JOACHIMSTHAL'S THEOREM REVISITED AND EXTENDED

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ABSTRACT. We revisit a plane geometry theorem due to F. JOACHIMSTHAL (1848), which deals with the intersection points of a circle and an ellipse. We take this opportunity to extend the result to "tangent cases," *i.e.*, those where the circle and the ellipse can be tangent at one or two points. These are excellent illustrations of the interaction between plane geometry (conics) and algebra (solving polynomial equations).

1. First observations and Joachimsthal's theorem[1]

Let

- $\mathscr{E}$ be a true (*i.e.* not a circle) ellipse centered at the origin with equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, (a>b>0)$.
- $\mathscr{C}$ be the circle with center $(\alpha,\beta)\in\mathbb{R}^2$ and radius $R>0$, thus with equation $(x-\alpha)^2+(y-\beta)^2=R^2$ or $x^2+y^2-2\alpha x-2\beta y+\gamma=0$ with $\gamma=\alpha^2+\beta^2-R^2$.

• A point $M_t=\begin{pmatrix}a\cos(t)\\ b\sin(t)\end{pmatrix}, (0\le t<2\pi)$, on the ellipse $\mathscr{E}$ will also be on the circle $\mathscr{C}$ if and only if

$$a^2\cos^2(t)+b^2\sin^2(t)-2\alpha a\cos(t)-2\beta b\sin(t)+\gamma=0, \tag{1}$$

or alternatively, by replacing the real functions sin and cos with their complex exponential form, a short calculation quickly yields:

$$\frac{a^2-b^2}{4}e^{4it}+(ib\beta-a\alpha)e^{3it}+\left(\frac{a^2+b^2+2\gamma}{2}\right)e^{2it}-(a\alpha+ib\beta)e^{it}+\frac{a^2-b^2}{4}=0.$$

Consequently, if with every (ellipse, circle) pair $(\mathscr{E},\mathscr{C})$ we associate the following 4^{th}- degree polynomial:

$$Q_{\mathscr{E},\mathscr{C}}(X)=(a^2-b^2)X^4+4(ib\beta-a\alpha)X^3+2\left(a^2+b^2+2\gamma\right)X^2-4(a\alpha+ib\beta)X+a^2-b^2,$$

we will have

$$M_t=\begin{pmatrix}a\cos(t)\\ b\sin(t)\end{pmatrix}\in\mathscr{E}\cap\mathscr{C}\quad\Longleftrightarrow\quad Q_{\mathscr{E},\mathscr{C}}(e^{it})=0. \tag{2}$$

Throughout the following, we will maintain these notations: M_t for the point with coordinates $\begin{pmatrix}a\cos(t)\\ b\sin(t)\end{pmatrix}$ on the ellipse and $Q_{\mathscr{E},\mathscr{C}}$ for the polynomial associated with the geometric pair $(\mathscr{E},\mathscr{C})$. In fact, $\mathscr{E}$ will remain fixed (and thus the coefficients a and b as well) and we will vary the circle $\mathscr{C}$ (that is, the parameters α,β,γ); hence the alternative notation $Q_{\alpha,\beta,\gamma}$ for $Q_{\mathscr{E},\mathscr{C}}$.

• Some consequences of equivalence (2):

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[1]FERDINAND JOACHIMSTHAL was a German mathematician born in 1818 who died young at the age of 43. His contributions concern geometry (conics, surfaces); several theorems bearing his name have been passed down to posterity. There is also a small town named JOACHIMSTHAL in the region of Brandenburg, Germany.

– The number of points in $\mathscr{E}\cap\mathscr{C}$ is exactly the number of roots of modulus 1 of the polynomial $Q_{\mathscr{E},\mathscr{C}}$.
– Since the ellipse $\mathscr{E}$ is not degenerate into a circle (as $a \neq b$), the polynomial $Q_{\mathscr{E},\mathscr{C}}$ is of degree 4: a circle therefore meets an ellipse in at most 4 points, which is easily illustrated (Figure 1 below) by a small drawing.

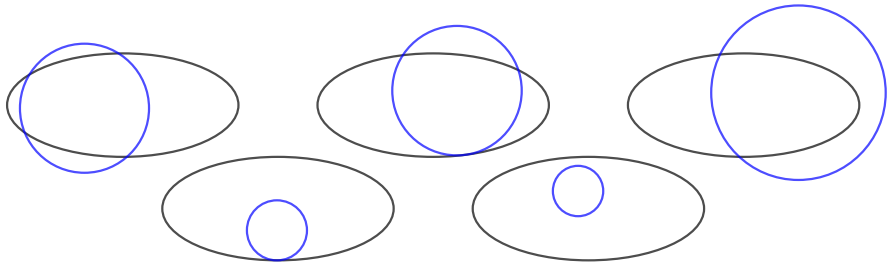


FIGURE 1. The circle $\mathscr{C}$ meets the ellipse $\mathscr{E}$ at $4, 3, 2, 1$ or 0 points.

– The constant coefficient and the leading coefficient of $Q_{\mathscr{E},\mathscr{C}}$ are equal (to a^2-b^2). Consequently, the product of the roots of $Q_{\mathscr{E},\mathscr{C}}$ is 1. In particular, if $\mathscr{C}$ meets $\mathscr{E}$ at 4 **distinct** points $M_{t_1},...,M_{t_4}$, then $Q_{\mathscr{E},\mathscr{C}}(X) = (a^2-b^2)\prod_{j=1}^4(X-e^{it_j})$ and $\prod_{j=1}^4(e^{it_j}) = e^{i(t_1+t_2+t_3+t_4)} = 1$, which implies $t_1+t_2+t_3+t_4 \equiv 0$ (modulo 2π). To be precise, with the choice we made for the parameters t_i (namely $0 \leq t_i < 2\pi$), the condition $t_1+t_2+t_3+t_4 \equiv 0$ (modulo 2π) translates to $t_1+t_2+t_3+t_4 = 2\pi$, 4π or 6π; we will see an example of each case below. What we have just stated about the t_i is the very essence of **Joachimsthal's Theorem** [2] and it turns out that the converse is true. More specifically:

Theorem 1. *(cocyclicity-coellipticity of 4 distinct points)*
The circle $\mathscr{C}$ meets the ellipse $\mathscr{E}$ at 4 distinct points $M_{t_1},\ldots,M_{t_4}$ if, and only if,

$$t_1+t_2+t_3+t_4 \equiv 0\ (2\pi). \tag{3}$$

Proof.
• We have already shown above that condition (3) is necessary, namely: "if $M_{t_1},\ldots,M_{t_4} \in \mathscr{E}\cap\mathscr{C}$, then $t_1+t_2+t_3+t_4 \equiv 0\ (2\pi)$." This is the forward part of the announced equivalence, which is the easiest to prove.
• Let us show that the mentioned condition (3) is also sufficient. Let $M_{t_1},...,M_{t_4}$ be four **distinct** points of the ellipse $\mathscr{E}$ for which $t_1+t_2+t_3+t_4$ is a multiple of 2π; we must show that these points lie on the same circle (they are cocyclic) with center (α,β) and radius $R>0$.

– A first approach: *Existence and uniqueness of such a circle (without concerning ourselves with* (α,β) *and* $R>0$*).*

The following is a quick way to prove the cocyclicity of the four points $M_{t_1},...,M_{t_4}$ when $t_1+t_2+t_3+t_4 \equiv 0\ (2\pi)$; it consists of repeating the reasoning from the proof of the forward part of Theorem 1 (*cf.* beginning of §1), starting from a circle passing through three of the points

[2] In [3, page 96], F. JOACHIMSTHAL writes (in French): *"This theorem is very useful, and its proof offers no difficulties."*

among $M_{t_1}, ..., M_{t_4}$. This is what is usually done in exercise books or discussion forums on the subject, though with a deficiency that we highlight at the end of this approach.

Let, for example, $\mathscr{C}$ be the unique circle passing through M_{t_1}, M_{t_2}, and M_{t_3}, a circle perfectly defined since the three points in question are not collinear (as they lie on the ellipse). We know that $\mathscr{C}$ then intersects the ellipse $\mathscr{E}$ at either four or three distinct points. Let us examine the two possibilities.

– Suppose that $\mathscr{C} \cap \mathscr{E}$ contains 4 (distinct) points, *i.e.*, $\mathscr{C} \cap \mathscr{E} = \{M_{t_1}, M_{t_2}, M_{t_3}, M_\theta\}$. Then, the forward part of Theorem 1 indicates that $t_1 + t_2 + t_3 + \theta \equiv 0\ (2\pi)$ necessarily. Yet, we knew that $t_1 + t_2 + t_3 + t_4 \equiv 0\ (2\pi)$. Consequently, $t_4 - \theta \equiv 0\ (2\pi)$, which means M_θ is none other than M_{t_4}.

– Suppose that $\mathscr{C} \cap \mathscr{E}$ contains only 3 points, say $\mathscr{C} \cap \mathscr{E} = \{M_{t_1}, M_{t_2}, M_{t_3}\}$, a situation whose impossibility we wish to demonstrate. We now refer to a result proven later (Proposition 2), which, while geometrically and intuitively clear, nonetheless requires a formal proof: $\mathscr{C}$ is tangent to $\mathscr{E}$ at one of the points t_1, t_2, t_3, for example at t_1 (this is the Geometric view of the situation), and e^{it_1} is a double root of the polynomial $Q_{\mathscr{E},\mathscr{C}}$ (this is the Algebraic view). But then, again thanks to the forward part of Theorem 1, $2t_1 + t_2 + t_3 \equiv 0\ (2\pi)$. However, we started with the relation $t_1 + t_2 + t_3 + t_4 \equiv 0\ (2\pi)$. Consequently, $t_1 - t_4 \equiv 0\ (2\pi)$, so $M_{t_1} = M_{t_4}$, which is impossible since these two points are assumed to be distinct.

