An overview of accelerated methods in convex optimization

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Plan

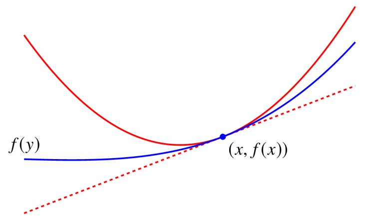
Framework and motivations

Minimization problem

$$\min_{x\in\mathbb{R}^N} F(x) = f(x) + h(x),$$

where:

- f is a convex differentiable function having a L-Lipschitz gradient,



- h is a convex proper lower semicontinuous function,
- F has a non-empty set of minimizers X^*.

Motivations

$$\min_{x \in \mathbb{R}^N} F(x),$$

Which algorithm is the most efficient according to the **assumptions** satisfied by F and the **expected accuracy**?

$\rightarrow$ **Convergence analysis** of the numerical schemes:

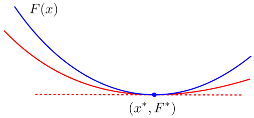
How fast does $F(x_k) - F^*$ decreases?

Framework and motivations

Classical geometry assumptions

- **Strong convexity ($\mathcal{SC}_\mu$):**
  F is μ-strongly convex if for all $x \in \mathbb{R}^N$, $g : x \mapsto F(x) - \frac{\mu}{2}\|x\|^2$ is convex.
- **Quadratic growth condition ($\mathcal{G}_\mu^2$):**
  F has a quadratic growth around its set of minimizers if

$$\exists \mu > 0,\ \forall x \in \mathbb{R}^N,\ \frac{\mu}{2} d(x, X^*)^2 \leqslant F(x) - F^*.$$



Example: LASSO problem:

$$F(x) = \frac{1}{2}\|Ax - y\|^2 + \lambda \|x\|_1.$$

Framework and motivations

Classical algorithms

Gradient Descent/Forward-Backward:

$$\forall k > 0, x_k = x_{k-1} - s\nabla F(x_{k-1}).$$

Nesterov's accelerated gradient/FISTA (Beck and Teboulle, 2009):

$$\forall k > 0, \begin{cases} x_k = y_{k-1} - s\nabla F(y_{k-1}), \\ y_k = x_k + \dfrac{k-1}{k+\alpha-1}(x_k - x_{k-1}), \end{cases}$$

where $\alpha > 0$. In general, $\alpha = 3$.
Heavy-Ball type schemes:

$$\forall k > 0, \begin{cases} x_k = y_{k-1} - s\nabla F(x_{k-1}) \text{ or } x_k = y_{k-1} - s\nabla F(y_{k-1}), \\ y_k = x_k + \alpha(x_k - x_{k-1}), \end{cases}$$

where $\alpha \in (0,1)$.

Plan

$\rightarrow$ **Key tool in convergence analysis**: Link numerical schemes to dynamical systems.

Gradient descent $\rightarrow$ Gradient flow

$$x_k = x_{k-1} - s\nabla F(x_{k-1})$$

The continuous setting: a guideline for the discrete analysis

$\rightarrow$ **Key tool in convergence analysis**: Link numerical schemes to dynamical systems.

Gradient descent$\rightarrow$ Gradient flow

$$x_k = x_{k-1} - s\nabla F(x_{k-1})$$

$$\iff \frac{x_k - x_{k-1}}{s} = -\nabla F(x_{k-1})$$

The continuous setting: a guideline for the discrete analysis

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$$\downarrow$$

$$\dot{x}(t) + \nabla F(x(t)) = 0.$$

The continuous setting: a guideline for the discrete analysis

Nesterov's accelerated gradient→Asymptotic vanishing damping system (Su, Boyd and Candès,2014)

$$\forall k > 0, \begin{cases} x_k = y_{k-1} - s\nabla F(y_{k-1}), \\ y_k = x_k + \dfrac{k-1}{k+\alpha-1}(x_k - x_{k-1}) \end{cases}$$

$$\downarrow$$

$$\ddot{x}(t) + \frac{\alpha}{t}\dot{x}(t) + \nabla F(x(t)) = 0$$

Heavy-Ball schemes→ Heavy-Ball Friction system

$$\forall k > 0, \begin{cases} x_k = y_{k-1} - s\nabla F(x_{k-1}) \text{ or } x_k = y_{k-1} - s\nabla F(y_{k-1}), \\ y_k = x_k + \alpha(x_k - x_{k-1}), \end{cases}$$

$$\downarrow$$

$$\ddot{x}(t) + \alpha\dot{x}(t) + \nabla F(x(t)) = 0$$

The continuous setting: a guideline for the discrete analysis

Why is this relevant?

- easier computations (derivatives),
- most of the time, convergence properties of the trajectories can be extended to the iterates of the related scheme.

