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WHEN SOME VARIATIONAL PROPERTIES FORCE CONVEXITY

M. Volle[1], J.-B. Hiriart-Urruty[2] and C. Zălinescu[3,4]

Abstract. The notion of adequate (resp. strongly adequate) function has been recently introduced to characterize the essentially strictly convex (resp. essentially firmly subdifferentiable) functions among the weakly lower semicontinuous (resp. lower semicontinuous) ones. In this paper we provide various necessary and sufficient conditions in order that the lower semicontinuous hull of an extended real-valued function on a reflexive Banach space is essentially strictly convex. Some new results on nearest (farthest) points are derived from this approach.

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1. Introduction

It is known that the convexity of a lower semicontinuous (lsc) extended real-valued function J on a Banach space X can be derived from the essential Fréchet differentiability of the Legendre–Fenchel conjugate J^* of J; this is also true for a weakly lsc function J on a weakly sequentially complete Banach space X, provided J^* is just essentially Gâteaux differentiable ([22], Thm. 2.1, [23], Thm. 1, [28], Thm. 3.9.2, [7], Thm. 4.5.1, Cor. 4.5.2). In the same spirit, and for X reflexive, it has been recently proved ([25], [Thm. 1) that a weakly lsc function J is essentially strictly convex if and only if J is adequate, a property we denote here by (A). Reinforcing the property (A), in [26], Theorem 1 it is shown that a lsc function J is essentially firmly subdifferentiable if and only if J is strongly adequate, a property we denote here by (A_s^+). This property is linked to the so-called Tychonov well-posedness of the minimization of the shifted functions $J - x^*$, where the x^*'s are appropriate continuous linear forms on X. In this paper we consider the property (A_w^+), lying between (A) and (A_s^+), obtained by replacing the norm topology in (A_s^+) by the weak topology.

We prove that if J, non-necessarily lsc, satisfies a certain property (A_0), (A_0) weaker than (A), then J satisfies (A_w^+) if and only if the lsc hull of J is essentially strictly convex (Cor. 3.7). Other related facts (Thms. 3.1, 3.6) are also established.

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[1] Department of Mathematics, University of Avignon, France. michel.volle@univ-avignon.fr

[2] Institut de Mathématiques, Université Paul Sabatier, Toulouse, France. jbhu@cict.fr

[3] Faculty of Mathematics, University Al. I. Cuza, Iaşi, Romania.

[4] Octav Mayer Institute of Mathematics, Romanian Academy, Romania. zalinesc@uaic.ro

The results we obtain are applied to the nearest and farthest point problems. We prove for instance that a remotal set S (in a Hilbert space) such that the square of the largest distance to S is Gâteaux differentiable is a singleton (Cor. 3.4), a result we have not found in the literature. We also prove that the farthest point problem is Tychonov well-posed (resp. weakly Tychonov well-posed) if and only if S is a singleton, or, if and only if the antiprojection mapping is norm to norm (resp. weak) continuous (Cor. 3.8), completing in this way well known results ([6, 14, 18], ...). Other classical facts in this field are revisited in the light of our conditions (A_s^+) and (A_w^+) (Cors. 3.3, 3.9). The results presented here can also be applied to nearest and farthest point problems with respect to Bregman distances as in [25, 27], a topic we do not tackle in this paper.

2. Notation and preliminaries

Given a Banach space X, we denote by $F(X)$ the set of functions $J : X \to \mathbb{R} \cup \{+\infty\}$ finite somewhere (*i.e.*, $\operatorname{dom} J := \{x \in X \mid J(x) < \infty\} \neq \emptyset$). To each $J \in F(X)$ we associate its Legendre–Fenchel conjugate J^* defined on the topological dual space X^* of X as

$$J^*(x^*) := \sup_{x \in X} \left(\langle x, x^* \rangle - J(x) \right), \quad x^* \in X^*.$$

The biconjugate J^{**} of J is defined on X^{**}, the bidual of X, then restricted to X by $J^{**}(x) := \sup_{x^* \in X^*}(\langle x, x^* \rangle - J^*(x^*))$. As usual, we set $\Gamma(X) := \{H \in F(X) \mid H = H^{**}\}$. Given $J \in F(X)$, let us introduce the multifunction $MJ : X^* \rightrightarrows X$ defined by:

$$MJ(x^*) = \begin{cases} \arg\min(J - x^*) & \text{if } J^*(x^*) \in \mathbb{R}, \\ \emptyset & \text{otherwise.} \end{cases}$$

