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An open global volume minimization problem

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Abstract We present an open global minimization problem: it concerns the minimization of the volume among the compact convex sets in $\mathbb{R}^3$ of a given constant thickness.

Keywords Global minimization · Convex compact sets · Thickness of convex sets · Volume function

Consider a compact convex set C in the 3-dimensional space $\mathbb{R}^3$, of constant thickness $l > 0$, that is to say satisfying the following property:

$$\text{For all unitary vectors } d \text{ in } \mathbb{R}^3, \quad \max_{x \in C} \langle x, d \rangle - \min_{x \in C} \langle x, d \rangle = l.$$

Here, $\langle u, v \rangle$ denotes the usual scalar (or inner) product in $\mathbb{R}^3$. In geometrical terms, the condition above means that the distance between two (arbitrary) parallel tangent planes to C always equals l. If necessary, the condition can be parameterized by using spherical coordinates (or others) for the unit vectors $d \in \mathbb{R}^3$. Such compact convex sets of constant thickness in $\mathbb{R}^3$ are sometimes called *spheroforms* in the literature.

A question which remains open since almost eighty years now is : what are the convex bodies (in $\mathbb{R}^3$) of constant thickness l and **of minimal volume?**

The counterpart maximization problem does not offer difficulties (as suspected, the spheres of diameter l are the solutions), but, in spite of nice properties of the volume function (continuity, its 3-rd root is concave (theorem of BRUNN- MINKOWSKI), etc.), we do not know the solutions of the minimization problem. What is known about this global minimization problem?

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- It indeed has solutions: the continuity of the volume function and the usual topologies on families of compact convex sets are sufficient ingredients to ensure the existence of solutions. From one solution, one gets another one by rotating it.
- By taking the constant thickness l equal to 1, a theoretical lower bound for the volume is known to be $\pi \frac{3\sqrt{6}-7}{3} \approx 0.365$.
- The suspected candidates as global minimizers are the so-called **Meissner's spheroforms** (sort of inflated tetrahedrons, where curved faces contain pieces of spheres and toroidal parts). Contrary to one might think at the first glance, MEISSNER's tetrahedrons do not consist of spherical pieces built up from regular tetrahedrons (such compact convex sets are not of constant thickness), but they look like them. Neither are they rotated REULEAUX' curvilinear triangles. Visualizations of MEISSNER's spheroforms can be seen on the following websites: http:www.lama.univ-savoie.fr/~lachand/Spheroforms.html, http:www.swisseduc.ch/mathematik/glichclick/index.html.
- Surprisingly enough, for convex bodies of constant width in $\mathbb{R}^3$, minimizing the volume and minimizing the surface area are *equivalent* problems (due to an old extraordinary theorem by BLASCHKE). To get an idea, for $l = 1$: the volume of the sphere is approximatively 0.52, while that of MEISSNER's spheroform is 0.42 (a reduction of about 20%); concerning the surface area, the reduction is of about 7%.

Recently, some additional properties have been derived about compact convex sets in $\mathbb{R}^3$ of constant thickness, but without solving the above global minimization problem [1,2]. The situation is completely understood in the 2-dimensional context: compact convex sets of $\mathbb{R}^2$ of constant width minimizing the area are the so-called REULEAUX' curvilinear triangles. These questions as well as properties of compact convex sets of constant width (in $\mathbb{R}^2$) or of constant thickness (in $\mathbb{R}^3$) are surveyed in [3].

Our question here is not theoretical but practical: **what "numerical" answer would be provided by some massive (and even brute) minimization process?** In other words, would some global minimization process help to grasp the solutions? to confirm or not the candidacy of MEISSNER's spheroforms? Beware: we are faced with a minimization problem which bears the same burdens as the minimization of a concave function over a convex set. Clearly, in any attempt, the starting configuration should not be the sphere (we will be stuck at a global maximizer, at least with "local" methods), but a possible initial configuration is the REULEAUX's (inflated) tetrahedron.

References

1. Lachand-Robert, Th., Oudet, E.: (2007) Bodies of constant width in arbitary dimension. Math. Nachr. **280**(7), 740–750
2. Bayen, T., Lachand-Robert, Th., Oudet, E.: (2007) Analytic parametrization and volume minimization of three dimensional bodies of constant width. Arch. Ration. Mech. Anal. **186**, 225–249
3. Bayen, T., Hiriart-Urruty, J.-B.: Objets convexes de largeur constante (en 2D) ou d'épaisseur constante (en 3D): du neuf avec du vieux (2008), submitted