note that the technical “compatibility” assumption \((T)\) is satisfied

- trivially if \(V \cap W = \{O_E\}\) (in that case it amounts to \(0 = 0\)).
- in the cases where \(V = L^p(\mathbb{R})\), \(W = L^q(\mathbb{R})\) (indeed, convergence of \((f_k)_k\) towards \(f\) in \(L^p(\mathbb{R})\) implies convergence almost everywhere of a subsequence of \((f_k)_k\) towards \(f\)).
- when the “holdall” vector space \(E\) is a Hausdorff topological vector space in which \(V\) and \(W\) are continuously imbedded.

We suppose that \((T)\) is in force for the rest of the section.

The vector space \(V + W\), equipped with the norm \(N\) as defined in (1), inherits some properties of \((V, \|\cdot\|_1)\) and \((W, \|\cdot\|_2)\). Here is one.

**Theorem 2** If \((V, \|\cdot\|_1)\) and \((W, \|\cdot\|_2)\) are Banach spaces, then so is \((V + W, N)\).

**Proof:** The proof of this theorem offers the opportunity to use a characterization of completeness of normed vector spaces which is not well-known. Let \((X, \|\cdot\|)\) be a normed space. We have indeed

\[ ((X, \|\cdot\|) \text{ is complete } \iff \left( \begin{array}{l} \text{Every series in } X, \text{ of general term } a_k, \text{ for which } \sum_{k=0}^{\infty} \|a_k\| < +\infty \text{ does converges in } X \end{array} \right) \right). \]

The implication \((\Rightarrow)\) is classical, it is the most often used. The converse implication \((\Leftarrow)\) is not often used, but we have an opportunity to do that here. For a proof of the equivalence (7), see for example ([6], Theorems 2-XIV-2.1
and 2.2, pages 164-165), ([4], page 20) or ([7], pages 262 and 270); it is also
sketched in ([1], page 24). The proof is not very difficult however, but readers
are encouraged to work through it themselves. We now proceed in this manner
to a proof of Theorem 2.

Consider a series in $V + W$, of general term $u_k$ for which $\sum_{k=0}^{\infty} N(u_k) < \infty$. We have to prove that $\sum_{k=0}^{n} u_k$ converges as $n \to +\infty$, to some element $u \in V + W$. In view of the definition (1) of $N$, for all positive integer $k$, there exist $v_k \in V$ and $w_k \in W$ satisfying

$$u_k = v_k + w_k, \quad \text{(8)}$$

Thus, $\sum_{k=0}^{\infty} \|v_k\|_1 + \|w_k\|_2 \leq N(u_k) + \frac{1}{k^2}$. \quad \text{(9)}$

Let $u := v + w \in V + W$. Let us check that, as expected, $\sum_{k=0}^{n} u_k$ converges to $u$ in $(V+W, N)$. Indeed,

$$N \left( u - \sum_{k=0}^{n} u_k \right) = N \left( u - \sum_{k=0}^{n} (v_k + w_k) \right) \leq \left\| v - \sum_{k=0}^{n} v_k \right\|_1 + \left\| w - \sum_{k=0}^{n} w_k \right\|_2. \quad \text{(10)}$$

It remains to apply (10) to get the desired result, $N \left( u - \sum_{k=0}^{n} u_k \right) \to 0$. \(\blacksquare\)

Comments : The construction of the norm $\nu$ on $V \cap W$ and $N$ on $V + W$
deserves some geometrical interpretation. Even if $\| \cdot \|_1$ (resp. $\| \cdot \|_2$) is only defined on $V \subset E$ (resp. on $W \subset E$), we can extend it to the whole of $E$ by setting $\|u\|_1 = +\infty$ if $u \notin V$ (resp. $\|u\|_2 = +\infty$ if $u \notin W$). We still denote by $\| \cdot \|_1$ and $\| \cdot \|_2$ the extended functions.