In fact MJ is nothing else but the inverse of the subdifferential of the (not necessarily convex) function $J \in F(X)$: $MJ = (\partial J)^{-1}$, the subdifferential of J at a point $x \in X$ being the set

$$\partial J(x) := \{x^* \in X^* \mid J(u) \geq J(x) + \langle u - x, x^* \rangle \quad \text{for all } u \in X\}.$$

The subdifferential of J^* will be understood with respect to the duality between X^* and the bidual X^{**} of X:

$$\partial J^*(x^*) = \{x^{**} \in X^{**} \mid J^*(u^*) \geq J^*(x^*) + \langle u^* - x^*, x^{**} \rangle \text{ for all } u^* \in X^*\}.$$

We thus have, for any $x^* \in X^*$:

$$MJ(x^*) \subset MJ^{**}(x^*) = X \cap \partial J^*(x^*) \subset \partial J^*(x^*) \subset X^{**}. \tag{2.1}$$

According to [5], Definition 5.2, one says that $H \in \Gamma(X)$ is *essentially strictly convex* if MH is locally bounded and H is strictly convex on the line segments in $\operatorname{dom} \partial H$; H is said to be *essentially smooth* (or *essentially Gâteaux differentiable)* if $\operatorname{dom} \partial H$ is open and ∂H is single-valued on $\operatorname{dom} \partial H$. Of course, if $H \in \Gamma(X)$ is essentially Gâteaux differentiable then $\operatorname{dom} \partial H = \operatorname{int}(\operatorname{dom} H)$ and H is Gâteaux differentiable on $\operatorname{dom} \partial H$. We thus have:

Theorem 2.1 ([5], Thm. 5.4)**.** *Assume X is reflexive and let $H \in \Gamma(X)$. Then H is essentially strictly convex if and only if H^* is essentially Gâteaux differentiable.*

We now introduce some general notions about well-posed optimization problems (see *e.g.* [10]). Given $I \in F(X)$, the problem $\min_X I$ is said to be (weakly) Tychonov well-posed (TWP) if I has a unique global minimizer over X toward which each minimizing sequence of I (weakly) converges. The problem $\min_X I$ is said to be (weakly) well-posed in the generalized sense (WPGS) if $\arg\min_X I$ is nonempty and every minimizing sequence of I has a subsequence (weakly) converging toward some element of $\arg\min_X I$. Of course (see [10], p. 24), the problem $\min_X I$ is (weakly) TWP if and only if it is (weakly) WPGS and $\arg\min_X I$ is a singleton. Given $J \in F(X)$, the following assumption will be intensively used in the paper:

$$(A_0): \quad \operatorname{dom} MJ = \operatorname{dom} \partial J^* \text{ is nonempty and open.}$$

Observe that if $J \in F(X)$ admits a continuous affine minorant function, then $\operatorname{dom} \partial J^*$ is necessarily nonempty. In the case when J is cofinite (*i.e.*, J^* is real-valued), (A_0) amounts to $\operatorname{dom} MJ = X^*$. If X is reflexive, any weakly lsc $J \in F(X)$ such that $\lim_{\|x\| \to \infty} J(x)/\|x\| = \infty$ satisfies (A_0) (because $J - x^*$ admits a global minimizer for each $x^* \in X^*$).

Since J^* is subdifferentiable at each point of $\operatorname{int}(\operatorname{dom} J^*)$, the condition (A_0) entails:

$$\emptyset \neq \operatorname{dom} \partial J^* = \operatorname{int}(\operatorname{dom} J^*) = \operatorname{dom} MJ. \tag{2.2}$$

Reinforcing (A_0) for $J \in F(X)$, let us consider the new assumption

$$(A): \quad J \text{ satisfies } (A_0) \text{ and } MJ \text{ is single-valued on its domain.}$$

In such a case we introduce the mapping

$$\mathfrak{m}_J : \operatorname{int}(\operatorname{dom} J^*) \to X \quad \text{with } MJ(x^*) = \{\mathfrak{m}_J(x^*)\}.$$

Condition (A) amounts to the notion of *adequate function* introduced in [25], for reflexive X. The main result about this notion is the following.