Back to the discrete setting

Challenging for the following reasons:

- no more derivative,
- several possible discretization choices,
- which condition on the stepsize?

The continuous setting: a guideline for the discrete analysis

Convergence results

Convergence rates of $F(x(t)) - F^*$:

	F convex	F μ-strongly convex
Gradient flow (Gradient descent)	$O\left(t^{-1}\right)$	$O\left(e^{-\mu t}\right)$
Heavy-Ball friction (Heavy-Ball schemes)	$O\left(t^{-1}\right)$	$O\left(e^{-2\sqrt{\mu}t}\right)$ if F is C^2
Asymptotic Vanishing Damping (Nesterov's accelerated gradient)	$O(t^{-2})$	$O\left(t^{-\frac{2\alpha}{3}}\right)$ Actually faster in finite time (see [1])

[1] J-F Aujol, Charles Dossal, Aude Rondepierre. FISTA is an automatic geometrically optimized algorithm for strongly convex functions. 2021. ⟨hal-03491527⟩

Plan

Restart strategies

About inertia

Recall the definition of Nesterov's accelerated gradient/FISTA:

$$\forall k > 0, \begin{cases} x_k = y_{k-1} - s\nabla F(y_{k-1}), \\ y_k = x_k + \frac{k-1}{k+2}(x_k - x_{k-1}) \end{cases}$$

$\rightarrow$ taking in account the previous iterates generates inertia.

Issue

Under growth assumptions such as $\mathcal{SC}_\mu$ or $\mathcal{G}^2_\mu$, inertial methods have to be **parametrized according to the growth parameter** μ to reach fast convergence rates.

$\rightarrow$ μ **is rarely computable!!**

Restart strategies

Restarting FISTA, why?

- to take advantage of inertia,
- to avoid oscillations.

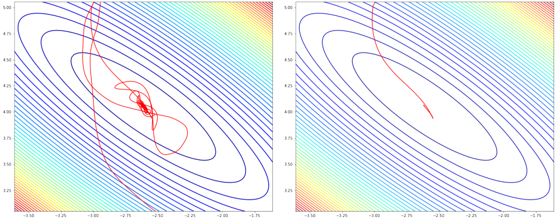


Figure: Trajectory of the iterates of FISTA (left) and FISTA restart (right) for a least-squares problem ($N = 20$).

Restart strategies

Restarting FISTA, how?

Algorithm 1 : FISTA restart

Require: $x_0 \in \mathbb{R}^N,\ y_0 = x_0,\ k = 0,\ i = 0.$
  repeat
    $k = k+1, i = i+1$
    $x_k = y_{k-1} - s\nabla f(y_{k-1})$
    if Restart condition is *True* **then**
      $i = 1$
    end if
    $y_k = x_k + \frac{i-1}{i+2}(x_k - x_{k-1})$
  until Exit condition is *True*

$\rightarrow$ Cutting inertia is equivalent to restarting the algorithm from the last iterate.

Restart strategies

Objective: get a restart condition that

- does not require to know the growth parameter μ,
- ensures a fast convergence of the method: $F(x_k) - F^* = O(e^{-K\sqrt{\frac{\mu}{L}}k})$,
- is not computationnaly expensive,
- is easy to implement.

Restart strategies

Empiric FISTA restart (O'Donoghue and Candès, 2015, Beck and Teboulle, 2009)

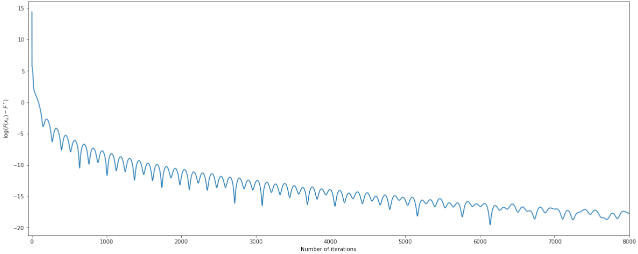
Restart under some exit condition

- on F:

$$F(x_k) > F(x_{k-1}),$$

- on ∇F:

$$\langle \nabla F(x_k), x_k - x_{k-1} \rangle > 0.$$



Restart strategies

Fixed FISTA restart (Necoara et al., 2019)

Restart every k^* iterations where k^* is defined according to the growth parameter μ. If $k^* = \left\lfloor 2e\sqrt{\frac{L}{\mu}} \right\rfloor$:

$$F(x_k) - F^* = O\left(e^{-\frac{1}{e}\sqrt{\frac{\mu}{L}}k}\right).$$

Restart strategies

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Adaptive FISTA restart (Alamo et al., 2019, Fercoq and Qu, 2019)

Restart according to the geometry of F and previous iterations.

