Minimality locus of weighted degrees for tame automorphisms

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Abstract

Given f a polynomial automorphism of $\mathbf{k}^2$, or a polynomial automorphism of $\mathbf{k}^3$ given as a product of an elementary automorphism and a permutation, we study the minimality locus of $\nu(f)$ for valuations ν in a countable union of apartments.

1 Introduction

We start by recalling the definition of a valuation.

Definition 1.1 (Valuation)**.** *A (real) valuation on the ring* $\mathbf{k}[x_1, ..., x_n]$ *is a function* $\nu : \mathbf{k}[x_1, \dots, x_n] \to \mathbf{R} \cup \{\infty\}$ *satisfying the following properties, for all* $P, Q \in \mathbf{k}[x_1, \dots, x_n]$*,* $c \in \mathbf{k}^*$*:*

1. $\nu(P+Q) \geqslant \min\{\nu(P), \nu(Q)\}$
2. $\nu(PQ) = \nu(P) + \nu(Q)$
3. $\nu(c) = 0$
4. $\nu(0) = \infty \Leftrightarrow P = 0$

In particular for deg the usual polynomial degree, $-\deg$ is a valuation on $\mathbf{k}[x_1, \dots, x_n]$. One may also consider a weight $\omega = (\omega_1, \dots, \omega_n)$, with ω_i positive real numbers, and the weighted degree

$$\deg_\omega(P) = \max\{\omega_1 i_1 + \ldots + \omega_n i_n \mid (i_1, \dots, i_n) \in \mathrm{Supp}(P)\}$$

such that $-\deg_\omega$ is a valuation. These valuations play a central role in this article, called *monomial valuations*:

Definition 1.2 (Monomial valuation)**.** *A tame valuation on* $\mathbf{k}[x_1, \dots, x_n]$ *is a valuation of the form* $\nu_{\mathrm{id},\omega} = -\deg_\omega$*, for a weight* $\omega \in (\mathbf{R}_+^*)^n$*.*

Polynomial automorphisms of $\mathbf{k}^n$ are of interest, as they act naturally on valuations.

Definition 1.3 (Polynomial endomorphism/automorphism of $\mathbf{k}^n$)**.**

$$\begin{array}{rcl} \mathrm{End}(\mathbf{k}^n) & = & \{f = (f_1, \dots, f_n) \mid f_i \in \mathbf{k}[x_1, \dots, x_n]\} \\ \vee & & \\ \mathrm{Aut}(\mathbf{k}^n) & = & \{f \in \mathrm{End}(\mathbf{k}^n) \mid f \textit{ invertible and } f^{-1} \in \mathrm{End}(\mathbf{k}^n)\} \end{array}$$

In this framework, we look at the left action of the group $\mathrm{Aut}(\mathbf{k}^n)$ on the space of valuations

$$\forall f \in \mathrm{Aut}(\mathbf{k}^n), \forall P \in \mathbf{k}[x_1, \dots, x_n] : f \cdot \nu(P) = \nu(P \circ f).$$

Many authors have studied the particular subgroup of such automorphisms generated by the triangular and linear ones, classically called *tame.* More particularly, in this paper we will examine triangular automorphisms and permutations.

Definition 1.4 (Triangular/elementary automorphism)**.** *An automorphism* $t \in \mathrm{Aut}(\mathbf{k}^n)$ *is triangular if:*

$$t = (a_1x_1 + P_1(x_2, \dots, x_n), a_2x_2 + P_2(x_3, \dots, x_n), \dots, a_nx_n + c)$$

where $a_1, \dots, a_n \in \mathbf{k}^*$*,* $c \in \mathbf{k}$*, and* $P_i \in \mathbf{k}[x_1, \dots, x_n]$*. We call* T *the subgroup of trianguar automorphisms.*

An automorphism $e \in \mathrm{Aut}(\mathbf{k}^n)$ *is elementary if:*

$$e = (x_1 + P(x_2, \dots, x_n), \dots, x_n)$$

where $P \in \mathbf{k}[x_{i+1}, \dots, x_n]$*. We call* E *the subgroup of elementary automorphisms.*

In the above, a polynomial P only appears in the first component, but we may consider automorphisms such as

$$(x_1, \dots, x_{i-1}, x_i + P(x_{i+1}, \dots, x_n), x_{i+1}, \dots, x_n)$$

that are of a similar form, but not elementary as in the definition above. They are conjugate of $e = (x_1 + P(x_{i+1}, \dots, x_n), \dots, x_i, \dots, x_n)$ by the permutation $p = (1i)$.

In this paper, a composition tp of a permutation $p \in \mathfrak{S}_n(\mathbf{k})$ with a triangular automorphism is called a *triangular-permutation* automorphism, and a composition ep of a permutation by an elementary automorphism an *elementary-permutation.*

Definition 1.5 (Tame automorphism)**.** $\mathrm{Tame}(\mathbb{A}^n_{\mathbf{k}}) := \langle \mathrm{GL}_n(\mathbf{k}), \mathrm{E}\rangle$.

Definition 1.6 (Tame valuation)**.** *A valuation* ν *is tame if* $\nu = g \cdot \nu_{\mathrm{id},\omega}$ *for a monomial valuation* $\nu_{\mathrm{id},\omega} = -\deg_\omega$ *and a tame automorphism* g*.*

We usually write $g \cdot \nu_{\mathrm{id},\omega} = \nu_{g,\omega}$.

Definition 1.7 (ν-degree)**.** *For every endomorphism* $f = (f_1, ..., f_n)$ *and valuation* ν

$$\nu(f) = \sup\left\{\frac{\nu(P \circ f)}{\nu(P)} \text{ with } P \in \mathbf{k}[x_1, \dots, x_n] \text{ non-constant}\right\}.$$

For a tame valuation $\nu = \nu_{g,\omega}$ it follows that

$$\nu_{g,\omega}(f) = \nu_{\mathrm{id},\omega}(g^{-1} \circ f \circ g).$$

This holds as a consequence of Lemma 3.2.

This gives rise to the classical notion of dynamical degree $\lambda(f)$, defined as the limit of $\nu_{\mathrm{id},\omega}(f^k)^{\frac{1}{k}}$ as $k \to \infty$, for any weight $\omega \in (\mathbf{R}^*_+)^n$. We recall that the dynamical degree of a polynomial automorphism is well defined and invariant under conjugation (Lemma 3.5). Furthermore $\lambda(f)$ equals the minimum of $\nu_{\mathrm{id},\omega}(f)$ on the set of monomial valuations $\nu_{\mathrm{id},\omega}$, if and only if f is algebraically stable with respect to ω (Definition 3.12).

Definition 1.8 (Apartment of monomial valuations)**.** *We consider the* apartment $\mathbf{E}_{\mathrm{id}}$ *of classes of such valuations* $\nu_{\mathrm{id},\omega}$ *under scaling:* $\omega \sim \lambda\omega$ *for any* $\lambda \in \mathbf{R}^*_+$*.*

Given a tame automorphism g*, we also denote by* $\mathbf{E}_g$ *the apartment of those tame valuations of the form* $\nu_{g,\omega}$ *modulo the relation* $\nu_{g,\omega} \sim \nu_{g,\lambda\omega}$*,* $\lambda \in \mathbf{R}^*_+$*.*

For any tame automorphism g, the quotient space $\mathbf{E}_g$ is naturally homeomorphic to the open simplex $\nabla \subset \mathbb{P}^{n-1}_{\mathbf{R}}$ of those points $[\omega] = [\omega_1 : \cdots : \omega_n]$ with only positive coefficients. Elements of $\mathbf{E}_g$ may be written as $\nu_{g,[\omega]}$. When there is no risk of confusion, we may write $\nu_{g,\omega}$ instead to lighten the notation, in other words designate $\nu_{g,[\omega]}$ by one of its representatives. We make the comment that $\nu_{g,\omega}(f)$ doesn't depend on the choice of a representative. Moreover, the action of $\mathrm{Aut}(\mathbf{k}^n)$ on valuations descends to an action on the space of classes of valuations. Now if we fix an automorphism f of $\mathbf{k}^n$ and a word $f = \mathfrak{m}$, we may act upon $\mathbf{E}_{\mathrm{id}}$ via the "letters" of $\mathfrak{m}$:

Definition 1.9. *For $f \in \langle \mathrm{E}, \mathfrak{S}_n \rangle$, and for any reduced word $f = \mathfrak{m} = g_1 \dots g_k$, with each g_i elementary or a permutation, we write*

$$\mathcal{E}^{\mathfrak{m}} := \ldots \cup \mathbf{E}_{f^{-1}} \cup \ldots \cup \mathbf{E}_{g_k^{-1} g_{k-1}^{-1}} \cup \mathbf{E}_{g_k^{-1}} \cup \mathbf{E}_{\mathrm{id}} \cup \mathbf{E}_{g_1} \cup \mathbf{E}_{g_1 g_2} \cup \ldots \cup \mathbf{E}_f \cup \mathbf{E}_{f g_1} \cup \ldots$$

In this order, any apartment intersects the next one and the previous one. As an example, $\mathbf{E}_{g_1} \cap \mathbf{E}_{g_1 g_2} = g_1 \cdot (\mathbf{E}_{\mathrm{id}} \cap \mathbf{E}_{g_2})$.

1.1 Main result

Our theorem focuses on the cases $n = 2$ and $n = 3$.

Theorem 1.10. *Let f be a tame automorphism of $\mathbf{k}^n$. For any reduced word $\mathfrak{m}$ of minimal length such that $f = \mathfrak{m}$, we consider the set $\mathcal{M}$ of valuations $\nu \in \mathcal{E}^{\mathfrak{m}}$ such that $\nu(f) = \min\{\mu(f) \mid \mu \in \mathcal{E}^{\mathfrak{m}}\}$.*

1. *If $n = 2$, then up to conjugating f by an affine automorphism, the locus $\mathcal{M} \subset \mathcal{E}^{\mathfrak{m}}$ is homeomorphic to $\mathbf{R}$. Moreover, $\nu(f)$ equals the dynamical degree of f for all $\nu \in \mathcal{M}$.*
2. *If $n = 3$, we assume that f is an automorphism of type elementary-permutation whose dynamical degree is not an integer, and that the dynamical degree of f is reached on $\mathcal{M}$. Then $\mathcal{M}$ is homeomorphic to $\mathbf{R}$.*

See Sections 4 and 5, more previsely Theorems 4.3 and 5.2, for the proof of our theorem. Basic facts, definitions and results about the dynamical degree and algebraic stability are given in Section 3. In Section 6, we give examples of

- an automorphism of $\mathbf{k}^3$ of type elementary-permutation for which the dynamical degree is not reached on $\mathbf{E}_{\mathrm{id}}$, but on its boundary,
- an automorphism of $\mathbf{k}^3$ of type triangular-permutation $f = tp$ for which the minimality locus $\mathcal{M}$ in $\mathcal{E}^{\mathfrak{m}p}$ is homeomorphic to $\mathbf{R}$,
- a triangular-permutation automorphism for which the minimality locus has dimension 2.

In the last two cases, we exhibit the fact that the intersection of $\mathcal{M}$ with $\bigcup_{m \in \mathbf{Z}} \mathbf{E}_{f^m}$ is disconnected.

2 Preliminary definitions and results

2.1 Confined valuations

Definition 2.1 (Confined valuations)**.** *Let $f \in \mathrm{Tame}(\mathbf{k}^n)$ and an apartment $\mathbf{E}_g$. A valuation $\nu' \in \mathbf{E}_g$ is confined by f if there exist weights $\nu \in \mathbf{E}_g$ such that $f \cdot \nu' = \nu$. We denote it by:*

$$\mathrm{Conf}(f, \mathbf{E}_g) := \{\nu_{g,\omega'} \in \mathbf{E}_g \mid \exists \omega, f \cdot \nu_{g,\omega'} = \nu_{g,\omega}\}$$

In other words, $\mathbf{E}_{fg} \cap \mathbf{E}_g$ is the set of all images via f of those monomial valuations that are confined by f.

Note that, for an automorphism f, the valuation $\nu_{g,\omega}$ being confined by f in $\mathbf{E}_g$ is equivalent to the valuation $\nu_{\mathrm{id},\omega}$ being confined by $g^{-1} f g$ in $\mathbf{E}_{\mathrm{id}}$.

For this reason, we simply call $\mathrm{Conf}(f) := \mathrm{Conf}(f, \mathbf{E}_{\mathrm{id}})$ the set of confined monomial valuations.

