Training of students for mathematics competitions (by JBHU)

Academic year 2024 – 2025

The statements of the exercises, as well as their solutions, are rather short.

Start searching without looking at the proposed hints.

Exercise 1. *As a warm up... Basic geometry in the plane*

On every side of a triangle ABC, one builds a square ; see the picture below.

1°) Prove that the three colored yellow triangles have the same area.

2°) What is the area of the hexagon built up in such a way, in terms of lengths a, b, c of the sides of the initial triangle ABC?

Hints. 1°) This question is (much) simpler than one may think at the first glance.

2°) Use HÉRON's formula giving the area of a triangle in (sole) terms of the lengths a, b, c of its sides.

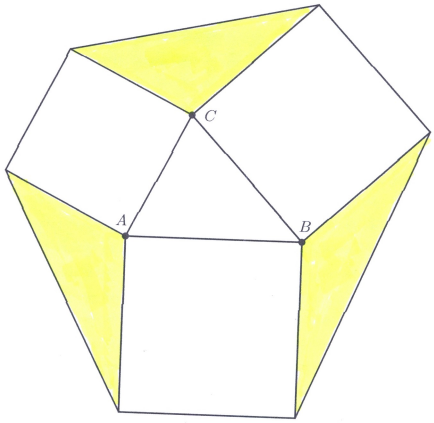


Figure 1: The three triangles

Exercise 2. *Number theory*

An integer (positive or negative) is called a *flower-squares* (un *flori-carrés* in French) if it can be expressed as sums of first squares up to some rank p, each square assigned with a sign $+$ or $-$: $\pm 1^2 \pm 2^2 \pm 3^2 \ ... \pm p^2$.

For example, the integer 2 is a flower-squares since $2 = -1^2 - 2^2 - 3^2 + 4^2$.

Is an integer (positive or negative) always a flower-squares?

Hints. - For any integer a, one has : $4 = a^2 - (a+1)^2 - (a+2)^2 + (a+3)^2$ (called generic decomposition of 4).

- Begin with considering the first integers $1, 2, 3, 4$.

Exercise 3. *Matrix analysis*

Let A be an (n, n) invertible matrix satisfying $A + A^{-1} = I_n$.

Express $A^k + A^{-k}$ for all $k \in \mathbb{N}$.

Hint. $A^k + A^{-k} = c_k.I_n$, where c_k involves a trigonometric function.

Exercise 4. *Analysis (functions of a real variable).* Source: Yerevan University (Armenia).

Find all twice differentiable functions $f : \mathbb{R} \to (0, +\infty)$ satisfying

$$f''(x)f(x) \geqslant 2\left(f'(x)\right)^2 \text{ for all } x \in \mathbb{R}. \qquad (\mathcal{I})$$

Hints. - A twice differentiable function $g : I \to \mathbb{R}$ is convex on (the open interval) I if and only if

$$g''(x) \geqslant 0 \text{ for all } x \in I.$$

- A convex function $g : \mathbb{R} \to \mathbb{R}$ bounded from above (on the whole of $\mathbb{R}$!) is necessarily constant (a powerful result indeed!).

Exercise 5. *Analysis (functions of a real variable).* Source: journal Quadrature (France).

Let $f : \mathbb{R} \to (0, +\infty)$ be a differentiable function satisfying

$$|f'(x) - f'(y)|^{2024} \leqslant |x - y| \text{ for all } x, y \text{ in } \mathbb{R}. \qquad (\mathcal{L})$$

Prove that

$$|f'(u)|^{2025} < 2025.f(u) \text{ for all } u \text{ in } \mathbb{R}. \qquad (\mathcal{K})$$

Hint. If one had put 2025 instead of 2024 in $(\mathcal{L})$, one would have obtained 2026 instead of 2025 in $(\mathcal{K})$. So, there is something more general to consider than the specific numbers $(2024, 2025)$ or $(2025, 2026)$...

Exercise 6. *Analysis (functions of several variables)*

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. We suppose that f is surjective, that is to say $f(\mathbb{R}^n) = \mathbb{R}$.

Let S be the set of x such that $f(x) = 0$ (a nonempty closed set indeed).

Assuming that $n \geqslant 2$, prove that S is unbounded (*i.e.*, is not compact).

Hint. The result clearly is untrue if $n = 1$..., so some topological property in $\mathbb{R}^n$ when $n \geqslant 2$ should enter into the picture...