In summary: as soon as the arguments t_i of four distinct points $M_{t_1}, M_{t_2}, M_{t_3}, M_{t_4}$ on the ellipse $\mathscr{E}$ satisfy $t_1 + t_2 + t_3 + t_4 \equiv 0\ (2\pi)$, these four points are cocyclic.

– A second approach: *Explicit calculations of such a circle (that is, expressions for* (α, β) *and* $R > 0$ *as functions of the data).*

Using the notations and observations seen above, we must prove the existence of three real numbers α, β, γ such that $\alpha^2 + \beta^2 - \gamma > 0$ (this is the quantity that will give rise to the radius $R > 0$), and

$$Q_{\alpha,\beta,\gamma}(e^{it_k}) = 0 \text{ for all } k = 1, ..., 4, \tag{4}$$

where

$$Q_{\alpha,\beta,\gamma}(X) = \left(a^2 - b^2\right) X^4 + 4(ib\beta - a\alpha)X^3 + 2(a^2 + b^2 + 2\gamma)X^2 - 4(a\alpha + ib\beta)X + a^2 - b^2.$$

The real numbers α, β, γ we are looking for are such that:

$$Q_{\alpha,\beta,\gamma}(X) = \left(a^2 - b^2\right)(X - e^{it_1})(X - e^{it_2})(X - e^{it_3})(X - e^{it_4}). \tag{5}$$

Let us address this problem and analyze the situation.

Identities (4)-(5) give rise, via NEWTON's "coefficient-root" relations for a polynomial, to the following linear system with unknowns α, β and γ:

$$(\mathscr{S}) \qquad \begin{cases} (a^2 - b^2)\sigma_1 &= -4(ib\beta - a\alpha) \\ (a^2 - b^2)\sigma_2 &= 2(a^2 + b^2 + 2\gamma) \\ (a^2 - b^2)\sigma_3 &= 4(a\alpha + ib\beta) \\ (a^2 - b^2)\sigma_4 &= a^2 - b^2, \end{cases}$$

with the usual notations:

$$\sigma_1 = \sum_k e^{it_k},\ \sigma_2 = \sum_{k<l} e^{it_k}e^{it_l},\ \sigma_3 = \sum_{k<l<m} e^{it_k}e^{it_l}e^{it_m},\ \sigma_4 = e^{it_1}e^{it_2}e^{it_3}e^{it_4}. \tag{6}$$

We must, therefore, ensure that $(\mathscr{S})$ admits a solution (α, β, γ) **in** $\mathbb{R}^3$ (not in $\mathbb{C}^3$).

Let us exploit the hypothesis "$t_1 + t_2 + t_3 + t_4 = 0\ (2\pi)$" and derive the consequences:

– The product σ_4 of the e^{it_k} is none other than $e^{i(t_1+t_2+t_3+t_4)}$ and is thus 1. Thus, the fourth equation of $(\mathscr{S})$ is satisfied; it provides no information about (α, β, γ).

– Clearly, $e^{it_1}e^{it_2} = e^{i(t_1+t_2)}$ equals $e^{-i(t_3+t_4)}$, which is the conjugate of $e^{i(t_3+t_4)}$; thus, in the sum leading to σ_2, $e^{it_1}e^{it_2}$ is grouped with its conjugate $e^{i(t_3+t_4)}$. What has just been said for the

pair $(1,2)$ is true for all pairs (k,l) as soon as $k<l$. Consequence: σ_2 *is real*. According to the second equation of $(\mathscr{S})$, the knowledge of σ_2 leads directly to that of γ:

$$\gamma=\frac{(a^2-b^2)\sigma_2}{4}-\frac{a^2+b^2}{2}=\frac{(a^2-b^2)\,\mathrm{Re}\,\sigma_2}{4}-\frac{a^2+b^2}{2}.$$

– In a manner somewhat similar to the previous calculation, $e^{it_1}e^{it_2}e^{it_3}=e^{i(t_1+t_2+t_3)}$ equals e^{-it_4}; the same applies to coefficients of the same type. Consequently, by comparing the sums leading to σ_1 and σ_3: σ_1 *and* σ_3 *are conjugates.*

– Finally, from the first and third equations of $(\mathscr{S})$, the conjunction of which constitutes a linear (CRAMER) system of two equations with two unknowns, α and β, we obtain

$$\left\{\begin{array}{l}\alpha=\frac{a^2-b^2}{8a}(\sigma_1+\sigma_3),\\ \beta=\frac{a^2-b^2}{8ib}(\sigma_3-\sigma_1),\end{array}\right.$$

that is, since σ_1 and σ_3 are conjugates,

$$\left\{\begin{array}{l}\alpha=\frac{a^2-b^2}{4a}\,\mathrm{Re}(\sigma_3)=\frac{a^2-b^2}{4a}\,\mathrm{Re}(\sigma_1)\in\mathbb{R},\\ \beta=\frac{a^2-b^2}{4b}\,\mathrm{Im}(\sigma_3)=\frac{b^2-a^2}{4b}\,\mathrm{Im}(\sigma_1)\in\mathbb{R}.\end{array}\right. \tag{7}$$

Our analysis led to solving $(\mathscr{S})$; we found one and only one possible solution α,β,γ. Upon synthesis, this triplet (α,β,γ) indeed leads to the desired relation (5).

There is an (important) point to verify: $x^2+y^2-2\alpha x-2\beta y+\gamma=0$ is the equation of a circle of radius $R>0$ only if $E=\alpha^2+\beta^2-\gamma>0$; R will then be $\sqrt{E}$. This is not easy nor immediate to prove, and the first approach (above), which ensures the existence and uniqueness of this circle, would allow us to dispense with it. However, to make the second approach complete and independent of the first, let us prove that indeed $E=\alpha^2+\beta^2-\gamma>0$. This is done in several steps:

– ***Step*** 1. ***Expression of*** $E=\alpha^2+\beta^2-\gamma$ ***as a function of the data*** a,b ***and*** t_1,t_2,t_3,t_4

The starting point is:

$$(\mathscr{T})\qquad\left\{\begin{array}{lcl}\alpha&=&\frac{a^2-b^2}{4a}\left[\cos(t_1)+\cos(t_2)+\cos(t_3)+\cos(t_4)\right]\\ \beta&=&\frac{b^2-a^2}{4b}\left[\sin(t_1)+\sin(t_2)+\sin(t_3)+\sin(t_4)\right]\\ \sigma_2&=&\cos(t_1+t_2)+\cos(t_1+t_3)+\cos(t_1+t_4)\\ &&\quad+\cos(t_2+t_3)+\cos(t_2+t_4)+\cos(t_3+t_4)\\ \gamma&=&\frac{1}{4}\left[(a^2-b^2)\sigma_2-2(a^2+b^2)\right].\end{array}\right.$$

The fact that $t_1+t_2+t_3+t_4\equiv 0\ (2\pi)$ leads to a slightly more compact form of σ_2, namely

$$\sigma_2=2\cos(t_1+t_2)+2\cos(t_1+t_3)+2\cos(t_1+t_4),$$

but this does not help much in this first step.

Using the expressions in $(\mathscr{T})$, we obtain the following expansion:

$$\begin{array}{rcl}16a^2b^2E&=&4a^2b^2\left\{\begin{array}{c}2a^2+2b^2+(b^2-a^2)[\cos(t_1+t_2)+\cos(t_1+t_3)+\cos(t_1+t_4)\\ +\cos(t_2+t_3)+\cos(t_2+t_4)+\cos(t_3+t_4)]\end{array}\right\}\\ &&+a^2(a^2-b^2)^2[\sin(t_1)+\sin(t_2)+\sin(t_3)+\sin(t_4)]^2\\ &&+b^2(a^2-b^2)^2[\cos(t_1)+\cos(t_2)+\cos(t_3)+\cos(t_4)]^2.\end{array}$$

For simplicity, let

$$\begin{array}{rcl}S&=&[\sin(t_1)+\sin(t_2)+\sin(t_3)+\sin(t_4)]^2,\\ C&=&[\cos(t_1)+\cos(t_2)+\cos(t_3)+\cos(t_4)]^2,\\ D&=&\cos(2t_1)+\cos(2t_2)+\cos(2t_3)+\cos(2t_4).\end{array}$$

Note for later that $S\geq 0$ and $D\leq 4$.

– ***Step*** 2. ***Expression of*** $16a^2b^2E$ ***using the functions*** S,C ***and*** D

First observation: Thanks to basic formulas $\cos(t_i+t_j)=\cos(t_i)\cos(t_j)-\sin(t_i)\sin(t_j)$ and $\cos(2t_i)=\cos^2(t_i)-\sin^2(t_i)$, we have

$$\begin{aligned} 2\sigma_2 &= 2\left[\cos(t_1+t_2)+\cos(t_1+t_3)+\cos(t_1+t_4)+\cos(t_2+t_3)+\cos(t_2+t_4)+\cos(t_3+t_4)\right] \\ &= C-S-D. \end{aligned}$$

Consequently, from the expression of $16a^2b^2E$ above, we get

$$16a^2b^2E = 2a^2b^2\left[4a^2+4b^2+D(a^2-b^2)\right]+Sa^2(a^4-b^4)-Cb^2(a^4-b^4). \tag{8}$$

– ***Step*** 3. ***Lower bounding the parts of the expansion of*** $16a^2b^2E$ ***in*** (8) ***containing*** S ***and*** C

First, since $a>b$, we have $Sa^2(a^4-b^4)\geq 0$.

