KÄHLER-RICCI FLOWS ON SINGULAR VARIETIES

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Abstract. These are lecture notes from a mini-course given by the authors in Toulouse (France), in june 2011. The goal was to illustrate the smoothing properties of the Kähler-Ricci flow on compact Kähler manifolds and to define the former on mildly singular varieties, following the recent works by Szekelehydi-Tosatti [SzTo11] and Song-Tian [ST09].

Contents

Introduction 2
1. The Kähler-Ricci flow on a singular variety 2
2. Toolbox 6
3. Smoothing property of the Kähler-Ricci flow 9
4. A priori estimates for parabolic Monge-Ampère equations 17
5. Proof of Theorem 1.10 23
References 27

Bibliography 27

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Introduction

The Kähler-Ricci flow $\dot{\omega}_t = -\mathrm{Ric}(\omega_t)$, defined on a compact Kähler manifold X endowed with an initial Kähler form ω_0, has been the object of intensive study over the last decades. In particular, it is known that the flow is defined as long as the cohomology class $[\omega_t] = [\omega_0] + t[K_X]$ stays in the Kähler cone of X. The flow is thus defined on a time interval interval $[0, T[$ with either $T = +\infty$, in which case K_X is nef and X is thus *minimal* by definition, or $T < +\infty$ and $[\omega_0]+T[K_X]$ lies on the boundary of the Kähler cone.

In [ST09], J.Song and G.Tian proposed to use the Minimal Model Program (MMP for short) to continue the flow beyond time T. At least when $[\omega_0]$ is a rational class (and hence X is projective), the MMP allows to find a mildly singular projective variety X' birational to X such that $[\omega_0] + t[K_X]$ induces a Kähler class on X' for $t > T$ sufficiently close to T. It is therefore natural to try and continue the flow on X', but new difficulties arise due to the singularities of X'. After blowing-up X' to resolve these singularities, the problem boils down to showing the existence of a unique solution to a certain degenerate parabolic Monge-Ampère equation, whose initial data is furthermore singular.

The purpose of these notes is to survey Song and Tian's solution to this problem. Along the way, a regularizing property of parabolic Monge-Ampère equations is exhibited, which can in turn be applied to prove the regularity of weak solutions to certain elliptic Monge-ampère equations, following [SzTo11].

Nota Bene. These notes are written after the lecture the authors delivered at the second ANR-MACK meeting (8-10 june 2011, Toulouse, France). As the audience consisted of non specialists, we have tried to make these lecture notes accessible with only few prerequisites.

1. The Kähler-Ricci flow on a singular variety

1.1. **Forms and currents with potentials.** Let X be a complex analytic variety with normal singularities. Since closed $(1,1)$-forms on X are not necessarily locally dd^c-exact, we introduce the following terminology (compare [EGZ09, §5.2]). Let $\mathcal{D}'_X$, $\mathcal{C}^\infty_X$ and $\mathcal{PH}_X = \ker dd^c$ denote respectively the sheaves of germs of distributions, smooth and pluriharmonic functions on X.

Definition 1.1. A $(1,1)$-form (resp. $(1,1)$-current) *with potentials* on X is defined to be a section of the quotient sheaf $\mathcal{C}^\infty_X/\mathcal{PH}_X$ (resp. $\mathcal{D}'_X/\mathcal{PH}_X$). We also set

$$H^{1,1}_{dd^c}(X) := H^1(X, \mathcal{PH}_X).$$

Concretely, a $(1,1)$-form with potentials is thus a closed $(1,1)$-form θ that is locally of the form $\theta = dd^c u$ for some smooth function u. We say that θ is a *Kähler form* if u is strictly psh. Similarly, a $(1,1)$-current with potentials T is locally of the form $dd^c\varphi$ where φ is a distribution. Note that T is positive iff φ is a psh function (see for instance [Dem85] for the basic facts about psh functions on complex varieties).

The space $H^{1,1}_{dd^c}(X)$ is isomorphic to the usual dd^c-cohomology space computed using either $(1,1)$-forms or currents with potentials, since the sheaves $\mathcal{C}^\infty_X$ and $\mathcal{D}'_X$ are both soft, hence acyclic.

Proposition 1.2. *Let $\alpha \in H^{1,1}_{dd^c}(X)$ and let T be a closed positive $(1,1)$-current on $X_{\rm reg}$ representing the dd^c-class $\alpha|_{X_{\rm reg}}$.*

(i) *There exists a unique positive $(1,1)$-current with potentials on X extending T, and its dd^c-cohomology class is α.*

(ii) *If X is compact Kähler and T has locally bounded potential on $X_{\rm reg}$ then $\int_{X_{\rm reg}} T^n$ is finite, bounded above by $\mathrm{vol}(\alpha)$.*

Proof. Let θ be a $(1,1)$-form with potentials on X representing α. We then have $T = \theta|_{X_{\rm reg}} + dd^c\varphi$ for some quasi-psh function φ on $X_{\rm reg}$. If U is a small enough neighborhood of a given point of X then $\theta = dd^c u$ for some smooth function u on U, and $u + \varphi$ is a psh function on $U_{\rm reg}$. By the Riemann extension theorem for psh function [GR56], $u + \varphi$ automatically extends to a psh function on U, and (i) easily follows. (ii) is then a consequence of [BEGZ10]. □

We will also use the following simple fact.

Lemma 1.3. *Let $\mu : X \to X'$ be a birational morphism between compact normal varieties, let $A \subset X$ and $A' \subset X'$ be closed analytic subsets of codimension at least 2, and let u be a psh function on $\mu^{-1}(X' \setminus A') \cap X \setminus A$. Then u is constant.*

Proof. By [GR56] u extends to a psh function on $\mu^{-1}(X'\setminus A')$, hence descends to a psh function u' on $X'\setminus A'$ since μ has connected fibers by Zariski's main theorem. By [GR56] again, u' extends to a psh function on X', hence is constant. □

1.2. **Log terminal singularities.** Recall that a normal variety X is $\mathbb{Q}$-*Gorenstein* if its canonical divisor K_X exists as a $\mathbb{Q}$-line bundle, which means that there exists $r \in \mathbb{N}$ and a line bundle L on X such that $L|_{X_{\rm reg}} = rK_{X_{\rm reg}}$. If X is compact Kähler and $\mathbb{Q}$-Gorenstein, it follows from Proposition 1.2 that any Kähler form ω on $X_{\rm reg}$ with dd^c-cohomology class $c_1(\pm K_{X_{\rm reg}})$ (in particular, $\pm$ the Ricci form of a Kähler metric on $X_{\rm reg}$) automatically has finite volume.

Let X be a $\mathbb{Q}$-Gorenstein variety and choose a log resolution of X, i.e. a projective birational morphism $\pi : X' \to X$ which is an isomorphism over $X_{\rm reg}$ and whose exceptional divisor $E = \sum_i E_i$ has simple normal crossings. There is a unique collection of rational numbers a_i, called the *discrepancies* of X (with respect to the chosen log resolution) such that

$$K_{X'} \sim_{\mathbb{Q}} \pi^* K_X + \sum_i a_i E_i.$$

By definition, X has *log terminal singularities* if $a_i > -1$ for all i. This definition is independent of the choice of a log resolution; this will be a consequence of the following analytic interpretation of log terminal singularities as a *finite volume* condition.

After replacing X with a small open set, we may choose a non-zero section σ of the line bundle rK_X for some $r \in \mathbb{N}^*$. Restricting to $X_{\rm reg}$ we get a smooth positive volume form by setting

$$\mu_\sigma := \left(i^{rn^2}\sigma\wedge\bar\sigma\right)^{1/r} \tag{1.1}$$

Such measures are called *adapted measures* in [EGZ09]. The key fact fact is then:

Lemma 1.4. *Let z_i be a local equation of E_i, defined on a neighborhood $U\subset X'$ of a given point of E. Then*

$$(\pi^*\mu_\sigma)_{U\setminus E} = \prod_i |z_i|^{2a_i} dV$$

for some smooth volume form dV on U.

As a consequence we see that a $\mathbb{Q}$-Gorenstein variety X has log terminal singularities iff every adapted measure μ_σ has locally finite mass near points of X_{sing}. The construction of adapted measures can be globalized as follows: let ϕ be a smooth metric on the $\mathbb{Q}$-line bundle K_X. Then

$$\mu_\phi := \left(\frac{i^{rn^2}\sigma\wedge\bar\sigma}{|\sigma|_{r\phi}}\right)^{1/r} \tag{1.2}$$

becomes independent of the choice of a local non-zero section σ of rK_X, hence defines smooth positive volume form on X_{reg}, which has locally finite mass at infinity (i.e. near points of X_{sing}) iff X is log terminal.

Remark 1.5. In [ST09] the authors define a smooth volume form on X to be a measure of the form μ_ϕ for a smooth metric ϕ on K_X. We prefer to avoid this terminology, which has the drawback that ω^n might not be smooth in this sense even if ω is a (smooth) Kähler form on X.

The following simple result illustrates why log terminal singularities are natural in the context of Kähler geometry.

Proposition 1.6. *Let X be a $\mathbb{Q}$-Gorenstein compact Kähler variety, and assume that there exists a Kähler form ω on X_{reg} with non-negative Ricci curvature and which extends as a $(1,1)$-current with potentials on X. Then X necessarily has log terminal singularities.*

Proof. Let $\phi = \log\omega^n$ be the smooth metric of $K_{X_{\text{reg}}}$ corresponding to the volume form ω^n. The curvature of $-\phi$ is equal to $\text{Ric}(\omega)$, which is non-negative by assumption. It follows that $-\phi$ is psh, hence extends to a psh metric on $-K_X$ by Proposition 1.2. If σ is a local generator of mK_X near a given point of X as above, then $-\log|\sigma|_{m\phi}$ is the corresponding local weight of $m\phi$, and is thus bounded below. This means that $\mu_\sigma \le C\omega^n$ for some $C>0$, which shows that μ_σ has finite mass near the given point of X by Proposition 1.2. □

Remark 1.7. Using the techniques of Remmert-Shiffman and Skoda-El mir, one can show that ω automatically extends to X as a closed positive $(1,1)$-current (that might however not have local potentials near singular points). It is likely that the assumption that this extension has local potentials in Proposition 1.6 is superfluous.

1.3. **Kähler-Ricci flows on singular varieties.** The following results are a mild generalization of [ST09]. The first one deals with the non-normalized Kähler-Ricci flow.