Clearly, $\| \cdot \|_1$ and $\| \cdot \|_2$ are convex positively homogeneous functions on $E$. Modern convex analysis accepts and can handle convex functions possibly taking the value $+\infty$ ([5]). An important geometrical object associated with a convex function $f : E \to \mathbb{R} \cup \{+\infty\}$ is its so-called strict epigraph

$$\text{epi}_s f := \{(x, r) \in E \times \mathbb{R} : f(x) < r\}$$

(literally, what is strictly above the graph of $f$). In our situation, $K_1 := \text{epi}_s \| \cdot \|_1$ and $K_2 := \text{epi}_s \| \cdot \|_2$ are open convex cones of $E$. So, what are the strict epigraphs of the norm functions $\nu$ and $N$ ? One easily checks the following

$$\text{epi}_s \nu = (\text{epi}_s \| \cdot \|_1) \cap (\text{epi}_s \| \cdot \|_2), \quad \text{and} \quad \text{epi}_s N = (\text{epi}_s \| \cdot \|_1) + (\text{epi}_s \| \cdot \|_2). \quad \text{(11)}$$
The sets where $\nu$ (resp. $N$) is finite, called the domain of $\nu$ (resp. of $N$) in convex analysis, is just $V \cap W$ (resp. $V + W$).

The binary operation which builds a convex function $f$ from two other ones $f_1$ and $f_2$, via the geometric construction

$$\text{epi}_s f = \text{epi}_s f_1 + \text{epi}_s f_2$$

is called the \textit{infimal convolution} of $f_1$ and $f_2$ ([2],[5]). This operation enjoys properties similar to the usual (integral) convolution in classical analysis.

In brief, the norm $N$ has been designed as an infimal convolution of the norms $\| \cdot \|_1$ and $\| \cdot \|_2$.

Returning to our particular setting $E = L^r(R), V = L^p(R)$ and $W = L^q(R)$ with $1 \leq p, q < +\infty$, the vector space $L^p(R) + L^q(R)$ can be equipped with a norm that we denote $\| \cdot \|_{p,q}$ as follows,

$$\| f \|_{p,q} = \inf \{ \| g \|_p + \| h \|_q : f = g + h \text{ with } g \in L^p(R) \text{ and } h \in L^q(R) \}. \quad (12)$$

As proved in Theorem 2, $(L^p(R) + L^q(R), \| \cdot \|_{p,q})$ is a Banach space.

3 Comparing $L^r(R)$ with $L^p(R) + L^q(R)$

We know that the Lebesgue spaces $L^r(R)$ and $L^s(R)$ (for $1 \leq r \neq s < +\infty$) cannot be compared. Neither $L^r(R)$ is contained in $L^s(R)$ nor the converse. A direct comparison is however possible if we deal with the sum of these spaces. Here is the main result of this section.

\textbf{Theorem 3} If $1 \leq p < q < +\infty$, then we have the following :

1. $L^r(R)$ is contained in $L^p(R) + L^q(R)$ whenever $r \in [p,q],$

2. if $1 \leq r < +\infty$ does not lie in $[p,q]$, then $L^r(R)$ is not contained in $L^p(R) + L^q(R)$.

\textbf{Proof :} (1) Let $f \in L^r(R)$ and consider $X := \{ x \in R : |f(x)| > 1 \}$ (a measurable set defined within a set of null measure) as well as $X^c = R \setminus X$ (the complementary set of $X$ in $R$). We decompose $f$ as follows :

$$f = f_1 + f_2, \text{ with } f_1 = f \cdot 1_X \text{ and } f_2 = f \cdot 1_{X^c}. \quad (13)$$

We claim that (13) provides an explicit decomposition of $f$ in $L^p(R) + L^q(R)$, that is to say $f_1 \in L^p(R)$ and $f_2 \in L^q(R)$. We first prove that,

$$\int_R |f_1(x)|^p d\lambda(x) = \int_X |f(x)|^p d\lambda(x) = \int_X |f(x)|^{p-r} \cdot |f(x)|^r d\lambda(x). \quad (14)$$
For \( x \in X \), \( |f(x)| > 1 \) and, since the exponent \( p-r \) is nonpositive, \( |f(x)|^{p-r} \leq 1 \). Consequently, the last integral in the string of equalities (14) is bounded above by \( \int_X |f(x)|^{r}d\lambda(x) \). Finally,

\[
\int_{\mathbb{R}} |f_1(x)|^p d\lambda(x) \leq \int_X |f(x)|^r d\lambda(x) \leq \int_{\mathbb{R}} |f(x)|^r d\lambda(x) < +\infty.
\]

We thus have proved that \( f_1 \in L^p(\mathbb{R}) \).