Next, the Cauchy & Schwarz inequality in $\mathbb{R}^4$ with the usual scalar product, applied to the vectors $(1,1,1,1)$ and $(\cos(t_1),\cos(t_2),\cos(t_3),\cos(t_4))$, leads to

$$\left[\cos(t_1)+\cos(t_2)+\cos(t_3)+\cos(t_4)\right]^2 \leq (1+1+1+1)\left[\cos^2(t_1)+\cos^2(t_2)+\cos^2(t_3)+\cos^2(t_4)\right]^2,$$

that is

$$C\leq 4\left[\cos^2(t_1)+\cos^2(t_2)+\cos^2(t_3)+\cos^2(t_4)\right]^2.$$

Thanks to the basic formula $\cos^2(t_i)=\frac{1+\cos(2t_i)}{2}$, the inequality above becomes

$$C\leq 2\left[\cos(2t_1)+\cos(2t_2)+\cos(2t_3)+\cos(2t_4)\right]+8,$$

so $C\leq 2D+8$.

– ***Step*** 4. ***Lower bounding*** $16a^2b^2E$***, then final lower bounding of*** E

Substituting the two lower bounds obtained in Step 3 into expression (8) for $16a^2b^2E$, and simplifying by b^2 along the way, we obtain

$$\begin{aligned} 16a^2E &\geq 2a^2\left[4a^2+4b^2+D(a^2-b^2)\right]-(a^4-b^4)(2D+8) \\ &= 8b^4+8a^2b^2-2b^2(a^2-b^2)D. \end{aligned}$$

We also know that $D\leq 4$, so

$$16a^2E\geq 8b^4+8a^2b^2-8b^2(a^2-b^2)=16b^4. \tag{9}$$

Finally, $(R^2=)E\geq\frac{b^4}{a^2}$ and

$$R\geq\frac{b^2}{a}. \tag{10}$$

We note from this second approach that, besides illustrating once more the use of Newton's algebraic "coefficient-root" relations for a polynomial in a geometric problem (*cf.* system $(\mathscr{S})$), it leads to an *explicit and calculable formulation* of the coordinates (α,β) of the center of the searched circle as well as its radius R. As has already been observed, we indeed have

$$(\mathscr{T}')\qquad \begin{cases} \alpha &= \frac{a^2-b^2}{4a}\left[\cos(t_1)+\cos(t_2)+\cos(t_3)+\cos(t_4)\right] \\ \beta &= \frac{b^2-a^2}{4b}\left[\sin(t_1)+\sin(t_2)+\sin(t_3)+\sin(t_4)\right] \\ \sigma_2 &= 2\cos(t_1+t_2)+2\cos(t_1+t_3)+2\cos(t_1+t_4) \\ \gamma &= \frac{1}{4}\left[(a^2-b^2)\sigma_2-2(a^2+b^2)\right] \\ R &= \sqrt{\alpha^2+\beta^2-\gamma}\,. \end{cases}$$

■

With the help of $(\mathscr{T}')$, here are some illustrative examples of explicit determinations of circles containing four distinct points $M_{t_1},...,M_{t_4}$ (with $t_1+t_2+t_3+t_4\equiv 0\ (2\pi)$) on an ellipse.

Some examples: For the starting ellipse $\mathscr{E}$, $a=36, b=17$, semi-axes common to the 3 examples illustrated in Figure 2.

(1) Data: $t_1 = 0$, $t_2 = \frac{\pi}{3}$, $t_3 = \frac{2\pi}{3}$, $t_4 = \pi$. Here, $t_1 + t_2 + t_3 + t_4 = 2\pi$. The symmetry of the data with respect to the y-axis suggests that α will be zero. Indeed,

$$\alpha = 0,\ \beta = -\frac{1007\sqrt{3}}{68} \simeq -25,65,\ R = \frac{\sqrt{9034851}}{68} \simeq 44,20.$$

(2) Data: $t_1 = \frac{\pi}{2}$, $t_2 = \frac{5\pi}{6}$, $t_3 = \frac{7\pi}{6}$, $t_4 = \frac{3\pi}{2}$. Here, $t_1 + t_2 + t_3 + t_4 = 4\pi$. The symmetry of the data with respect to the x-axis leads us to guess β will be zero. Indeed,

$$\alpha = -\frac{1007\sqrt{3}}{144} \simeq -12,11,\ \beta = 0,\ R = \frac{\sqrt{9034851}}{144} \simeq 20,87.$$

(3) Data: $t_1 = \pi$, $t_2 = \frac{9\pi}{6}$, $t_3 = \frac{5\pi}{3}$, $t_4 = \frac{11\pi}{6}$. Here, $t_1 + t_2 + t_3 + t_4 = 6\pi$. Then,

$$\alpha \simeq 2,56,\ \beta \simeq 35,04,\ R \simeq 52,1.$$

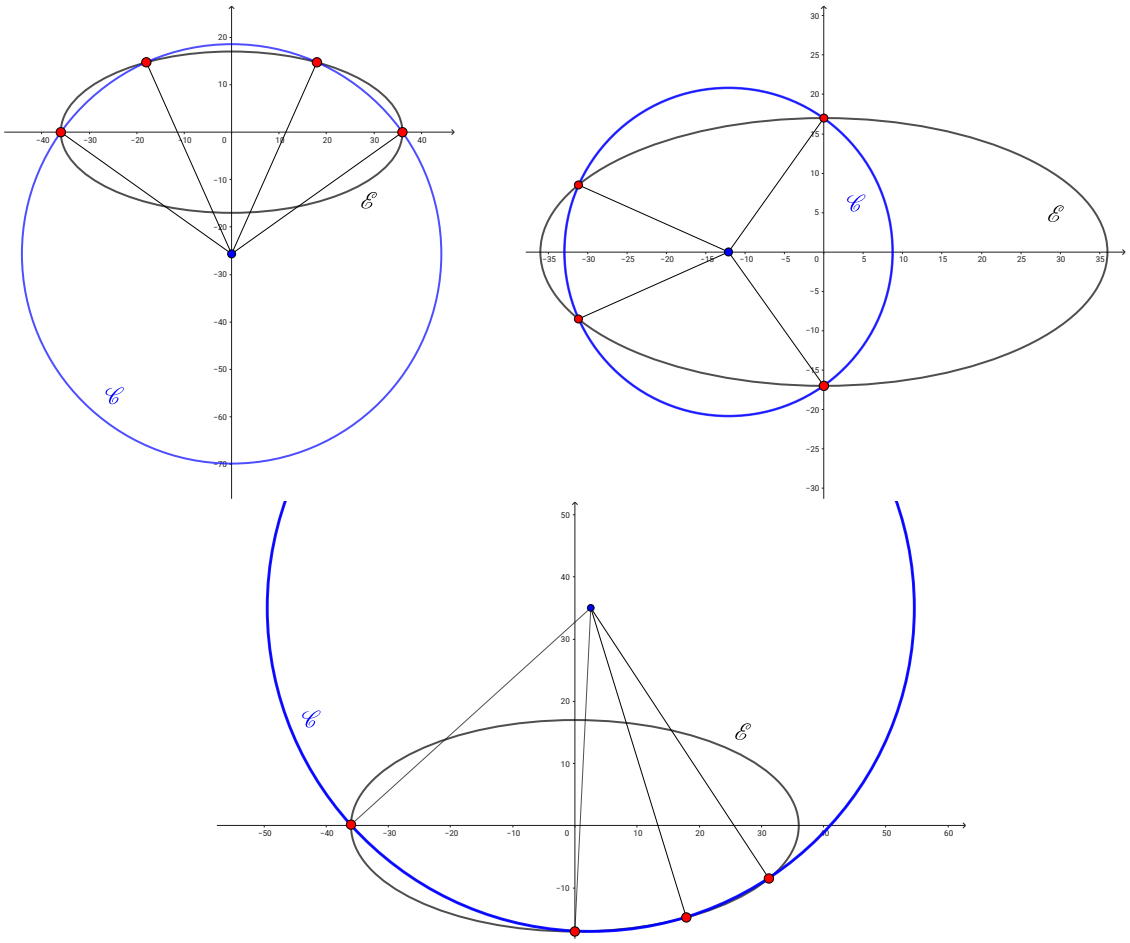


FIGURE 2. The three examples.

• **Some observations.** Let $\varepsilon > 0$ be small, intended to tend toward zero, and let

$$t_1 = \varepsilon,\ t_2 = 2\varepsilon,\ t_3 = 2\pi - 2\varepsilon,\ t_4 = 2\pi - \varepsilon.$$

Then, the four corresponding points $M_{t_1}, M_{t_2}, M_{t_3}, M_{t_4}$ are close to the vertex $(a, 0)$ of the ellipse $\mathscr{E}$, and will indeed tend toward this vertex as ε tends toward 0. As $t_1 + t_2 + t_3 + t_4 = 4\pi$, Theorem 1 ensures that the points $M_{t_1}, M_{t_2}, M_{t_3}, M_{t_4}$ are cocyclic (in one and only one way); let us say they lie on the circle $\mathscr{C}_\varepsilon$.

We now let ε tend toward 0: a short calculation shows that the circle $\mathscr{C}_\varepsilon$ tends toward the circle with center $\left(\frac{a^2-b^2}{a}, 0\right)$ and radius b^2/a, which is none other than **the osculating circle** (or **the circle of curvature**) to the ellipse $\mathscr{E}$ at the vertex $(a, 0)$.