- Adaptive restart by Alamo et al.: $F(x_k) - F^* = O\left(e^{-\frac{1}{16}\sqrt{\frac{\mu}{L}}k}\right)$.
- Adaptive restart by Fercoq and Qu: $F(x_k) - F^* = o\left(e^{-\frac{\sqrt{2}-1}{2\sqrt{e}(2-\sqrt{\frac{\mu}{\mu_0}})}\sqrt{\frac{\mu}{L}}k}\right)$.

Restart strategies

Strategy of our scheme:

- to estimate the growth parameter μ at each restart,
- to adapt the number of iterations of the following restart according to this estimation.
- to stop the algorithm when the exit condition $\|\nabla F(r_j)\| \leqslant \varepsilon$ is satisfied.

Algorithm 2 : Automatic FISTA restart

Require: $r_0 \in \mathbb{R}^N, j = 1$

$n_0 = \lfloor 2C \rfloor$

$r_1 = \text{FISTA}(r_0, n_0)$

$n_1 = \lfloor 2C \rfloor$

repeat

$j = j + 1$

$r_j = \text{FISTA}(r_{j-1}, n_{j-1})$

$\tilde{\mu}_j = \min_{\substack{i \in \mathbb{N}^* \\ i<j}} \frac{4L}{(n_{i-1}+1)^2} \frac{F(r_{i-1}) - F(r_j)}{F(r_i) - F(r_j)}$ — Estimation of the parameter μ.

if $n_{j-1} \leqslant C\sqrt{\frac{L}{\tilde{\mu}_j}}$ **then**

$n_j = 2n_{j-1}$ — Update of the number of iterations per restart.

end if

until $\|\nabla F(r_j)\| \leqslant \varepsilon$

Restart strategies

Theorem (Aujol, Dossal, L., Rondepierre, 2021)

If F satisfies the assumptions stated before and $C > 4$, then

$$F(r_j^+) - F^* = O\left(e^{-\frac{\log\left(\frac{C^2}{4}-1\right)}{4C}\sqrt{\frac{\mu}{L}}\sum_{i=0}^{j} n_i}\right).$$

Let $C = 6.38$, then

$$F(r_j^+) - F^* = O\left(e^{-\frac{1}{12}\sqrt{\frac{\mu}{L}}\sum_{i=0}^{j} n_i}\right).$$

Restart strategies

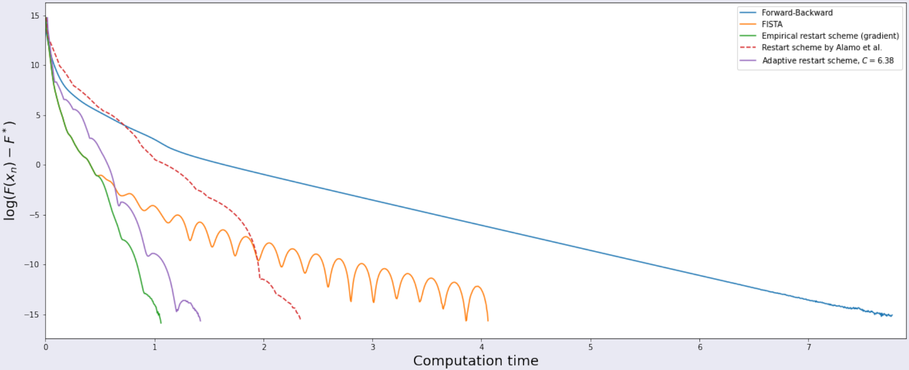
Image inpainting:

$$\min_x F(x) := \frac{1}{2}\|Mx - y\|^2 + \lambda\|Tx\|_1,$$

where M is a mask operator and T is an orthogonal transformation ensuring that Tx^0 is sparse.

Restart strategies

Image inpainting:



Restart strategies

Summary:

Algorithm	Convergence rate
Forward-Backward	$O\left(e^{-\frac{\mu}{L}k}\right)$
Optimal FISTA restart	$O\left(e^{-\frac{1}{e}\sqrt{\frac{\mu}{L}}k}\right)$
Empirical FISTA restart	$O(k^{-2})$
FISTA restart by Fercoq and Qu	$O\left(e^{-\frac{\sqrt{2}-1}{2\sqrt{e}(2-\sqrt{\frac{\mu}{\mu_0}})}\sqrt{\frac{\mu}{L}}k}\right)$
FISTA restart by Alamo et al.	$O\left(e^{-\frac{1}{16}\sqrt{\frac{\mu}{L}}k}\right)$
Automatic FISTA restart	$O\left(e^{-\frac{1}{12}\sqrt{\frac{\mu}{L}}k}\right)$

Restart strategies

Adding backtracking on L (joint work with Luca Calatroni, to be submitted)

This restart strategy can be extended to functions whose Lipschitz constant L **cannot be estimated** or poorly: this involves a variant of FISTA which estimates L using **backtracking**.