Theorem 1.8. *Let X be a compact Kähler variety with log terminal singularities and let $f : X \to Y$ be a birational morphism with Y compact normal, such that K_X is f-ample. Let $\alpha \in H^{1,1}_{dd^c}(Y)$ be a Kähler class on Y and set*

$$T := \sup\{t > 0 \mid f^*\alpha + t[K_X] \text{ Kähler class on } X\}.$$

Set $\Omega := X_{\mathrm{reg}} \cap f^{-1}(Y_{\mathrm{reg}})$ and let $\omega_0 \in f^*\alpha$ be a positive $(1,1)$-current on X with continuous potentials. Then there exists a unique family of $(1,1)$-currents $\omega_t \in f^*\alpha + t[K_X]$, $t \in]0, T[$, such that

(i) *ω_t has uniformly bounded potentials w.r.t. to $t \in]0, T'[$ for each $T' < T$.*
(ii) *on $\Omega\times]0, T[$ ω_t is smooth and satisfies $\dot\omega_t = -\mathrm{Ric}(\omega_t)$.*
(iii) *the potentials of ω_t converge to those of ω_0 in $\mathcal{C}^0(\Omega)$.*

For the so-called normalized Kähler-Ricci flow, the result implies:

Corollary 1.9. *Let X be a compact variety with log terminal singularities and $\pm K_X$ ample (and hence X projective). Given a positive $(1,1)$-current ω_0 with continuous potentials such that $[\omega_0] = [\pm K_X]$, there exists a unique unique family of $(1,1)$-currents $\omega_t \in [\pm K_X]$, $t \in]0, +\infty[$, such that*

(i) *ω_t has uniformly bounded potentials w.r.t. to $t \in]0, T'[$ for each $T' < +\infty$.*
(ii) *on $X_{\mathrm{reg}}\times]0, +\infty[$ ω_t is smooth and satisfies $\dot\omega_t = -\mathrm{Ric}(\omega_t)$ on $\Omega\times]0, +\infty[$.*
(iii) *the potentials of ω_t converge to those of ω_0 in $\mathcal{C}^0(X_{\mathrm{reg}})$.*

A simple change of variable reduces Corollary 1.9 to a special case of Theorem 1.8. More specifically, setting

$$\widetilde{\omega}_s := (1 \pm s)\omega_{\pm\log(1\pm s)}$$

transforms a solution of $\dot\omega_t = -\mathrm{Ric}(\omega_t) \mp \omega_t$ on $X_{\mathrm{reg}}\times]0, +\infty[$ into a solution of $\dot{\widetilde{\omega}}_s = -\mathrm{Ric}(\widetilde{\omega}_s)$ on $X_{\mathrm{reg}}\times]0, T[$, with $[\omega_s] = [\omega_0] + s[K_X]$ where $T = +\infty$ if K_X is ample and $T = 1$ if $-K_X$ is ample.

1.4. **Reduction to a parabolic Monge-Ampère equation.** As a first step, we will reduce Theorem 1.8 to a degenerate parabolic Monge-Ampère equation on a log resolution of X.

With the notation of Theorem 1.8, let $\theta_0 \in f^*\alpha$ be a smooth representative, so that $\omega_0 = \theta_0 + dd^c\varphi_0$ for some θ_0-psh function $\varphi_0 \in C^0(X)$. Let also ϕ be a smooth metric on K_X, with curvature form χ and associated adapted measure $\mu = \mu_\phi$ as in (1.2). For any Kähler form ω on X_{reg}, it follows from the definitions that

$$-dd^c \log(\omega^n/\mu) = \chi + \mathrm{Ric}(\omega) \tag{1.3}$$

holds on X_{reg}. Setting

$$\theta_t := \theta_0 + t\chi,$$

we are looking for a family of positive currents of the form $\omega_t = \theta_t + dd^c\varphi_t$ where φ_t is smooth on $\Omega\times]0, T[$. Using (1.3), the equation $\dot\omega_t = -\mathrm{Ric}(\omega_t)$ reads

$$dd^c(\dot\varphi_t - \log(\omega_t^n/\mu)) = 0$$

on Ω for each $t \in]0, T[$, which implies that $\dot{\varphi}_t - \log(\omega_t^n/\mu)$ is constant by Lemma 1.3. After writing this constant $c(t)$ as a time derivative to absorb it in φ_t, we end up with the parabolic Monge-Ampère equation on $\Omega \times]0, T[$

$$\dot{\varphi}_t = \log\left(\theta_t + dd^c\varphi_t\right)^n/\mu.$$

with $\theta_t := \theta_0 + t\chi$. Now let as in §1.2 $\pi : X' \to X$ be a log resolution. By Lemma 1.4 $\mu' := \pi^*\mu$ is of the form

$$\mu' := e^{\psi^+ - \psi^-} dV$$

where dV is a smooth volume form on X', $\psi^\pm$ are quasi-psh functions with logarithmic poles along the exceptional divisor E, smooth on $X \setminus E$, and such that $e^{-\psi^-} \in L^p$ for some $p > 1$.

If we set $\theta'_t := \pi^*\theta_t$ and $\varphi'_0 := \pi^*\varphi_0$ then θ'_t is an affine path of closed $(1,1)$-forms on X' with semipositive class, and the ample locus Ω of $[\theta'_0]$ is contained in $X' \setminus E$. After dropping the primes, we are thus reduced to proving the following theorem:

Theorem 1.10. *Let X be a compact Kähler manifold. Assume given the following data:*

- *an affine path $\theta_t = \theta_0 + t\chi$, $t \in [0, T[$, of closed $(1,1)$-forms such that the* cohomology class *of θ_t is semipositive and big for $t \in [0, T[$.*
- *a positive measure μ of the form*

$$\mu = e^{\psi^+ - \psi^-} dV$$

  where $\psi^\pm$ are quasi-psh functions that are smooth on a Zariski open subset Ω of the ample locus of $[\theta_0]$ and such that $e^{-\psi^-} \in L^p$ for some $p > 1$.
- *a function $\varphi_0 \in C^0(X) \cap \mathrm{PSH}(X, \theta_0)$.*

Then there exists a unique family φ_t, $t \in]0, T[$, of functions on X such that:

(i) *φ_t is θ_t-psh and uniformly bounded w.r.t. $t \in]0, T'[$ for each $T' < T$.*
(ii) *on $\Omega \times]0, T[$ φ_t is smooth and satisfies $\dot{\varphi}_t = \log\left(\theta_t + dd^c\varphi_t\right)^n/\mu$.*
(iii) *$\varphi_t \to \varphi_0$ uniformly on compact subsets of Ω as $t \to 0$.*

2. Toolbox

2.1. **The maximum principle.** The following simple maximum principle will the main tool to establish upper an lower bounds.

Proposition 2.1. *Let X be a (not necessarily compact) Kähler manifold, let ω_t, $t \in [0, T]$, be a smooth family of Kähler metrics on X, and denote by $\Delta_t = \mathrm{tr}_{\omega_t} dd^c$ the Laplacian with respect to ω_t. Assume that $H \in C^\infty(X \times [0, T])$ satisfies*

$$\left(\frac{\partial}{\partial t} - \Delta_t\right) H \geq 0$$

or

$$\frac{\partial}{\partial t} H \geq \log\left[\frac{(\omega_t + dd^c H_t)^n}{\omega_t^n}\right],$$

and assume also that $H \to +\infty$ near $\partial X \times [0, T]$ if X is not compact. Then $\inf_X H_t \geq \inf_X H_0$ for all $t \in [0, T]$. If we replace $\geq$ with $\leq$ and assume that $H \to -\infty$ near $\partial X \times [0, T]$ then $\sup_X H_t \leq \sup_X H_0$.

Proof. Upon replacing H with $H \pm \varepsilon t$ with $\varepsilon > 0$, we may assume in each case that the inequality is strict. The properness assumption guarantees that H achieves its infimum (resp. supremum) at some point $(x_0, t_0) \in X \times [0, T]$, and the strict differential inequality implies that t_0 is necessarily 0, since we would have $\frac{\partial}{\partial t} H \le 0$ (resp. ≥ 0) and $dd^c H \ge 0$ (resp. ≤ 0) at (x_0, t_0) otherwise. □

2.2. **A Laplacian inequality.** If θ, ω are $(1,1)$-forms with ω Kähler, recall that the *trace* of θ with respect to ω is defined by

$$\operatorname{tr}_\omega(\theta) := n \frac{\theta \wedge \omega^{n-1}}{\omega^n}.$$

At each point of X one can diagonalize θ with respect to ω, with eigenvalues $\lambda_1 \le ... \le \lambda_n$, and we then have $\operatorname{tr}_\omega(\theta) = \sum_i \lambda_i$. If φ is a function, its Laplacian with respect to ω is given by

$$\Delta_\omega \varphi = \operatorname{tr}_\omega(dd^c \varphi).$$

Proposition 2.2. *Let ω, ω' be two Kähler forms on a Kähler manifold X.*

(i) *We have*

$$\left(\frac{\omega'^n}{\omega^n}\right)^{\frac{1}{n}} \le \tfrac{1}{n} \operatorname{tr}_\omega(\omega') \le \left(\frac{\omega'^n}{\omega^n}\right) (\operatorname{tr}_{\omega'}(\omega))^{n-1}.$$

(ii) *If the holomorphic bisectional curvature of ω is bounded below by $B \in \mathbb{R}$, then*

$$\Delta_{\omega'} \log \operatorname{tr}_\omega(\omega') \ge -\frac{\operatorname{tr}_\omega \operatorname{Ric}(\omega')}{\operatorname{tr}_\omega(\omega')} + B \operatorname{tr}_{\omega'}(\omega).$$

The inequality in (ii) goes back to [Aub78, Yau78]; in the present form it is due to Siu [Siu87, pp. 97–99]. We include a proof for the reader's convenience.

Proof. The left-hand inequality in (i) amounts to the arithmetico-geometric inequality for the eigenvalues of ω' wrt ω; the right-hand inequality follows from similar elementary eigenvalue considerations.

We now prove (ii). Since this is a pointwise inequality, we can choose normal coordinates (z_i) at a given point $0 \in X$ so that $\omega = \sqrt{-1} \sum_{k,l} \omega_{kl} dz_k \wedge d\bar{z}_l$ and $\omega' = \sqrt{-1} \sum_{k,l} \omega'_{kl} dz_k \wedge d\bar{z}_l$ satisfy

$$\omega_{kl} = \delta_{kl} - \sum_{i,j} R_{ijkl} z_i \bar{z}_j + O(\|z\|^3).$$

near 0 and $\omega'_{kl} = \lambda_k \delta_{kl}$ at 0. Here R_{ijkl} denotes the curvature tensor of ω, δ_{kl} stands for the Kronecker symbol, and $\lambda_1 \le ... \le \lambda_n$ are the eigenvalues of ω' with respect to ω at 0.

Observe that the inverse matrix $(\omega^{kl}) = (\omega_{kl})^{-1}$ satisfies

$$\omega^{kl} = \delta_{kl} + \sum_{i,j} R_{ijkl} z_i \bar{z}_j + O(\|z\|^3). \tag{2.1}$$

Recall also that the curvature tensor of ω' is given in local coordinates (z_i) by

$$R'_{ijkl} = -\partial_i \bar{\partial}_j \omega'_{kl} + \sum_{p,q} \omega'_{pq} \partial_i \omega'_{kq} \bar{\partial}_j \omega'_{pl},$$

hence

$$R'_{ijkl} = -\partial_i \overline{\partial}_j \omega'_{kl} + \sum_p \lambda_p^{-1} \partial_i \omega_{kp} \overline{\partial}_j \omega'_{pl} \tag{2.2}$$

at the point 0. Setting $u := \mathrm{tr}_\omega(\omega')$ we have $dd^c \log u = u^{-1} dd^c u - u^{-2} du \wedge d^c u$, hence

$$\Delta_{\omega'} \log u = u^{-1} \Delta_{\omega'} u - u^{-2} \, \mathrm{tr}_{\omega'}(du \wedge d^c u).$$

Now we have at 0

$$\Delta_{\omega'} u = \sum_{ik} \lambda_i^{-1} \partial_i \overline{\partial}_i (\omega^{kk} \omega'_{kk})$$

and

$$\mathrm{tr}_{\omega'}\,(du \wedge d^c u) = \sum_{i,k,l} \lambda_i^{-1} \partial_i \omega'_{kk} \partial_i \omega'_{ll},$$

with

$$\partial_i \overline{\partial}_i (\omega^{kk} \omega'_{kk}) = \lambda_k R_{iikk} + \partial_i \overline{\partial}_i \omega'_{kk}$$

thanks to (2.1). It follows that

$$\Delta_{\omega'} \log u = u^{-1} \left(\sum_{ik} \lambda_i^{-1} \lambda_k R_{iikk} + \sum_{i,k} \lambda_i^{-1} \partial_i \overline{\partial}_i \omega'_{kk} \right) - u^{-2} \left(\sum_{i,k,l} \lambda_i^{-1} \partial_i \omega'_{kk} \partial_i \omega'_{ll} \right). \tag{2.3}$$