This is not a surprise because (we will find and generalize this result in the next section), according to [1, Theorem 4.12, p. 108], for a sufficiently smooth curve (like $\mathscr{E}$ in our example), the osculating circle at the point with parameter t_0 is also the limit, as t and t' tend toward t_0, of the circle passing through the three points with parameters t, t', t_0, or again the limit, as t tends toward t_0, of the circle passing through the two points with parameters t, t_0 and tangent to the curve at t_0.

Recall that the radius of curvature of the ellipse $\mathscr{E}$ at a point $M_\theta = \begin{pmatrix} a\cos(\theta) \\ b\sin(\theta) \end{pmatrix}$ is

$$R_{osc}(\theta) = \frac{\left[a^2 - (a^2-b^2)\cos^2(\theta)\right]^{3/2}}{ab},$$

so that

$$\min_{\theta\in[0,2\pi)} R_{osc}(\theta) = R_{osc}(0) = R_{osc}(\pi) = \frac{b^2}{a}, \tag{11}$$

$$\max_{\theta\in[0,2\pi)} R_{osc}(\theta) = R_{osc}(\pi/2) = R_{osc}(3\pi/2) = \frac{a^2}{b}. \tag{12}$$

What we have just observed allows us to say that the lower bound in (10) is optimal, *i.e.*, **the infimum of the radii of circles containing** 4 **distinct points of the ellipse** $\mathscr{E}$ **is** b^2/a. Furthermore, a slightly improved form of inequality (10) where S was kept (instead of being replaced by 0 as in Step 3) indicates that a sequence of radii of circles passing through the 4 points $M_{t_{1,n}}, M_{t_{2,n}}, M_{t_{3,n}}, M_{t_{4,n}}$ of the ellipse $\mathscr{E}$ is "minimizing" (that is, tends toward b^2/a) when $[\sin(t_{1,n}) + \sin(t_{2,n}) + \sin(t_{3,n}) + \sin(t_{4,n})] \to 0$ as $n \to +\infty$.

In a dual manner, we can show that **the supremum of the radii of circles containing** 4 **distinct points of the ellipse** $\mathscr{E}$ **is** a^2/b. Indeed, starting from the same point, namely the expression for $16a^2b^2E$ given here,

$$16a^2b^2E = 2a^2b^2\left[4a^2 + 4b^2 + D(a^2-b^2)\right] + Sa^2(a^4-b^4) - Cb^2(a^4-b^4), \tag{8bis}$$

we follow a path parallel to that of Steps 3 and 4 of the proof above.

– ***Step*** $3bis$***. Upper bounding the parts of the expansion of*** $16a^2b^2E$ ***in*** $(8bis)$ ***containing*** S ***and*** D

First, as $C \geq 0$ and $a > b$, we have $-Cb^2(a^4-b^4) \leq 0$.

Next, the Cauchy & Schwarz inequality in $\mathbb{R}^4$ with the usual scalar product applied to the vectors $(1,1,1,1)$ and $(\sin(t_1), \sin(t_2), \sin(t_3), \sin(t_4))$ leads to:

$$\left[\sin(t_1) + \sin(t_2) + \sin(t_3) + \sin(t_4)\right]^2 \leq (1+1+1+1)\left[\sin^2(t_1) + \sin^2(t_2) + \sin^2(t_3) + \sin^2(t_4)\right]^2,$$

that is

$$S \leq 4\left[\sin^2(t_1) + \sin^2(t_2) + \sin^2(t_3) + \sin^2(t_4)\right]^2.$$

Thanks to the basic formula $\sin^2(t_i) = \frac{1-\cos(2t_i)}{2}$, the inequality above becomes

$$S \leq -2\left[\cos(2t_1) + \cos(2t_2) + \cos(2t_3) + \cos(2t_4)\right] + 8,$$

that is $S \leq -2D + 8$.

– ***Step*** $4bis$***. Upper bounding*** $16a^2b^2E$***, then final upper bounding of*** E***.***

Substituting the two upper bounds obtained in step $3bis$ into the expression $(8bis)$ for $16a^2b^2E$ and simplifying by a^2 along the way, we obtain

$$\begin{aligned} 16b^2E &\leq 2b^2\left[4a^2 + 4b^2 + D(a^2-b^2)\right] + (a^4-b^4)(-2D+8) \\ &= 8a^4 + 8a^2b^2 - 2a^2(a^2-b^2)D. \end{aligned}$$

We also know that $D \geq -4$ and $a > b$, so

$$16b^2E \leq 8a^4 + 8a^2b^2 + 8a^2(a^2 - b^2) = 16a^4.$$

Finally, $(R^2 =)E \leq \frac{a^4}{b^2}$ and

$$R \leq \frac{a^2}{b}. \tag{13}$$

Furthermore, a slightly improved form of inequality (13) where C was kept (instead of being replaced by 0 as in Step 3*bis*), indicates that a sequence of radii of circles passing through the 4 points $M_{t_{1,n}}, M_{t_{2,n}}, M_{t_{3,n}}, M_{t_{4,n}}$ of the ellipse $\mathscr{E}$ is "maximizing" (that is, tends toward a^2/b) when $[\cos(t_{1,n}) + \cos(t_{2,n}) + \cos(t_{3,n}) + \cos(t_{4,n})] \to 0$ as $n \to +\infty$.

2. At the frontier of Joachimsthal's theorem

As seen in the first section, Joachimsthal's theorem tells us that a circle $\mathscr{C}$ meets an ellipse $\mathscr{E}$ at four distinct points $M_{t_j}, (j = 1, \ldots, 4)$ if and only if the roots of $Q_{\mathscr{E},\mathscr{C}}$ are points on the unit circle $e^{it_j}, (j = 1, \ldots, 4)$ satisfying $t_1 + t_2 + t_3 + t_4 \equiv 0\ (2\pi)$.

In this section, we will focus on the case where the points M_{t_j} are not necessarily distinct (or equivalently: the roots of modulus 1 of $Q_{\mathscr{E},\mathscr{C}}$ are not of multiplicity 1); we have not found in the literature any text where such cases are correctly adressed.

Let us begin with an example: if the circle $\mathscr{C}$ meets the ellipse $\mathscr{E}$ at exactly three points $M_{t_j}, (j = 1, 2, 3)$; the study of the polynomial $Q_{\mathscr{E},\mathscr{C}}$ carried out in the first paragraph ensures that it admits three distinct roots $e^{it_j}, (j = 1, 2, 3)$; being of degree 4, it also admits a fourth root $z \in \mathbb{C}$, namely:

$$Q_{\mathscr{E},\mathscr{C}}(X) = (a^2 - b^2)(X - e^{it_1})(X - e^{it_2})(X - e^{it_3})(X - z),$$

and since the product of the roots of $Q_{\mathscr{E},\mathscr{C}}$ is 1, the only alternative is that z is also on the unit circle, *i.e.*, $z = e^{i\theta}$. Finally, since we assumed that $\mathscr{C}$ meets $\mathscr{E}$ at exactly three points, the only possibility is that θ is equal to t_1, t_2 or t_3. Let us say $\theta = t_1$; eventually, as the product of the roots is 1, we have $2t_1 + t_2 + t_3 \equiv 0\ (2\pi)$. That is:

$$Q_{\mathscr{E},\mathscr{C}}(X) = (a^2 - b^2)(X - e^{it_1})^2(X - e^{it_2})(X - e^{it_3}) \quad \text{with} \quad 2t_1 + t_2 + t_3 \equiv 0\ (2\pi).$$

In other words: *"The circle $\mathscr{C}$ meets the ellipse $\mathscr{E}$ at exactly 3 points if and only if the roots of $Q_{\mathscr{E},\mathscr{C}}$, all have modulus 1, one being of multiplicity 2 and the other two being simple."*

When representing such a situation (where $\mathscr{C}$ meets the ellipse $\mathscr{E}$ at exactly three points), as shown in the figure below, there is necessarily a point where $\mathscr{C}$ is tangent to the ellipse $\mathscr{E}$. Since $Q_{\mathscr{E},\mathscr{C}}$ possesses a double root at e^{it_1}, the logical expectation would be that the point of tangency is at M_{t_1} as in Figure 3 below.

We shall see that this is indeed the case, and even better: more precisely, the results below specify the nature of the intersection $\mathscr{E} \cap \mathscr{C}$ according to the multiplicity of the roots $e^{i\theta}$ of $Q_{\mathscr{E},\mathscr{C}}$, with the help of the osculating circle (or circle of curvature).

Proposition 2. ***(Multiplicity greater than or equal to** 2**).*** *The circle $\mathscr{C}$ is tangent to the ellipse $\mathscr{E}$ at the point M_θ if and only if $e^{i\theta}$ is at least a double root of $Q_{\mathscr{E},\mathscr{C}}$.*

Proof.
• If $\mathscr{C}$ is tangent to $\mathscr{E}$ at the point M_θ, let us show that $e^{i\theta}$ is a double root of $Q_{\mathscr{E},\mathscr{C}}$: $Q_{\mathscr{E},\mathscr{C}}(e^{i\theta}) = Q'_{\mathscr{E},\mathscr{C}}(e^{i\theta}) = 0$.
Since $M_\theta \in \mathscr{E} \cap \mathscr{C}$, we already have $Q_{\mathscr{E},\mathscr{C}}(e^{i\theta}) = 0$ where:

$$Q_{\mathscr{E},\mathscr{C}}(X) = (a^2 - b^2)X^4 + 4(ib\beta - a\alpha)X^3 + 2\left(a^2 + b^2 + 2\gamma\right)X^2 - 4(a\alpha + ib\beta)X + a^2 - b^2.$$

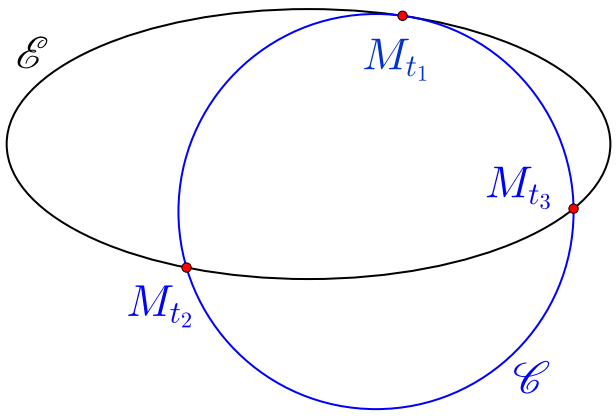


FIGURE 3. $\mathscr{C}$ meets the ellipse $\mathscr{E}$ at exactly three points.

