DYNAMICS OF QUADRATIC POLYNOMIAL MAPPINGS OF $\mathbb{C}^2$

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ABSTRACT. We classify up to conjugacy (by linear affine automorphisms) quadratic polynomial endomorphisms of $\mathbb{C}^2$ and test on them two natural questions. Do they admit an algebraically stable compactification ? Do they admit a unique measure of maximal entropy ?

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INTRODUCTION

A remarkable feature of one-dimensional complex dynamics is the proeminent role played by the "quadratic family" $P_c(z) = z^2 + c$. The latter has revealed an exciting source of study and inspiration for the study of general rational mappings $f : \mathbb{P}^1 \to \mathbb{P}^1$ as well as for more general dynamical systems [Ly 00]. Our purpose here is to introduce several quadratic families of polynomial self-mappings of $\mathbb{C}^2$ which may hopefully be the complex two-dimensional counterpart to the celebrated quadratic family.

We partially classify quadratic polynomial endomorphisms of $\mathbb{C}^2$ (*section 2*) using some numerical invariants (dynamical degrees $\lambda_1(f), d_t(f)$, dynamical Lojasiewicz exponent $DL_\infty(f)$) which we define in *section 1.* We then use this classification to test two related questions.

Question 1. *Does there exist a unique invariant probability measure of maximal entropy ?*

Question 2. *Does there exist an algebraically stable compactification ?*

Simple examples show that there may be infinitely many invariant probability measures of maximal entropy when $d_t(f) = \lambda_1(f)$. When $d_t(f) >$

$\lambda_1(f)$, it is proved in [G 02b] that the Russakovskii-Shiffman measure μ_f is the unique measure of maximal entropy. We push further the study of μ_f, when f is quadratic, by showing that it is compactly supported in $\mathbb{C}^2$ (*section 4*). Moreover every plurisubharmonic function is in $L^1(\mu_f)$ (*section 5*) and the "exceptional set" is algebraic (*section 6*).

When $d_t(f) < \lambda_1(f)$, one also expects the existence of a unique measure of maximal entropy (this is the case when f is a complex Hénon mapping [BLS 93a]). If f is algebraically stable on some smooth compactification of $\mathbb{C}^2$, one can then construct invariant currents T_+, T_- such that $f^*T_+ = \lambda_1(f)T_+$ and $f_*T_- = \lambda_1(f)T_-$ (see [G 02]). It is usually difficult to define the invariant measure $\mu_f = T_+ \wedge T_-$. This can be done however when f is polynomial in $\mathbb{C}^2$, since T_+ admits continuous potentials off a finite set of points. We briefly discuss Question 1 for quadratic mappings with $d_t(f) < \lambda_1(f)$ in *section 3*: the answer is positive for an open set of parameters, but unknown in general.

1. Numerical invariants

1.1. **Algebraic stability.** Let $f : \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial mapping. We always assume f is *dominating*, i.e. the jacobian Jf of f does not vanish identically. Let us denote by $d_t(f)$ the topological degree of f (i.e. the number of preimages of a generic point) and by $\delta_1(f)$ its algebraic degree (i.e. the degree of the preimage of a generic line in $\mathbb{C}^2$). If $f = (P, Q)$ in coordinates, then $\delta_1(f) = \max(\deg P, \deg Q)$. Clearly d_t behaves well both under iteration ($d_t(f^j) = [d_t(f)]^j$) and under conjugacy ($d_t(f) = d_t(\Phi^{-1} \circ f \circ \Phi)$). Concerning δ_1 we also have a straightforward inequality

$$\delta_1(f \circ g) \leq \delta_1(f) \cdot \delta_1(g), \quad (*)$$

however equality fails in general. Nevertheless $(*)$ shows the sequence $(\delta_1(f^j))$ is submultiplicative, therefore we can define

$$\lambda_{\mathbf{1}}(\mathbf{f}) := \lim \left[\delta_1(f^j)\right]^{1/j} =: \text{first dynamical degree of } f.$$

It follows again from $(*)$ that $\lambda_1(f)$ is invariant under conjugacy.

In order to compute $\lambda_1(f)$, one needs to compute $\delta_1(f^j)$ for all $j \geq 1$. Although this can be achieved "by hand" in some simple situations, there is a subtler way of computing $\lambda_1(f)$ which moreover yields interesting information about the dynamics. Let $X = \mathbb{C}^2 \cup Y_\infty$ be a smooth compactification of $\mathbb{C}^2$, where Y_∞ denotes the divisor at infinity. We still denote by f the meromorphic extension of f to X and let $I_f \subset Y_\infty$ be the indeterminacy set of f, i.e. the finite number of points at which f is not holomorphic.