On the one hand, the assumption on the holomorphic bisectional curvature of ω reads $R_{iikk} \geq B$ for all i, k, hence

$$\sum_{ik} \lambda_i^{-1} \lambda_k R_{iikk} \geq B(\sum_i \lambda_i^{-1})(\sum_k \lambda_k) = B \, \mathrm{tr}_{\omega'}(\omega) u. \tag{2.4}$$

On the other hand, (2.2) yields

$$\sum_{i,k} \lambda_i^{-1} \partial_i \overline{\partial}_i \omega'_{kk} = -\sum_{i,k} \lambda_i^{-1} R'_{iikk} + \sum_{i,k,p} \lambda_i^{-1} \lambda_p^{-1} |\partial_i \omega'_{kp}|^2.$$

Note that $\sum_{i,k} \lambda_i^{-1} R'_{iikk} = \mathrm{tr}_\omega \mathrm{Ric}(\omega')$, while

$$\sum_{i,k,p} \lambda_i^{-1} \lambda_p^{-1} |\partial_i \omega'_{kp}|^2 \geq \sum_{i,k} \lambda_i^{-1} \lambda_k^{-1} |\partial_i \omega'_{kk}|^2 \geq u^{-1} \sum_{i,k,l} \lambda_i^{-1} \partial_i \omega'_{kk} \partial_i \omega'_{ll}$$

by the Cauchy-Schwarz inequality. Combining this with (2.3) and (2.4) yields the desired inequality. □

2.3. **Existence theorem for parabolic Monge-Ampère equations.** The following result is basically due to Cao [Cao85], Tsuji [Tsu88] and Tian-Zhang [TZha06].

Theorem 2.3. *Let X be a compact Kähler manifold and μ be a smooth positive volume form on X. Let also $(\omega_t)_{t \in [0,T[}$ be a smooth family of Kähler forms. Then every $\varphi_0 \in C^\infty(X)$ such that $\omega_0 + dd^c \varphi_0 > 0$ uniquely extends to a solution $\varphi \in C^\infty(X \times [0,T[)$ of*

$$\frac{\partial}{\partial t} \varphi = \log \left[\frac{(\omega_t + dd^c \varphi_t)^n}{\mu} \right].$$

Uniqueness follows from the maximum principle (Proposition 2.1). We refer to the lecture notes by Song and Weinkove for a proof of existence, which amounts to proving a priori estimates similar to §4 below.

3. Smoothing property of the Kähler-Ricci flow

By analogy with the regularizing properties of the Heat equation, it is expected that the Kähler-Ricci flow can be started from a degenerate initial data (say a positive current, rather than a Kähler form), instantaneously smoothing out the latter.

The goal of this section is to illustrate positively this expectation by explaining the proof of the following result of Szekelyhidi-Tosatti [SzTo11]:

Theorem 3.1. *Let (X,ω) be a n-dimensional compact Kähler manifold. Let $F:\mathbb{R}\times X\to\mathbb{R}$ be a smooth function and assume $\psi_0\in PSH(X,\omega)$ is continuous*[1] *and satisfies*

$$(\omega+dd^c\psi_0)^n=e^{-F(\psi_0,x)}\omega^n.$$

Then $\psi_0\in\mathcal{C}^\infty(X)$ is smooth.

As the reader will realize later on, the proof is a good warm up, as the arguments are similar to the ones we are going to use when proving Theorem 1.10.

Let us recall that such equations contain as a particular case the Kähler-Einstein equation. Namely when the cohomology class $\{\omega\}$ is proportional to the first Chern class of X[2], $\lambda\{\omega\}=c_1(X)$ for some $\lambda\in\mathbb{R}$, then the above equation is equivalent to

$$Ric(\omega+dd^c\psi_0)=\omega+dd^c\psi_0,$$

when taking

$$F(\varphi,x)=\lambda\varphi+h(x)$$

with $h\in\mathcal{C}^\infty(X)$ such that $\mathrm{Ric}(\omega)=\lambda\omega+dd^ch$. Szekelyhidi-Tosatti's result is thus particularly striking since there isn't uniqueness[3] of the solutions to such equations (when one exists).

The interest in such regularity results stems for example from the recent works [BBGZ09, EGZ11] which provide new tools to construct weak solutions to such complex Monge-Ampère equations.

The idea of the proof is both simple and elegant, and goes as follows: assume we can run a complex Monge-Ampère flow

$$\frac{\partial\varphi_t}{\partial t}=\log\left[\frac{(\omega+dd^c\varphi_t)^n}{\omega^n}\right]+F(\varphi_t,x)$$

with an initial data $\varphi_0\in PSH(X,\omega)\cap\mathcal{C}^0(X)$ in such a way that

(1) $(x,t)\mapsto\varphi_t$ is continuous on $X\times[0,T]$,

[1] The authors state their result assuming that ψ_0 is merely bounded, but they use in an essential way the continuity of ψ_0, which is nevertheless known in this context by [Koł98].

[2] This of course assumes that $c_1(X)$ has a definite sign.

[3] In the Kähler-Einstein Fano case, a celebrated result of Bando and Mabuchi [BM87] asserts that any two solutions are connected by the flow of a holomorphic vector field.

(2) $(x,t) \mapsto \varphi_t$ is $\mathcal{C}^\infty$-smooth on $X \times]0,T]$.

Then ψ_0 will be a fixed point of such a flow hence if ψ_t denotes the flow originating from ψ_0, $\psi_0 \equiv \psi_t$ has to be smooth !

To simplify our task, we will actually give full details only in case

$$F(s,x) = -G(s) + h(x) \text{ with } s \mapsto G(s) \text{ being } \textit{convex}$$

and merely briefly indicate what extra work has to be done to further establish the most general result. Note that this particular case nevertheless covers the Kähler-Einstein setting.

In the sequel we consider the above flow starting from a smooth initial potential φ_0 and establish various a priori estimates that eventually will allow us to start from a poorly regular initial data. We fix once and for all a finite time $T > 0$ (independent of φ_0) such that all flows to be considered are well defined on $X \times [0,T]$: it is standard that the maximal interval of time on which such a flow is well defined can be computed in cohomology, hence depends on the cohomology class of the initial data rather than on the (regularity properties of the) chosen representative.

3.1. **A priori estimate on φ_t.** We consider in this section on $X \times [0,T]$ the complex Monge-Ampère flow $(CMAF)$

$$\frac{\partial \varphi_t}{\partial t} = \log \left[\frac{(\omega + dd^c \varphi_t)^n}{\omega^n} \right] + F(\varphi_t, x)$$

with initial data $\varphi_0 \in PSH(X,\omega) \cap \mathcal{C}^\infty(X)$. Our aim is to bound $\|\varphi_t\|_{L^\infty(X \times [0,T])}$ in terms of $\|\varphi_0\|_{L^\infty(X)}$ and T.

3.1.1. *Heuristic control.* Set $M(t) = \sup_X \varphi_t$. It suffices to bound $M(t)$ from above, the bound from below for $m(t) := \inf_X \varphi_t$ will follow by symmetry.

Assume that we can find $t \in [0,T] \mapsto x(t) \in X$ a differentiable map such that $M(t) = \varphi_t(x(t))$. Then M is differentiable and satisfies

$$M'(t) = \frac{\partial \varphi_t}{\partial t}(x(t)) \leq F(\varphi_t(x(t)), x(t)) \leq \overline{F}(M(t)),$$

where

$$\overline{F}(s) := \sup_{x \in X} F(s,x)$$

is a Lipschitz map.

It follows therefore from the Cauchy-Lipschitz theory of ODE's that $M(t)$ is bounded from above on $[0,T]$ in terms of $T, M(0) = \sup_X \varphi_0$ and $\overline{F}$ (hence F).

3.1.2. *A precise bound.* We now would like to establish a more precise control under a simplifying assumption:

Lemma 3.2. *Assume φ_t, ψ_t are smooth families of ω-psh functions such that*

$$\frac{\partial \varphi_t}{\partial t} \leq \log \left[(\omega + dd^c \varphi_t)^n / \omega^n \right] + F(\varphi_t, x)$$

and

$$\frac{\partial \psi_t}{\partial t} \geq \log \left[(\omega + dd^c \psi_t)^n / \omega^n \right] + F(\psi_t, x),$$

where

$$F(s,x)=\lambda s-G(s,x) \text{ with } s\mapsto G(s,\cdot) \text{ non-decreasing.}$$

Then for all $t\in[0,T]$,

$$\sup_X(\varphi_t-\psi_t)\leq e^{\lambda T}\max\{\sup_X(\varphi_0-\psi_0),0\}.$$

Proof. Set $u(x,t):=e^{-\lambda t}(\varphi_t-\psi_t)(x)-\varepsilon t\in\mathcal{C}^0(X\times[0,T])$, where $\varepsilon>0$ is fixed (arbitrary small). Let $(x_0,t_0)\in X\times[0,T]$ be a point at which u is maximal.

If $t_0=0$, then $u(x,t)\leq(\varphi_0-\psi_0)(x_0)\leq\sup_X(\varphi_0-\psi_0)$ and we obtain the desired upper bound by letting $\varepsilon>0$ decrease to zero.

Assume now that $t_0>0$. Then $\dot{u}\geq 0$ at this point, hence

$$0\leq-\varepsilon-\lambda e^{-\lambda t}(\varphi_t-\psi_t)+e^{-\lambda t}(\dot{\varphi}_t-\dot{\psi}_t).$$

On the other hand $dd^c_x u\leq 0$, hence $dd^c_x\varphi_t\leq dd^c_x\psi_t$ and

$$\begin{aligned}\dot{\varphi}_t-\dot{\psi}_t &\leq F(\varphi_t,x)-F(\psi_t,x)+\log\left[\frac{(\omega+dd^c\varphi_t)^n}{(\omega+dd^c\psi_t)^n}\right]\\ &\leq F(\varphi_t,x)-F(\psi_t,x).\end{aligned}$$

Recall now that $F(s,x)=\lambda s-G(s,x)$. Previous inequalities therefore yield

$$G(\varphi_t,x)<G(\psi_t,x) \text{ at point } (x,t)=(x_0,t_0).$$

Since $s\mapsto G(s,\cdot)$ is assumed to be non-decreasing, we infer $\varphi_{t_0}(x_0)\leq\psi_{t_0}(x_0)$, so that for all $(x,t)\in X\times[0,T]$,

$$u(x,t)\leq u(x_0,t_0)\leq 0.$$

Letting ε decrease to zero yields the second possibility for the upper bound. □

By reversing the roles of φ_t,ψ_t, we obtain the following useful:

Corollary 3.3. *Assume* φ_t,ψ_t *are solutions of* $(CMAF)$ *with* F *as above. Then*

$$\|\varphi_t-\psi_t\|_{L^\infty(X\times[0,T])}\leq e^{\lambda T}\|\varphi_0-\psi_0\|_{L^\infty(X)}.$$

As a consequence, if $\varphi_{0,j}$ is a sequence of smooth ω-psh functions decreasing to $\varphi_0\in PSH(X,\omega)\cap\mathcal{C}^0(X)$, and $\varphi_{t,j}$ are the corresponding solutions to $(CMAF)$ on $X\times[0,T]$, then the sequence $(\varphi_{t,j})_j$ uniformly converges towards φ_t on $X\times[0,T]$ as $j\to+\infty$ with $\varphi_t\in\mathcal{C}^0(X\times[0,T])$.