Theorem 2.2 ([25], Thm. 1)**.** *Assume X is reflexive, and let $J \in F(X)$ be weakly lsc. Then J satisfies (A) if and only if J is essentially strictly convex.*

We now reinforce (A) by introducing:

$$(A_s^+) \quad \begin{cases} J \text{ satisfies } (A_0) \text{ and, for every } x^* \in \operatorname{dom} MJ, \text{ the problem} \\ \min_X (J - x^*) \text{ is TWP.} \end{cases}$$

In fact, (A_s^+) coincides with the notion of *strongly adequate function* introduced in [26], Definition 1. In order to recall the main results in [26], we need some definitions: $H \in \Gamma(X)$ is said to be *essentially Fréchet differentiable* (see [24, 26], Prop. 2) if H is Fréchet differentiable on $\operatorname{dom} \partial H$. Setting

$$\Gamma_0 := \{\psi : \mathbb{R}_+ \to [0, \infty] \mid \psi \text{ convex lsc, } \psi(t) = 0 \text{ only for } t = 0\},$$

a mapping $H \in \Gamma(X)$ is *essentially firmly subdifferentiable* ([26], Def. 4), if for any $x \in \operatorname{dom} \partial H$, any $x^* \in \partial H(x)$, there exists $\psi \in \Gamma_0$ such that $H(u) \geq H(x) + \langle u - x, x^* \rangle + \psi(\|u - x\|)$, for any $u \in X$. We thus have:

Theorem 2.3 ([26], Prop. 3)**.** *Let $J \in F(X)$ be lsc. Then J satisfies (A_s^+) if and only if J^* is essentially Fréchet differentiable.*

Theorem 2.4 ([26], Thm. 1)**.** *Let $J \in F(X)$ be lsc. If J satisfies (A_s^+), then J is essentially firmly subdifferentiable. The converse holds for X reflexive.*

Let us complete the result above with the following:

Corollary 2.5. *Assume X is reflexive, and let $J \in F(X)$ be lsc. Then the following are equivalent:*

(*i*) *J satisfies (A_s^+);*
(*ii*) *J is essentially firmly subdifferentiable;*
(*iii*) *J^* is essentially Fréchet differentiable;*

(iv) J satisfies (A) and $\mathfrak{m}_J$ is (norm to norm) continuous.

Proof. Theorem 2.4 says that (i) $\Leftrightarrow$ (ii), and Theorem 2.3 ensures that (i) $\Leftrightarrow$ (iii). According to (2.1), $\mathfrak{m}_J$ is a selection of ∂J^*, and [20], Proposition 2.8, says that (iii) $\Leftrightarrow$ (iv). $\square$

We now illustrate the situation with two classical examples. For this we need to recall further definitions. Given $S \subset X$, we denote by ι_S the indicator function of S : $\iota_S(x) := 0$ if $x \in S$, $\iota_S(x) := +\infty$ if $x \in X \setminus S$; $\operatorname{conv} S$ stands for the convex hull of S, and $\overline{S}$ for its closure; $d_S(y) := \inf_{x \in S} \|y - x\|$ denotes the distance from $y \in X$ to S, and $\Delta_S(y) := \sup_{x \in S} \|y - x\|$ the largest deviation from y to S. Needless to say, one has

$$d_S = d_{\overline{S}}, \quad \Delta_S = \Delta_{\overline{S}}. \tag{2.3}$$

Given $f, g \in F(X)$, we denote by $y \longmapsto (f \Box g)(y) := \inf_{x \in X} (f(y - x) + g(x))$ the infimal convolution of f and g. One has for instance $d_S = \|\cdot\| \Box \iota_S$.

Example 2.6. Given a nonempty set S in a Hilbert space X, let us consider $J_S := \frac{1}{2}\|\cdot\|^2 + \iota_S$, which is always cofinite. The function J_S satisfies (A_0) if and only if S is *proximinal*, which means: any point of X admits a nearest point in S. Condition (A) amounts to saying that S is Tchebychev: any point of X has a single nearest point in S. One says that S is *approximately compact* [11] if for any $y \in X$, any sequence $(x_n) \subset S$ such that $\lim_{n\to\infty} \|y - x_n\| = d_S(y)$ contains a subsequence converging to an element of S. Thus, J_S satisfies (A_s^+) if and only if S is Tchebychev and approximately compact. For a Tchebychev S, we denote by $p_S(y)$ the projection of $y \in X$ onto S. We thus have $p_S = \mathfrak{m}_{J_S}$.

Lemma 2.7. *Let S be a nonempty subset in a Hilbert space X. The following then are equivalent:*

(i) S is closed and convex;
(ii) J_S is essentially firmly subdifferentiable;
(iii) J_S is essentially strictly convex;
(iv) $J_S \in \Gamma(X)$.

Proof. (i) $\Rightarrow$ (ii) By the Moreau–Rockafellar theorem (see *e.g.* [28], Thm. 2.8.7) one has $J_S^* = (\frac{1}{2}\|\cdot\|^2)^* \Box \iota_S^* = \frac{1}{2}\|\cdot\|^2 \Box \iota_S^*$, which is Fréchet differentiable on X ([17], Prop. 7.d). By Corollary 2.5, J_S is thus essentially firmly subdifferentiable. The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i) are obvious. $\square$

Applying Theorem 2.2 and Lemma 2.7 to J_S, we recover Klee's theorem [15]: *a nonempty weakly closed subset of a Hilbert space is convex if and only if it is Tchebychev*. Applying Corollary 2.5 and Lemma 2.7 to J_S, we obtain that a Tchebychev set is approximately compact if and only if it is convex ([11], Thm. 3).