Definition

1.1. *One says f is* **algebraically stable** *in X if for every curve C of X and every $j \geq 1$, $f^j(C \setminus I_{f^j}) \notin I_f$, where I_{f^j} denotes the indeterminacy set of f^j.*

It is known [PS 91] that every smooth compactification of $\mathbb{C}^2$ is a projective algebraic surface $X = \mathbb{C}^2 \cup Y_\infty$, where the divisor at infinity $Y_\infty = C_1 \cup \cdots \cup C_s$ consists of a finite number of rational curves $C_1, \ldots, C_s$. Since we are dealing with polynomial mappings, $f(\mathbb{C}^2) \subset \mathbb{C}^2$ and the indeterminacy set I_f is located inside Y_∞. Therefore the only curves that can be contracted to a point of indeterminacy are the C_j's. So the condition of algebraic stability is quite easy to check here.

Question

1.2. *Let $f : \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial dominating mapping. Can one always* *find a smooth compactification of* $\mathbb{C}^2$ *on which* f *becomes algebraically stable ?*

We will see below that the answer is positive when $\delta_1(f) = 2$. The answer is negative in general for rational mappings [F 02]. The point is that if $f : X \to X$ is algebraically stable in X, then $\lambda_1(f)$ equals the spectral radius of the linear action induced by pull-back by f on the cohomology vector space $H^{1,1}(X,\mathbb{R})$. Moreover, this is is the starting point for the construction of invariant currents (see [G 02]).

1.2. **Dynamical Lojasiewicz exponent.** A general principle is that the behaviour of f at infinity governs its dynamics at bounded distance. Recall that the Lojasiewicz exponent $L_\infty(f)$ of f at infinity is defined by

$$L_\infty(f) = \sup\{\nu \in \mathbb{R} \,/\, \exists C, R > 0, ||m|| \geq R \Rightarrow ||f(m)|| \geq C||m||^\nu\}.$$

It is known that $L_\infty(f)$ is always a rational number (possibly $-\infty$) which is positive iff f is proper. Moreover there are explicit formulae which yield $L_\infty(f)$ by simple computation [CK 92].

Lemma

1.3. *Let* $f, g : \mathbb{C}^2 \to \mathbb{C}^2$ *be polynomial dominating mappings. Then the following holds:*

i) $L_\infty(f) \leq \delta_1(f)$ *with equality iff* f *extends holomorphically to* $\mathbb{P}^2$.

ii) $L_\infty(f \circ g) \leq \delta_1(f) \cdot L_\infty(g)$ *if* g *is proper.*

iii) $L_\infty(f) \cdot L_\infty(g) \leq L_\infty(f \circ g)$ *if* g *is proper.*

Proof. Let us denote by ω the Fubini-Study Kähler form on $\mathbb{P}^2$.

i) Set $d = \delta_1(f)$. Then $f = (P, Q)$, where P, Q are polynomials such that $d = \max(\deg P, \deg Q)$, so there exists $C_1 > 0$ such that

$$||m|| \geq 1 \Rightarrow ||f(m)|| \leq C_1||m||^d.$$

This yields $L_\infty(f) \leq d$.

Assume $||f(m)|| \geq C||m||^d$ for $||m|| \geq R$. It then follows from Taylor's lemma (see lemma 1.5 below) that

$$d_t(f) = \int_{\mathbb{C}^2} f^*(\omega^2) \geq d \int_{\mathbb{C}^2} f^*\omega \wedge \omega = d^2 = \int_{\mathbb{P}^2} (\tilde{f})^*(\omega^2).$$

Therefore the meromorphic extension $\tilde{f}$ of f to $\mathbb{P}^2$ has no point of indeterminacy, i.e. f extends holomorphically to $\mathbb{P}^2$. Conversely, if $\tilde{f}$ is holomorphic on $\mathbb{P}^2$, then f has non degenerate homogeneous components of degree d hence $L_\infty(f) = d$.