3.2. **A priori estimate on $\dot{\varphi}_t$.** We assume here again that on $X\times[0,T]$

$$\frac{\partial\varphi_t}{\partial t}=\log\left[\frac{(\omega+dd^c\varphi_t)^n}{\omega^n}\right]+F(\varphi_t,x)$$

with initial data $\varphi_0\in PSH(X,\omega)\cap\mathcal{C}^\infty(X)$.

Lemma 3.4. *There exists* $C>0$ *which only depends on* $\|\varphi_0\|_{L^\infty(X)}$ *such that for all* $t\in[0,T]$,

$$\|\dot{\varphi}_t\|_{L^\infty(X)}\leq e^{CT}\|\dot{\varphi}_0\|_{L^\infty(X)}.$$

Let us stress that such a bound requires both that the initial potential φ_0 is uniformly bounded and that the initial density

$$f_0 = \frac{(\omega + dd^c\varphi_0)^n}{\omega^n} = \log \dot{\varphi}_0 - F(\varphi_0, x)$$

is uniformly bounded away from zero and infinity. We shall consider in the sequel more general situations with no a priori control on the initial density f_0.

Proof. Observe that

$$\frac{\partial \dot{\varphi}_t}{\partial t} = \Delta_t \dot{\varphi}_t + \frac{\partial F}{\partial s}(\varphi_t, x)\dot{\varphi}_t,$$

where Δ_t denotes the Laplace operator associated to $\omega_t = \omega + dd^c\varphi_t$.

Since F is $\mathcal{C}^1$-smooth, we can find a constant $C > 0$ which only depends on (F and) $\|\varphi_t\|_{L^\infty(X\times[0,T])}$ such that

$$-C < \frac{\partial F}{\partial s}(\varphi_t, x) < +C.$$

Consider $H_+(x,t) := e^{-Ct}\dot{\varphi}_t(x)$ and let (x_0, t_0) be a point at which H_+ realizes its maximum on $X \times [0,T]$. If $t_0 = 0$, then $\dot{\varphi}_t(x) \leq e^{CT} \sup_X \varphi_0$ for all $(x,t) \in X \times [0,T]$. If $t_0 > 0$, then

$$0 \leq \left(\frac{\partial}{\partial t} - \Delta_t\right)(H_+) = e^{-Ct}\left[\frac{\partial F}{\partial s}(\varphi_t, x) - C\right]\dot{\varphi}$$

hence $\dot{\varphi}_{t_0}(x_0) \leq 0$, since $\frac{\partial F}{\partial s}(\varphi_t, x) - C < 0$. Thus $\dot{\varphi}_t(x) \leq 0$ in this case. All in all, this shows that

$$\dot{\varphi}_t \leq e^{CT} \max\left\{\sup_X \dot{\varphi}_0, 0\right\}.$$

Considering the minimum of $H_-(x,t) := e^{+Ct}\dot{\varphi}_t(x,t)$ yields a similar bound from below and finishes the proof since $\max\{\sup_X \dot{\varphi}_0, -\inf_X \dot{\varphi}_0\} \geq 0$. □

3.3. **A priori estimate on $\Delta\varphi_t$.** Recall that we are considering on $X \times [0,T]$

$$\frac{\partial \varphi_t}{\partial t} = \log\left[\frac{(\omega + dd^c\varphi_t)^n}{\omega^n}\right] + F(\varphi_t, x)$$

with initial data $\varphi_0 \in PSH(X,\omega) \cap \mathcal{C}^\infty(X)$. Our aim in this section is to establish an upper bound on $\Delta_\omega\varphi_t$, which is uniform as long as $t > 0$ and is allowed to blow up when t decreases to zero.

3.3.1. *A convexity assumption.* To simplify our task, we shall assume that

$$F(s,x) = -G(s) + h(x), \text{ with } s \mapsto G(s) \text{ being convex.}$$

This assumption allows us to bound from above $\Delta_\omega F(\varphi, x)$ as follows:

Lemma 3.5. *There exists $C > 0$ which only depends on $\|\varphi_0\|_{L^\infty(X)}$ such that*

$$\Delta_\omega\left(F(\varphi_t, x)\right) \leq C\left[1 + \operatorname{tr}_\omega(\omega_t)\right],$$

where $\omega_t = \omega + dd^c\varphi_t$.

Recall here that for any smooth function h and $(1,1)$-form β,

$$\Delta_\omega h := n\frac{dd^c h\wedge\omega^{n-1}}{\omega^n} \quad \text{while} \quad \operatorname{tr}_\omega\beta := n\frac{\beta\wedge\omega^{n-1}}{\omega^n}.$$

Proof. Observe that

$$dd^c\left(F(\varphi,x)\right) = -G''(\varphi)d\varphi\wedge d^c\varphi - G'(\varphi)dd^c\varphi \leq -G'(\varphi)dd^c\varphi$$

since G is convex. Now $dd^c\varphi = (\omega + dd^c\varphi) - \omega = \omega_\varphi - \omega = \omega_t - \omega$ is a difference of positive forms and $-C \leq -G'(\varphi) \leq +C$, therefore

$$dd^c\left(F(\varphi,x)\right) \leq C\left(\omega_t + \omega\right),$$

which yields the desired upper bound. □

Our simplifying assumption thus yields a bound from above on $\Delta_\omega\left(F(\varphi,x)\right)$ which depends on $\operatorname{tr}_\omega(\omega_\varphi)$ (and $\|\varphi_0\|_{L^\infty(X)}$) but not on $\|\nabla\varphi_t\|_{L^\infty(X\times[\varepsilon,T])}$. A slightly more involved bound from above is available in full generality, which relies on Blocki's gradient estimate [Bło09]. We refer the reader to the proofs of [SzTo11, Lemmata 2.2 and 2.3] for more details.

3.3.2. *The estimate.*

Proposition 3.6. *Assume that $F(s,x) = -G(s) + h(x)$, with $s \mapsto G(s)$ convex. Then*

$$0 \leq \operatorname{tr}_\omega(\omega_t) \leq C\exp\left(C/t\right)$$

where $C > 0$ depends on $\|\varphi_0\|_{L^\infty(X)}$ and $\|\dot\varphi_0\|_{L^\infty(X)}$.

Proof. We set $u(x,t) := \operatorname{tr}_\omega(\omega_t)$ and

$$\alpha(x,t) := t\log u(x,t) - A\varphi_t(x),$$

where $A > 0$ will be specified later. The desired inequality will follow if we can uniformly bound α from above. Our plan is to show that

$$\left(\frac{\partial}{\partial t} - \Delta_t\right)(\alpha) \leq C_1 + (Bt + C_2 - A)\operatorname{tr}_{\omega_t}(\omega)$$

for uniform constants $C_1, C_2 > 0$ which only depend on $\|\varphi_0\|_{L^\infty(X)}$, $\|\dot\varphi_0\|_{L^\infty(X)}$.

Observe that

$$\left(\frac{\partial}{\partial t} - \Delta_t\right)(\alpha) = \log u + \frac{t}{u}\frac{\partial u}{\partial t} - A\dot\varphi_t - t\Delta_t\log u + A\Delta_t\varphi_t.$$

The last term yields $A\Delta_t\varphi_t = An - A\operatorname{tr}_{\omega_t}(\omega)$. The for to last one is estimated thanks to Proposition 2.2,

$$-t\Delta_t\log u \leq Bt\ \operatorname{tr}_{\omega_t}(\omega) + t\frac{\operatorname{tr}_\omega(Ric(\omega_t))}{\operatorname{tr}_\omega(\omega_t)}.$$

It follows from Lemma 3.5 that

$$\begin{aligned}\frac{t}{u}\frac{\partial u}{\partial t} &= \frac{t}{u}\Delta_t\left(\log\frac{\omega_t^n}{\omega^n}\right)+\frac{t}{u}\Delta_\omega F(\varphi_t,x)\\ &= \frac{t}{u}\left\{-\operatorname{tr}_\omega(Ric\,\omega_t)+\operatorname{tr}_\omega(Ric\,\omega)\right\}+\frac{t}{u}\Delta_\omega F(\varphi_t,x)\\ &\leq -t\frac{\operatorname{tr}_\omega(Ric\,\omega_t)}{\operatorname{tr}_\omega(\omega_t)}+C\frac{(1+u)}{u}.\end{aligned}$$

We infer

$$-t\Delta_t\log u+\frac{t}{u}\frac{\partial u}{\partial t}\leq Bt\ \operatorname{tr}_{\omega_t}(\omega)+C_1,$$

using that u is uniformly bounded below as follows from Proposition 2.2 again.

To handle the remaining (first and third) terms, we simply note that $\dot\varphi_t$ is uniformly bounded below, while

$$\log u\leq\log\left[C\operatorname{tr}_{\omega_t}(\omega)^{n-1}\right]\leq C_2+C_3\operatorname{tr}_{\omega_t}(\omega)$$

by Proposition 2.2 and the elementary inequality $\log x<x$. Altogether this yields

$$\left(\frac{\partial}{\partial t}-\Delta_t\right)(\alpha)\leq C_4+(Bt+C_3-A)\operatorname{tr}_{\omega_t}(\omega)\leq C_4,$$

if we choose $A>0$ so large that $Bt+C_3-A<0$. The desired inequality now follows from the maximum principle. □

3.4. **Proof of Theorem 3.1.**

3.4.1. *Higher order estimates.* Using the complex parabolic Evans-Krylov theory together with Schauder's estimates, it follows from our previous estimates that the following higher order a priori estimates hold:

Proposition 3.7. *For each fixed $\varepsilon>0$ and $k\in\mathbb{N}$, there exists $C_k(\varepsilon)>0$ which only further depends on $\|\varphi_0\|_{L^\infty(X)}$ and $\|\dot\varphi_0\|_{L^\infty(X)}$ such that*

$$\|\varphi_t\|_{\mathcal{C}^k(X\times[\varepsilon,T])}\leq C_k(\varepsilon).$$

3.4.2. *A stability estimate.* Let $0\leq f,g\in L^2(\omega^n)$ be densities such that

$$\int_X f\omega^n=\int_X g\omega^n=\int_X\omega^n.$$

It follows from the celebrated work of Kolodziej [Koł98] that there exists unique continuous ω-psh functions φ,ψ such that

$$(\omega+dd^c\varphi)^n=f\omega^n,(\omega+dd^c\psi)^n=g\omega^n\quad\text{and}\quad\int_X(\varphi-\psi)\omega^n=0.$$

We shall need the following stability estimates:

Theorem 3.8. *There exists $C>0$ which only depends on $\|f\|_{L^2},\|g\|_{L^2}$ such that*

$$\|\varphi-\psi\|_{L^\infty(X)}\leq C\|f-g\|_{L^2(X)}^\gamma,$$

for some uniform exponent $\gamma>0$.

Such stability estimates go back to the work of Kolodziej [Koł03] and Blocki [Bło03]. Much finer stability results are available by now (see [DZ10, GZ11]). We sketch a proof of this version for the convenience of the reader.