Since $J_S^* = \frac{1}{2}\left(\|\cdot\|^2 - d_S^2\right)$ (see [3, 13]), J_S^* is essentially Fréchet differentiable if and only if d_S^2 is Fréchet differentiable on X. We thus infer from Corollary 2.5 and Lemma 2.7 that for any nonempty closed set S it holds (see *e.g.* [28], Thm. 3.9.3, or also [12], Thm. 4.3):

$$d_S^2 \text{ is Fréchet differentiable} \iff \overline{S} \text{ is convex.} \tag{2.4}$$

Finally, Corollary 2.5 and Lemma 2.7 allow us to recover that a Tchebychev (hence closed) set is convex if and only if the projection mapping $p_S : X \to S$ is (norm to norm) continuous ([2], p. 237).

Example 2.8. With each nonempty bounded set S in a Hilbert space X let us associate the mapping $J^S := -\frac{1}{2}\|\cdot\|^2 + \iota_{-S}$, which is always cofinite. One easily sees that J^S satisfies (A_0) if and only if S (equivalently $-S$) is *remotal* ([9, 18, 21], ...): any point of X admits a farthest point in S; J^S satisfies (A) if and only if S is uniquely remotal: each point of X has a single farthest point in S; S is said to be *nearly compact* (or M-compact, or Δ compact [6, 18, 19], ...) if for any $u \in X$, any sequence $(x_n) \subset S$ such that $\lim_{n\to\infty} \|u - x_n\| = \Delta_S(u)$ contains a subsequence converging to an element of S. Of course, a compact set is nearly compact, but a nearly compact set does not need to be closed. Observe that J^S satisfies (A_s^+) if and only if S is uniquely remotal and nearly compact. If S is uniquely remotal, we denote by $q_S(u)$ the point of S the farthest from u and call q_S an antiprojection S ([1]). We thus have $q_S(u) = -\mathfrak{m}_{J^S}(u)$ for $u \in X$.

Lemma 2.9. *For any nonempty bounded set S in a Hilbert space, the following are equivalent:*

(i) S *is a singleton;*
(ii) J^S *is essentially firmly subdifferentiable;*
(iii) J^S *is essentially strictly convex;*
(iv) J^S *is convex.*

Proof.

(i) $\Rightarrow$ (ii) If $S = \{a\}$, then $J^S = \iota_{\{-a\}} - \frac{1}{2}\|a\|^2$ is clearly essentially firmly subdifferentiable.
(ii) $\Rightarrow$ (iii) Because J^S is essentially firmly subdifferentiable, S is closed (and convex); hence J^S is lsc. By Theorem 2.4 J^S satisfies (A_s^+), and so, J^S satisfies (A_0). By Theorem 2.3 J^S is essentially strictly convex.
The implications (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i) are easy. □

Applying Corollary 2.5 and Lemma 2.9 to the function $J^{\overline{S}}$ we obtain that a uniquely remotal set is a singleton if and only if it is nearly compact ([6]). Since $(J^S)^* = \frac{1}{2}\left(\Delta_S^2 - \|\cdot\|^2\right)$ (see [14]), $(J^S)^*$ is essentially Fréchet differentiable if and only if Δ_S^2 is Fréchet differentiable on X. We thus have by Corollary 2.5 and Lemma 2.9 that, for any nonempty closed bounded set $S \subset X$:

$$\Delta_S^2 \text{ is Fréchet differentiable on } X \iff S \text{ is a singleton},$$

or, by (2.3), for any nonempty bounded set $S \subset X$:

$$\Delta_S^2 \text{ Fréchet differentiable on } X \iff S \text{ singleton} \iff \overline{S} \text{ singleton} \tag{2.5}$$

(see [14]). Finally, Corollary 2.5 and Lemma 2.9 allow us to retrieve that a closed uniquely remotal set is a singleton if and only if $q_S : X \to S$ is (norm to norm) continuous ([6]).

3. The main results

In this section X is a Banach space.

Given $J \in F(X)$, we denote by $\overline{J}$ the lsc hull (or closure) of J.