Lemma 2.2. *Let $t \in \mathrm{T}$ be a triangular automorphism. Then, for weights $\omega = (\omega_1, \ldots, \omega_n)$ and $\omega' = (\omega'_1, \ldots, \omega'_n)$ such that $t \cdot \nu_{\mathrm{id},\omega'} = \nu_{\mathrm{id},\omega}$ we have $\omega' = \omega$. In other words, any monomial valuation confined by t is fixed by t.*

Proof: As t and t^{-1} are triangular, their linear parts l_t and $l_{t^{-1}}$ are invertible upper triangular matrices. More precisely, if

$$t = (a_1x_1 + P_1(x_2, \dots, x_n), a_2x_2 + P_2(x_3, \dots, x_n), \dots, a_nx_n + c),$$

with L_i being the linear part of P_i, that is, the sum of its degree 1 monomials, then

$$l_t = (a_1x_1 + L_1, a_2x_2 + L_2, \dots, a_nx_n)$$

where $a_ix_i + L_i = (l_t)_i$. We write $g = t^{-1}$. On one side, we have, evaluating in t_i for all i:

$$\nu_{\mathrm{id},\omega'}(t_i) = \nu_{\mathrm{id},\omega}(x_i)$$

and, on the other side, we evaluate in g_i for all i:

$$\nu_{\mathrm{id},\omega'}(x_i) = \nu_{\mathrm{id},\omega}(g_i)$$

In particular, as we deal with monomial valuations, this implies on one side for all i:

$$\omega_i \geqslant -\nu_{\mathrm{id},\omega'}((l_t)_i)$$

and on the other side:

$$\omega'_i \geqslant -\nu_{\mathrm{id},\omega}((l_g)_i)$$

As $(l_t)_i$ and $(l_g)_i$ are invertible they both depend at least on x_i, and, as we are dealing with monomial valuations:

$$-\nu_{\mathrm{id},\omega'}((l_t)_i) = \max\{\omega'_i, \omega'_j : j \neq i \mid x_j \subset (l_t)_i\}$$

And same for l_g:

$$-\nu_{\mathrm{id},\omega}((l_g)_i) = \max\{\omega_i, \omega_j : j \neq i \mid x_j \subset (l_g)_i\}$$

hence, the inequalities imply $\omega_i \geqslant \omega'_i$ on one hand, and $\omega'_i \geqslant \omega_i$, on the other hand, and we get the result. ■

Remark 2.3. The same proof also works for $p^{-1}tp$, for any permutation p.

2.2 For a triangular automorphism

Here we assume $f = t = (x_1 + P_1, \dots, x_{n-1} + P_{n-1}, x_n)$, with each P_i a polynomial in the variables $x_{i+1}, \dots, x_n$. This section is dedicated to prove the following:

Theorem 2.4. *Valuations that are fixed by* t *are the classes of* $\nu_{\mathrm{id},\omega}$ *for* $\omega = (\omega_1, ..., \omega_n)$ *satisfying*

$$\omega_i \geq \deg_\omega P_i, \, \forall i \in \{1, ..., n-1\}.$$

In other words, all apartments $\mathbf{E}_{t^m}$ intersect along $\{\nu_{\mathrm{id},\omega} \mid \omega_i \geq \deg_\omega P_i\}$, which is a subset $\mathbf{E}_{\mathrm{id}}$ delimited by hyperplanes.

We fix $t \in \mathrm{T}$ and define a subset $C_t = \{\omega_i \geqslant \deg_\omega(P_i) \mid 1 \leqslant i \leqslant n\}$ of strictly positive weights. A precomposition by an affine diagonal automorphism, of the form $(\alpha_1x_1 + \beta_1, \dots, \alpha_nx_n + \beta_n)$ does not affect C_t, hence we can suppose that $t = (x_1 + P_1(x_2, \dots, x_n), x_2 + P_2(x_3, \dots, x_n), \dots, x_n)$.

Lemma 2.5. *We have* $\omega \in C_t$ *if and only if* $\omega \in C_{t^{-1}}$.

Proof: For $t = (x_1 + P_1(x_2, \dots, x_n), x_2 + P_2(x_3, \dots, x_n), \dots, x_n)$, we have $t^{-1} = (x_1 - \overline{P_1}(x_2, \dots, x_n), \dots, x_{n-1} - \overline{P_{n-1}}(x_n), x_n)$ where the $\overline{P_i}$'s are defined recursively, with $\overline{P_{n-1}}(x_n) = P_{n-1}(x_n)$ and $\overline{P_{i-1}}(x_i, \dots, x_n) = P_{i-1}(x_i - \overline{P_i}, \dots, x_{n-1} - \overline{P_{n-1}}, x_n)$.

We prove $C_t \subset C_{t^{-1}}$ by backward induction. We suppose that this is true up to the relations i to n, and let ω_{i-1} such that $\omega_{i-1} \geqslant \deg_\omega(P_{i-1})$. We want to prove that $\omega_{i-1} \geqslant \deg_\omega(\overline{P_{i-1}})$. But this is equivalent to $\omega_j \geqslant \deg_\omega(\overline{P_j})$ for $j \geqslant i$ and $\omega_{i-1} \geqslant \deg_\omega(P_{i-1})$ which is true by induction hypothesis and the supposition.

We prove the reverse inclusion by taking t^{-1} instead of t. ■

Lemma 2.6. *Let* $t \in \mathrm{T}$ *and* $m = x_1^{a_1} \dots x_n^{a_n}$ *a monomial. Then for any* $\omega \in C_t$*, we have* $\deg_\omega(m) = \deg_\omega(m \circ t)$.

Proof: By computation, we find that there is a finite family $\mathcal{J}$ of indices $J \neq (a_1, \dots, a_n)$ such that

$$m \circ t = (x_1 + P_1)^{a_1}(x_2 + P_2)^{a_2} \dots x_n^{a_n} = x_1^{a_1} \dots x_n^{a_n} + \sum_{J \in \mathcal{J}} b_J x^J.$$

Hence, we have that $m \subset m \circ t$ (m actually appears as a monomial in $m \circ t$) and, since $\omega \in C_t$, m is of maximal degree amongst all the monomials of $m \circ t$: $\deg_\omega(m \circ t) = \deg_\omega(m)$. ∎

Proof of Theorem 2.4: Let $\omega \in C_t$. We want to prove that t fixes $\nu_{\mathrm{id},\omega}$. Let $P \in \mathbf{k}[x_1, \dots, x_n]$. By additive property of degrees (or valuations), there exists $m \subset P$ such that $\deg_\omega(P \circ t) \leqslant \deg_\omega(m \circ t)$. By Lemma 2.6 we have $\deg_\omega(m \circ t) = \deg_\omega(m) \leqslant \deg_\omega(P)$ (this last inequality being because we deal with monomial valuations, or weighted degrees).

In conclusion, $\deg_\omega(P \circ t) \leqslant \deg_\omega(P)$.

By Lemma 2.5, we have $\omega \in C_{t^{-1}}$. and we can apply the inequality just proved for $Q = P \circ t$ and t^{-1}: $\deg_\omega(Q \circ t^{-1}) \leqslant \deg_\omega(Q)$ thus $\deg_\omega((P \circ t) \circ t^{-1}) = \deg_\omega(P) \leqslant \deg_\omega(P \circ t)$ and we get the reverse inequality. Thus, $\deg_\omega(P) = \deg_\omega(P \circ t)$ which is $t \cdot \nu_{\mathrm{id},\omega} = \nu_{\mathrm{id},\omega}$.

Conversely, if there is a j such that $\omega_j < \deg_\omega(P_j)$, then $t \cdot \nu_{\mathrm{id},\omega}$ is not equal to $\nu_{\mathrm{id},\omega}$. One only has to consider $t \cdot \nu_{\mathrm{id},\omega}(x_j) = \nu_{\mathrm{id},\omega}(P_j)$ (by additive property of valuations). Thus $t \cdot \nu_{\mathrm{id},\omega}(x_j) > \nu_{\mathrm{id},\omega}(x_j)$ in this situation. ∎

We end this section by recalling the following result from [LP19]:

Proposition 2.7 (Stabilizer of a valuation). *Let $\omega = (\omega_1, \dots, \omega_n)$ with $\omega_1 > \dots > \omega_n$ and $\nu = \nu_{\mathrm{id},\omega}$. Then, the stabilizer of ν in $\mathrm{Tame}(\mathbf{k}^n)$ is formed by elements $(a_1 x_1 + P_1(x_2, \dots x_n), a_2 x_2 + P_2(x_3, \dots, x_n), \dots, a_n x_n + c)$ such that $\deg_\omega(P_i) \leqslant \omega_i$.*

Proof: We do the case $n = 2$. Let $f \in \mathrm{Stab}(\nu)$.

As a polynomial automorphism of $\mathbf{k}$ is also a polynomial automorphism of $\overline{\mathbf{k}}$, we can assume that $\mathbf{k}$ is algebraically closed. We can suppose $\omega = (\omega_1, 1)$. We have

$$1 = \deg_{\nu_{f,\omega}}(x_2) = \deg_{\nu_{\mathrm{id},\omega}}(f_2)$$

Hence, f_2 only depends on x_2 and is linear in this variable: $f_2 = cx_2 + d$. Moreover, if we fix y, we have $f_1(\cdot, y)$ automorphism of $\mathbb{A}^1$, hence of the form $f_1 = Q(y)x + P(y)$.

If Q was not constant, one could find a root r of Q, and $f_1(\cdot, r) = P(r)$ would not be an automorphism. Hence, we have $Q(y) = a$ and $f_1 = ax + P(y)$.

We proceed by recursion for n general, and get that $f = (a_1 x_1 + P_1(x_2, \dots x_n), a_2 x_2 + P_2(x_3, \dots, x_n), \dots, a_n x_n + c)$.

Moreover, we must have that $\nu_{f,\omega}(x_i) = \nu_{\mathrm{id},\omega}(x_i)$ for all i. Hence $\nu(f_i) = \nu(a_i x_i + P_i) = \nu(x_i)$. In particular,

$$-\nu(P_i) \leqslant -\nu(f_i) = -\nu(x_i) = \omega_i$$

The fact that any element of this form fixes $\nu_{\mathrm{id},\omega}$ has been shown in Theorem 2.4. ∎

2.3 For a permutation

If $f = p \in \mathfrak{S}_n$.

We recall that permutations act on the basis of $\mathbf{k}^n$. There is a similar action on the space of weights:

Definition 2.8 (Action of permutations over the space of weights). *Let $p \in \mathfrak{S}_n$ and $[\omega] \in \nabla$. We set:*

$$p \cdot [\omega] := [\omega_{p^{-1}(1)} : \dots : \omega_{p^{-1}(n)}]$$

Lemma 2.9. *Let $p \in \mathfrak{S}_n$ be a permutation, $g \in \mathrm{Tame}(\mathbf{k}^n)$, and $\nu_{\mathrm{id},[\omega]}$ a (an homotopy class of) monomial valuation. Then we have:*

$$\nu_{gp,[\omega]} = \nu_{g,p\cdot[\omega]}$$

Proof: We recall that $p = (x_{p^{-1}(1)}, \dots, x_{p^{-1}(n)})$.

Let $P \in \mathbf{k}[x_1, \dots, x_n]$. We write $Q = P \circ g = \sum_{I \in \mathbf{N}} a_I x^I$. The left-hand side gives:

$$\nu_{gp,[\omega]}(P) = \nu_{\mathrm{id},[\omega]}(Q \circ p) = \nu_{\mathrm{id},[\omega]}(Q(x_{p^{-1}(1)}, \dots, x_{p^{-1}(n)}))$$

The right-hand side gives:

$$\nu_{g,p\cdot[\omega]}(P) = \nu_{\mathrm{id},p\cdot[\omega]}(Q(x_1, \dots, x_n))$$

Both are:

$$\max\{i_1\omega_{p^{-1}(1)} + \ldots + i_n\omega_{p^{-1}(n)} \mid a_I \neq 0\}$$

■

As direct consequences, one can note the following result (to be related to Remark 2.3)

Corollary 2.10. *Let $f \in \mathrm{Tame}(\mathbf{k}^n)$. Then $[\omega] \in \mathrm{Fix}(f)$ if and only if $p \cdot [\omega] \in \mathrm{Fix}(pfp^{-1})$.*

Proof: We take $\nu = \nu_{\mathrm{id},\omega}$ such that $f \cdot \nu = \nu$. We have:

$$\begin{aligned} f \cdot \nu_{\mathrm{id},\omega} = \nu_{id,\omega} &\Leftrightarrow pfp^{-1} \cdot (p \cdot \nu) = p \cdot \nu \\ &\Leftrightarrow pfp^{-1} \cdot \nu_{\mathrm{id},p\cdot\omega} = \nu_{\mathrm{id},p\cdot\omega} \end{aligned}$$

■

Corollary 2.11. *The automorphism p confines every monomial valuation.*

Proof: It is a direct consequence from Lemma 2.9. ■

In other words, all $p^m \cdot \nu_{\mathrm{id},\omega}$ are monomial, and $\mathcal{E}^p = \mathbf{E}_{\mathrm{id}}$.