Proof. The proof decomposes in two main steps. We first claim that

$$\|\varphi-\psi\|_{L^2(X)} \leq C\|f-g\|_{L^2(X)}^{\frac{1}{2^n-1}}, \tag{3.1}$$

for some appropriate $C>0$. Indeed we are going to show that

$$\int_X d(\varphi-\psi)\wedge d^c(\varphi-\psi)\wedge\omega^{n-1} \leq C_1 I(\varphi,\psi)^{2^{-(n-1)}}, \tag{3.2}$$

where

$$I(\varphi,\psi) := \int_X (\varphi-\psi)\left\{(\omega+dd^c\psi)^n-(\omega+dd^c\varphi)^n\right\} \geq 0$$

is non-negative, as the reader can check that an alternative writing is

$$I(\varphi,\psi) = \sum_{j=0}^{n-1}\int_X d(\varphi-\psi)\wedge d^c(\varphi-\psi)\wedge\omega_\varphi^j\wedge\omega_\psi^{n-1-j}.$$

In our case the Cauchy-Schwarz inequality yields

$$I(\varphi,\psi) = \int_X (\varphi-\psi)(g-f)\omega^n \leq \|\varphi-\psi\|_{L^2}\|f-g\|_{L^2},$$

therefore (3.1) is a consequence of (3.2) and Poincaré's inequality.

To prove (3.2), we write $\omega=\omega_\varphi-dd^c\varphi$ and integrate by parts to obtain,

$$\begin{aligned}
&\int d(\varphi-\psi)\wedge d^c(\varphi-\psi)\wedge\omega^{n-1}\\
&\quad=\int d(\varphi-\psi)\wedge d^c(\varphi-\psi)\wedge\omega_\varphi\wedge\omega^{n-2}-\int d(\varphi-\psi)\wedge d^c(\varphi-\psi)\wedge dd^c\varphi\wedge\omega^{n-2}\\
&\quad=\int d(\varphi-\psi)\wedge d^c(\varphi-\psi)\wedge\omega_{\varphi_1}\wedge\omega^{n-2}+\int d(\varphi-\psi)\wedge d^c\varphi\wedge(\omega_\varphi-\omega_\psi)\wedge\omega^{n-2}
\end{aligned}$$

We take care of the last term by using Cauchy-Schwarz inequality, which yields

$$\int d(\varphi-\psi)\wedge d^c\varphi\wedge\omega_\varphi\wedge\omega^{n-2} \leq A\left(\int d(\varphi-\psi)\wedge d^c(\varphi-\psi)\wedge\omega_\varphi\wedge\omega^{n-2}\right)^{1/2},$$

where

$$A^2=\int d\varphi\wedge d^c\varphi\wedge\omega_\varphi\wedge\omega^{n-2}$$

is uniformly bounded from above, since φ is uniformly bounded in terms of $\|f\|_{L^2(X)}$ by the work of Kolodziej [Koł98]. Similarly

$$-\int d(\varphi-\psi)\wedge d^c\varphi\wedge\omega_\psi\wedge\omega^{n-2} \leq B\left(\int d(\varphi-\psi)\wedge d^c(\varphi-\psi)\wedge\omega_\psi\wedge\omega^{n-2}\right)^{1/2},$$

where

$$B^2=\int d\varphi\wedge d^c\varphi\wedge\omega_\psi\wedge\omega^{n-2}$$

is uniformly bounded from above. Note that both terms can be further bounded from above by the same quantity by bounding from above ω_φ (resp. ω_ψ) by $\omega_\varphi + \omega_\psi$.

Going on this way by induction, replacing at each step ω by $\omega_\varphi + \omega_\psi$, we end up with a control from above of $\int d(\varphi-\psi)\wedge d^c(\varphi-\psi)\wedge\omega^{n-1}$ by a quantity that is bounded from above by $CI(\varphi,\psi)^{2^{-(n-1)}}$ (there are $(n-1)$-induction steps), for some uniform constant $C>0$. This finishes the proof of the first step.

The second step consists in showing that

$$\|\varphi-\psi\|_{L^\infty(X)} \leq C_2\|\varphi-\psi\|^\gamma_{L^2(X)}$$

for some constants $C_2, \gamma > 0$. We are not going to dwell on this second step here, as it would take us too far. It relies on the comparison techniques between the volume and the Monge-Ampère capacity, as used in [Koł98]. □

3.4.3. *Conclusion.* We are now in position to conclude the proof of Theorem 3.1 (at least in case $F(s,x) = -G(s) + h(x)$, with G convex). Let $\psi_0 \in PSH(X,\omega)$ be a *continuous* solution to

$$(\omega + dd^c\psi_0)^n = e^{-F(\psi_0,x)}\omega^n.$$

Fix $u_j \in \mathcal{C}^\infty(X)$ arbitrary smooth functions which uniformly converge to ψ_0 and let $\psi_j \in PSH(X,\omega)\cap\mathcal{C}^\infty(X)$ be the unique smooth solutions of

$$(\omega + dd^c\psi_j)^n = c_j e^{-F(u_j,x)}\omega^n,$$

normalized by $\int_X(\psi_j - \psi_0)\omega^n = 0$. Here $c_j \in \mathbb{R}$ are normalizing constants wich converge to 1 as $j \to +\infty$, such that

$$c_j \int_X e^{-F(u_j,x)}\omega^n = \int_X \omega^n,$$

and the existence (and uniqueness) of the ψ_j's is provided by Yau's celebrated result [Yau78]. It follows from the stability estimate (Theorem 3.8) that

$$\|\psi_j - \psi_0\|_{L^\infty(X)} \longrightarrow 0 \text{ as } j \to +\infty,$$

hence

$$\|\psi_j - u_j\|_{L^\infty(X)} \longrightarrow 0 \text{ as } j \to +\infty.$$

Consider the complex Monge-Ampère flows

$$\frac{\partial\varphi_{t,j}}{\partial t} = \log\left[\frac{(\omega + dd^c\varphi_{t,j})^n}{\omega^n}\right] + F(\varphi_{t,j},x) - \log c_j,$$

with initial data $\varphi_{0,j} := \psi_j$. It follows from Lemma 3.2 that

$$\|\varphi_{t,j} - \varphi_{t,k}\|_{L^\infty(X\times[0,T])} \leq e^{\lambda T}\|\psi_j - \psi_k\|_{L^\infty(X)} + |\log c_j - \log c_k|,$$

thus $(\varphi_{t,j})_j$ is a Cauchy sequence in the Banach space $\mathcal{C}^0(X\times[0,T])$. We set

$$\varphi_t := \lim_{j\to+\infty}\varphi_{t,j} \in \mathcal{C}^0(X\times[0,T]).$$

Note that $\varphi_t \in PSH(X,\omega)$ for each $t \in [0,T]$ fixed and $\varphi_0 = \psi_0 = \lim\varphi_{0,j}$ by continuity. Proposition 3.7 shows moreover that $(\varphi_{t,j})_j$ is a Cauchy sequence in the Fréchet space $\mathcal{C}^\infty(X\times]0,T])$, hence $(x,t)\mapsto \varphi_t(x)\in\mathcal{C}^\infty(X\times]0,T])$. Observe that

$$\|\dot\varphi_{0,j}\|_{L^\infty(X)} = \|F(\psi_j,x)-F(u_j,x)\|_{L^\infty(X)} \le C\|\psi_j-u_j\|_{L^\infty(X)} \to 0.$$

Lemma 3.4 therefore yields for all $t>0$,

$$\|\dot\varphi_t\|_{L^\infty(X)} = \lim_{j\to+\infty}\|\dot\varphi_{t,j}\|_{L^\infty(X)} \le C\lim_{j\to+\infty}\|\dot\varphi_{0,j}\|_{L^\infty(X)} = 0.$$

This shows that $t\mapsto\varphi_t$ is constant on $]0,T]$, hence constant on $[0,T]$ by continuity. Therefore $\psi_0\equiv\varphi_t$ is smooth, as claimed.

4. A priori estimates for parabolic Monge-Ampère equations

In this section (X,ω) denotes a compact Kähler manifold endowed with a reference Kähler form with volume form dV. Let $(\omega_t)_{t\in[0,T]}$ be a smooth path of Kähler forms, a smooth positive volume form $\mu=fdV$. Our goal is to provide *a priori* estimates on a solution $\varphi\in C^\infty(X\times[0,T])$ to

$$\frac{\partial}{\partial t}\varphi = \log\left[\frac{(\omega_t+dd^c\varphi_t)^n}{\mu}\right]. \tag{4.1}$$

that only depend on

- the C^0-norm of φ_0;
- a given semipositive and big $(1,1)$-form θ such that $\omega_t\ge\theta$ for $t\in[0,T]$;
- the L^p-norm and certain Hessian bounds for the density f of μ.

4.1. C^0-bound.

Lemma 4.1. *Let θ be a semipositive and big $(1,1)$-form and $C>0$, $p>1$ such that*

(i) $0\le\theta\le\omega_t\le C\omega$ *for* $t\in[0,T]$.
(ii) $C^{-1}\le\int\mu$ *and* $\int f^p dV\le C$.
(iii) $\sup_X|\varphi_0|\le C$.

Then there exists $A>0$ only depending on θ, T, p and C such that

$$\sup_{X\times[0,T]}|\varphi|\le A.$$

Proof. During the proof we shall say that a constant is *under control* if it only depends on the desired quantities.

Step 0: an auxiliary construction. The following construction will also be used in the proof of Lemma 4.3 below. For $\varepsilon\in]0,1]$ we introduce the Kähler form

$$\eta_\varepsilon := (1-\varepsilon)\theta+\varepsilon^2\omega$$

and set $c_\varepsilon := \log\frac{\int\eta_\varepsilon^n}{\int\mu}$. Since ω_t is a continuous family of Kähler forms, we can fix $0<\varepsilon\ll 1$ such that $\omega_t\ge\varepsilon\omega$ for all $t\in[0,T]$. Note that c_ε is under control (even though ε itself is not!). Observe that $\omega_t\ge(1-\varepsilon)\theta+\varepsilon\omega_t$, hence

$$\omega_t\ge\eta_\varepsilon \text{ for } t\in[0,T]. \tag{4.2}$$

By [Yau78] there exists a unique smooth η_ε-psh function ρ_ε such that

$$(\eta_\varepsilon+dd^c\rho_\varepsilon)^n = e^{c_\varepsilon}\mu \tag{4.3}$$

and normalized by $\sup_X \rho_\varepsilon = 0$. The L^p-norm of the right-hand side is under control and $1/2\theta \leq \eta_\varepsilon \leq (C+1)\omega$, so that the uniform version of Kolodziej's L^∞-estimates [EGZ09] shows that $\sup_X |\rho_\varepsilon|$ is under control.

Step 1: Bounding φ_t from above. By non-negativity of the relative entropy of the probability measure $\mu/\int\mu$ with respect to $(\omega_t + dd^c\varphi_t)^n/\int\omega_t^n$ we have

$$\int\left(-\dot\varphi_t + \log\left(\frac{\int\omega_t^n}{\int\mu}\right)\right)\mu \geq 0.$$

It follows that $\frac{d}{dt}\left(\int\varphi_t\mu\right) \leq A_1$ with A_1 under control, hence $\int\varphi_t\mu \leq A_2$ since $\sup_X|\varphi_0|$ is under control. On the other hand there exists $\delta > 0$ and $B_1 > 0$ such that $\int e^{-\delta\psi}\omega^n \leq B_1$ for all normalized θ-psh functions φ, by Skoda's uniform integrability theorem [Zer01]. By Hölder's inequality it follows that $\int e^{-\delta'\varphi}\mu \leq B_2$ where $\delta' := \delta/q$ with q the conjugate exponent of p, and we get a uniform mean value inequality

$$\sup_X \varphi \leq \frac{\int\varphi\mu}{\int\mu} + B_3$$

for all θ-psh functions φ. Applying this to $\varphi = \varphi_t$ yields the desired upper bound on φ_t.