Now let us verify that $Q'_{\mathscr{E},\mathscr{C}}(e^{i\theta}) = 0$: $\mathscr{C}$ is tangent to $\mathscr{E}$ at point M_θ (Figure 4) if and only if $\overrightarrow{\Omega M_\theta} \cdot \overrightarrow{u} = 0$, where Ω is the center of $\mathscr{C}$ and $\overrightarrow{u}$ is a tangent vector to $\mathscr{E}$ at M_θ.

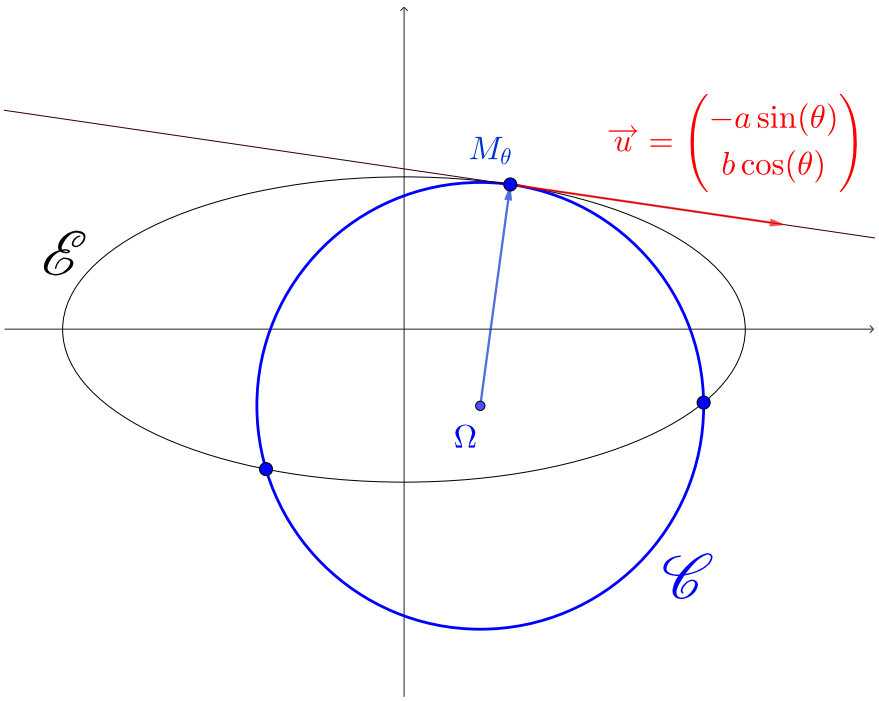


FIGURE 4. $\mathscr{C}$ is tangent to $\mathscr{E}$ at point M_θ.

Translated into coordinates, the orthogonality relation above is written:

$$0 = \begin{pmatrix} a\cos(\theta) - \alpha \\ b\sin(\theta) - \beta \end{pmatrix} \cdot \begin{pmatrix} -a\sin(\theta) \\ b\cos(\theta) \end{pmatrix} = (b^2 - a^2)\cos(\theta)\sin(\theta) + a\alpha\sin(\theta) - b\beta\cos(\theta),$$

which can be reduced, after conversion to exponential form, to:

$$(b^2 - a^2)e^{4i\theta} + 2e^{3i\theta}(a\alpha - ib\beta) - 2e^{i\theta}(a\alpha + ib\beta) + a^2 - b^2 = 0.$$

In summary: $\mathscr{C}$ is tangent to $\mathscr{E}$ at the point M_θ if and only if $e^{i\theta}$ is a root of the polynomial:

$$H(X) = (b^2 - a^2)X^4 + 2(a\alpha - ib\beta)X^3 - 2(a\alpha + ib\beta)X + a^2 - b^2.$$

But:

$$\begin{aligned} Q_{\mathscr{E},\mathscr{C}}(X) - H(X) &= 2(a^2 - b^2)X^4 + 6(ib\beta - a\alpha)X^3 + 2\left(a^2 + b^2 + 2\gamma\right)X^2 - 2(a\alpha + ib\beta)X \\ &= 2X\left[(a^2 - b^2)X^3 + 3(ib\beta - a\alpha)X^2 + \left(a^2 + b^2 + 2\gamma\right)X - (a\alpha + ib\beta)\right] \\ &= \frac{XQ'_{\mathscr{E},\mathscr{C}}(X)}{2}. \end{aligned}$$

In particular, if $X = e^{i\theta}$:

$$e^{i\theta}Q'_{\mathscr{E},\mathscr{C}}(e^{i\theta}) = 2Q_{\mathscr{E},\mathscr{C}}(e^{i\theta}) - 2H(e^{i\theta}) = 0 \quad \Longrightarrow \quad Q'_{\mathscr{E},\mathscr{C}}(e^{i\theta}) = 0,$$

and $e^{i\theta}$ is indeed a double root of $Q_{\mathscr{E},\mathscr{C}}$. QED.

• Conversely, if $Q_{\mathscr{E},\mathscr{C}}(e^{i\theta}) = Q'_{\mathscr{E},\mathscr{C}}(e^{i\theta}) = 0$, then as we have just seen that:

$$H(x) = Q_{\mathscr{E},\mathscr{C}}(X) - \frac{XQ'_{\mathscr{E},\mathscr{C}}(X)}{2},$$

we immediately have $H(e^{i\theta}) = 0$, which (see the beginning of the proof) is equivalent to saying that $\mathscr{C}$ is tangent to $\mathscr{E}$ at point M_θ. QED. ■

Let us now move to the case where $Q_{\mathscr{E},\mathscr{C}}$ has a root of modulus 1 with multiplicity 3 or 4.

Proposition 3. ***(Multiplicity greater than or equal to*** 3***).***

(1) *If $e^{i\theta}$ is a root of multiplicity* 3 *or* 4 *of $Q_{\mathscr{E},\mathscr{C}}$, then $\mathscr{C}$ is the osculating circle to the ellipse $\mathscr{E}$ at the point M_θ.*
(2) *$e^{i\theta}$ is a root of multiplicity* 4 *of $Q_{\mathscr{E},\mathscr{C}}$ if and only if M_θ is a vertex of the ellipse $\mathscr{E}$, i.e., if and only if $\theta \in \{0, \pi/2, \pi, 3\pi/2\}$, and $\mathscr{C}$ is the osculating circle to the ellipse $\mathscr{E}$ at the point M_θ.*

Proof.

(1) Suppose that $e^{i\theta}$ is a root of multiplicity greater than or equal to 3 (thus, 3 or 4) of $Q_{\mathscr{E},\mathscr{C}}$. Let us verify that $\mathscr{C}$ is the osculating circle to the ellipse $\mathscr{E}$ at M_θ. We already know that $\mathscr{C}$ is tangent to $\mathscr{E}$ at M_θ (from Proposition 2, since $e^{i\theta}$ is a root of multiplicity at least 2); it therefore suffices to show that $\mathscr{C}$ and the osculating circle to $\mathscr{E}$ at M_θ have the same radius. Let (see the previous paragraph):

$$R = R_{osc} := \frac{\left[a^2 - (a^2 - b^2)\cos^2(\theta)\right]^{3/2}}{ab}.$$

We propose two proofs.

• *First proof.* We can verify this formula by direct calculation since we know the radius of $\mathscr{C}$: indeed (by Theorem 1), it suffices to show that:

$$\begin{aligned} R^2 = \frac{(a^2 - b^2)^2}{16a^2b^2}&\left[b^2\left(3\cos(\theta) + \cos(3\theta)\right)^2 + a^2\left(3\sin(\theta) - \sin(3\theta)\right)^2\right] \\ &+ \frac{(a^2 + b^2)}{2} - \frac{(a^2 - b^2)}{4} \times 6\cos(2\theta) = R_{\text{osc}}^2 := \frac{\left[a^2 - (a^2 - b^2)\cos^2(\theta)\right]^3}{a^2b^2}. \end{aligned}$$

With (given the hypotheses: $\theta_i = \theta, (i = 1, 2, 3)$ and $\theta_4 = -3\theta$ (2π)):

$$\mathrm{Re}(\sigma_1) = 3\cos(\theta) + \cos(3\theta) = 4\cos^3(\theta) \quad \mathrm{Im}(\sigma_1) = 3\sin(\theta) - \sin(3\theta) = 4\sin^3(\theta), \quad \sigma_2 = 6\cos(2\theta).$$