Theorem 3.1. *Assume $J \in F(X)$ satisfies (A_0) and J^* is essentially Gâteaux differentiable. Then $\overline{J} = J^{**}$ and $\overline{J}$ is essentially strictly convex.*

Proof. Since $J^{**} \le \overline{J} \le J$, it suffices to prove that $\overline{J}(x) = J^{**}(x)$ for any $x \in \operatorname{dom} J^{**}$. From (A_0) it follows that J^{**} is proper. By the Brøndsted–Rockafellar Theorem ([28], Thm. 3.1.2) there exists a sequence $((x_n, x_n^*))_{n\ge1} \subset \partial J^{**}$ such that $\|x_n - x\| \to 0$ and $J^{**}(x_n) \to J^{**}(x)$. Since $x_n^* \in \partial J^{**}(x_n)$ and $J^{***} = J^*$, one has $x_n \in \partial J^*(x_n^*)$, and so $x_n = \nabla J^*(x_n^*) \in X$. By using (A_0) we get, for any $n \ge 1$, $\emptyset \ne MJ(x_n^*) \subset \partial J^*(x_n^*) = \{x_n\}$, whence $x_n \in MJ(x_n^*)$ and, consequently, $J(x_n) = J^{**}(x_n)$. We thus have

$$\overline{J}(x) \le \liminf_{n\to\infty} J(x_n) = \lim_{n\to\infty} J^{**}(x_n) = J^{**}(x) \le \overline{J}(x),$$

and finally $\overline{J}(x) = J^{**}(x)$. Hence $\overline{J} = J^{**}$.

Let us prove that J^{**} is essentially strictly convex. To this end we first observe that $(\partial J^{**})^{-1} = \partial J^*$; in fact if $x^{**} \in \partial J^*(x^*)$ then $x^* \in \operatorname{dom} \partial J^* = \operatorname{dom} MJ$, and there exists $x \in X$ such that $x \in MJ(x^*) \subset \partial J^*(x^*)$. Since J^* is Gâteaux differentiable at x^* we have that $x = \nabla J^*(x^*) = x^{**}$. Therefore, $(\partial J^{**})^{-1} = \partial J^*$ is locally bounded ([5], Cor. 2.19). It remains to prove that J^{**} is strictly convex on the line segments in $\operatorname{dom} \partial J^{**}$ or, equivalently ([5], Lem. 5.1), that

$$\partial J^{**}(x) \cap \partial J^{**}(y) \ne \emptyset \Rightarrow x = y.$$

So, assume that $x^* \in \partial J^{**}(x) \cap \partial J^{**}(y) \ne \emptyset$. Then x and y belong to $\partial J^*(x^*)$ and $x = y = \nabla J^*(x^*)$. □

Corollary 3.2. *With the same hypothesis as in Theorem 3.1, J is strictly convex on every convex subset of* $\operatorname{dom}\partial J = \operatorname{dom}\partial J^{**}$.

Proof. We first check that $\operatorname{dom}\partial J = \operatorname{dom}\partial J^{**}$. The inclusion $\subset$ is easy (see *e.g.* [28], Thm. 2.4.1). Conversely, for any $x \in \operatorname{dom}\partial J^{**}$ there exists $x^* \in \partial J^{**}(x)$, and so $x \in \partial J^*(x^*)$. Since J^* is essentially Gâteaux differentiable, we get $x = \nabla J^*(x^*)$. By (A_0) we have that $x^* \in \operatorname{dom} MJ$ and so $\emptyset \neq MJ(x^*) \subset \partial J^*(x^*) = \{x\}$. Consequently, $MJ(x^*) = \{x\}$, whence $x^* \in \partial J(x)$, and so $x \in \operatorname{dom}\partial J$.

By Theorem 3.1 we know that J^{**} is essentially strictly convex, and so strictly convex on every convex subset of $\operatorname{dom}\partial J^{**} = \operatorname{dom}\partial J$. Since J and J^{**} coincide on $\operatorname{dom}\partial J$ (see *e.g.* [28], Thm. 2.4.1), we have proved that J is strictly convex on every convex subset of $\operatorname{dom}\partial J^{**}$. $\square$

Note that even for J lower semicontinuous, Theorem 3.1 can not be derived from [7], Cor. 4.5.2: (a) can not be applied because J^* is not Fréchet differentiable, while (b) can not be applied because J is not sequentially weakly lsc. However, in the case $\dim X < \infty$ and J lsc, Theorem 3.1 follows from [27], Fact 2.7 because Fréchet and Gâteaux differentiability coincide for convex functions.

Applying Theorem 3.1 in the context of Example 2.6, we obtain the following result which is stated in an equivalent form in [8], Corollary 4.7.

Corollary 3.3. *Let S be proximinal in a Hilbert space X. Then, S is convex if and only if d_S^2 is Gâteaux differentiable on X.*

Proof. Necessity: since S is convex, we know that d_S^2 is Fréchet (hence Gâteaux) differentiable on X (see (2.4)).