Step 2: Bounding φ_t from below. Consider η_ε and ρ_ε as in Step 0, and set $H_t := \varphi_t - \rho_\varepsilon - c_\varepsilon t$. By (4.3) and (4.2) we get

$$\frac{\partial}{\partial t}H_t = \log\frac{(\omega_t + dd^c\rho_\varepsilon + dd^cH_t)^n}{(\eta_\varepsilon + dd^c\rho_\varepsilon)^n} \geq \log\frac{(\omega_t + dd^c\rho_\varepsilon + dd^cH_t)^n}{(\omega_t + dd^c\rho_\varepsilon)^n}$$

on $X\times[0,T]$, hence $\inf_X H_t \geq \inf_X H_0$ by Proposition 2.1. Since c_ε and $\sup_X|\rho_\varepsilon| \leq M$ are under control, this concludes the proof of Lemma 4.1. □

Remark 4.2. Let us stress, as a pedagogical note to the non expert reader, that this parabolic $\mathcal{C}^0$-estimate thus follows from

- the elementary maximum principle (Proposition 2.1)
- Skoda's uniform integrability theorem
- Kolodziej's uniform elliptic estimate [Koł98, EGZ09]

4.2. **Bounding the time derivative.**

Lemma 4.3. *With the notation and assumptions of Lemma 4.1, assume furthermore that ω_t is an* affine *path, so that $\dot\omega_t = \chi$ is independent of t. Then there exists $A > 0$ only depending on θ, C p and T such that*

$$\sup_X \dot\varphi_t \leq At^{-1} \textit{ for } t \in]0,T].$$

For each $T' < T$ there exists A' only depending on θ, C, p and T' such that

$$\inf_X \dot\varphi_t \geq -A't^{-1} \textit{ for } t \in]0,T'].$$

Proof. We have $\omega_t = \omega_0 + t\chi$. Set $\omega'_t := \omega_t + dd^c\varphi_t$ and let $\Delta'_t = \mathrm{tr}_{\omega'_t} dd^c$ be the Laplacian with respect to ω'_t. We trivially have

$$\Delta'_t\varphi_t = n - \mathrm{tr}_{\omega'_t}(\omega_t), \tag{4.4}$$

while applying $\frac{\partial}{\partial t}$ to $\dot\varphi_t = \log(\omega'^n_t/\mu)$ yields

$$\left(\frac{\partial}{\partial t} - \Delta'_t\right)\dot\varphi_t = \mathrm{tr}_{\omega'_t}\chi. \tag{4.5}$$

Step 1: bounding $\dot\varphi_t$ from above. Set $H_t := t\dot\varphi_t - \varphi_t - nt$. Then

$$\left(\frac{\partial}{\partial t} - \Delta'_t\right)H_t = \mathrm{tr}_{\omega'_t}(t\chi - \omega_t) = \mathrm{tr}_{\omega'_t}(-\omega_0) \le 0$$

on $X \times [0, T]$. Proposition 2.1 yields $\sup_X H_t \le \sup_X H_0$ for $0 \le t \le T$, hence the desired upper bound on $t\dot\varphi_t$ since $\sup_X |\varphi_t|$ is under control by Lemma 4.1.

Step 2: bounding $\dot\varphi_t$ from below. Recall η_ε, ρ_ε from Step 0 of the proof of Lemma 4.1. Consider

$$H_t := t\dot\varphi_t + A\varphi_t - \rho_\varepsilon + Bt$$

where $A, B > 0$ will be specified afterwards. Using (4.4) and (4.5) we get

$$\left(\frac{\partial}{\partial t} - \Delta_t\right)H_t = \mathrm{tr}_{\omega_t}(t\chi + A\omega_t + dd^c\rho_\varepsilon) + (1 + A)\dot\varphi_t - An + B.$$

We now fix $A \gg 1$ under control such that $(A+1)T'/A < T$. We then have for $t \in [0, T']$

$$A\omega_t + t\chi = A\omega_{(A+1)t/A} \ge A\eta_\varepsilon \ge \eta_\varepsilon$$

by (4.2), hence

$$\left(\frac{\partial}{\partial t} - \Delta'_t\right)H_t \ge \mathrm{tr}_{\omega'_t}(\eta_\varepsilon + dd^c\rho_\varepsilon) + (A+1)\dot\varphi_t - An + B$$
$$\ge ne^{nc_\varepsilon}\left(\frac{\mu}{\omega'^n_t}\right)^n + (A+1)\log\left(\frac{\omega'^n_t}{\mu}\right) - An + B$$

using (4.3) and the arithmetico-geometric inequality. Since

$$ne^{nc_\varepsilon}x^{1/n} - (A+1)\log x \ge -C$$

is bounded below by a constant under control for $x \in]0, +\infty[$, we may now choose $B > 0$ under control such that $\left(\frac{\partial}{\partial t} - \Delta_t\right)H_t \ge 0$ on $X \times [0, T']$. Proposition 2.1 therefore yields $\inf_X H_t \ge \inf_X H_0$ for $t \in [0, T']$, which concludes the proof of Lemma 4.3 since $\sup_X |\rho_\varepsilon|$ and $\sup_X |\varphi_t|$ are under control. □

4.3. **Bounding the Laplacian on the ample locus.**

Lemma 4.4. *With the notation and assumptions of Lemma 4.1, assume that* $\dot\omega_t \le C\omega$. *Assume also that the volume form* μ *is written as*

$$\mu = e^{\psi^+ - \psi^-}\omega^n \tag{4.6}$$

where $\psi^\pm \in C^\infty(X)$ *satisfy*

(i) $dd^c\psi^+ \ge -C\,\omega$ *and* $-C \le \sup_X \psi^+ \le C$.

(ii) *$dd^c\psi^- \geq -C\,\omega$, $\sup_X \psi^- \leq C$, and $\|e^{-\psi^-}\|_{L^p} \leq C$ for a given $p > 1$.*

Let also K be a compact subset of the ample locus of the big class $[\theta]$ and $0 < T' < T$. Then there exists $A > 0$ only depending on θ, C, p, T' and K such that

$$\sup_K |\Delta\varphi_t| e^{\psi^-} \leq e^{At^{-1}} \text{ for } t \in]0, T'].$$

Proof. We first observe that the estimate (ii) of Lemma 4.1 in fact follows from (i) and (ii). Indeed the upper bound follows from Hölder's inequality. To get the lower bound, it is enough to show by Jensen's inequality that $\int \psi^+ \omega^n$ is under control, which follows from the mean value inequality for $C\omega$-psh functions.

From now on we work on the ample locus $\Omega \subset X$ of $[\theta]$. We may choose a θ-psh function $\psi_\theta \leq 0$ such that $\psi_\theta \to -\infty$ near $\partial\Omega$ and

$$\widetilde{\omega} := (\theta + dd^c\psi_\theta)|_\Omega$$

extends to a Kähler form on a compactification $\widetilde{X}$ of Ω dominating X. The latter condition implies that there exists $C_1 > 0$ under control such that $\omega \leq C_1\widetilde{\omega}$ and the holomorphic bisectional curvature of $\widetilde{\omega}$ is bounded below by $-C_1$. By Proposition 2.2 we thus have

$$-\Delta'_t \log \operatorname{tr}_{\widetilde{\omega}}(\omega'_t) \leq \frac{\operatorname{tr}_{\widetilde{\omega}} \operatorname{Ric}(\omega'_t)}{\operatorname{tr}_{\widetilde{\omega}}(\omega'_t)} + C_1 \operatorname{tr}_{\omega'_t}(\widetilde{\omega}). \tag{4.7}$$

Now $\omega'^n_t = e^{\dot{\varphi}_t}\mu$ implies $\operatorname{Ric}(\omega'_t) = \operatorname{Ric}(\mu) - dd^c\dot{\varphi}_t$. Combining this with

$$\frac{\partial}{\partial t} \log \operatorname{tr}_{\widetilde{\omega}}(\omega'_t) = \frac{\operatorname{tr}_{\widetilde{\omega}}(\dot{\omega}_t + dd^c\dot{\varphi}_t)}{\operatorname{tr}_{\widetilde{\omega}}(\omega'_t)},$$

we get

$$\left(\frac{\partial}{\partial t} - \Delta'_t\right) \log \operatorname{tr}_{\widetilde{\omega}}(\omega'_t) \leq \frac{\operatorname{tr}_{\widetilde{\omega}}\left(\operatorname{Ric}(\mu) + \dot{\omega}_t\right)}{\operatorname{tr}_{\widetilde{\omega}}(\omega'_t)} + C_1 \operatorname{tr}_{\omega'_t}(\widetilde{\omega}).$$

Now $\operatorname{Ric}(\mu) = -dd^c\psi^+ + dd^c\psi^- + \operatorname{Ric}(\omega) \leq C_2\widetilde{\omega} + dd^c\psi^-$ for some $C_2 > 0$ under control, and $\dot{\omega}_t \leq C\omega$ by assumption, hence

$$\left(\frac{\partial}{\partial t} - \Delta'_t\right) \log \operatorname{tr}_{\widetilde{\omega}}(\omega'_t) \leq \frac{C_3 + \Delta_{\widetilde{\omega}}\psi^-}{\operatorname{tr}_{\widetilde{\omega}}(\omega'_t)} + C_1 \operatorname{tr}_{\omega'_t}(\widetilde{\omega}).$$

In order to absorb ψ^- in the left-hand side, write

$$0 \leq C\omega + dd^c\psi^- \leq CC_1\widetilde{\omega} + dd^c\psi^- \leq \operatorname{tr}_{\omega'_t}(CC_1\widetilde{\omega} + dd^c\psi^-)\omega'_t,$$

which yields

$$0 \leq \frac{nCC_1 + \Delta_{\widetilde{\omega}}\psi^-}{\operatorname{tr}_{\widetilde{\omega}}(\omega'_t)} \leq CC_1 \operatorname{tr}_{\omega'_t}(\widetilde{\omega}) + \Delta_t\psi^-.$$

Using the trivial inequality $\operatorname{tr}_{\widetilde{\omega}}(\omega'_t)\operatorname{tr}_{\omega'_t}(\widetilde{\omega}) \geq n$ we arrive at

$$\left(\frac{\partial}{\partial t} - \Delta'_t\right) \left(\log \operatorname{tr}_{\widetilde{\omega}}(\omega'_t) + \psi^-\right) \leq C_4 \operatorname{tr}_{\omega'_t}(\widetilde{\omega}) \tag{4.8}$$

with $C_4 > 0$ under control.