Let us show that $16a^2b^2R^2 = 16a^2b^2R_{\text{osc}}^2$:

$$
\begin{aligned}
&16a^2b^2R^2 = \\
&= (a^2-b^2)^2\left[b^2\left(3\cos(\theta)+\cos(3\theta)\right)^2 + a^2\left(3\sin(\theta)-\sin(3\theta)\right)^2\right] \\
&\qquad\qquad + 8a^2b^2(a^2+b^2) - 24a^2b^2(a^2-b^2)\cos(2\theta) \\
&= (a^2-b^2)^2\left(16b^2\cos^6(\theta)+16a^2\sin^6(\theta)\right) + 8a^2b^2(a^2+b^2) - 24a^2b^2(a^2-b^2)\cos(2\theta) \\
&= 8\left[(a^2-b^2)^2\left(2b^2\cos^6(\theta)+2a^2\sin^6(\theta)\right) + a^2b^2(a^2+b^2) - 3a^2b^2(a^2-b^2)(2\cos^2(\theta)-1)\right] \\
&= 8\left[(a^2-b^2)^2\left(2b^2\cos^6(\theta)+2a^2[1-\cos^2(\theta)]^3\right)\right. \\
&\qquad\qquad \left. +a^4b^2 + a^2b^4 - 3a^2b^4 + 3a^4b^2 - 6a^4b^2\cos^2(\theta) + 6a^2b^4\cos^2(\theta)\right] \\
&= 8\left[(a^2-b^2)^2\left(2b^2\cos^6(\theta)+2a^2(1-3\cos^2(\theta)+3\cos^4(\theta)-\cos^6(\theta)\right)\right. \\
&\qquad\qquad \left. +4a^4b^2 - 2a^2b^4 - 6a^4b^2\cos^2(\theta) + 6a^2b^4\cos^2(\theta)\right] \\
&= 8\left[(a^2-b^2)^2\left(2b^2\cos^6(\theta)-6a^2\cos^2(\theta)+6a^2\cos^4(\theta)-2a^2\cos^6(\theta)\right)\right. \\
&\qquad\qquad \left. +2a^2(a^4-2a^2b^2+b^4) + 4a^4b^2 - 2a^2b^4 - 6a^4b^2\cos^2(\theta) + 6a^2b^4\cos^2(\theta)\right] \\
&= 8\left[(a^2-b^2)^2\left(2(b^2-a^2)\cos^6(\theta)-6a^2\cos^2(\theta)+6a^2\cos^4(\theta)\right)\right. \\
&\qquad\qquad \left. +2a^6 - 4a^4b^2 + 2a^2b^4 + 4a^4b^2 - 2a^2b^4 - 6a^4b^2\cos^2(\theta) + 6a^2b^4\cos^2(\theta)\right] \\
&= 8\left[(a^2-b^2)^2\left(2(b^2-a^2)\cos^6(\theta)-6a^2\cos^2(\theta)+6a^2\cos^4(\theta)\right)\right. \\
&\qquad\qquad \left. +2a^6 - 6a^4b^2\cos^2(\theta) + 6a^2b^4\cos^2(\theta)\right] \\
&= 16\left[(a^2-b^2)^2\left((b^2-a^2)\cos^6(\theta)-3a^2\cos^2(\theta)+3a^2\cos^4(\theta)\right) + a^6 - 3a^2b^2\cos^2(\theta)(a^2-b^2)\right] \\
&= 16\left[a^6 - (a^2-b^2)^3\cos^6(\theta) - 3a^2(a^2-b^2)^2\cos^2(\theta)\right. \\
&\qquad\qquad \left. +3a^2(a^2-b^2)^2\cos^4(\theta) - 3a^2b^2\cos^2(\theta)(a^2-b^2)\right] \\
&= 16\left[a^6 - (a^2-b^2)^3\cos^6(\theta) + 3a^2(a^2-b^2)^2\cos^4(\theta)\right. \\
&\qquad\qquad \left. -3a^2(a^2-b^2)^2\cos^2(\theta) - 3a^2b^2\cos^2(\theta)(a^2-b^2)\right] \\
&= 16\left[a^6 - (a^2-b^2)^3\cos^6(\theta) + 3a^2(a^2-b^2)^2\cos^4(\theta) - 3a^2(a^2-b^2)\cos^2(\theta)[a^2-b^2+b^2]\right] \\
&= 16\left[a^6 - 3a^4(a^2-b^2)\cos^2(\theta) + 3a^2(a^2-b^2)^2\cos^4(\theta) - (a^2-b^2)^3\cos^6(\theta)\right] \\
&= 16\left(a^2-(a^2-b^2)\cos^2(\theta)\right)^3 = 16a^2b^2R_{\text{osc}}^2.
\end{aligned}
$$

This calculation is somewhat tedious; we prefer another method.

- *Second proof.* If $e^{i\theta}$ is a root of multiplicity 3 of $Q_{\mathscr{E},\mathscr{C}}$, then $\mathscr{C}$ is tangent to $\mathscr{E}$ at M_θ (and intersects the ellipse again at M_φ, where $\varphi = 2\pi - 3\theta$, by Theorem 1).
Let us denote, for $\varepsilon > 0$ by $\mathscr{C}_\varepsilon$, the circle meeting $\mathscr{E}$ at points $M_{\theta-\varepsilon}, M_\theta, M_{\theta+\varepsilon}, M_\varphi$ (it always exists by Theorem 1). Then, if we let ε tend to 0:

– $\mathscr{C}\varepsilon$ tends to $\mathscr{C}$ by the continuity of $x \mapsto M_x$ and the continuity of the radius with respect to the arguments of the intersection points (as per the radius formula in the proof of Theorem 1).

– But we also know that $\mathscr{C}_\varepsilon$ tends to $\mathscr{C}_\theta$, the osculating circle to the ellipse at M_θ, because (see [1], Theorem 4.12, page 108) for a sufficiently regular curve, the osculating circle at point t_0 is the limit, as t and t' tend to t_0, of the circle passing through points t, t', and t_0.
Finally $\mathscr{C} = \mathscr{C}_\theta$, hence

$$R = R_{osc} := \frac{\left[a^2-(a^2-b^2)\cos^2(\theta)\right]^{3/2}}{ab}.$$

(2) • $e^{i\theta}$ being a root of multiplicity 4 of $Q_{\mathscr{E},\mathscr{C}}$ is equivalent to $Q_{\mathscr{E},\mathscr{C}}(X) = (a^2-b^2)(X-e^{i\theta})^4$, which implies $4\theta \equiv 0\ (2\pi)$, hence $\theta \in \{0, \pi/2, \pi, 3\pi/2\}$, corresponding to the 4 vertices of the ellipse. Based on the first step, we obtain the osculating circle at M_θ (see Figure 5 below) .
For example, for $\theta = \pi/2$ (the same applies to $\theta = 0$), the radius of circle $\mathscr{C}$ is:

$$R^2 = \frac{(a^2-b^2)^2}{16a^2b^2}\left[b^2\mathrm{Re}^2(\sigma_1) + a^2\mathrm{Im}^2(\sigma_1)\right] + \frac{(a^2+b^2)}{2} - \frac{(a^2-b^2)\sigma_2}{4},$$

and since $\theta = \pi/2$ gives $\sigma_1 = 4$ and $\sigma_2 = 6$, so:

$$\begin{aligned} R^2 &= \frac{(a^2-b^2)^2}{16a^2b^2}\left(16b^2\right) + \frac{(a^2+b^2)}{2} - \frac{6(a^2-b^2)}{4} = \frac{(a^2-b^2)^2}{a^2} + \frac{(a^2+b^2)}{2} - \frac{3(a^2-b^2)}{2} \\ &= \frac{2(a^2-b^2)^2 + a^2(a^2+b^2) - 3a^2(a^2-b^2)}{2a^2} = \frac{b^4}{a^2}. \end{aligned}$$

Thus $R = \frac{b^2}{a}$. We indeed find the radius of the osculating circle to the ellipse $\mathscr{E}$ at the point $M_{\pi/2}$.

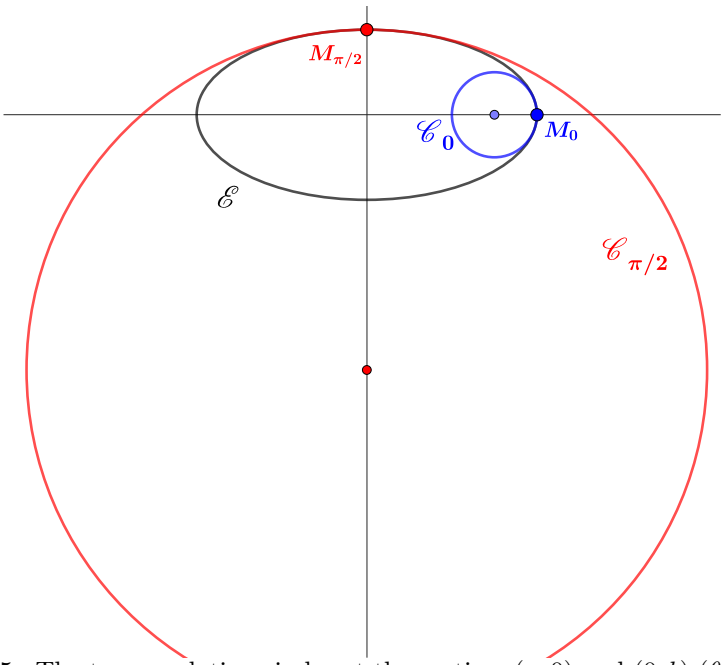


FIGURE 5. The two osculating circles at the vertices $(a, 0)$ and $(0, b)$ $(\theta = 0, \pi/2)$ with respective radii b^2/a and a^2/b.