Second, we prove that \( f_2 \in L^q(\mathbb{R}) \). Indeed,

\[
\int_{\mathbb{R}} |f_2(x)|^q d\lambda(x) = \int_{X^c} |f(x)|^q d\lambda(x) = \int_X |f(x)|^{q-r} \cdot |f(x)|^r d\lambda(x). \tag{15}
\]

For \( x \in X^c \), \( |f(x)| \leq 1 \) and, since the exponent \( q-r \) is nonnegative, \( |f(x)|^{q-r} \leq 1 \). Again, the last integral in the string of equalities (15) is bounded above by \( \int_{X^c} |f(x)|^r d\lambda(x) \). As a result,

\[
\int_{\mathbb{R}} |f_2(x)|^q d\lambda(x) = \int_{X^c} |f(x)|^r d\lambda(x) = \int_{\mathbb{R}} |f(x)|^r d\lambda(x) < +\infty. \tag{16}
\]

We therefore, have proved that \( f_2 \in L^q(\mathbb{R}) \).

(2) The second part of Theorem 3 is a bit harder to prove (like most of the negative results in mathematics). We actually have to distinguish two cases for \( r \) in the segment \([p, q) : r < p \) and \( r > q \).

**Case 1 :** \( r < p \). Choose \( \alpha \) satisfying \( 1/p < \alpha < 1/r \) and let \( f \) be defined on \( \mathbb{R} \) by \( f(x) = x^{-\alpha}1_{(0,1)}(x) \).

Since \( |f(x)|^r = x^{-\alpha r} \) for \( x \in (0, 1] \) and 0 elsewhere, the choice of \( \alpha \) implies that \( f \in L^r(\mathbb{R}) \) (since \( \alpha r < 1 \)). The same argument shows that \( f \notin L^q(\mathbb{R}) \) (since \( \alpha p > 1 \)).

Suppose now that \( L^r(\mathbb{R}) \subset L^p(\mathbb{R}) + L^q(\mathbb{R}) \). Then

\[
f \in L^r(\mathbb{R}) \subset L^p(\mathbb{R}) + L^q(\mathbb{R}) \subset L^p([0, 1]) + L^q([0, 1]). \tag{17}
\]

But since \([0, 1] \) is of Lebesgue finite measure and \( p < q \), \( L^q([0, 1]) \) is contained in \( L^p([0, 1]) \), so that (17) yields that \( f \in L^p([0, 1]) \). This is not the case.

Thus we have proved that \( L^r(\mathbb{R}) \) is not contained in \( L^p(\mathbb{R}) + L^q(\mathbb{R}) \).

**Case 2 :** \( r > q \). Our proof in this case relies on a technical lemma that we present separately.

**Lemma 1** Let \( 1 \leq p < q < +\infty \), let \( \Omega \) be a measurable subset of \( \mathbb{R} \), and let \( f \in L^p(\Omega) + L^q(\Omega) \). Then \( f \in L^q(\Omega) \) whenever it is essentially bounded on \( \Omega \).

**Proof of the Lemma :** Let \( f \) be decomposed as \( f = f_p + f_q \), with \( f_p \in L^p(\Omega) \) and \( f_q \in L^q(\Omega) \). So, to prove that \( f \in L^q(\Omega) \) amounts to proving that \( f_p \in L^q(\Omega) \).
Let \( X := \{ x \in \mathbb{R} : |f_p(x)| > 1 \} \). To show that \( \int_\Omega |f_p(x)|^q d\lambda(x) \) is finite, we cut it into two pieces: \( \int_{\Omega \cap X} |f_p(x)|^q d\lambda(x) \) and \( \int_{\Omega \cap X^c} |f_p(x)|^q d\lambda(x) \).

Consider the first piece. Since \( f_p \in L^p(\mathbb{R}) \), the set \( X \) is of finite (Lebesgue) measure. Now with the definition of \( \text{Consequently,} \)
\[
\int_{\Omega \cap X^c} |f_p(x)|^q d\lambda(x) = \int_{\Omega \cap X^c} |f_p(x)|^{q-p} \cdot |f_p(x)|^p d\lambda(x) \\
\leq \int_{\Omega \cap X^c} |f_p(x)|^p d\lambda(x) \leq \int_{\mathbb{R}} |f_p(x)|^p d\lambda(x) < +\infty.
\]
This concludes the argument for the first piece.