2.4 For a tame automorphism of type "triangular–permutation"

Let $f = tp$ with p a nontrivial permutation and t a triangular automorphism.

Lemma 2.12. *Let $f = tp \in \mathrm{Tame}(\mathbf{k}^n)$, with $t \in \mathrm{T}$ and $p \in \mathfrak{S}_n$. Then, for weights $\omega = (\omega_1, \dots, \omega_n)$ and $\omega' = (\omega'_1, \dots, \omega'_n)$ such that $f \cdot \nu_{\mathrm{id},\omega'} = \nu_{\mathrm{id},\omega}$ we have $\omega = p \cdot \omega'$.*

Proof: We have $tp \cdot \nu_{\mathrm{id},\omega'} = t \cdot \nu_{\mathrm{id},p.\omega'}$, by Lemma 2.9. By Lemma 2.2, we have $p \cdot \omega' = \omega$. ■

Proposition 2.13. *Let $f, g \in \mathrm{Tame}(\mathbf{k}^n)$. There exist a permutation p such that for any ω, ω' such that $\nu_{f,\omega} = \nu_{g,\omega'}$ we have $\omega = p \cdot \omega'$.*

Proof: Up to the action by g^{-1}, we only have to prove the Proposition for any $f \in \mathrm{Tame}(\mathbf{k}^n)$ and $g = \mathrm{id}$. As only linear parts are involved in proof of Lemma 2.2, we use the Bruhat decomposition of l_f the linear part, this decomposition being $l_f = t_2pt_1$, with $p \in \mathfrak{S}_n$ and $t_1, t_2 \in \mathrm{T}$.

But $t_2pt_1 = t_2pt_1p^{-1}p$ while the proof of Lemma 2.2 works for t_2 and pt_1p^{-1}. Hence, by Corollary 2.12 if $f \cdot \nu_{\mathrm{id},\omega'} = \nu_{\mathrm{id},\omega}$, we have $\omega = p.\omega'$. ■

We examine the images via f of those monomial valuations that are confined by f, in other words at the set $\mathbf{E}_f \cap \mathbf{E}_{\mathrm{id}}$:

Lemma 2.14. *We have :*

$$\mathbf{E}_f \cap \mathbf{E}_{\mathrm{id}} = \mathbf{E}_t \cap \mathbf{E}_{\mathrm{id}} = \mathrm{Fix}(t) \cap \mathbf{E}_{\mathrm{id}}.$$

Moreover,

$$\mathbf{E}_{f^{-1}} \cap \mathbf{E}_{\mathrm{id}} = p^{-1} \cdot \mathbf{E}_t \cap \mathbf{E}_{\mathrm{id}} = p^{-1} \cdot \mathrm{Fix}(t) \cap \mathbf{E}_{\mathrm{id}}.$$

Proof: Let $\nu \in \mathbf{E}_t \cap \mathbf{E}_{\mathrm{id}}$. As t fixes valuations (Lemma 2.2), we have $t \cdot \nu = \nu$. We write ν' such that $p \cdot \nu' = \nu$. By Lemma 2.9, we have $tp \cdot \nu' = \nu$.

$$\begin{aligned} \nu \in \mathbf{E}_t \cap \mathbf{E}_{\mathrm{id}} &\Leftrightarrow t \cdot \nu = \nu \\ &\Leftrightarrow tp \cdot \nu' = \nu \\ &\Leftrightarrow \nu \in \mathbf{E}_f \cap \mathbf{E}_{\mathrm{id}} \end{aligned}$$

For the second part, taking the same notations:

$$\begin{aligned} \nu \in \mathbf{E}_t \cap \mathbf{E}_{\mathrm{id}} &\Leftrightarrow tp \cdot \nu' = \nu \\ &\Leftrightarrow \nu' = f^{-1} \cdot \nu \\ &\Leftrightarrow \nu' \in \mathbf{E}_{f^{-1}} \cap \mathbf{E}_{\mathrm{id}} \end{aligned}$$

Hence the result. ■

In other words, $\mathbf{E}_{\mathrm{id}}$ meets both $\mathbf{E}_f$ and $\mathbf{E}_{f^{-1}}$, respectively along $\mathrm{Fix}(t)$ and $p^{-1} \cdot \mathrm{Fix}(t)$. Both these subsets are delimited by hyperplanes (Theorem 2.4).

Lemma 2.15. *For all $m \in \mathbf{Z}$,*

$$\mathbf{E}_{f^{m+1}} \cap \mathbf{E}_{f^m} = \mathrm{Fix}(t) \cap \mathbf{E}_{f^m}$$

and

$$\mathbf{E}_{f^{m-1}} \cap \mathbf{E}_{f^m} = p^{-1} \cdot \mathrm{Fix}(t) \cap \mathbf{E}_{f^m}.$$

Proof: For any m, we have an homeomorphism $\mathbf{E}_{f^m} \simeq \mathbf{E}_{\mathrm{id}} \simeq \nabla$, equivariant under the action of t. So the result follows from the previous lemma. ■

Corollary 2.16. *If p is such that*

$$(p^{-1} \cdot \mathrm{Fix}(t)) \cap \mathrm{Fix}(t) \cap \mathbf{E}_{\mathrm{id}} = \varnothing,$$

then each apartment $\mathbf{E}_{f^m}$ meets the apartments $\mathbf{E}_{f^{m-1}}$ and $\mathbf{E}_{f^{m+1}}$, but not the other $\mathbf{E}_{f^n}$.

For sanity, we write the result of the action of f (and f^{-1}) over the set of confined valuations:

Corollary 2.17. *We have*

$$\mathbf{E}_{f^{m+1}} \cap \mathbf{E}_{f^m} = p \cdot \mathrm{Conf}(f, \mathbf{E}_{f^m})$$

And

$$\mathbf{E}_{f^{m-1}} \cap \mathbf{E}_{f^m} = p^{-1} \cdot \mathrm{Conf}(f^{-1}, \mathbf{E}_{f^m})$$

Example: We write those sets explicitly for $f = ep$, for $e = (x_1 + P(x_2, x_3), x_2, x_3)$ elementary and $p = (x_3, x_1, x_2)$. We have by Theorem 2.4:

$$\mathrm{Fix}(e) \cap \mathbf{E}_{\mathrm{id}} = \bigcap_{(a_2,a_3) \in \mathrm{Supp}(P)} \{[\omega] \in \nabla \mid \omega_1 \geqslant a_2\omega_2 + a_3\omega_3\}$$

Hence, by Lemma 2.14:

$$\mathbf{E}_f \cap \mathbf{E}_{\mathrm{id}} = \bigcap_{(a_2,a_3) \in \mathrm{Supp}(P)} \{[\omega] \in \nabla \mid \omega_1 \geqslant a_2\omega_2 + a_3\omega_3\}$$

and:

$$\mathbf{E}_{f^{-1}} \cap \mathbf{E}_{\mathrm{id}} = \bigcap_{(a_2,a_3) \in \mathrm{Supp}(P)} \{[\omega'] \in \nabla \mid \omega'_{p^{-1}(1)} \geqslant a_2\omega'_{p^{-1}(2)} + a_3\omega'_{p^{-1}(3)}\}$$

3 Algebraic stability with respect to ν and dynamical degree

3.1 Computation of the degree

We have given in our Introduction (Definition 1.7) the definition of the ν-degree of an endomorphism, for ν a valuation. We note that this degree is the same for valuations that are equivalent up to scaling (Definition 1.8), therefore it is well-defined over apartments. We see a way to compute it for monomial valuations:

Proposition 3.1. *Let $[\omega] := [\omega_1 : \ldots : \omega_n]$. For any class of monomial valuation $\nu_{\mathrm{id},[\omega]} \in \mathbf{E}_{\mathrm{id}}$ and $f = (f_1, \ldots, f_n) \in \mathrm{End}(\mathbf{k}^n)$, we have:*

$$\nu_{\mathrm{id},[\omega]}(f) = \max_i \left\{ \frac{\nu_{\mathrm{id},[\omega]}(f_i)}{\nu_{\mathrm{id},[\omega]}(x_i)} \right\} = \max_i \left\{ \frac{-\nu_{\mathrm{id},[\omega]}(f_i)}{\omega_i} \right\}$$

Proof: We write $\nu := \nu_{\mathrm{id},[\omega]}$.

The inequality

$$\max_i \frac{\nu(f_i)}{\nu(x_i)} \leqslant \deg_\nu(f) = \sup_{P \in \mathbf{k}[x_1,\ldots,x_n]\setminus\{0\}} \frac{\nu(f^*P)}{\nu(P)}$$

is obvious. We now prove the converse inequality.

For $P \in \mathbf{k}[x_1, \ldots, x_n]$, we can write P as a finite sum of its distinct monomials (with a non-zero coefficient): $P = \sum_{m \subset P} c_m.m$. As ν is a monomial valuation, we have:

$$-\nu(P) = \max_{m \subset P} \{-\nu(m)\}$$

We have, by the property of valuations:

$$-\nu(P \circ f) \leqslant \max_{m \subset P} \{-\nu(m \circ f)\}$$

So:

$$\frac{\nu(P \circ f)}{\nu(P)} \leqslant \frac{\max_{m\subset P} \{-\nu(m \circ f))\}}{\max_{m\subset P} \{-\nu(m)\}}$$

Let m' be a monomial such that $-\nu(m' \circ f) = \max_{m\subset P} \{-\nu(m \circ f)\}$. Then we have:

$$\frac{\max_{m\subset P} \{-\nu(m \circ f))\}}{\max_{m\subset P} \{-\nu(m)\}} \leqslant \frac{\nu(m' \circ f)}{\nu(m')}$$

For a monomial, $m' = x_1^{a_1} \ldots x_n^{a_n}$, we have $m' \circ f = f_1^{a_1} \ldots f_n^{a_n}$, so: $\nu(m' \circ f) = \sum a_i \nu(f_i)$ and:

$$\frac{\nu(m' \circ f)}{\nu(m')} = \frac{\sum a_i \nu(f_i)}{\sum a_i \nu(x_i)} \leqslant \max_j \frac{\nu(f_j)}{\nu(x_j)}$$

The last inequality follows from the fact that, for any values $x_i, y_i > 0$, $1 \leqslant i \leqslant n$, we have $\frac{x_1+\ldots+x_n}{y_1+\ldots+y_n} \leqslant \max_i \frac{x_i}{y_i}$.

In conclusion, we end up with:

$$\sup_{P \in \mathbf{k}[x_1,\ldots,x_n]\setminus\{0\}} \frac{\nu_{\mathrm{id},\omega}(P \circ f)}{\nu_{\mathrm{id},\omega}(P)} \leqslant \max_j \frac{\nu(f_j)}{\nu(x_j)}$$

■

Example: For a monomial valuation ν with weight $[\omega_1 : \ldots : \omega_n]$ and a monomial endomorphism

$$m_i = (0, \ldots, 0, x_1^{a_1} \ldots x_n^{a_n}, 0, \ldots, 0)$$

with zeroes everywhere except in the ith component, we have:

$$\nu(m_i) = \frac{a_1\omega_1 + \ldots + a_n\omega_n}{\omega_i}$$

Lemma 3.2. *For any $g, h \in \mathrm{Tame}(\mathbf{k}^n)$, $f \in \mathrm{End}(\mathbf{k}^n)$ we have:*

$$g \cdot \nu_{h,[\omega]}(gfg^{-1}) = \nu_{h,[\omega]}(f)$$

Proof: Using the change of coordinates $P' = P \circ g$, we compute:

$$\begin{aligned} g \cdot \nu_{h,[\omega]}(gfg^{-1}) &= \max_{P \in \mathbf{k}[x_1,\ldots,x_n]\setminus\{0\}} \left\{ \frac{g \cdot \nu_{h,[\omega]}(P \circ (gfg^{-1}))}{g \cdot \nu_{h,[\omega]}(P)} \right\} \\ &= \max_{P \in \mathbf{k}[x_1,\ldots,x_n]\setminus\{0\}} \left\{ \frac{\nu_{h,[\omega]}(P \circ (gf))}{\nu_{h,[\omega]}(P \circ g)} \right\} \\ &= \max_{P' \in \mathbf{k}[x_1,\ldots,x_n]\setminus\{0\}} \left\{ \frac{\nu_{h,[\omega]}(P' \circ f)}{\nu_{h,[\omega]}(P')} \right\} \\ &= \nu_{h,[\omega]}(f) \end{aligned}$$

■

Remark 3.3. To compute the degree over an arbitrary apartment, we will make use of the following expression, from Lemma 3.2:

$$\nu_{g,[\omega]}(f) = \nu_{\mathrm{id},[\omega]}(g^{-1}fg)$$

3.2 Dynamical degree

We recall the definition of the dynamical degree:

Definition 3.4 (First dynamical degree)**.** *Let $f \in \mathrm{End}(\mathbf{k}^n)$ and $\deg(f)$ be the standard degree.*

$$\lambda(f) := \lim_{n\to\infty} (\deg(f^n))^{\frac{1}{n}}$$

It is well-defined because of submultiplicativity of the degree (see Lemma 3.6). We recall that the dynamical degree is a conjugation invariant.