Sufficiency: since S is proximinal, S is closed. Moreover $J_S^* = \frac{1}{2}\big(\|\cdot\|^2 - d_S^2)\big)$ is Gâteaux differentiable on X. By Theorem 3.1 we infer that $\overline{J_S} = J_S$ is convex, and $S = \operatorname{dom} J_S$ is convex too (see also Lem. 2.7). $\square$

We now apply Theorem 3.1 to the farthest points problem. Recall that a remotal set is necessarily bounded but not necessarily closed. We have not found the next result in the literature. It has to be compared with (2.4) and (2.5).

Corollary 3.4. *Let S be remotal in a Hilbert space X. Then, S is a singleton if and only if Δ_S^2 is Gâteaux differentiable on X.*

Proof. Necessity is clear. Sufficiency: we know that $(J^S)^* = \frac{1}{2}\big(\Delta_S^2 - \|\cdot\|^2\big)$ is Gâteaux differentiable on X. By Theorem 3.1 it follows that $\overline{J^S} = -\frac{1}{2}\|\cdot\|^2 + \iota_{-\overline{S}}$ is strictly convex. This is only possible if $-\overline{S}$ (hence S) is a singleton (see also Lem. 9). $\square$

In order to give further applications of Theorem 3.1, we must now consider the following question: given $J \in F(X)$ satisfying (A_0), when is J^* essentially Gâteaux differentiable? To this end we first state an important consequence of [4], Corollary 6, corresponding to the bornology generated by the singletons (see also [28], Thm. 3.9.1). We adopt the same method as in [16], Proposition 4 for the case of the Fréchet bornology.

Lemma 3.5. *Let $K \in F(X)$ be finite and weakly lsc at a given $x \in X$, and let $x^* \in \operatorname{int}(\operatorname{dom} K^*)$. Then the following are equivalent:*

(i) *The problem $\min_X(K - x^*)$ is weakly TWP with solution x;*
(ii) *K^* is Gâteaux differentiable at x^* with $\nabla K^*(x^*) = x$.*

Proof. According to [4], Corollary 6, we just have to verify the condition

$$\liminf_{\lambda\to\infty} \lambda^{-1} \inf_{\|v\|>\lambda} (K(x+v) - K(x) - \langle v, x^*\rangle) > 0. \tag{3.1}$$

Since K^* is finite and continuous at $x^* \in \operatorname{int}(\operatorname{dom} K^*)$, there exist $r \in \mathbb{R}$ and $\rho > 0$ such that

$$K^* \leq r + \iota_{B_*(x^*,\rho)}, \tag{3.2}$$

where $B_*(x^*, \rho)$ is the closed dual ball of center x^* and radius ρ. Taking the conjugates of both sides in (3.2) we get $K \geq K^{**} \geq \rho \|\cdot\| + x^* - r$. We thus have

$$K(x+v) - K(x) - \langle v, x^* \rangle \geq \rho \|x+v\| + \langle x, x^* \rangle - r - K(x).$$

Consequently,

$$\lambda^{-1} \inf_{\|v\|>\lambda} (K(x+v) - K(x) - \langle v, x^* \rangle) \geq \rho + \lambda^{-1} s,$$

where $s := \langle x, x^* \rangle - \rho \|x\| - r - K(x)$; hence (3.1) holds. □

We now provide a necessary condition for having J^* essentially Gâteaux differentiable. For that we introduce the following property, which is weaker than (A_s^+) :

$$(A_w^+) \quad \begin{cases} J \text{ satisfies } (A_0) \text{ and, for every } x^* \in \operatorname{dom} MJ, \text{ the problem} \\ \min_X (J - x^*) \text{ is weakly TWP.} \end{cases}$$

Theorem 3.6. *Let $J \in F(X)$ satisfy (A_0). Then J^* is essentially Gâteaux differentiable if and only if J verifies (A_w^+).*

Proof. Observe first that $\operatorname{dom} MJ = \operatorname{dom} \partial J^* = \operatorname{int}(\operatorname{dom} J^*)$ because (A_0) holds.

Assume that J^* is essentially Gâteaux differentiable and take $x^* \in \operatorname{dom} MJ$. Then $\emptyset \neq MJ(x^*) \subset \partial J^*(x^*) = \{\nabla J^*(x^*)\}$. It follows that $x := \nabla J^*(x^*) \in \operatorname{dom} \partial J$, and so J is weakly lsc at x. By Lemma 3.5 we obtain that $J - x^*$ is weakly TWP.

