Training of students for mathematics competitions (by JBHU)
Academic year 2023 – 2024

Exercise 1. *As a warm up... or cool down (this summer)*

Let $A_0, A_1, ..., A_{n-1}$ be the n vertices of a regular polygon inscribed in the unit circle in the plane, with $A_0 = (1, 0)$ for example. We consider the mean value of the lengths $A_0A_1, A_0A_2, ..., A_0A_{n-1}$:

$$L_n = \frac{1}{n-1} \left(A_0A_1 + A_0A_2 + ... + A_0A_{n-1} \right).$$

Calculate the limit of L_n as $n \to +\infty$.

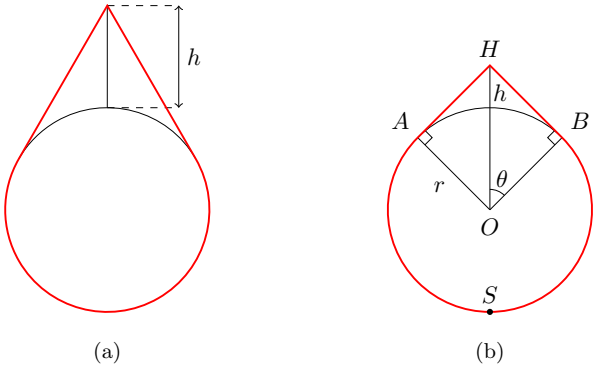
Exercise 2. (*from Iranian mathematics competitions,* 1978)

In the plane, a point, whose Cartesian coordinates are denoted by x and y, is called *rational* if both of its coordinates are rational.

Prove that if the center of a given circle in the plane is not rational, then there are at most 2 rational points on the circle.

Exercise 3. *Surprising but true...*

Let us take the Earth assimilated to a sphere of radius $r = 6400$ km. A tight rope surrounds it at the level of a plane passing through the two poles, which cuts the sphere in two. This rope is long, more than 40 000 km. We extend the rope by only 1 meter. We then pull the rope to a point: it remains in the meridian plane which cuts the Earth in two, but is stretched so as to lift off the surface of the Earth as much as possible. See the figure below (which is not to scale, of course). Question: How high (h) does the rope take off?



(a) (b)

Exercise 4. *Fair division of a pizza into eight parts*

Let us consider a circular pizza that we want to divide into eight parts, four for yourself, four for your friend, both getting the same amount of pizza. One usually does that using a knife and making four straights cuts from the center of the pizza. Suppose now that the

center for cuttings is a point P in the pizza, not necessarily the center: with four adjacent cuts rotated by an angle of $\pi/4 = 45^\circ$, we get eight different pieces, similar to sectors of a circle but they are not really ones; however we call them "sectors" for simplicity reasons. You take every second "sector" (for example the white ones in Figure 1) and your friend the remaining ones (slight red in Figure 1). So, each has got four pieces of pizza, of different shapes of course.

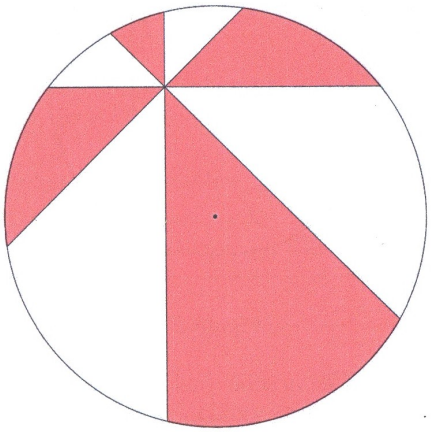


Figure 1

Prove that the pizza has been equally divided, that is to say: the total area of the four pieces of pizza is the same for you and your friend!

Exercise 5. *A mean value theorem... which is not of mean value*

Fot a differentiable function $f : \mathbb{R} \to \mathbb{R}$, we know the following classical mean value theorem, ("Théorème des accroissements finis" in French): For any $a < b$, there exists c lying strictly between a and b such that

$$\frac{f(b) - f(a)}{b - a} = f'(c). \tag{1}$$

For example, if the function is quadratic, that is to say $f : x \mapsto f(x) = \alpha x^2 + \beta x + \gamma$, with $\alpha \neq 0$, then such a c is unique and equals $\frac{a+b}{2}$.

(i) Converse. Suppose that for all a, b, the c for which (1) holds true *is unique and equals* $\frac{a+b}{2}$. Prove that f is necessarily quadratic.

(ii) Generalization. Suppose that your friend prefers another barycentric combination than with 1/2. Let $0 < p < 1$, $p \neq \frac{1}{2}$, and $q = 1 - p$ (thus also $\neq \frac{1}{2}$). Suppose that for all a, b, the c for which (1) holds true *is unique and equals* $pa + qb$. Could you find such functions f?