Lemma 3.5. *Let $f \in \mathrm{Tame}(\mathbf{k}^n)$, $g \in \mathrm{Aut}(\mathbf{k}^n)$. We have:*

$$\lambda(f) = \lambda(g^{-1}fg)$$

Proof: We have $\deg((g^{-1}fg)^n) = \deg(g^{-1}f^ng) \leqslant \deg(g^{-1})\deg(f^n)\deg(g)$. Taking the n-th root of both sides of the inequality and letting n go to infinity, we get $\lambda(g^{-1}fg) \leqslant \lambda(f)$. We get the reverse inequality from the first, with g^{-1} instead of g and $g^{-1}fg$ instead of f. ■

We have submultiplicativity for the degree for tame valuations in general:

Lemma 3.6. *Let $f, g \in \mathrm{End}(\mathbf{k}^n)$ and ν (a class of) tame valuation, we have*

$$\nu(g \circ f) \leqslant \nu(g)\nu(f)$$

In particular $\nu(f^n) \leqslant \nu(f)^n$.

Proof: We start with Definition 1.7

$$\deg_\nu(g \circ f) := \sup_{P \in \mathbf{k}[x_1,\ldots,x_n]} \left\{ \frac{\nu(P \circ g \circ f)}{\nu(P)} \right\}$$

For any $P \in \mathbf{k}[x_1, \ldots, x_n]$, we have

$$\frac{\nu(P \circ f \circ g)}{\nu(P)} = \frac{\nu(P \circ f \circ g)}{\nu(P \circ g)} . \frac{\nu(P \circ g)}{\nu(P)}$$

$$\frac{\nu(P \circ f \circ g)}{\nu(P)} \leqslant \sup_{Q \in \mathbf{k}[x_1,\ldots,x_n]} \left\{ \frac{\nu(Q \circ f)}{\nu(Q)} \right\} \frac{\nu(P \circ g)}{\nu(P)}$$

Therefore, we have

$$\sup_{P \in \mathbf{k}[x_1,\ldots,x_n]} \left\{ \frac{\nu(P \circ f \circ g)}{\nu(P)} \right\} \leqslant \sup_{P \in \mathbf{k}[x_1,\ldots,x_n]} \left\{ \frac{\nu(P \circ f)}{\nu(P)} \right\} \sup_{P \in \mathbf{k}[x_1,\ldots,x_n]} \left\{ \frac{\nu(P \circ g)}{\nu(P)} \right\}$$

Hence the result. ■

Lemma 3.7. *Let $f \in \mathrm{End}(\mathbf{k}^n)$. For $\nu = \nu_{g,[\omega]}$ with $[\omega] \in \nabla$ and $g \in \mathrm{Aut}(\mathbf{k}^n)$, the limit*

$$\lim_{m\to\infty} (\nu(f^m))^{\frac{1}{m}}$$

exists and is denoted by $\lambda_\nu(f)$.

Proof: By Lemma 3.6, $a_n := \log(\nu(f^n))$ is a subadditive sequence. By Fekete's Lemma, the limit of $\frac{a_n}{n}$ exists, and is equal to its infimum. ■

In particular, we have $\nu(f) \geqslant \lambda_\nu(f)$. Moreover, we see that this limit is always the dynamical degree, for any tame valuation with non-zero weight:

Proposition 3.8. *Let $f \in \mathrm{End}(\mathbf{k}^n)$. For any $g \in \mathrm{Tame}(\mathbf{k}^n)$ and any $[\omega] \in \nabla$ we have:*

$$\lambda(f) = \lim_{m\to\infty} \left(\nu_{g,[\omega]}(f^m)\right)^{\frac{1}{m}}$$

In particular, for all ν, $\lambda_\nu = \lambda$.

Proof: We fix a valuation ν and we set $\Omega = \frac{\min_i(\omega_i)}{\max_i(\omega_i)} > 0$. We have:

$$\Omega \deg(f^m) \leqslant \nu_{\mathrm{id},[\omega]}(f^m) \leqslant \frac{1}{\Omega}\deg(f^m)$$
$$\Omega^{\frac{1}{m}}(\deg(f^m))^{\frac{1}{m}} \leqslant (\nu_{\mathrm{id},[\omega]}(f^m))^{\frac{1}{m}} \leqslant \frac{1}{\Omega^{\frac{1}{m}}}(\deg(f^m))^{\frac{1}{m}}$$

Taking the limit when m goes to infinity, we get the result, as $\Omega^{\frac{1}{m}} \to 1$.

To get the result for $\nu_{g,[\omega]}$, we use the fact that the dynamical degree is an invariant of conjugation:

$$\begin{aligned}\lambda(f) &= \lambda(g^{-1}fg)\\ &= \lim_{m\to\infty}\left(\nu_{\mathrm{id},[\omega]}(g^{-1}f^m g)\right)^{\frac{1}{m}}\\ &= \lim_{m\to\infty}\left(\nu_{g,[\omega]}(f^m)\right)^{\frac{1}{m}}\end{aligned}$$

■

3.3 ν-algebraic stability

In order to be able to introduce ν-maximal homogeneous components of a polynomial/endomorphism, we fix the following notation:

Definition 3.9 (Monomial endomorphism). *Given a monomial $m = x_1^{a_1}\dots x_n^{a_n}$ and an integer $0 \leqslant i \leqslant n$, the monomial endomorphism m_i is*

$$(0,\dots,0,x_1^{a_1}\dots x_n^{a_n},0,\dots,0) \in \mathrm{End}(\mathbb{A}^n_{\mathbf{k}})$$

with the monomial m in the i-th coordinate and zeros everywhere else.

We denote monomial endomorphisms as m_i with an *index* throughout this text. In our convention, a monomial m (respectively m_i) has coefficient 1.

Terminology. In this paper, we say that m *belongs* to $P \in \mathbf{k}[x_1,\dots,x_n]$ (respectively m_i to $f \in \mathrm{End}(\mathbf{k}^n)$) and we write $m \subset P$ (respectively $m_i \subset f$) for a monomial $m = x_1^{a_1}x_2^{a_2}\dots x_n^{a_n}$, if it belongs to the support of P, *i.e.* if this monomial features in the polynomial P with a non-zero coefficient (respectively if the associated monomial endomorphism features in f with a non-zero coefficient).

In particular, for any polynomial P (resp. any polynomial endomorphism f) there exist coefficients $c_m \in \mathbf{k}^*$ (resp. c_{m_i}) such that

$$P = \sum_{m\subset P} c_m m \text{ (respectively, } f = \sum_{m_i\subset f} c_{m_i} m_i).$$

More generally, we say that an endomorphism f *contains* an endomorphism g if all the monomials contained in g are also contained in f, and we write $g \subset f$. We sometimes say that g is a *component* of f in such a situation.

Definition 3.10. *Let $P \in \mathbf{k}[x_1, \dots, x_n]$. We call ω-maximal homogeneous component and denote by P_ω the sum of monomials $m \subset P$ such that* $\deg_\omega(m) = \deg_\omega(P)$.

Definition 3.11 (ω-maximal homogeneous component)**.** *Let $\nu = \nu_{\mathrm{id},[\omega]} \in \mathbf{E}_{\mathrm{id}}$ and $f \in \mathrm{End}(\mathbf{k}^n)$. We define the maximal (or leading) homogeneous component of f for ν as being the endomorphism $f_{[\omega]}$ of f which is the sum of the monomials $m_i \subset f$ such that $\nu(m_i) = \nu(f)$, along with their coefficients.*

The link between the degree and the dynamical degree is algebraic stability:

Definition 3.12 (ν-algebraic stability)**.** *Let $f \in \mathrm{End}(\mathbf{k}^n)$.*

*For a given valuation ν, we say that f is ν-*algebraically stable *if, for all n, we have:*

$$\nu(f^n) = (\nu(f))^n$$

If $\nu = \nu_{\mathrm{id},[\omega]}$, we say that f is $[\omega]$-algebraically stable.

Lemma 3.13. *Let $f, g \in \mathrm{End}(\mathbf{k}^n)$. We have $\nu_{\mathrm{id},[\omega]}(g \circ f) < \nu_{\mathrm{id},[\omega]}(g) \cdot \nu_{\mathrm{id},[\omega]}(f)$ if and only if $g_{[\omega]} \circ f_{[\omega]} = 0$.*

Proof: If $g_{[\omega]} \circ f_{[\omega]} \neq 0$, then, as $g_{[\omega]}$ and $f_{[\omega]}$ are ω-homogeneous, $g_{[\omega]} \circ f_{[\omega]}$ is ω-homogemenous of degree $\nu_{\mathrm{id},[\omega]}(g) \cdot \nu_{\mathrm{id},[\omega]}(f)$. So we have

$$\nu_{\mathrm{id},[\omega]}(g \circ f) = \nu_{\mathrm{id},[\omega]}(g) \cdot \nu_{\mathrm{id},[\omega]}(f).$$

If $g_{[\omega]} \circ f_{[\omega]} = 0$, in other words $(g \circ f)_{[\omega]} \neq g_{[\omega]} \circ f_{[\omega]}$, then

$$\nu_{\mathrm{id},[\omega]}((g \circ f)_{[\omega]}) < \nu_{\mathrm{id},[\omega]}(g_{[\omega]}) \cdot \deg_{\nu_{\mathrm{id},[\omega]}}(f_{[\omega]}).$$

This means

$$\nu_{\mathrm{id},[\omega]}(g \circ f) = \nu_{\mathrm{id},[\omega]}((g \circ f)_{[\omega]}) < \nu_{\mathrm{id},[\omega]}(g) \cdot \deg_{\nu_{\mathrm{id},[\omega]}}(f).$$

■

Lemma 3.14. *Let $f \in \mathrm{Tame}(\mathbf{k}^n)$, and $[\omega] \in \nabla$. The following statements are equivalent:*

(1) f is $[\omega]$-algebraically stable

(2) $f_{[\omega]}$ is such that $(f_{[\omega]})^n \neq 0$ for all n

(3) $\nu_{\mathrm{id},[\omega]}(f) = \nu_{\mathrm{id},[\omega]}(f_{[\omega]}) = \lambda(f)$

and imply the fact that $\nu_{\mathrm{id},[\omega]}(f)$ is minimal among monomial valuations.

Proof: We prove (1) $\Leftrightarrow$ (2). Suppose that f is not algebraically stable. Then there exists some $r > 1$ minimal such that $\nu_{\mathrm{id},[\omega]}(f^r) < (\nu_{\mathrm{id},[\omega]}(f))^r$. By Proposition 3.13, we get that $(f_{[\omega]})^r = 0$. Conversely, assume that there exists r such that $(f_{[\omega]})^r = 0$. Then $\nu_{\mathrm{id},[\omega]}(f^r) < \nu_{\mathrm{id},[\omega]}(f_{[\omega]})^r$, again by Proposition 3.13, hence f is not algebraically stable.

We prove (1) $\Leftrightarrow$ (3). Algebraic stability implies $\nu_{\mathrm{id},[\omega]}(f) = \lambda(f)$ by Proposition 3.8. Conversely, if f is not $[\omega]$-algebraically stable, there exist r, $\varepsilon > 0$ such that $\nu_{\mathrm{id},[\omega]}(f)^r - \varepsilon > \nu_{\mathrm{id},[\omega]}(f^r)$. By submultiplicativity of the degree and induction, we have for all k:

$$(\nu_{\mathrm{id},[\omega]}(f)^r - \varepsilon)^k > \nu_{\mathrm{id},[\omega]}(f^r)^k \geqslant \nu_{\mathrm{id},[\omega]}(f^{rk})$$

We take the power $\frac{1}{kr}$ of both sides of the inequality and let k go to infinity. We get:

$$(\nu_{\mathrm{id},[\omega]}(f)^r - \varepsilon)^{\frac{1}{r}} \geqslant \lambda(f)$$

As $(\nu_{\mathrm{id},[\omega]}(f)^r - \varepsilon)^{\frac{1}{r}} < \nu_{\mathrm{id},[\omega]}(f)$ for any $r \geqslant 1$, we get that $\nu_{\mathrm{id},[\omega]}(f) > \lambda(f)$.