Conversely, assume that J satisfies (A_w^+) and take $x^* \in \operatorname{dom} \partial J^* = \operatorname{dom} MJ$. Let $x \in MJ(x^*)$. Then $x^* \in \partial J(x)$, and so J is weakly lsc at x and x is a minimum point of $J - x^*$. By (A_w^+) we have that $J - x^*$ is weakly TWP with solution x. Using again Lemma 3.5 we have that J^* is Gâteaux differentiable at x^*. □

Corollary 3.7. *Assume that X is reflexive, and let $J \in F(X)$ be satisfying (A_0). The following are then equivalent:*

(*i*) *J satisfies (A_w^+);*
(*ii*) *J^* is essentially Gâteaux differentiable;*
(*iii*) *J satisfies (A) and $\mathfrak{m}_J$ is norm to weak continuous;*
(*iv*) *$\overline{J}$ is essentially strictly convex.*

Proof.

(i) ⇔ (ii) comes from Theorem 3.6. Theorem 3.1 says that (ii) ⇒ (iv). Since $J^* = (\overline{J})^*$, Theorem 2.1 says that (iv) ⇒ (ii).
(ii) ⇒ (iii) Since J satisfies (A_0) and J^* is essentially Gâteaux differentiable, it follows from (2.1) that $\mathfrak{m}_J = \nabla J^*$, and ∇J^* is norm to weak continuous by [20], Proposition 2.8.
(iii) ⇒ (ii) By (2.1), $\mathfrak{m}_J$ is a norm to weak selection of ∂J^*, and [20], Proposition 2.8, says that J^* is Gâteaux differentiable on $\operatorname{int}(\operatorname{dom} J^*) = \operatorname{dom} \partial J^*$. □

We now return to Examples 2.6 and 2.8.

Corollary 3.8. *Let S be a nonempty subset of a Hilbert space X. Then the following are equivalent:*

(*i*) *for any $u \in X$, the problem $\min_{x \in S} \|u - x\|$ is weakly TWP;*
(*ii*) *for any $u \in X$, the problem $\min_{x \in S} \|u - x\|$ is TWP;*
(*iii*) *S is closed and convex;*
(*iv*) *S is Tchebychev and p_S is norm to norm continuous;*
(*v*) *S is Tchebychev and p_S is norm to weak continuous.*

Proof.

(i) $\Rightarrow$ (ii) Let (x_n) be such that $\|u - x_n\| \to d_S(u)$. By (i), there exists $x \in S$ such that $\|u - x\| = d_S(s)$, and (x_n) converges weakly to x. Now, $\frac{1}{2}\|x_n - x\|^2 = \frac{1}{2}\|x_n - u\|^2 + \frac{1}{2}\|u - x\|^2 + \langle x_n - u, u - x\rangle$, and, since $\langle x_n, u - x\rangle \to \langle x, u - x\rangle$ we obtain that $\frac{1}{2}\|x_n - x\|^2 \to 0$ and, finally, $\lim x_n = x$.

(ii) $\Rightarrow$ (iii) Observe that S is proximinal (even Tchebychev), hence closed. The function J_S is lsc and satisfies (A_s^+). By Theorem 2.4 (or Cor. 2.5) J_S is convex, and $S = \operatorname{dom} J_S$ is convex, too.

(iii) $\Rightarrow$ (iv) By Lemma 2.7, J_S is essentially firmly subdifferentiable, and by Corollary 2.5, J_S satisfies (A_s^+) or, equivalently, the condition (iv).

(iv) $\Rightarrow$ (v) is obvious.

(v) $\Rightarrow$ (i) The assertion (v) amounts to saying that J_S satisfies (A) and $\mathfrak{m}_{J_S} = p_S$ is norm to weak continuous. By Corollary 3.7, J_S satisfies (A_w^+) or, equivalently, condition (i). □

Corollary 3.9. *For any nonempty bounded set S in a Hilbert space X, the following are equivalent:*

(*i*) *for any $u \in X$, the problem $\max_{x\in S}\|u - x\|$ is TWP;*
(*ii*) *for any $u \in X$, the problem $\max_{x\in S}\|u - x\|$ is weakly TWP;*
(*iii*) *S is uniquely remotal and q_S is norm to weak continuous;*
(*iv*) *S is a singleton;*
(*v*) *S is uniquely remotal and q_S is norm to norm continuous.*

Proof.