• We can directly verify the case of multiplicity 4: if $Q^{(k)}_{\mathscr{E},\mathscr{C}}(e^{it_1}) = 0$, for $k = 0, 1, 2, 3$, then

$$\begin{cases} 0 &= (a^2-b^2)e^{4it_1} + 4(ib\beta - a\alpha)e^{3it_1} + 2\left(a^2+b^2+2\gamma\right)e^{2it_1} - 4(a\alpha + ib\beta)e^{it_1} + a^2 - b^2, \\ 0 &= 4(a^2-b^2)e^{3it_1} + 12(ib\beta - a\alpha)e^{2it_1} + 4\left(a^2+b^2+2\gamma\right)e^{it_1} - 4(a\alpha + ib\beta), \\ 0 &= 12(a^2-b^2)e^{2it_1} + 24(ib\beta - a\alpha)e^{it_1} + 4\left(a^2+b^2+2\gamma\right), \\ 0 &= 24(a^2-b^2)e^{it_1} + 24(ib\beta - a\alpha). \end{cases}$$

From the fourth equation, we substitute $(a^2-b^2)e^{it_1} = -(ib\beta - a\alpha)$ into the third:

$$0 = 12(a^2-b^2)e^{2it_1} - 24(a^2-b^2)e^{2it_1} + 4\left(a^2+b^2+2\gamma\right)$$

which yields $3(a^2-b^2)e^{2it_1} = \left(a^2+b^2+2\gamma\right)$, hence, e^{2it_1} is real. Now, the second equation becomes:

$$\begin{aligned} 0 &= 4(a^2-b^2)e^{3it_1} + 12(ib\beta - a\alpha)e^{2it_1} + 4\left(a^2+b^2+2\gamma\right)e^{it_1} - 4(a\alpha+ib\beta) \\ &= 4(a^2-b^2)e^{3it_1} - 12(a^2-b^2)e^{3it_1} + 12(a^2-b^2)e^{3it_1} - 4(a\alpha+ib\beta) \\ &= 4(a^2-b^2)e^{3it_1} - 4(a\alpha+ib\beta), \end{aligned}$$

i.e., $(a^2-b^2)e^{3it_1} = a\alpha + ib\beta$. But since $(a^2-b^2)e^{it_1} = -(ib\beta - a\alpha)$, it remains:

$$a\alpha + ib\beta = (a^2-b^2)e^{3it_1} = e^{2it_1}(a\alpha - ib\beta).$$

As e^{2it_1} is real, by taking real and imaginary parts:

$$e^{2it_1}a\alpha = a\alpha, \quad -e^{2it_1}ib\beta = ib\beta \quad \text{and} \quad e^{2it_1} \in \mathbb{R},$$

which gives (since a and b are non-zero):

$$t_1 \in \{0,\pi\} \text{ and } \beta = 0 \quad \text{or} \quad t_1 \in \{\pi/2, 3\pi/2\} \text{ and } \alpha = 0.$$

This returns to the four vertices of the ellipse and the circle of curvature. QED. ■

Some Consequences.

• In the case of two points of tangency as in the figure below, Proposition 2 ensures that:

$$Q_{\mathscr{E},\mathscr{C}}(X) = (a^2-b^2)(X-e^{it_1})^2(X-e^{it_2})^2.$$

Since $2t_1 + 2t_2 \equiv 0\ (2\pi)$, M_{t_1} and M_{t_2} are symmetric with respect to the horizontal axis, and thus the center of $\mathscr{C}$ lies on this axis (as shown in Figure 6).

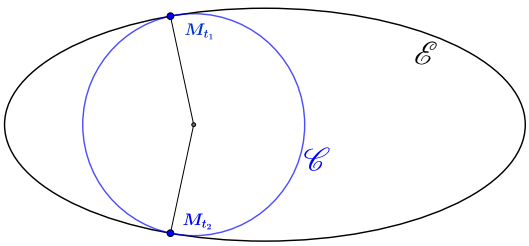


FIGURE 6. $\mathscr{C}$ is tangent to $\mathscr{E}$ with multiplicity 2 at two points.

• If $\mathscr{C} \cap \mathscr{E}$ is reduced to a single point with tangency (figure7):

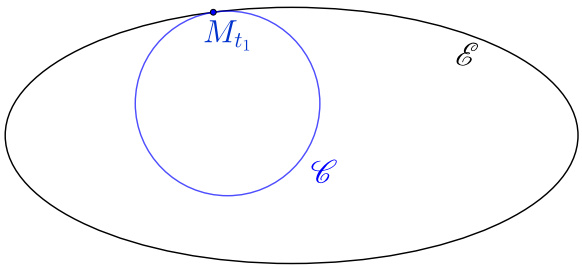


FIGURE 7. A single point with tangency of multiplicity 2.

then e^{it_1} is the unique root of $Q_{\mathscr{E},\mathscr{C}}$ on the unit circle, and it is of multiplicity 2 or 4. Indeed, it is at least 2 by Proposition 2; it cannot be 3 because then the fourth root would also have modulus 1 and would be either e^{it_1} (multiplicity 4) or another root $e^{it_2} \neq e^{it_1}$, in which case $\mathscr{C} \cap \mathscr{E}$ would consist of two points.

In summary, when $\mathscr{C} \cap \mathscr{E}$ is reduced to a single point M_{t_1}, then:

– Either e^{it_1} is a root of $Q_{\mathscr{E},\mathscr{C}}$ with multiplicity exactly 4 (and M_{t_1} is a vertex of the ellipse, and $\mathscr{C}$ is the osculating circle):

$$Q_{\mathscr{E},\mathscr{C}}(X) = (a^2 - b^2)(X - e^{it_1})^4, \quad t_1 \in \{0, \pi, \pi/2, 3\pi/2\}.$$

– Or e^{it_1} is a root of $Q_{\mathscr{E},\mathscr{C}}$ with multiplicity exactly 2 (as in Figure 7), of the form:

$$Q_{\mathscr{E},\mathscr{C}}(X) = (a^2 - b^2)(X - e^{it_1})^2(X - z_1)(X - z_2)$$

with $|z_i| \neq 1$, and as the product of the roots is 1, we have $|z_1| \cdot |z_2| = 1$.

• It is important to observe that when the roots of $Q_{\mathscr{E},\mathscr{C}}$ are not all of modulus 1 (*i.e.*, either a single double root of modulus 1; or two distinct roots of modulus 1), it is very easy to construct circles intersecting the ellipse at these roots for any $R \in]0, +\infty[$. Possible values for R thus describe the entire interval $]0, +\infty[$. This is a notable difference compared to the case where all roots have modulus 1, where we demonstrated that $b^2/a \leq R \leq a^2/b$.

• In the case of a triple root, Proposition 3 ensures that, apart from the four vertices of the ellipse, the osculating circle at M_θ crosses the ellipse to intersect it again at $M_{2\pi-3\theta}$. This provides a simple method for drawing the osculating circle, as its center is at the intersection of the normal to the ellipse at M_θ and the perpendicular bisector of the segment with endpoints M_θ and $M_{2\pi-3\theta}$. For example, in the Figure 8 below, we have drawn the circle passing through $M_{\pi/4}$ with order 3 and through $M_{2\pi-3\theta} = M_{5\pi/4}$ with order 1 for the ellipse $x^2/4 + y^2 = 1$. The formulas in the proof of Theorem 1 give: $\alpha \simeq 0,53, \beta \simeq -1,06, R \simeq 2,74$, and we indeed recover the osculating circle to the ellipse at $M_{\pi/4}$.

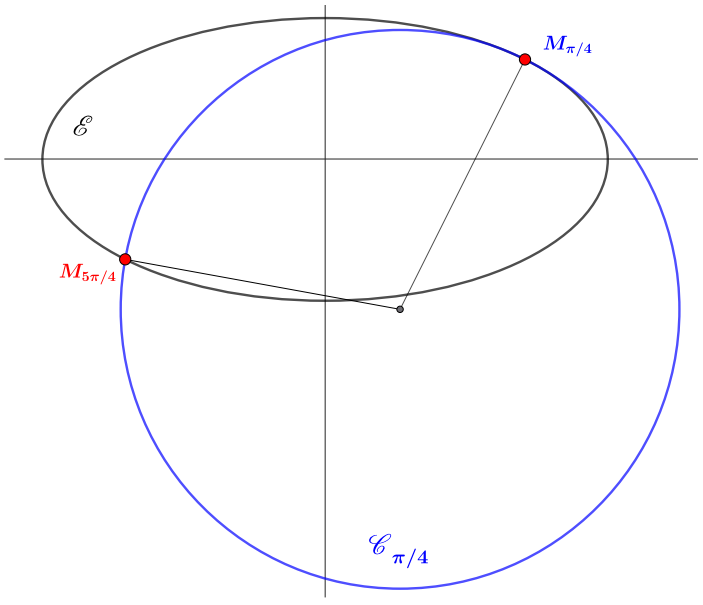


FIGURE 8. Triple root at $e^{i\pi/4}$ = osculating circle at $M_{\pi/4}$.