The dynamical degree is a lower bound for the degree over tame valuations by Lemma 3.7. Hence, any valuation that reaches it is minimal. ■

It is worth noticing an upper bound for the rank of nilpotent endomorphisms:

Lemma 3.15. *We consider $h \in \mathrm{End}(\mathbf{k}^n)$ such that $h((0, \dots, 0)) = 0$. We moreover suppose that h is nilpotent and we denote by r the smallest integer such that $h^r = 0$. Then $r \leqslant n$.*

Proof: We consider some $0 \leqslant i < n$, and we denote by $X_i := \overline{h^i(\mathbf{k}^n)}$ the closure of the image of h^i. It is an irreducible variety. We also denote by $g_i = h|_{\overline{h^i(\mathbf{k}^n)}}$ the restriction (of domain and target) of h to X_i. As $g_i : X_i \to X_i$ is still nilpotent, it cannot be dominant, if $\dim(X_i) > 0$. Hence, X_{i+1} must be a closed irreducible variety strictly contained in X_i. Hence, if $\dim(X_{i+1}) \neq 0$, we must have $\dim(X_{i+1}) < \dim(X_i)$. As $\dim(X_0) = \dim(\mathbf{k}^n) = n$, we must have $\dim(X_i) \leqslant n - i$ Hence, we must have $\dim(h^n(\mathbf{k}^n)) = 0$, hence $h^n = 0$, as h fixes the origin. ■

Lemma 3.16. *Let $f \in \mathrm{End}(\mathbf{k}^n)$. For each $l \in \mathbf{R}_{>0}$, the set $\{[\omega] \in \nabla \mid \nu_{\mathrm{id},[\omega]}(f) \leqslant l\}$ is a convex polytope in ∇.*

Proof: We write $P = \{[\omega] \in \nabla \mid \nu_{\mathrm{id},[\omega]}(f) \leqslant l\}$. Then

$$P = \bigcap_{1 \leqslant i \leqslant n} \{[\omega] \mid \deg_\omega(f_i) \leqslant l\omega_i\}$$

By construction, $\deg_\omega(f_i) = \deg_\omega(m)$ for (at least) one monomial m involved in f_i. Then the subset $\{\deg_\omega(f_i) \leqslant l\omega_i\}$ is given by linear constraints, as much as there are such dominant monomials. Hence P is the intersection of finitely many closed semi-spaces, with frontier hyperplanes, hence a convex polygon. ■

In particular, the minimality locus is a convex polytope. Its dimension may be smaller than $n-1$: as an example when $n = 3$, it may be a point or a segment.

4 Minimality locus in the union of apartments over $\mathbf{k}^2$

We recall the Theorem of Jung for the structure of automorphisms of $\mathbf{k}^2$:

Theorem 4.1 ([Jung42])**.**

$$\mathrm{Aut}(\mathbf{k}^2) = \mathrm{Tame}(\mathbf{k}^2) = \mathrm{E} * \mathrm{A}$$

Where A denotes the group of affine automorphisms and $\mathbf{E} * \mathbb{A}$ the amalgamated product of elementary automorphisms and affine automorphisms along their intersection $\mathrm{E} \cap \mathrm{A}$.

Let $e = (ax_1 + P(x_2), bx_2 + c) \in \mathrm{E}$ and the non-trivial permutation $p = (x_2, x_1) \in \mathfrak{S}_2$. The composition $h := ep = (ax_2 + P(x_1), bx_1 + c)$ is said of Hénon type.

Corollary 4.2. *Let $f \in \mathrm{Tame}(\mathbf{k}^2)$, not affine nor elementary. Up to affine conjugation, there exists $r \in \mathbf{N}$ and Hénon automorphisms h_i, for $i = 1, \dots, r$ such that $f = h_1 \dots h_r$.*

Having $f = h_1 \dots h_r$, we consider this set of tame valuations:

$$\mathcal{E}^{h_1 \dots h_r} = \dots \cup \mathbf{E}_{f^{-1}} \cup \dots \cup \mathbf{E}_{h_r^{-1} h_{r-1}^{-1}} \cup \mathbf{E}_{h_r^{-1}} \cup \mathbf{E}_{\mathrm{id}} \cup \mathbf{E}_{h_1} \cup \mathbf{E}_{h_1 h_2} \cup \dots \cup \mathbf{E}_f \cup \mathbf{E}_{f h_1} \cup \dots$$

It is the framework of Definition 1.9. This section is dedicated to prove the following:

Theorem 4.3. *Let $f \in \mathrm{Tame}(\mathbf{k}^2)$, such that $f = h_1 \dots h_r$ for h_i Hénon automorphisms. Then the minimality locus $\mathcal{M}$ of $\nu(f)$ for $\nu \in \mathcal{E}^{h_1 \dots h_r}$ is connected, homeomorphic to $\mathbf{R}$.*

We first compute the degree over $\mathbf{E}_{\mathrm{id}}$ for any product for Hénon automorphisms.

Lemma 4.4. *Let $f = h_1 \dots h_r$ with $h_i = (a_i x_2 + P_i(x_1), b_i x_1 + c_i)$ and $d_i = \deg(P_i)$.*

Over $\mathbf{E}_{\mathrm{id}}$, the only monomials that matter for the computation of the degree are $(x_1^{d_1 \dots d_r} + x_2^{d_1 \dots d_{r-1}}, x_1^{d_2 \dots d_r}) \subset f$. In other words,

$$\nu_{\mathrm{id},[\omega_1:\omega_2]}(f) = \nu_{\mathrm{id},[\omega_1:\omega_2]}(x_1^{d_1 \dots d_r} + x_2^{d_1 \dots d_{r-1}}, x_1^{d_2 \dots d_r}),\ \forall [\omega_1 : \omega_2] \in \nabla.$$

In particular, if $[\omega_1 : \omega_2] = [\omega : 1]$:

$$\nu_{\mathrm{id},[\omega:1]}(h_1 \dots h_r) = \max\{\frac{d_1 \dots d_{r-1}}{\omega}, d_1 \dots d_r, d_2 \dots d_r \omega\}.$$

Proof: If $r = 1$, we have $f = h_1$. We have, for any $\omega > 0$, $(P_1)_\omega = x_1^{d_i}$ and $\deg_\omega(P_1) = d_1\omega$. Hence:

$$\nu_{\mathrm{id},[\omega:1]}(h_1) = \nu_{\mathrm{id},[\omega:1]}(x_2 + x_1^{d_1}, x_1) = \max\{\frac{1}{\omega}, d_1, \omega\},$$

as required.

We suppose that the result holds up to $r-1$ Hénon automorphisms. Hence we only consider $(x_1^{d_1\dots d_{r-1}} + x_2^{d_1\dots d_{r-2}}, x_1^{d_2\dots d_{r-1}}) \subset h_1 \dots h_{r-1}$. We also only have to consider $(x_2 + x_1^{d_r}, x_1) \subset h_r$, as those are sums of dominant monomials, by the induction hypothesis. We compute:

$$\begin{aligned}&(x_1^{d_1\dots d_{r-1}} + x_2^{d_1\dots d_{r-2}}, x_1^{d_2\dots d_{r-1}}) \circ (x_2 + x_1^{d_r}, x_1)\\ =&((x_2 + x_1^{d_r})^{d_1\dots d_{r-1}} + x_1^{d_1\dots d_{r-2}}, (x_2 + x_1^{d_r})^{d_2\dots d_{r-1}}) \ (\star)\end{aligned}$$

We note, as the various monomials appear only once, there is no cancellation in this composition. Having set all nonzero coefficients as equal to 1 does not change the value of the degree, so

$$\nu_{\mathrm{id},[\omega:1]}(h_1 \dots h_r) = \nu_{\mathrm{id},[\omega:1]}(\star) = \max_{\omega>0}\{\frac{d_1 \dots d_{r-1}}{\omega}, d_1 \dots d_r, \omega d_2 \dots d_r\}$$

and these correspond to the maximum component of $(x_2^{d_1\dots d_{r-1}} + x_1^{d_1\dots d_r}, x_1^{d_2\dots d_r})$. ■

Corollary 4.5. *Let $f = h_1 \dots h_r$, $\omega > 0$ and $\nu = \nu_{\mathrm{id},[\omega:1]}$. Then*

$$f \text{ is } \nu\text{-algebraically stable} \Leftrightarrow \frac{1}{d_r} \leqslant \omega \leqslant d_1$$

In that case, the degree, equal to $\deg(h_1)\dots\deg(h_r)$*, is minimal, equal to the dynamical degree of f.*

Proof: By Lemma 4.4, if $\frac{1}{d_r} < \omega < d_1$, we have $\nu(f) = d_1 \dots d_r$ and the maximal homogeneous component associated to it is $f_{[\omega:1]} = (x_1^{d_1\dots d_r}, 0)$, which does not vanish after iteration. Therefore by Lemma 3.14, f is $\nu_{\mathrm{id},[\omega]}$-algebraically stable for $\frac{1}{d_r} < \omega < d_1$, and we have $\deg_\omega f = \lambda(f)$. This also holds for $\frac{1}{d_r} \leqslant \omega \leqslant d_1$ by continuity of the degree, so f is $\nu_{\mathrm{id},[\omega:1]}$-algebraically stable if $\frac{1}{d_r} \leqslant \omega \leqslant d_1$, as an application of Lemma 3.14.

Moreover, if $\omega > d_1$, we have $f_{[\omega:1]} = (0, x_1^{d_1\dots d_{r-1}})$ and f is not algebraically stable as $(f_{[\omega:1]})^2 = (0,0)$ and by Lemma 3.14.

Similarly, if $\omega < \frac{1}{d_r}$, we have $f_{[\omega:1]} = (x_2^{d_2\dots d_r}, 0)$ and f is not algebraically stable for the same reason. ■

The reason why the minimality locus is homeomorphic to $\mathbf{R}$ inside $\mathcal{E}^{h_1\dots h_r}$ is the following:

Lemma 4.6. *Borders of the intersection loci $\partial(\mathbf{E}_{h_1} \cap \mathbf{E}_{\mathrm{id}})$, $\partial(\mathbf{E}_{h_r^{-1}} \cap \mathbf{E}_{\mathrm{id}})$ coincide with change of dominating monomials for $f = h_1 \dots h_r$. In addition f is ν-algebraically stable, with $\nu \in \mathbf{E}_{\mathrm{id}}$, if and only if ν does not belong to the interior of $\mathbf{E}_{h_1} \cap \mathbf{E}_{\mathrm{id}}$, nor to the interior of $\mathbf{E}_{h_r^{-1}} \cap \mathbf{E}_{\mathrm{id}}$.*

Proof: Given h a Hénon automorphism, we denote $h = ep$ with e elementary and p the nontrivial permutation. We denote by d its degree. To compute $\mathbf{E}_h \cap \mathbf{E}_{\mathrm{id}}$ (and $\mathbf{E}_{h^{-1}} \cap \mathbf{E}_{\mathrm{id}}$), it suffices to compute $\mathrm{Fix}(e) \cap \mathbf{E}_{\mathrm{id}}$ by Lemma 2.14. By Theorem 2.4, we have:

$$\mathrm{Fix}(e) \cap \mathbf{E}_{\mathrm{id}} = \{\nu_{\mathrm{id},[\omega:1]} \in \mathbf{E}_{\mathrm{id}} \mid \omega \geqslant d\}$$

Hence:

$$\mathbf{E}_h \cap \mathbf{E}_{\mathrm{id}} = \{\nu_{\mathrm{id},[\omega:1]} \in \mathbf{E}_{\mathrm{id}} \mid \omega \geqslant d\}$$

And:

$$\mathbf{E}_{h^{-1}} \cap \mathbf{E}_{\mathrm{id}} = p^{-1}(\mathbf{E}_h \cap \mathbf{E}_{\mathrm{id}}) = \{\nu_{\mathrm{id},[\omega:1]} \in \mathbf{E}_{\mathrm{id}} \mid \omega \leqslant \frac{1}{d}\}$$

But we see by Lemma 4.4, that if $\frac{1}{d_r} < \omega < d_1$ then $(x_1^{d_1\dots d_r}, 0)$ is dominating algebraically stable, and if $\omega > d_1$ then $(0, x_1^{d_1\dots d_{r-1}})$ is dominating non-algebraically stable. Hence, $\{\nu_{\mathrm{id},[d_1:1]}\} = \partial(\mathbf{E}_{h_1} \cap \mathbf{E}_{\mathrm{id}})$ corresponds to a change of dominating monomial.