(i) $\Rightarrow$ (ii), (iv) $\Rightarrow$ (v), and (v) $\Rightarrow$ (i) are clear. Condition (ii) amounts to saying that J^S satisfies (A_w^+). By Corollary 3.7 we thus have: (ii) implies that $q_S = -\mathfrak{m}_{J^S}$ is norm to weak continuous which, in turn, is equivalent to stating that $\overline{J^S} = J^{\overline{S}}$ is essentially strictly convex. By Lemma 2.9 this amounts to having that $\overline{S}$ is a singleton or, equivalently, that S is a singleton. Consequently (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv). □

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References

[1] E. Asplund, Fréchet differentiability of convex functions. *Acta Math.* **121** (1968) 31–47.
[2] E. Asplund, Čebysev sets in Hilbert spaces. *Trans. Amer. Math. Soc.* **9** (1969) 235–240.
[3] E. Asplund, Differentiability of the metric projection in finite-dimensional Euclidean spaces. *Proc. Amer. Math. Soc.* **38** (1973) 218–219.
[4] E. Asplund and R.T. Rockafellar, Gradients of convex functions. *Trans. Amer. Math. Soc.* **139** (1969) 443–467.
[5] H.H. Bauschke, J.M. Borwein and P.L. Combettes, Essential smoothness, essential strict convexity, and Legendre functions in Banach spaces. *Commun. Contemp. Math.* **3** (2001) 615–647.
[6] J. Blatter, Weiteste Punkte und nächste Punkte. *Rev. Roum. Math. Pures Appl.* **14** (1969) 615–621.
[7] J. M. Borwein and J. D. Vanderwerff, Convex Functions: Constructions, Characterizations and Counterexamples. *Encyclopedia of Mathematics and its Applications*, vol. 109. Cambridge University Press, Cambridge (2010).
[8] F.H. Clarke, R.J. Stern and P.R. Wolenski, Proximal smoothness and the lower-C^2 property. *J. Conv. Anal.* **2** (1995) 117–144.
[9] S. Cobzaş, Geometric properties of Banach spaces and the existence of nearest and farthest points. *Abstract Appl. Anal.* **3** (2005) 259–285.
[10] A.L. Dontchev and T. Zolezzi, *Well-Posed Optimization Problems*. Springer-Verlag, Berlin (1993).
[11] N.V. Efimov and S.B. Steckin, Approximative compactness and Čebysev sets. *Soviet Math. Dokl.* **2** (1961) 1226–1228.
[12] S. Fitzpatrick, Metric projections and the differentiability of distance functions. *Bull. Austral. Math. Soc.* **22** (1980) 291–312.
[13] J.-B. Hiriart-Urruty, Ensembles de Tchebychev vs. ensembles convexes: l'état de la situation vu via l'analyse convexe non lisse. *Ann. Sc. Math. Québec* **22** (1998) 47–62.
[14] J.-B. Hiriart-Urruty, La conjecture des points les plus eloignés revisitée. *Ann. Sci. Math. Québec* **29** (2005) 197–214.
[15] V. Klee, Convexity of Chebyshev sets. *Math. Ann.* **142** (1961) 292–304
[16] M. Lassonde, *Asplund spaces*, Stegall variational principle and the RNP. *Set-Valued Var. Anal.* **17** (2009) 183–193.
[17] J.-J. Moreau, *Fonctionnelles Convexes*, Collège de France, 1966. Republished by the "Tor Vergata" University, Rome (2003).

[18] T.D. Narang, A study of farthest points. *Nieuw Arch. Voor Wiscunde* **3** (1977) XXV 54–79.
[19] B.B. Panda and O.P. Kapoor, On farthest points of sets. *J. Math. Anal. Appl.* **62** (1978) 345–353.
[20] R.R. Phelps, Convex Functions, Monotone Operators and Differentiability. *Lect. Notes Math.,* vol. 1364. Springer-Verlag (1989).
[21] T. Precupanu, Relationships between farthest point problem and best approximation problem. *Anal. Sci. Univ. AI. I. Cuza, Mat.* **57** (2011) 1–12.
[22] V. Soloviov, Duality for nonconvex optimization and its applications. *Anal. Math.* **19** (1993) 297–315.
[23] V. Soloviov, *Characterization of convexity in terms of smoothness*. Unpublished report, Moscow Aviation Institute (1995).
[24] T. Stromberg, Duality between Fréchet differentiability and strong convexity. *Positivity* **15** (2011) 527–536.
[25] M. Volle and J.-B. Hiriart-Urruty, A characterization of essentially strictly convex functions in reflexive Banach spaces. *Nonlinear Anal.* **75** (2012) 1617–1622.
[26] M. Volle and C. Zălinescu, On strongly adequate functions on Banach spaces. *J. Convex Anal.* (to appear).
[27] X. Wang, On Chebyshev functions and Klee functions. *J. Math. Anal. Appl.* **368** (2010) 293–310.
[28] C. Zălinescu, *Convex Analysis in General Vector Spaces*. World Scientific, River Edge, N.J. (2002).