Consider now the second piece. Since \( f \) has been assumed essentially bounded on \( \Omega \),
\[
|f_p(x)| \leq |f(x) - f_q(x)| \leq \|f\|_\infty + |f_q(x)| \quad \text{for almost all } x \in \Omega.
\]
Consequently,
\[
\int_{\Omega \cap X} |f_p(x)|^q d\lambda(x) \leq \int_{\Omega \cap X} (\|f\|_\infty + |f_q(x)|)^q d\lambda(x).
\]
(18)
The convexity of the function \( t \mapsto t^q \) on \( [0, +\infty) \) implies that \( (\|f\|_\infty + |f_q(x)|)^q \leq 2^{q-1} (\|f\|_\infty^q + |f_q(x)|^q) \). So, we pursue the string of inequalities (18) with
\[
\int_{\Omega \cap X} |f_p(x)|^q d\lambda(x) \leq 2^{q-1} \int_{\Omega \cap X} (\|f\|_\infty^q + |f_q(x)|^q) d\lambda(x) \\
\leq 2^{q-1} \|\lambda(X)\|_{\ell_q}^q + (\|f_q\|_q)^q.
\]
To summarize, we have proved that
\[
\int_\Omega |f_p(x)|^q d\lambda(x) = \int_{\Omega \cap X} |f_p(x)|^q d\lambda(x) + \int_{\Omega \cap X^c} |f_p(x)|^q d\lambda(x) < +\infty.
\]
Thus, \( f_p \in L^p(\Omega) \), which was our objective. That concludes the proof of the technical lemma.

Let us go back to the second part of the proof of Theorem 3, the case where \( r > q \). Choose \( \alpha \) satisfying \( 1/r < \alpha < 1/q \) and let \( f \) be defined on \( \mathbb{R} \) by \( f(x) = x^{-\alpha}1_{[1, +\infty)}(x) \).

Since \( |f(x)|^r = x^{-\alpha r} \) for \( x \in [1, +\infty) \) and 0 elsewhere, the choice of \( \alpha \) implies that \( f \in L^{r}(\Omega) \) (since \( \alpha r > 1 \)). But also \( f \) is essentially bounded on \( \mathbb{R} \). If \( f \) were in \( L^p(\mathbb{R}) + L^q(\mathbb{R}) \), the technical lemma would imply that \( f \in L^r(\mathbb{R}) \). But, this is not the case since \( |f(x)|^q = x^{-\alpha q} \) for \( x \in [1, +\infty) \) and 0 elsewhere, the choice of \( \alpha \) implies that \( \alpha q < 1 \).

Thus, again in the case where \( r > q \), \( L^r(\mathbb{R}) \) is not contained in \( L^p(\mathbb{R}) + L^q(\mathbb{R}) \).

We end this note with the following observation, which links sections 2 and 3. In the first part of Theorem 3, we have proved that \( L^r(\mathbb{R}) \subset L^p(\mathbb{R}) + L^q(\mathbb{R}) \)
whenever $r \in [p, q]$. In the course of its proof, a simple explicit decomposition of $f \in L^r(\mathbb{R})$ as $f = f_1 + f_2$, with $f_1 \in L^p(\mathbb{R})$ and $f_2 \in L^q(\mathbb{R})$ has been provided (see (13) and the upper bounds (14') and (16)). Indeed, as a consequence of (14) and (16),

$$
\|f\|_{p,q} \leq \|f_1\|_p + \|f_2\|_q \leq \|f\|_{r/p}^r + \|f\|_{r/q}^r. \tag{19}
$$

Hence, the injection of $L^r(\mathbb{R})$ into $L^p(\mathbb{R}) + L^q(\mathbb{R})$ is continuous; therefore, there exists $C > 0$ such that

$$
\|f\|_{p,q} \leq C\|f\|_r. \tag{20}
$$

Indeed, using the inequality (19), we can get an upper bound for the norm of this injection (a somewhat complicated expression in terms of $p, q, r$). This result complements a more classical one which says that, when $r \in [p, q]$, $L^p(\mathbb{R}) \cap L^q(\mathbb{R})$ is contained in $L^r(\mathbb{R})$, then the injection is continuous.

References


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