We also have that if $\omega < \frac{1}{d_r}$ then $(x_2^{d_2\dots d_r}, 0)$ is dominating non-algebraically stable. Hence $\{\nu_{\mathrm{id},[1:d_r]}\} = \partial(\mathbf{E}_{h_r^{-1}} \cap \mathbf{E}_{\mathrm{id}})$ also corresponds to a change of dominating monomials. ■

As we have denoted by $\mathcal{M}$ the minimality locus of $\nu(f)$ among $\nu \in \mathcal{E}^{h_1 \dots h_r}$, as a summary of the above we obtain:

Corollary 4.7. *$\mathcal{M} \cap \mathbf{E}_{\mathrm{id}}$ is a segment with vertices $\partial(\mathbf{E}_{h_1} \cap \mathbf{E}_{\mathrm{id}}) = \{\nu_{\mathrm{id},[d_1:1]}\}$ and $\partial(\mathbf{E}_{h_r^{-1}} \cap \mathbf{E}_{\mathrm{id}}) = \{\nu_{\mathrm{id},[1:d_r]}\}$.*

Proof: By Corollary 4.5, the minimality locus in $\mathbf{E}_{\mathrm{id}}$ is equal to $\{\nu_{\mathrm{id},[\omega:1]} \mid \frac{1}{d_r} \leqslant \omega \leqslant d_1\}$. By Lemma 4.6, we see that it intersects $\mathbf{E}_{h_1} \cap \mathbf{E}_{\mathrm{id}} = \{\nu_{\mathrm{id},[\omega:1]} \in \mathbf{E}_{\mathrm{id}} \mid \omega \geqslant d_1\}$ and $\mathbf{E}_{h_r^{-1}} \cap \mathbf{E}_{\mathrm{id}} = \{\nu_{\mathrm{id},[\omega:1]} \in \mathbf{E}_{\mathrm{id}} \mid \omega \leqslant \frac{1}{d_r}\}$ exactly at their border point. ■

We now proceed with a proof of our Theore 4.3.

Proof of Theorem 4.3: For each apartment $\mathbf{E}_\eta$ in $\mathcal{E}^{h_1 \dots h_r}$, we have either

- $\eta = f^m h_1 \dots h_i$, $m \geqslant 0$ and $1 \leqslant i \leqslant r$, or
- $\eta = f^m h_r^{-1} \dots h_i^{-1}$, $m \leqslant 0$ and $1 \leqslant i \leqslant r$.

Acting upon $\mathbf{E}_{\mathrm{id}}$ via η, by Lemma 3.2, computing the degree of f on $\mathbf{E}_\eta$ amounts to computing the degree of $\eta^{-1} f \eta$ on $\mathbf{E}_{\mathrm{id}}$. By Corollary 4.7, the minimal degree of $\eta^{-1} f \eta$ on $\mathbf{E}_{\mathrm{id}}$ is reached on the segment delimited by

- $\mathbf{E}_{h_{i+1}} \cap \mathbf{E}_{\mathrm{id}}$ and $\mathbf{E}_{h_i^{-1}} \cap \mathbf{E}_{\mathrm{id}}$, if $\eta = f^m h_1 \dots h_i$, $m \geqslant 0$ (in which case $\eta^{-1} f \eta = h_{i+1} \dots h_r h_1 \dots h_i$),
- $\mathbf{E}_{h_i} \cap \mathbf{E}_{\mathrm{id}}$ and $\mathbf{E}_{h_{i-1}^{-1}} \cap \mathbf{E}_{\mathrm{id}}$, if $\eta = f^m h_r^{-1} \dots h_i^{-1}$, $m \leqslant 0$ (in which case $\eta^{-1} f \eta = h_i \dots h_r h_1 \dots h_{i-1}$).

Via the action of η, we conclude that $\nu(f)$ reaches its minimum for $\nu \in \mathbf{E}_\eta$, if and only in ν belongs to the segment delimited by

- $\mathbf{E}_{\eta h_{i+1}} \cap \mathbf{E}_\eta$ and $\mathbf{E}_{\eta h_i^{-1}} \cap \mathbf{E}_\eta$, if $\eta = f^m h_1 \dots h_i$, $m \geqslant 0$,
- $\mathbf{E}_{\eta h_i} \cap \mathbf{E}_\eta$ and $\mathbf{E}_{\eta h_{i-1}^{-1}} \cap \mathbf{E}_\eta$, if $\eta = f^m h_r^{-1} \dots h_i^{-1}$, $m \leqslant 0$.

So $\mathcal{M}$ describes a segment in each apartment. Moreover, two consecutive segments of this form meet at exactly one border point. So $\mathcal{M}$ is an infinite line over $\mathcal{E}^{h_1 \dots h_r}$. ■

The locus $\mathcal{M}$ is depicted in purple in Figure 1 below.

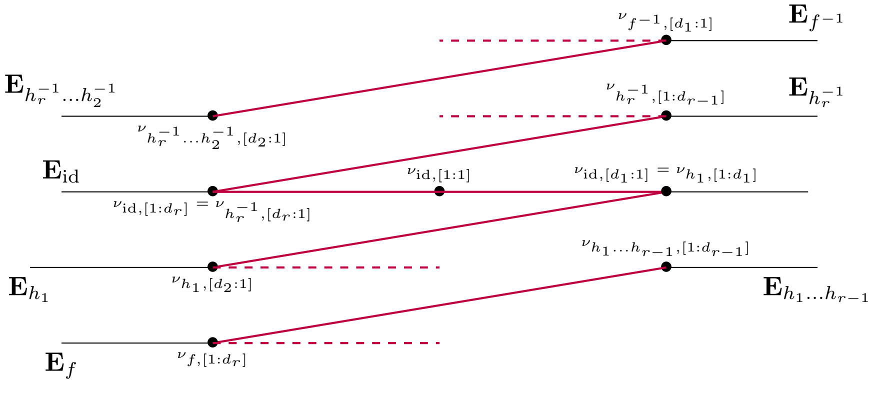


Figure 1: Minimality locus $\mathcal{M}$ for $h_1 \dots h_r$

5 The minimality locus in the union of all apartments over $\mathbf{k}^3$

In this section, we focus on the case $n = 3$ and we study the minimality locus of $\nu(f)$ on all $\nu \in \mathcal{E}^{ep}$ for f a tame automorphism of $\mathbf{k}^3$ of type $f = ep$ with e elementary and p a permutation.

We being by a remark on $\mathcal{E}^{ep}$.

Remark 5.1. As we can write $e = (x_1 + P_1(x_2, x_3), x_2, x_3)$, we have

$$f = (x_{p^{-1}(1)} + P(x_{p^{-1}(2)}, x_{p^{-1}(3)}), x_{p^{-1}(2)}, x_{p^{-1}(3)}).$$

We remark that there is no expression of f as $p'e'$ with e' an elementary automorphism and p' a permutation, unless $p' = (x_1, x_2, x_3)$ — otherwise the polynomial P would not appear in the first component of f. But then $p'e' = e''p'$ for some other elementary e''. Hence we will only consider an expression $f = ep$. We have:

$$\mathcal{E}^{ep} = \bigcup_{m \in \mathbf{Z}} \mathbf{E}_{f^m}.$$

Indeed, $\mathbf{E}_e = \mathbf{E}_f$, $\mathbf{E}_{fe} = \mathbf{E}_{f^2}$, and so on.

Theorem 5.2. *Assume that $f = ep$ is an elementary-permutation automorphism with $\lambda(f) \notin \mathbf{N}$ and that the dynamical degree $\lambda(f)$ is reached on $\mathbf{E}_{\mathrm{id}}$, i.e. there exists a monomial valuation $\nu_{\mathrm{id},[\omega]} \in \mathbf{E}_{\mathrm{id}}$ for which f is algebraically stable.*

Then the minimality locus

$$\mathcal{M} = \{\nu \in \mathcal{E}^{ep} \mid \nu(f) = \lambda(f)\}$$

is homeomorphic to $\mathbf{R}$.

5.1 Step-by-step proof of the theorem

We assume that the dynamical degree of f is not an integer and that it is met in $\mathbf{E}_{\mathrm{id}}$. In particular, as $\lambda(f) \neq 1$, we have $\deg(P) \geqslant 2$ with deg the standard degree.

Lemma 5.3. *Let $f = ep$, with $e = (x_1 + P(x_2, x_3), x_2, x_3)$ and p a permutation.*

We assume that its dynamical degree is not an integer and that it equal to $\nu(f)$ for $\nu \in \mathcal{M}$. Then p is of order 3. Moreover we may assume $f = ep$ with $p = (123) = (x_3, x_1, x_2)$, in other words:

$$f = ep = (x_1 + P(x_2, x_3), x_2, x_3)(x_3, x_1, x_2) = (x_3 + P(x_1, x_2), x_1, x_2)$$

The condition $\nu(f) = \lambda(f)$ for $\nu \in \mathcal{M}$ is equivalent to the existence of a monomial valuation $\nu_{\mathrm{id},[\omega]} \in \mathbf{E}_{\mathrm{id}}$ such that f is $\nu_{\mathrm{id},[\omega]}$-algebraically stable. In Section 6, we give examples of $f = ep$ that do not satisfy this condition, where p has order 2, $\lambda(f)$ is met on the boundary of $\mathbf{E}_{\mathrm{id}}$ and $\mathcal{M} = \varnothing$.

Proof of Lemma 5.5 : One can first remark that the two order 3 permutations are conjugated to by $\sigma = (x_1, x_3, x_2)$, which has x_1 as its invariant coordinate. The conjugate of e by the two-cycle $\sigma = (x_1, x_3, x_2)$ is an elementary e' and

$$ep = \sigma e \sigma^{-1} \sigma p \sigma^{-1} = e'(\sigma p \sigma^{-1}).$$

Hence, we can assume that $p = (x_3, x_1, x_2)$ is the permutation involved in the expression of f. Now we prove by contradiction that p has order 3:

If p is of order at most 2, we will see that the dynamical degree of ep is an integer. If $p = \mathrm{id}$, then f is elementary and $\lambda(f) = 1$.

If $p = (23) = (x_1, x_3, x_2)$, then we see that $(ep)^2$ is an elementary, hence, by the fact that $\lambda(f^2) = \lambda(f)^2$, the dynamical degree of f is the square root of that $(ep)^2$, i.e. equal to 1.

If $p = (12) = (x_2, x_1, x_3)$ — the case $p = (13) = (x_3, x_2, x_1)$ being conjugated by (23) — we have $ep = (x_2 + P(x_1, x_3), x_1, x_3)$. Given $[\omega]$ for which f is algebraically stable, we have $(P_\omega(x_1, x_3), 0, 0) \subset f_{[\omega]}$, otherwise we would have $f_{[\omega]} \subset (x_2, x_1, x_3)$; as (x_2, x_1, x_3) is an involution, this would mean $\lambda(f) = 1$. Moreover, $P_\omega(x_1, x_3)$ is divisible by x_3, or else there would be a monomial x_1^d in P_ω and we would have $\lambda(f) = \nu_{\mathrm{id},[\omega]}(x_1^d, 0, 0) = d \in \mathbf{N}$. So $(P_\omega(x_1, x_3), 0, 0)$ is nilpotent, hence $(P_\omega(x_1, x_3), 0, 0) \neq f_{[\omega]}$. We cannot have $(P_\omega(x_1, x_3), 0, x_3) \subset f_{[\omega]}$, since $\nu_{\mathrm{id},[\omega]}(0, 0, x_3) = 1$. Furthermore, we cannot have $(x_2 + P_\omega(x_1, x_3), 0, 0) = f_{[\omega]}$, as the left member is nilpotent. For the same reason, $(P_\omega(x_1, x_3), x_1, 0) \neq f_{[\omega]}$. So $f_{[\omega]} = (x_2 + P_\omega(x_1, x_3), x_1, 0)$, meaning

$$\frac{\omega_2}{\omega_1} = \nu_{\mathrm{id},[\omega]}(x_2, 0, 0) = \lambda(f) = \nu_{\mathrm{id},[\omega]}(0, x_1, 0) = \frac{\omega_1}{\omega_2}$$

leading to $\lambda(f) = 1$, a contradiction.