Now set

$$H_t := t(\log \operatorname{tr}_{\widetilde{\omega}}(\omega'_t) + \psi^-) + A(\psi_\theta - \varphi_t).$$

with $A := 2 + C_4 T$. Since

$$\widetilde{\omega} + dd^c(\varphi_t - \psi_\theta) = \theta + dd^c\varphi_t \le \omega_t + dd^c\varphi_t = \omega'_t$$

we have

$$\Delta'_t(\varphi_t - \psi_\theta) \le n - \mathrm{tr}_{\omega'_t}(\widetilde{\omega}),$$

which combines with (4.8) to yield

$$\left(\frac{\partial}{\partial t} - \Delta'_t\right) H_t \le \log \mathrm{tr}_{\widetilde{\omega}}(\omega'_t) + \psi^- - 2\,\mathrm{tr}_{\omega_t}(\widetilde{\omega}) - A\dot{\varphi}_t + An$$

$$\le \log \mathrm{tr}_{\widetilde{\omega}}(\omega'_t) + \psi^- - 2\,\mathrm{tr}_{\omega_t}(\widetilde{\omega}) + C_5 t^{-1}$$

since $\sup_X |t\dot{\varphi}_t|$ is under control. By (i) of Proposition 2.2 we get

$$\log \mathrm{tr}_{\widetilde{\omega}}(\omega'_t) + \psi^- \le (n-1)\log \mathrm{tr}_{\omega'_t}(\widetilde{\omega}) + C_6 t^{-1} \tag{4.9}$$

using $\psi^+ \le 0$ and the bound on $|t\dot{\varphi}_t|$, and we get

$$\left(\frac{\partial}{\partial t} - \Delta'_t\right) H_t \le -\,\mathrm{tr}_{\omega'_t}(\widetilde{\omega}) + C_7 t^{-1}$$

since $(n-1)\log x - 2x \le -x + O(1)$ for $x \in]0, +\infty[$.

We now follow the proof of the minimum principle; since $\psi_\theta \to -\infty$ near $\partial\Omega$, there exists $(x_0, t_0) \in \Omega \times [0, T]$ such that $H_{t_0}(x_0) = \sup_{(x,t)\in X\times[0,T']} H_t(x)$ for some $t_0 \in]0, T']$. If $t_0 > 0$ then $\left(\frac{\partial}{\partial t} - \Delta_t\right) H_t \ge 0$ at (x_0, t_0), hence $\mathrm{tr}_{\omega_t}(\widetilde{\omega}) \le C_7 t^{-1}$ at (x_0, t_0), and we get

$$t(\log \mathrm{tr}_{\widetilde{\omega}}(\omega'_t) + \psi^-) \le C_8$$

at (x_0, t_0) thanks to (4.9). Since $\psi_\theta \le 0$ and $|\varphi_t|$ is under control, we infer $H_t(x) \le C_9$ at (x_0, t_0), hence for all $(x, t) \in X \times [0, T']$. As a conclusion we obtain $A, B > 0$ under control such that

$$\mathrm{tr}_{\widetilde{\omega}}(\omega'_t) \le B e^{-\psi^- - At^{-1}\psi_\theta}$$

for $t \in [0, T']$, which concludes the proof of Lemma 4.4. □

4.4. **A stability estimate.** We next prove the following Lipschitz continuity property of solutions to (4.1).

Lemma 4.5. *Let ω_t^i, φ_t^i, $i = 1, 2$ satisfy the assumptions of Lemma 4.4 (with the same measure μ and semipositive and big form θ). Then for each $K \Subset \mathrm{Amp}\,(\theta)$ such that $\inf_K \psi^- \ge -C$ there exists $A_K > 0$ under control such that for all $t \in [0, T]$*

$$\sup_K \left|\varphi_t^1 - \varphi_t^2\right| \le A_K \left(\sup_X \left|\varphi_0^1 - \varphi_0^2\right| + \sup_{t\in[0,T]} \|\omega_t^1 - \omega_t^2\|\right)$$

where we have set for each real $(1,1)$-form α

$$\|\alpha\| = \inf\left\{s \ge 0 \mid \pm\alpha \le s\omega\right\}.$$

Proof. We write

$$N := \sup_X |\varphi_0^1 - \varphi_0^2|,\ M = \sup_{t\in[0,T]} \|\omega_t^1 - \omega_t^2\|.$$

If $M = 0$ then $\omega_t^1 = \omega_t^2$ for $t \in [0,T]$, and Proposition 2.1 easily yields the desired inequality with $A_K = 1$. We thus assume that $M > 0$ and set for $\lambda \in [0, M]$

$$\omega_t^\lambda := \left(1 - \tfrac{\lambda}{M}\right)\omega_t^1 + \tfrac{\lambda}{M}\omega_t^2.$$

Since ω_t^λ is a Kähler form for $t \in [0,T]$, Theorem 2.3 yields a unique solution $\varphi^\lambda \in C^\infty(X \times [0,T])$ to the parabolic Monge-Ampère equation

$$\begin{cases} \dfrac{\partial}{\partial t}\varphi^\lambda = \log\left[\dfrac{\left(\omega_t^\lambda + dd^c\varphi_t^\lambda\right)^n}{\mu}\right] \\ \varphi_0^\lambda = \left(1 - \tfrac{\lambda}{M}\right)\varphi_0^1 + \tfrac{\lambda}{M}\varphi_0^2 \end{cases} \tag{4.10}$$

and φ^λ futhermore depends smoothly on λ. We also note that $\sup_X |\varphi_t^\lambda|$ is uniformly under control for $t \in [0,T]$ and $\lambda \in [0,M]$, thanks to Lemma 4.1. Setting $\omega_t'^\lambda := \omega_t^\lambda + dd^c\varphi_t^\lambda$ we have

$$\left(\frac{\partial}{\partial t} - \Delta_t^\lambda\right)\left(\frac{\partial}{\partial\lambda}\varphi_t^\lambda\right) = M^{-1}\operatorname{tr}_{\omega_t'^\lambda}\left(\omega_t' - \omega_t\right) \le \operatorname{tr}_{\omega_t'^\lambda}(\omega) \tag{4.11}$$

where Δ_t^λ denotes the Laplacian with respect to $\omega_t'^\lambda$ and the right-hand inequality follows from the definition of M. Now introduce

$$H_t = e^{-At}\left(\frac{\partial}{\partial\lambda}\varphi_t^\lambda\right) - A^2\varphi_t^\lambda + A^2\psi_\theta + A\psi^-,$$

where $A > 0$ will be specified below. Recalling (4.6) we compute

$$\left(\frac{\partial}{\partial t} - \Delta_t^\lambda\right) H_t = -AH_t - A^3\varphi_t^\lambda + A^3\psi_\theta + A^2\psi^+ - A^2\log\frac{(\omega_t'^\lambda)^n}{\omega^n}$$

$$+e^{-At}\left(\frac{\partial}{\partial t} - \Delta_t^\lambda\right)\left(\frac{\partial}{\partial\lambda}\varphi_t^\lambda\right) + \operatorname{tr}_{\omega_t'^\lambda}\left(-A^2(\omega_t^\lambda + dd^c\psi_\theta) - Add^c\psi^-\right) + A^2 n$$

$$\le -AH_t + A^2 n\log\operatorname{tr}_{\omega_t'^\lambda}(\omega) + \operatorname{tr}_{\omega_t'^\lambda}\left(e^{-At}\omega - A^2(\omega_t^\lambda + dd^c\psi_\theta) - A\,dd^c\psi^-\right) + B_1,$$

using $\psi_\theta, \psi^+ \le 0$, the arithmetico-geometric inequality and the fact that $\sup_X |\varphi_t^\lambda|$ is under control. Using the lower bound $dd^c\psi^- \ge -C\omega$ we get

$$e^{-At}\omega - A^2(\omega_t^\lambda + dd^c\psi_\theta) - Add^c\psi^- \le (AC+1)\omega - A^2(\theta + dd^c\psi_\theta) \le -cA^2\omega$$

for $c > 0$ under control and all A large enough. It follows that

$$\operatorname{tr}_{\omega_t'^\lambda}\left(e^{-At}\omega - A^2(\omega_t^\lambda + dd^c\psi_\theta) - A\,dd^c\psi^-\right) + A^2 n\log\operatorname{tr}_{\omega_t'^\lambda}(\omega)$$

$$\le A^2\left(-c\operatorname{tr}_{\omega_t'^\lambda}(\omega) + n\log\operatorname{tr}_{\omega_t'^\lambda}(\omega)\right) \le A^2 B_2.$$

We conclude that $\left(\frac{\partial}{\partial t} - \Delta_t^\lambda\right) H_t \le -AH_t + B_3$ with $A, B_3 > 0$ under control. Now

$$H_0 = M^{-1}\left(\varphi_0^2 - \varphi_0^1\right) - A^2\varphi_0^\lambda + A^2\psi_\theta + A\psi^-,$$

hence $\sup_X H_0 \le M^{-1}N + B_4$, and the maximum principle yields $\sup_X H_t \le M^{-1}N + B_5$. It follows that

$$\sup_K \frac{\partial}{\partial\lambda}\varphi_t^\lambda \le B_6 M^{-1}N + B_7,$$

which integrates to

$$\sup_K \left(\varphi_t^2 - \varphi_t^1\right) \le B_6 N + B_7 M,$$

and the result follows by symmetry. □

5. Proof of Theorem 1.10

Let (X,ω) be a compact Kähler manifold endowed with a reference Kähler form. We assume given the following data:

- An affine path $\theta_t = \theta_0 + t\chi$, $t \in [0,T[$, of closed $(1,1)$-forms such that the *cohomology class* of θ_t is semipositive and big for $t \in [0,T[$.
- A positive measure μ of the form

$$\mu = e^{\psi^+ - \psi^-} dV$$

  where $\psi^\pm$ are quasi-psh functions that are smooth on a Zariski open subset Ω of the ample locus of $[\theta_0]$ and such that $e^{-\psi^-} \in L^p$ for some $p > 1$.
- A function $\varphi_0 \in C^0(X) \cap \mathrm{PSH}(X,\theta_0)$.

Our goal is to show the existence of a unique family φ_t of functions on X which satisfy the following properties:

(i) φ_t is θ_t-plurisubharmonic and bounded, uniformly with respect to $t \in]0,T'[$ for each $T' < T$.

(ii) on $\Omega\times]0,T[$ φ_t is smooth and satisfies there

$$\frac{\partial}{\partial t}\varphi = \log\left[\frac{(\theta_t + dd^c\varphi_t)^n}{\mu}\right]. \tag{5.1}$$

(iii) $\varphi_t \to \varphi_0$ uniformly on compact subsets of Ω as $t \to 0$.

As a first remark, we may assume that there exists a semipositive and big form θ with $\theta_t \ge \theta$ for $t \in [0,T]$. Indeed, by assumption there exists $u_0, u_T \in C^\infty(X)$ such that $\theta_0 + dd^c u_0$ and $\theta_T + dd^c u_T$ are both semipositive and big. If we set

$$u_t := \left(1 - \tfrac{t}{T}\right) u_0 + \tfrac{t}{T} u_T$$

then each $0 < \varepsilon \ll 1$ we then have

$$\theta_t + dd^c u_t = \left(1 - \tfrac{t}{T}\right)(\theta_0 + dd^c u_0) + \tfrac{t}{T}(\theta_T + dd^c v_T).$$

$$\ge \tfrac{\varepsilon}{T}(\theta_0 + dd^c u_0) =: \theta$$

for $t \in [0, T-\varepsilon]$. The reduction is now achieved by replacing T with $T-\varepsilon$, θ_0 with $\theta_0 + dd^c u_0$, χ with $\chi + T^{-1}dd^c(u_T - u_0)$, and ψ^+ with $\psi^+ + T^{-1}(u_T - u_0)$.

5.1. **Existence.** We regularize the data. By [Dem92], there exist two sequences $\psi_k^\pm \in C^\infty(X)$ such that

- $\psi_k^\pm$ decreases pointwise to $\psi^\pm$, and the convergence is in $C^\infty(\Omega)$;
- $dd^c\psi_k^\pm \geq -C\omega$ for some fixed $C > 0$.