• Theorem 1 allows us to quickly recover the following nice result by Vincenzo Janni (1862) [2]: *"Through any point M_θ on the ellipse pass 3 osculating circles to three distinct points $M_{t_i}, (i =$*

$1, 2, 3$)*, and the area of the triangle with these three points as vertices (Figure 9) is constant and equals* $3ab\sqrt{3}/4$*."*

Indeed, from the above, an osculating circle to the ellipse at M_t will pass through $M_\theta \in \mathscr{E}$ if and only if $\theta + 3t \equiv 0\ (2\pi)$, i.e., $t_1 = -\theta/3, t_2 = \frac{2\pi-\theta}{3}, t_3 = \frac{4\pi-\theta}{3}$. Then, if we set:

$$M_{t_i} = \begin{pmatrix} x_i = a\cos(t_i) \\ y_i = b\sin(t_i) \end{pmatrix}, \quad i = 1, 2, 3$$

the area of the triangle with vertices $M_{t_i}, (i = 1, 2, 3)$ will be:

$$\mathscr{A} = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} x_1 & x_2 & x_1+x_2+x_3 \\ y_1 & y_2 & y_1+y_2+y_3 \\ 1 & 1 & 3 \end{vmatrix}.$$

But $x_1 + x_2 + x_3 = 0 = y_1 + y_2 + y_3$ (elementary calculation), so that:

$$\begin{aligned} \mathscr{A} &= \begin{vmatrix} x_1 & x_2 & x_1+x_2+x_3 \\ y_1 & y_2 & y_1+y_2+y_3 \\ 1 & 1 & 3 \end{vmatrix} = \begin{vmatrix} x_1 & x_2 & 0 \\ y_1 & y_2 & 0 \\ 1 & 1 & 3 \end{vmatrix} = 3(x_1y_2 - x_2y_1) \\ &= 3ab\left[\cos\left(\frac{-\theta}{3}\right)\sin\left(\frac{2\pi-\theta}{3}\right) - \sin\left(\frac{-\theta}{3}\right)\cos\left(\frac{2\pi-\theta}{3}\right)\right] \\ &= \frac{3ab\sqrt{3}}{4}\left[\cos^2\left(\frac{\theta}{3}\right) + \sin^2\left(\frac{\theta}{3}\right)\right] = \frac{3ab\sqrt{3}}{4}. \end{aligned}$$

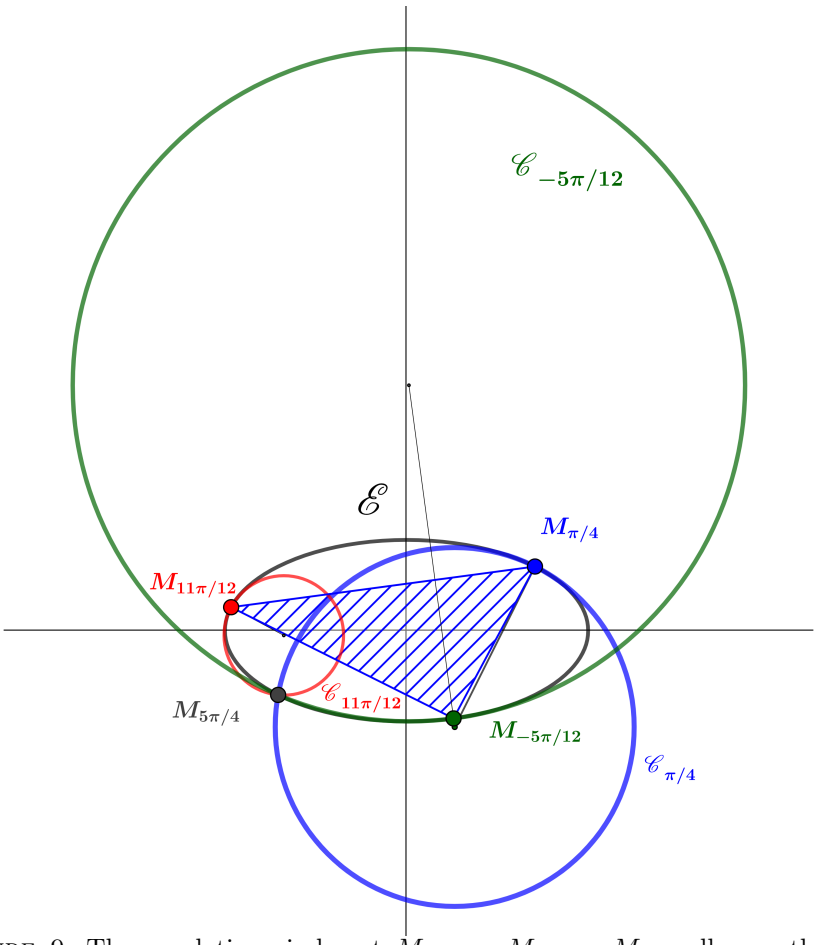


FIGURE 9. The osculating circles at $M_{-5\pi/12}, M_{11\pi/12}, M_{\pi/4}$ all pass through $M_{5\pi/4}$ and form a triangle of constant area equal to $\frac{3ab\sqrt{3}}{4}$.

• We also recover another result from Joachimsthal [3]:
"Through any point M_θ on the ellipse pass 3 osculating circles to three distinct points $M_{t_i}, (i = 1, 2, 3)$, and the center of gravity of the triangle with these three points as vertices is the center of the ellipse (see Figure 10)."

Once again, based on the above, through $M_\theta \in \mathscr{E}$ pass the three osculating circles to the ellipse at $M_{-\frac{\theta}{3}}$, $M_{\frac{2\pi-\theta}{3}}$ and $M_{\frac{4\pi-\theta}{3}}$. Consequently, the coordinates (x_g, y_g) of the center of gravity of the triangle passing through these three points satisfy:

$$x_g = \frac{x_{-\frac{\theta}{3}} + x_{\frac{2\pi-\theta}{3}} + x_{\frac{4\pi-\theta}{3}}}{3} = \frac{a}{3}\left[\cos\left(-\frac{\theta}{3}\right) + \cos\left(\frac{2\pi-\theta}{3}\right) + \cos\left(\frac{4\pi-\theta}{3}\right)\right]$$
$$= \frac{a}{3}\left[\cos(-\theta/3) - \cos(-\theta/3)/2 - \sqrt{3}\sin(-\theta/3)/2 - \cos(-\theta/3)/2 + \sqrt{3}\sin(-\theta/3)/2\right] = 0.$$

Similarly, $y_g = \frac{y_{-\frac{\theta}{3}} + y_{\frac{2\pi-\theta}{3}} + y_{\frac{4\pi-\theta}{3}}}{3} = 0$, hence $(x_g, y_g) = (0, 0)$. QED.

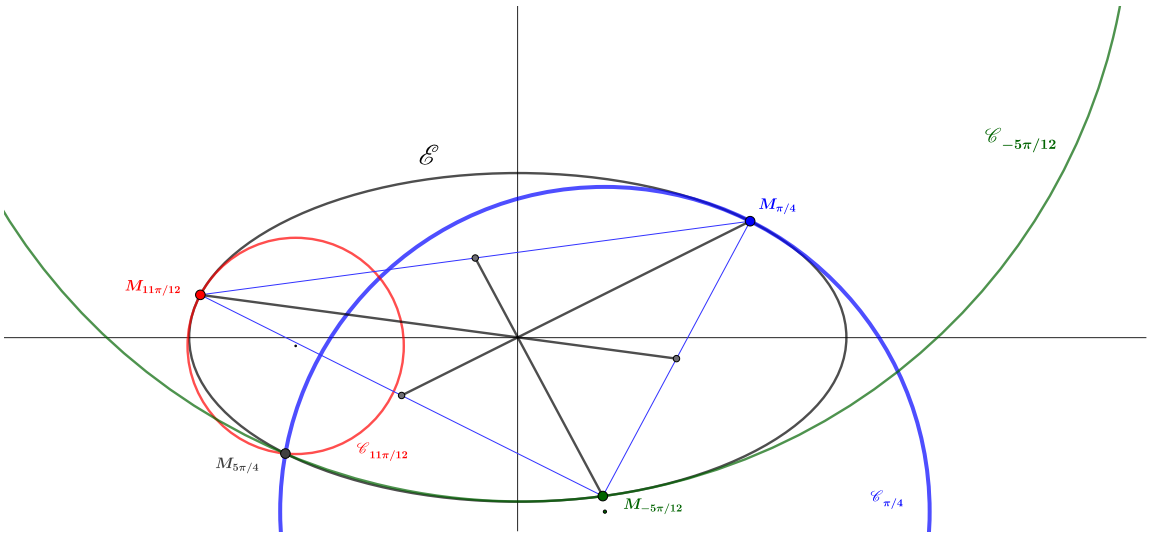


FIGURE 10. The center of gravity of the triangle with vertices $M_{-5\pi/12}, M_{11\pi/12}, M_{\pi/4}$ is the center of the ellipse.

Conclusion. In this article, we first revisited F. Joachimsthal's result (in paragraph 1) by providing complete proofs (necessary and sufficient conditions) and quantified bounds (on circle radii) for the situation where four points on an ellipse are cocyclic; then, in paragraph 2, we extended these results to situations with fewer than four intersection points, *i.e.*, cases of tangency. Synthetically, this is summarized as:

Theorem 4. *Let $\mathscr{E}$ be the centered ellipse, $\mathscr{C}$ a circle, and $Q_{\mathscr{E},\mathscr{C}}$ the associated polynomial.*

1. *$\mathscr{E} \cap \mathscr{C}$ is the set of roots of modulus 1 of $Q_{\mathscr{E},\mathscr{C}}$.*
2. *$\mathscr{C}$ is tangent to $\mathscr{E}$ at M_t if and only if e^{it} is at least a double root of $Q_{\mathscr{E},\mathscr{C}}$.*
3. *If e^{it} is at least a triple root of $Q_{\mathscr{E},\mathscr{C}}$, then $\mathscr{C}$ is the osculating circle to the ellipse at M_t.*
4. *Let $t_1, t_2, t_3, t_4 \in \mathbb{R}$ be **not necessarily distinct**. If $t_1 + t_2 + t_3 + t_4 \equiv 0\ (2\pi)$, then there exists a circle $\mathscr{C}$ such that $\mathscr{E} \cap \mathscr{C} = \{M_{t_i},\ i = 1, \dots, 4\}$ and the radius R of $\mathscr{C}$ satisfies:*

$$\inf_\theta R_{osc}(\theta) = \frac{b^2}{a} \leq R \leq \frac{a^2}{b} = \sup_\theta R_{osc}(\theta).$$

The converse is true if the M_{t_i} are pairwise distinct, and even if only three of them are distinct.

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