The conclusion follows that p has order 3, as required. ■

Lemma 5.4. *Let $f = ep$ an elementary-permutation automorphism satisfying the same assumptions as in Lemma 5.3.*

$$f = (x_3 + P(x_1, x_2), x_1, x_2).$$

Then we have $\deg(P) \geqslant 2$ *for the standard degree, and for any* $[\omega]$ *such that* f *is* ω*-algebraically stable:*

$$(P_\omega(x_1, x_2), x_1, 0) \subset f_{[\omega]}$$

Moreover $P_\omega(x_1, x_2)$ *is divisible by* x_2*.*

Proof: The fact that $(P_\omega(x_1, x_2), x_1, 0) \subset f_{[\omega]}$ can be proven in a similar way as in the proof of Lemma 5.3: if $[\omega]$ is a weight for which f is algebraically stable, we have $(P_\omega(x_1, x_3), 0, 0) \subset f_{[\omega]}$, otherwise we would have $f_{[\omega]} \subset (x_2, x_1, x_3)$, which has dynamical degree 1. $P_\omega(x_1, x_2)$ is divisible by x_2, or else there would be a monomial x_1^d in P_ω and we would have $\lambda(f) = \nu_{\mathrm{id},[\omega]}(x_1^d, 0, 0) = d \in \mathbf{N}$. We have $(P_\omega(x_1, x_2), x_1, 0) \subset f_{[\omega]}$, since $(x_3 + P_\omega(x_1, x_2), 0, x_2)$ is nilpotent. ■

Next, we identify the minimality locus in $\mathbf{E}_{\mathrm{id}}$:

Lemma 5.5. *Let* $f = ep$ *with* $e = (x_1 + P(x_2, x_3), x_2, x_3)$ *and* $p = (x_3, x_1, x_2)$ *satisfying the same assumptions as in Lemma 5.3.*

Then the minimality locus of $\nu(f)$ *among monomial valuations* ν *is a segment on the line* $\omega_1 = \lambda\omega_2$ *where* $\lambda = \lambda(f)$ *is the dynamical degree. Its vertices are intersection points with* $\omega_2 = \lambda\omega_3$*,* i.e. $[\lambda^2 : \lambda : 1]$ *and* $\omega_3 = \lambda\omega_1$*,* i.e. $[\lambda : 1 : \lambda^2]$*.*

Proof: We have $f = (x_3 + P(x_1, x_2), x_1, x_2)$.

We compute the degree of f, for a weight $[\omega] = [\omega_1 : \omega_2 : \omega_3]$:

$$\nu_{\mathrm{id},[\omega]}(f) = \max\left\{\frac{\omega_3}{\omega_1}, \frac{\deg_\omega(P(x_1, x_2))}{\omega_1}, \frac{\omega_1}{\omega_2}, \frac{\omega_2}{\omega_3}\right\}$$

Finding $\nu_{\mathrm{id},[\omega]}$ for which f is algebraically stable requires finding $[\omega]$ such that $(P_{[\omega]}(x_1, x_2), x_1, 0)$ is dominating, in other words $(P_{[\omega]}(x_1, x_2), x_1, 0) \subset f_{[\omega]}$, by Lemma 5.4.

So a necessary condition is

$$\frac{\deg_{[\omega]}(P_\omega(x_1, x_2))}{\omega_1} = \frac{\omega_1}{\omega_2} = \lambda(f) =: \lambda.$$

Moreover, we have $\frac{\omega_1}{\omega_2} \geqslant \frac{\omega_3}{\omega_1}$, or else the component $(P_\omega(x_1, x_2), x_1, 0)$ would be dominated by $(x_3, 0, 0)$. As $\omega_1 = \lambda\omega_2$, this yields $\omega_3 \leqslant \lambda\omega_1 = \lambda^2\omega_2$. We also have $\frac{\omega_1}{\omega_2} \geqslant \frac{\omega_2}{\omega_3}$. As $\omega_1 = \lambda\omega_2$, this yields $\omega_3 \geqslant \frac{1}{\lambda}\omega_2$. Note that the interval $[\frac{1}{\lambda}\omega_2, \lambda^2\omega_2]$ is nonempty, as $\lambda > 1$ and $\omega_2 > 0$.

Conversely, for such a weight $[\omega]$, if $\omega_3 \in]\frac{1}{\lambda}\omega_2, \lambda^2\omega_2[$, the maximal homogeneous part $f_{[\omega]}$ is $(P_{[\omega]}(x_1, x_2), x_1, 0)$.

For any non-zero polynomial Q, for $g = (Q(x_1, x_2), x_1)$ we have $g^2 \neq 0$, as the second coordinate of g^2 is $Q(x_1, x_2)$. Hence, by Lemma 3.15, $f_{[\omega]}$ never vanishes after iterations — otherwise, its nilpotent degree would be at most 2.

By Lemma 3.14, we have algebraic stability, and for such a weight $[\omega]$, $\nu_{\mathrm{id},[\omega]}(f)$ is equal to the dynamical degree, hence the minimum of $\nu(f)$ for $\nu \in \mathbf{E}_{\mathrm{id}}$.

As a conclusion, by continuity of $\nu_{\mathrm{id},[\omega]}$ with respect to $[\omega]$, any weight in the segment $\omega_1 = \lambda\omega_2$ and $\frac{\omega_2}{\lambda} \leqslant \omega_3 \leqslant \lambda^2\omega_2 = \lambda\omega_1$ is in the minimality locus. ■

Lemma 5.6. *Let* $f = ep$ *with* $e = (x_1 + P(x_2, x_3), x_2, x_3)$ *and* $p = (x_3, x_1, x_2)$ *satisfying the same assumptions as in Lemma 5.3. Then* $\mathbf{E}_f \cap \mathbf{E}_{\mathrm{id}}$ *(respectively* $\mathbf{E}_{f^{-1}} \cap \mathbf{E}_{\mathrm{id}}$*) meets the minimality locus, a segment, at one of its extremity points (respectively the other extremity point).*

Proof: For any elementary automorphism, confined valuations are fixed (Lemma 2.2). We determine the set of monomial valuations fixed by e, that is, the set $\mathbf{E}_f \cap \mathbf{E}_{\mathrm{id}}$: by Theorem 2.4, the condition is $\omega_1 \geqslant \deg_\omega P(x_2, x_3)$ for $\nu_{\mathrm{id},[\omega]}$.

Let $\nu_{\mathrm{id},[\omega]}$ be one of the extremity points of the minimality segment explained in Lemma 5.5, such that

$$\lambda = \frac{\deg_\omega P(x_1, x_2)}{\omega_1} = \frac{\omega_1}{\omega_2} = \frac{\omega_2}{\omega_3}.$$

In particular,

$$\frac{\deg_\omega P(x_1,x_2)}{\omega_1} = \frac{\deg_\omega P(x_2,x_3)}{\omega_2}.$$

We choose $x_2^{a_2}x_3^{a_3}$ a dominating monomial of $P(x_2,x_3)$, such that $\deg_\omega P(x_2,x_3) = a_2\omega_2 + a_3\omega_3$. The conditions above yield $\frac{\omega_1}{\omega_2} = \frac{a_2\omega_2+a_3\omega_3}{\omega_2}$ and $\lambda^2 = a_2\lambda + a_3$. Hence for this valuation $\nu_{\mathrm{id},[\omega]}$ we have

$$\lambda^2 = \frac{\omega_1}{\omega_2}\frac{\omega_2}{\omega_3} = a_2\frac{\omega_2}{\omega_3} + a_3,$$

yielding $\omega_1 = a_2\omega_2 + a_3\omega_3 = \deg_\omega P(x_2,x_3)$. We deduce that this valuation is a boundary point of $\mathbf{E}_f$. Conversely, any point in the interior of $\mathbf{E}_f \cap \mathbf{E}_{\mathrm{id}}$ satisfies

$$\omega_1 > \deg_\omega P(x_2,x_3) \iff \frac{\omega_1}{\omega_2} > \frac{\deg_\omega P(x_2,x_3)}{\omega_2} = \frac{\deg_\omega P(x_1,x_2)}{\omega_1},$$

which is not in $\mathcal{M}$ by Lemma 5.5.

Now, we determine the set of monomial valuations in $\mathbf{E}_{f^{-1}} \cap \mathbf{E}_{\mathrm{id}}$; by Lemma 2.14 and example 2.4, the condition for a weight to be in $\mathbf{E}_{p^{-1}e^{-1}} \cap \mathbf{E}_{\mathrm{id}}$ is $\omega_3 \geqslant \deg_\omega P(x_1,x_2)$.

Consider the monomial valuation $\nu_{\mathrm{id},[\omega]}$ situated at the other extremity point of the minimality segment, such that

$$\lambda = \frac{\deg_\omega P(x_1,x_2)}{\omega_1} = \frac{\omega_1}{\omega_2} = \frac{\omega_3}{\omega_1}.$$

This valuation satisfies $\omega_3 = \deg_\omega P(x_1,x_2)$, so it is on the boundary of $\mathbf{E}_{f^{-1}}$. Conversely, any valuation in the interior of $\mathbf{E}_{f^{-1}} \cap \mathbf{E}_{\mathrm{id}}$ satisfies $\omega_3 > \deg_\omega P(x_1,x_2)$, which means

$$\frac{\omega_3}{\omega_1} > \frac{\deg_\omega P(x_1,x_2)}{\omega_1}$$

which is not in $\mathcal{M}$ by Lemma 5.5. ∎

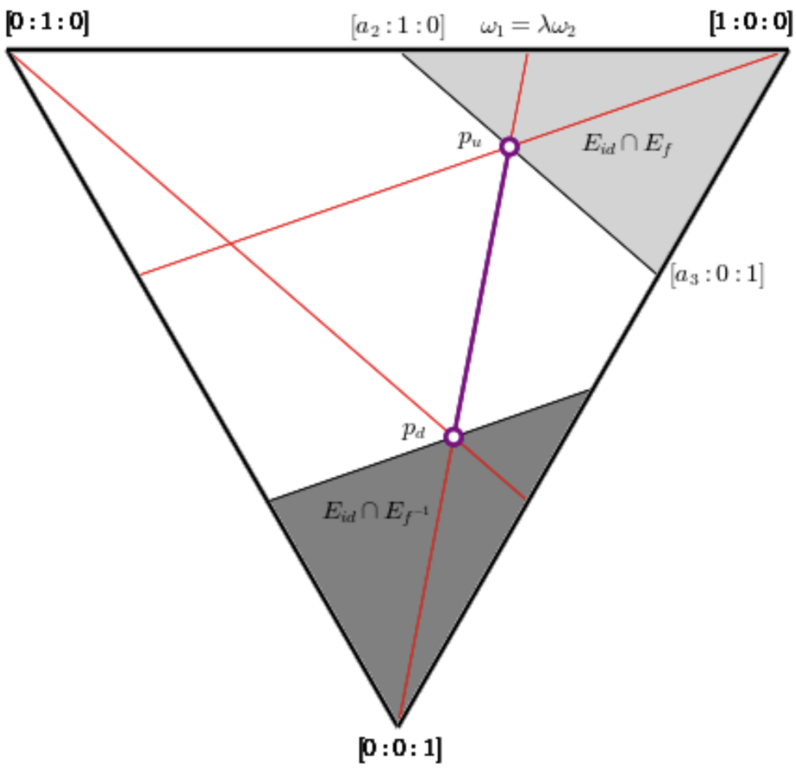


Figure 2: Minimality locus $\mathcal{M} \cap \mathbf{E}_{\mathrm{id}}$ for ep

Now we finish the proof of Theorem 5.2.

Proof of Theorem 5.2 : We know by Lemma 5.6 that $\mathcal{M}$ in $\mathbf{E}_{\text{id}}$ describes a segment that intersects $\mathbf{E}_{f^{-1}}$ and $\mathbf{E}_f$ exactly at its extremity points. This fact holds on all apartments $\mathbf{E}_{f^m}$ for $m \in \mathbf{Z}$. Indeed, we always have $\nu_{f^m,[\omega]}(f) = \nu_{\text{id},[\omega]}(f)$, and $\mathcal{M} \cap \mathbf{E}_{f^m}$ is the image of $\mathcal{M}$ under the action of f^m. Moreover $\mathbf{E}_{f^m} \cap \mathbf{E}_{f^{m+1}} = f^m \cdot (\mathbf{E}_{\text{id}} \cap \mathbf{E}_f)$ for all $m \in \mathbf{Z}$.

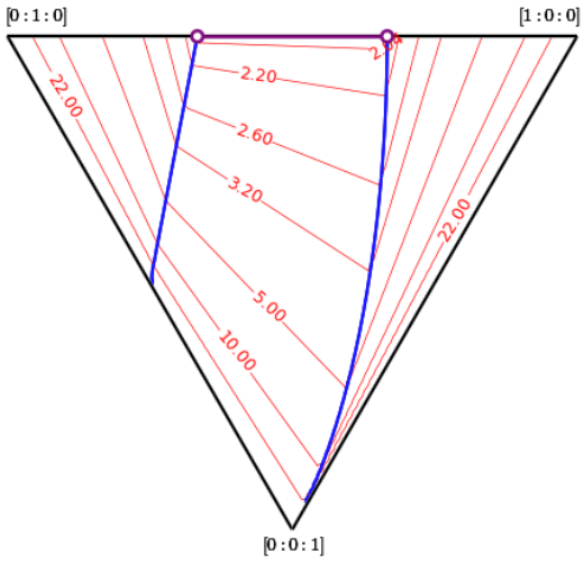
Moreover, we have $\mathbf{E}_f \cap \mathbf{E}_{\text{id}} \cap \mathbf{E}_{f^{-1}} = \varnothing$ as in Figure 2. The two apartments $\mathbf{E}_f$ and $\mathbf{E}_{f^{-1}}$ do not meet outside of $\mathbf{E}_{\text{id}}$, or else $\mathcal{E}^{ep}$ would not be simpply connected, in contradiction with [LP21]. More precisely $\mathbf{E}_{f^m} \cap \mathbf{E}_{f^{m+n}} = \varnothing$ unless $n = \pm 1$.