By Richberg's theorem we similarly get a decreasing sequence $\varphi_0^j \in C^\infty(X)$ such that $\delta_j := \sup_X \left|\varphi_0^j - \varphi_0\right| \to 0$ and $\theta_0 + dd^c\varphi_0^j > -\varepsilon_j\omega$ with $\varepsilon_j \to 0$. We then set

- $\theta_0^j := \theta_0 + \varepsilon_j\omega,\ \theta_t^j = \theta_0^j + t\chi$.
- $\mu_{k,l} = e^{\psi_k^+ - \psi_l^-}\omega^n$.

Since θ_t^j is a Kähler form for $t \in [0,T]$ and $\theta_0^j + dd^c\varphi_0^j > 0$, Theorem 2.3 yields a unique solution $\varphi^{j,k,l} \in C^\infty(X \times [0,T])$ to

$$\begin{cases} \dfrac{\partial}{\partial t}\varphi_t^{j,k,l} = \log\left[\dfrac{\left(\theta_t^j + dd^c\varphi^{j,k,l}\right)^n}{\mu_{k,l}}\right] \\ \varphi_0^{j,k,l} = \varphi_0^j \end{cases} \tag{5.2}$$

Lemma 5.1. *The sequence $(\varphi^{j,k,l})_{j,k,l}$ is bounded in the Fréchet space $C^\infty(\Omega\times]0,T[)$, and there exists $C > 0$ such that $\sup_{X\times[0,T]}\left|\varphi^{j,k,l}\right| \leq C$ for all j,k,l.*

Proof. The C^0-bound on $X \times [0,T]$ follows from Lemma 4.1. By Lemma 4.3 and 4.4, for each compact set $L \Subset \Omega\times]0,T[$ there exists a uniform constant $C_L > 0$ such that

$$\sup_L\left|\varphi^{j,k,l}\right| + \sup_L\left|\frac{\partial}{\partial t}\varphi^{j,k,l}\right| + \sup_L\left|\Delta\varphi^{j,k,l}\right| \leq C_L \tag{5.3}$$

for all j,k,l. The boundedness in C^∞-topology on $\Omega\times]0,T[$ follows by the parabolic version of the Evans-Krylov *a priori* estimates and parabolic boot-strapping (see e.g. [Gill11]). □

Lemma 5.2. *For each j fixed the sequence $\varphi^{j,k,l}$ is increasing (resp. decreasing) with respect to k (resp. l). For each $K \Subset \Omega$ there exists $A_K > 0$ such that*

$$\sup_{K\times[0,T]}\left|\varphi^{i,k,l} - \varphi^{j,k,l}\right| \leq A_K\left(\delta_i + \delta_j + \varepsilon_i + \varepsilon_j\right) \tag{5.4}$$

for all i,j,k,l.

Proof. The monotonicity with respect to k and l follows immediately from Proposition 2.1, while the last assertion is a consequence of Lemma 4.5. □

Using Lemma 5.1 and 5.2 we get the existence of

$$\varphi^{j,k} = \lim_{l\to\infty}\varphi^{j,k,l},\ \varphi^j = \lim_{k\to\infty}\varphi^{j,k}$$

in $C^\infty(\Omega\times]0,T[)$ by monotonicity. By Lemma 5.2 the sequence φ_j is Cauchy with respect to the sup-norm, hence the existence of

$$\varphi = \lim_{j\to\infty}\varphi_j$$

in $C^\infty(\Omega\times]0,T[)$ using again Lemma 5.1. By (5.2), φ satisfies

$$\frac{\partial}{\partial t}\varphi = \log\left[\frac{(\theta_t + dd^c\varphi_t)^n}{\mu}\right]$$

on $\Omega\times]0,T[$. Lemma 5.2 also shows that φ is bounded on $\Omega\times[0,T]$ and yields for each $K \Subset \Omega$ a constant $A_K > 0$ such that

$$\sup_{K\times]0,T]}\left|\varphi^{j,k,l} - \varphi\right| \le A_K(\delta_j + \varepsilon_j)$$

for all j,k,l. Since for each j,k,l fixed we have $\lim_{t\to 0}\varphi_t^{j,k,l} = \varphi_0^j$ it follows that $\varphi_t \to \varphi_0$ uniformly on compact subsets of Ω, so that φ satisfies (5.1).

5.2. **Uniqueness.** Let $\varphi' \in C^\infty(\Omega\times]0,T[)$ be another solution to (i), (ii) and (iii) above. Our goal is to prove $\varphi' = \varphi$ by the maximum principle. Fix $\psi \in C^\infty(\Omega)$ such that $\theta + dd^c\psi \ge 0$, $\psi \le 0$ and $\psi \to -\infty$ near $\partial\Omega$. We also fix $0 < c \ll 1$ with $c\theta \le \omega$, so that $\omega + c\, dd^c\psi \ge 0$.

Let us first prove $\varphi \ge \varphi'$. For a given index j set $H_t := \varphi_t^j - \varphi_t' - c\varepsilon_j\psi$. On $\Omega\times]0,T]$ we have

$$\frac{\partial}{\partial t}H = \log\frac{(\theta_t + dd^c\varphi_t' + dd^cH_t + \varepsilon_j(\omega + c\, dd^c\psi))^n}{(\theta_t + dd^c\varphi_t')^n}$$

$$\ge \log\frac{(\theta_t + dd^c\varphi_t' + dd^cH_t)^n}{(\theta_t + dd^c\varphi_t')^n}$$

hence $\inf_\Omega H_s \ge \inf_\Omega H_t$ for $s \ge t > 0$ by Proposition 2.1. Since φ_t^j and φ_t' are bounded on Ω independently of t and $\psi \to -\infty$ at $\partial\Omega$, there exists $K_j \Subset \Omega$ independent of $t \in]0,T[$ such that $\inf_\Omega H_t = \inf_{K_j} H_t$. Using the boundary conditions $\lim_{t\to 0}\varphi_t^j = \varphi_0^j$ and $\lim_{t\to 0}\varphi_t' = \varphi_0$ uniformly on compact sets of Ω, it follows that

$$\lim_{t\to 0}\inf_{K_j} H_t = \inf_{K_j}\left(\varphi_0^j - \varphi_0 - c\varepsilon_j\psi\right) \ge 0$$

since $\varphi_0^j \ge \varphi_0$ and $\psi \le 0$. We have thus shown that $\varphi^j \ge \varphi' + c\varepsilon_j\psi$ on $\Omega\times]0,T[$, hence $\varphi \ge \varphi'$ by letting $j \to \infty$.

In order to prove the converse inequality, we need to introduce yet another parameter in the construction of φ, in order to allow more flexibility. Fix $T' < T$ and choose $0 < \delta_0 \ll 1$ such that $T' \le (1-\delta_0)T$. For $\delta \in [0,\delta_0]$ and $t \in [0,T']$ set $\theta_t^\delta := (1-\delta)\theta_0 + t\chi$ and $\theta_t^{\delta,j} := \theta_t^\delta + \varepsilon_j\omega$, and note that

$$\theta_t^{\delta,j} \ge (1-\delta_0)\theta.$$

Since

$$(1-\delta)\theta_0 + \varepsilon_j\omega + (1-\delta)dd^c\varphi_0^j > 0,$$

Theorem 2.3 yields a unique solution $\varphi^{\delta,j,k,l} \in C^\infty(X \times [0, T'])$ to

$$\begin{cases} \frac{\partial}{\partial t}\varphi_t^{\delta,j,k,l} = \log \left[\frac{\left(\theta_t^{\delta,j} + dd^c \varphi^{\delta,j,k,l}\right)^n}{\mu_{k,l}} \right] \\ \varphi_0^{\delta,j,k,l} = (1-\delta)\varphi_0^j \end{cases} \tag{5.5}$$

Just as in Lemma 5.1 and 5.2, $\varphi^{j,k,l}$ is monotonic with respect to k and l, uniformly bounded on $X \times [0, T']$, the sequence $(\varphi^{j,k,l})_{j,k,l}$ is bounded in $C^\infty(\Omega \times]0, T'])$, and for each $K \Subset \Omega$ we have an estimate

$$\sup_{K\times[0,T']} \left|\varphi^{\delta,i,k,l} - \varphi^{\delta,j,k,l}\right| \le A_K(\varepsilon_i + \varepsilon_j + \delta_i + \delta_j)$$

independent of $\delta \in [0, \delta_0]$, i, j, k and l. We may thus consider

$$\varphi^{\delta,j,k} = \lim_{l\to\infty} \varphi^{\delta,j,k,l},\ \varphi^{\delta,j} = \lim_{k\to\infty} \varphi^{\delta,j,k},\ \varphi^{\delta} = \lim_{j} \varphi^{\delta,j}$$

in $C^\infty(\Omega \times]0, T[)$.

Since

$$\sup_X \left|\varphi_0^{j,k,l} - \varphi_0^{\delta,j,k,l}\right| = \delta \sup_X \left|\varphi_0^j\right|$$

and $\|\theta_0^j - \theta_0^{\delta,j}\| = \delta\|\theta\|$ are uniformly bounded, Lemma 4.5 shows that

$$\sup_{K\times[0,T']} \left|\varphi^{\delta,j,k,l} - \varphi^{j,k,l}\right| \le C_K \delta$$

for each $K \Subset \Omega$, with $C_K > 0$ independent of δ, j, k, l, and hence

$$\sup_K \left|\varphi^\delta - \varphi\right| \le C_K\delta \tag{5.6}$$

for all $\delta \in [0, \delta_0]$.

Now we introduce for a given $\delta \in]0, \delta_0]$ $H_t := \varphi_t' - \varphi_t^\delta - \delta\psi \in C^\infty(\Omega \times]0, T'[)$. We have

$$\frac{\partial}{\partial t} H = \log \frac{\left(\theta_t^\delta + dd^c\varphi_t^\delta + \delta(\theta_0 + dd^c\psi) + dd^c H_t\right)^n}{\left(\theta_t^\delta + dd^c\varphi_t^\delta\right)^n}$$

$$\ge \log \frac{\left(\theta_t^\delta + dd^c\varphi_t^\delta + dd^c H_t\right)^n}{\left(\theta_t^\delta + dd^c\varphi_t^\delta\right)^n}$$

hence $\inf_\Omega H_s \ge \inf_\Omega H_t$ for $s \ge t > 0$ by Proposition 2.1. Since φ^δ and φ are bounded, there exists $K_\delta \Subset \Omega$ such that $\inf_\Omega H_t = \inf_{K_\delta}(\varphi_t' - \varphi_t^\delta - \delta\psi)$, hence $\lim_{t\to 0} \inf_\Omega H_t = \inf_{K_\delta}(-\delta\varphi_0 - \delta\psi) \ge -\delta \sup_X |\varphi_0|$. We have thus shown that $\varphi' \ge \varphi^\delta + \delta\psi - \delta \sup_X |\varphi_0|$ on $\Omega \times]0, T']$, and we obtain $\varphi' \ge \varphi$ on Ω by letting $\delta \to 0$ thanks to (5.6).

Remark 5.3. Since $\dot\varphi_t$ is uniformly bounded for t in a compact set of $]0, T[$, [EGZ11] implies that $\varphi_t \in C^0(X)$ for each $t \in]0, T[$ and $\sup_X |\varphi_t - \varphi_s| \le C|t - s|$ for t, s in a compact set of $]0, T[$. It follows that $\varphi \in C^0(X \times]0, T[)$. Is it continuous on the whole $X \times [0, T[$?

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