$\rightarrow$ The method is fully automatic while ensuring a fast convergence rate.

Plan

Attenuating oscillations introducing Hessian-driven damping

Hessian-driven damping

(DIN-AVD) system (**Attouch, Peypouquet and Redont, 2016**)

$$\ddot{x}(t) + \frac{\alpha}{t}\dot{x}(t) + \beta H_F(x(t))\dot{x}(t) + \nabla F(x(t)) = 0.$$

- Attenuation of the oscillations through the introduction of a geometry-driven damping term.

Attenuating oscillations introducing Hessian-driven damping

Hessian-driven damping

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$$\ddot{x}(t) + \frac{\alpha}{t}\dot{x}(t) + \beta H_F(x(t))\dot{x}(t) + \nabla F(x(t)) = 0.$$

- Attenuation of the oscillations through the introduction of a geometry-driven damping term.

Integrability properties

- **Attouch, Peypouquet and Redont, 2016**: if F is convex and C^2, $\alpha \geqslant 3$ and $\beta > 0$:

$$\int_{t_0}^{+\infty} t^2 \|\nabla F(x(t))\|^2 dt < +\infty,$$

- **Aujol, Dossal, Hoàng, L. and Rondepierre, 2022**: if F is convex and C^2, satisfies $\mathcal{G}_\mu^2$ and has a unique minimizer. Then, for $\alpha \geqslant 3$ and $\beta > 0$:

$$\int_{t_0}^{+\infty} t^{\alpha-\varepsilon} \|\nabla F(x(t))\|^2 dt < +\infty, \forall \varepsilon \in (0,1).$$

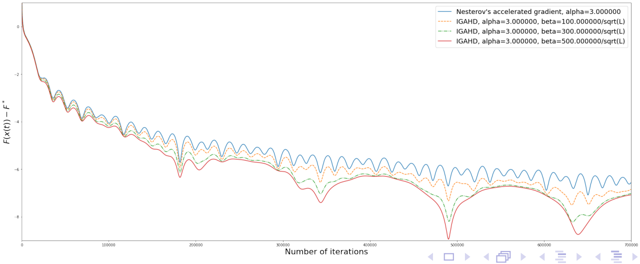
Attenuating oscillations introducing Hessian-driven damping

Derivating a numerical scheme: IGAHD (Attouch, Chbani, Fadili and Riahi, 2020)

$$\ddot{x}(t) + \frac{\alpha}{t}\dot{x}(t) + \beta H_F(x(t))\dot{x}(t) + \left(1 + \frac{\beta}{t}\right)\nabla F(x(t)) = 0.$$

$$\downarrow$$

$$\begin{cases} x_k = y_{k-1} - s\nabla F(y_{k-1}), \\ y_k = x_k + \dfrac{k-1}{k+\alpha-1}(x_k - x_{k-1}) - \beta\sqrt{s}(\nabla F(x_k) - \nabla F(x_{k-1})) - \dfrac{\beta\sqrt{s}}{k}\nabla F(x_{k-1}), \end{cases}$$



Attenuating oscillations introducing Hessian-driven damping

Summary

The Hessian-driven damping term is a **physical way** to attenuate oscillations. As this is a relatively recent subject of research, there are some limitations:

- the behavior of the numerical schemes derivated from (DIN-AVD) is not fully understood (current convergence rates hold if β is **small**),
- the dependency in β is not known,
- there is no proof showing that it is faster than classical inertial schemes.

Plan

Conclusion

Other comments/questions:

- How can high-resolution ODEs (see [2]) improve convergence analysis?
- Is it possible to adapt geometry parameter estimation to Heavy-Ball type methods?
- Can we combine restart with parallel calculations?
- Are inertial methods still fast without uniqueness of the minimizer? (current work)

[2] Shi, B., Du, S.S., Jordan, M.I. et al. Understanding the acceleration phenomenon via high-resolution differential equations. Math. Program. 195, 79–148 (2022). https://doi.org/10.1007/s10107-021-01681-8

Conclusion

Thank you for your attention!

Preprints:

- Jean-François Aujol, Charles Dossal, Hippolyte Labarrière, Aude Rondepierre. FISTA restart using an automatic estimation of the growth parameter. 2021. ⟨hal-03153525v4⟩
- Jean-François Aujol, Charles Dossal, Văn Hào Hoàng, Hippolyte Labarrière, Aude Rondepierre. Fast convergence of inertial dynamics with Hessian-driven damping under geometry assumptions. 2022. ⟨hal-03693218v2⟩

Website:
https://www.math.univ-toulouse.fr/~hlabarri/

I am open to post-doc offers!:)

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