The conclusion follows that $\mathcal{M}$ is a countable union of compact segment, with two consecutive segments meeting at an extremity point, and any segment meeting only the previous and the next one. So $\mathcal{M} \simeq \mathbf{R}$. ■

6 Examples and drawings.

6.1 Example of an elementary-permutation automorphism whose dynamical degree is not reached in the $\mathbf{E}_{\text{id}}$ apartment

We consider $f = ep$ with $e = (x_1 + x_2^2 x_3, x_2, x_3)$ and $p = (x_2, x_1, x_3)$. We note that $\nu(f)$ for $\nu \in \mathbf{E}_{\text{id}}$ does not have a minimum, as illustrated below:



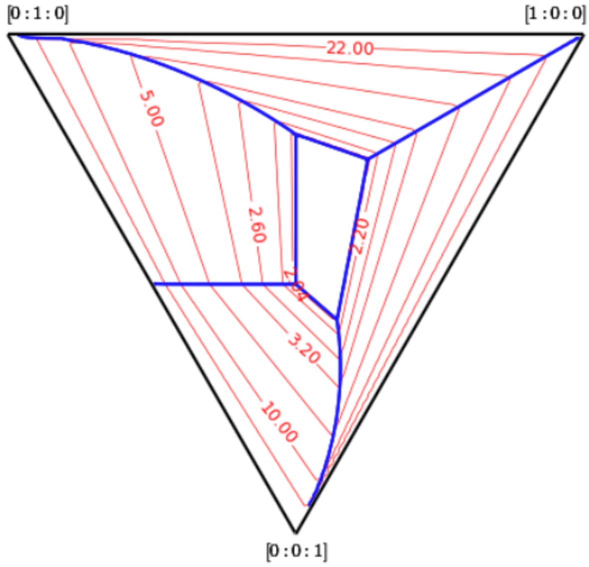
Level lines of $\nu(f)$ in $\mathbf{E}_{\text{id}}$.

We note that the dynamical degree of f is an integer, equal to 2 and that it is reached on the boundary of $\mathbf{E}_{\text{id}}$, not inside. However, it is not in the framework of this paper to consider weights with zero components. The reader can refer to [BvS22, Section 2] on this matter.

Note that here p is a permutation of order 2. For $f = ep$ with p a permutation of order 3, we expect that the dynamical degree is reached in the apartment of monomial valuations.

6.2 Example of a triangular-permutation automorphism with a minimality locus of dimension 2.

We consider $f = tp$ with $t = (x_1 + x_2 x_3 + x_3^2, x_2 + x_3, x_3)$ and $p = (x_3, x_2, x_1)$. Then the minimality locus in $\mathbf{E}_{\text{id}}$ has dimension two, as illustrated below:



Level lines of $\nu(f)$ in $\mathbf{E}_{\mathrm{id}}$.

The quadrilateral depicted at the center is $\mathcal{M} \cap \mathbf{E}_{\mathrm{id}}$. In this case, the dynamical degree is reached and is an integer.

6.3 Example of a triangular-permutation automorphism with minimality locus homeomorphic to R

We make a remark on the case where f is of triangular-permutation type, $f = tp$:

Remark 6.1. If $t = (x_1 + P_1(x_2, x_3), x_2 + P_2(x_3), x_3)$ is triangular non-elementary, then we can write $t = e_2 e_1 = \tilde{e}_1 \tilde{e}_2$, where $e_2 = (x_1, x_2 + P_2(x_3), x_3) = \tilde{e}_2$, $e_1 = (x_1 + P_1(x_2, x_3), x_2, x_3)$, and $\tilde{e}_1 = (x_1 + P_1(x_2 - P_2(x_2), x_3), x_2, x_3)$.

Lemma 6.2. *If $t \in \mathrm{T} \setminus \mathrm{E}$, then $t = e_2 e_1 = \tilde{e}_1 \tilde{e}_2$ as in Remark 5.1(6.1), and those are the only expressions of t with at most two elementary letters (that is, two elementary automprphisms, each having two invariant coordinates).*

Proof: As t is not elementary, one cannot write t with one letter. If $t = ee'$ then, as the third coordinate of t is invariant both e and e' have their third coordinate invariant. Moreover, both cannot have the two same invariant coordinates, otherwise the result would be elementary.

There are two cases left, where e and e' are uniquely determined by t. ■

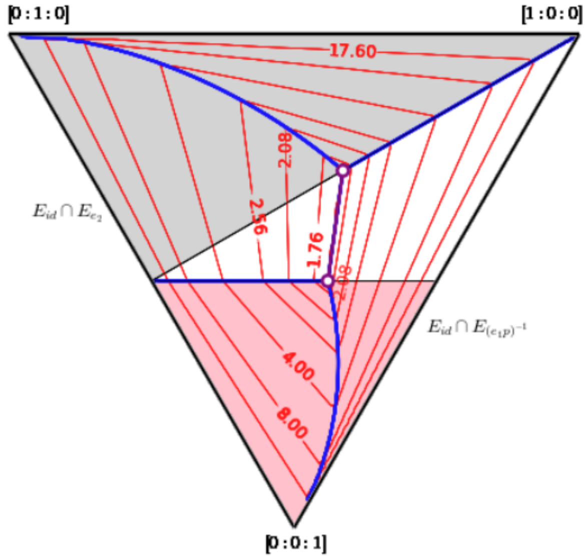
Then, for $f = tp$, where $t \in \mathrm{T} \setminus \mathrm{E}$, we will consider degree over $\mathcal{E}^{e_2 e_1 p}$ and $\mathcal{E}^{\tilde{e}_1 \tilde{e}_2 p}$.

We give here an explicit example of $f = tp$, with $t = e_2 e_1$ a nonelementary triangular automorphisms, for which $\mathcal{M}$ in $\mathcal{E}^{e_2 e_1 p}$ is homeomorphic to $\mathbf{R}$. However we will see that the minimality locus $\mathcal{M}$ does *not* meet $\mathbf{E}_f$ and $\mathbf{E}_{f^{-1}}$.

For $p = (x_3, x_2, x_1)$ and $t = (x_1 + x_2 x_3, x_2 + x_3, x_3)$, with $t = e_2 e_1$;

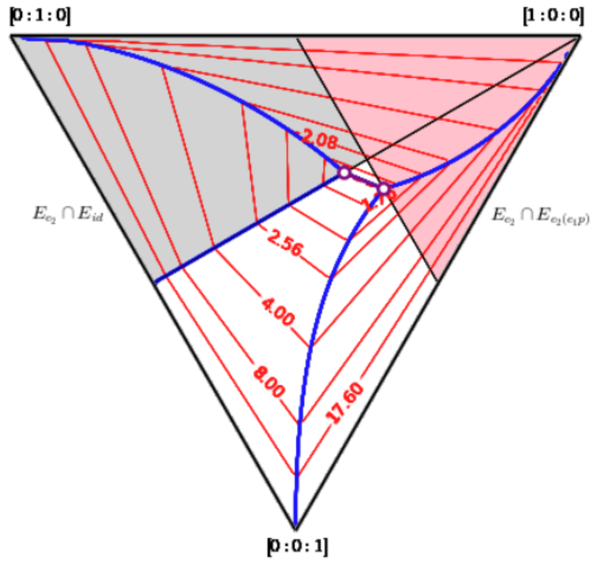
$$e_2 = (x_1, x_2 + x_3, x_3),\ e_1 = (x_1 + x_2 x_3, x_2, x_3),$$

the dynamical degree $\lambda(f)$ equals the golden ratio φ. This value is reached in $\mathbf{E}_{\mathrm{id}}$ – meaning that there exists a weight for which f is algebraically stable. On the apartment $\mathbf{E}_{\mathrm{id}}$, we draw in red some level lines of the degree $\nu(f)$, and in grey and red the respective intersections $\mathbf{E}_{e_2} \cap \mathbf{E}_{\mathrm{id}}$ and $\mathbf{E}_{p^{-1} e_1^{-1}} \cap \mathbf{E}_{\mathrm{id}}$:



Level lines of $\nu(f)$ in $\mathbf{E}_{\mathrm{id}}$ and the minimality locus.

Notice that the minimality locus in $\mathbf{E}_{\mathrm{id}}$, which is here the segment delimited by the two nodes, meets the previous and the next apartment of $\mathcal{E}^{e_2e_1p}$ only at its extremity points. Now we depict the minimality locus $\mathcal{M}$ in the next apartment $\mathbf{E}_{e_2}$, also here a segment:



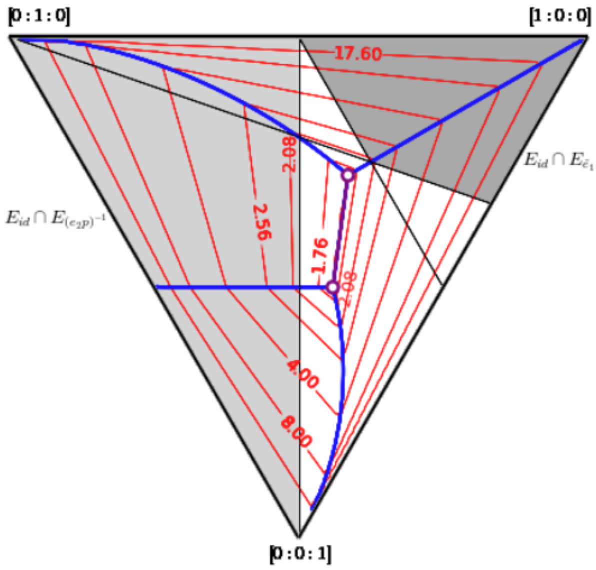
Level lines of $\nu(f)$ in $\mathbf{E}_{e_2}$ and the minimality locus.

This segment also meets the next apartment $\mathbf{E}_f$ and the previous one $\mathbf{E}_{\mathrm{id}}$ only at its extremity points. This fact holds on all the apartments involved in $\mathcal{E}^{e_2e_1p}$; the minimality locus $\mathcal{M}$ is a countable union of compact segments, with two consecutive segments meeting at an extremity. So it is homeomorphic to $\mathbf{R}$, as in Theorem 5.2.

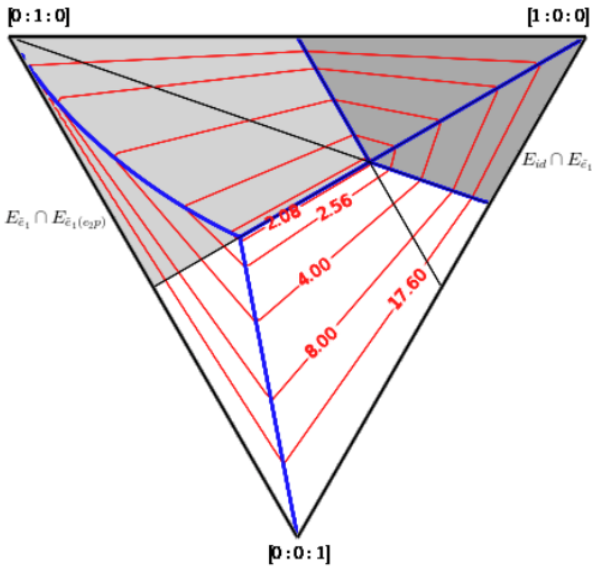
Furthermore, we also have $t = \tilde{e}_1\tilde{e}_1$ with

$$\tilde{e}_2 = e_2,\, \tilde{e}_1 = (x_1 + (x_2 - x_3)x_3, x_2, x_3)$$

and we may also look at the minimality locus in $\mathcal{E}^{\tilde{e}_1\tilde{e}_2p}$. On the apartments $\mathbf{E}_{\mathrm{id}}$ and $\mathbf{E}_{\tilde{e}_1}$, this gives:



Level lines of $\nu(f)$ in $\mathbf{E}_{\mathrm{id}}$



Level lines of $\nu(f)$ in $\mathbf{E}_{\tilde{e}_1}$

Notice that the dynamical degree of f is not reached on $\mathbf{E}_{\tilde{e_1}}$, as

$$\min\{\nu(f) \mid \nu \in \mathbf{E}_{\tilde{e_1}}\} = 2 \neq \varphi = \lambda(f),$$

and that the segment delimited by the two nodes $\mathcal{M} \cap \mathbf{E}_{\mathrm{id}}$ does not meet $\mathbf{E}_{\tilde{e}_1}$. Hence $\mathcal{M}$ in $\mathcal{E}^{\tilde{e}_1\tilde{e}_2p}$ is disconnected.

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