ii) Set again $d = \delta_1(f)$. Then there exists $C_1 > 0$ such that

$$||g(m)|| \geq 1 \Rightarrow ||f \circ g(m)|| \leq C_1 ||g(m)||^d.$$

When g is proper this reads, for every $R > 1$ large enough,

$$||m|| \geq R \Rightarrow ||f \circ g(m)|| \leq C_1 ||g(m)||^d.$$

The desired inequality follows.

iii) Assume $||f(m)|| \geq C_1 ||m||^\nu$ for $||m|| \geq R_1$. When g is proper we infer $||f \circ g(m)|| \geq C_2 ||g(m)||^\nu$ for $||m|| \geq R_2$. This yields $L_\infty(f \circ g) \geq \nu L_\infty(g)$, hence $L_\infty(f \circ g) \geq L_\infty(f) \cdot L_\infty(g)$. □

If f is not proper, then (ii) and (iii) of Lemma 1.1 are false, as simple examples show. The lemma shows the sequence $(L_\infty(f^j))_{j \in \mathbb{N}}$ is supmultiplicative when f is proper.

Definition

1.4. *Let $f : \mathbb{C}^2 \to \mathbb{C}^2$ be a proper polynomial mapping. The dynamical Lojasiewicz exponent of f at infinity is*

$$\mathbf{DL}_\infty(\mathbf{f}) := \lim \left[L_\infty(f^j)\right]^{1/j}.$$

Let us recall the following useful lemma [T 83].

Lemma

1.5. *Let S be a positive closed current of bidegree $(1,1)$ on $\mathbb{P}^2$. Let u be a locally bounded plurisubharmonic function in $\mathbb{C}^2$. Assume $u(m) \geq \nu \log^+ ||m|| + C$ on the support of S, for some $C, \nu > 0$. Then*

$$\int_{\mathbb{C}^2} S \wedge dd^c u \geq \nu \int_{\mathbb{C}^2} S \wedge \omega.$$

The proof is an integration by part argument (see proposition 4.3 in [G 02]).

Proposition

1.6. *Let $f : \mathbb{C}^2 \to \mathbb{C}^2$ be a proper polynomial mapping. Then the following holds:*

i) $L_\infty(f)$ is invariant under affine conjugacy, $DL_\infty(f)$ is invariant under polynomial conjugacy.

ii) $0 < L_\infty(f) \leq DL_\infty(f) \leq \lambda_1(f) \leq \delta_1(f)$.

iii) $L_\infty(f) \leq d_t(f)/\delta_1(f)$ so $DL_\infty(f) \leq d_t(f)/\lambda_1(f)$.

Remark

1.7. *All these inequalities are strict in general. Note that when $DL_\infty(f) > 1$ then infinity is an "attracting" set for f: there exists a neighborhood V of infinity in $\mathbb{C}^2$ and $l \geq 1$ such that $\overline{f^l V} \subset V$ and $\cap_{j \geq 0} f^j(V) = \emptyset$. Therefore every point $a \in \mathcal{B}^+(\infty) := \cup_{n \geq 0} f^{-n}(V)$ escapes to infinity in forward time, so the non wandering set of f is included in the compact set $K^+ := \{p \in \mathbb{C}^2 \,/\, (f^n(p))_{n \geq 0}$ is bounded$\} = \mathbb{C}^2 \setminus \mathcal{B}^+(\infty)$.*

Proof. Everything follows immediately from lemma 1.3 except iii). Assume $||f(m)|| \geq C||m||^\nu$ for $||m|| \geq R$, where $C, \nu, R > 0$. It follows from two applications of Lemma 1.5 that

$$d_t(f) = \int_{\mathbb{C}^2} f^*\omega \wedge f^*\omega \geq \nu \int_{\mathbb{C}^2} f^*\omega \wedge \omega = \nu \delta_1(f).$$

Therefore $d_t(f) \geq \delta_1(f) L_\infty(f)$ and by iteration $d_t(f) \geq \lambda_1(f) DL_\infty(f)$. $\square$

Examples

1.8.

1) Consider $f(z,w) = (P(w), Q(z)+R(w))$*, where* P, Q, R *are polynomials of degree* p, q, d *respectively, with* $d > \max(p,q)$*. We get* $\lambda_1(f) = \delta_1(f) = d$ *and* $d_t(f) = pq$*,* $L_\infty(f) = DL_\infty(f) = pq/d = d_t(f)/\lambda_1(f)$*.*

2) Consider $f(z,w) = (w, z^2 + aw + c)$*, where* $(a,b) \in \mathbb{C}^2$*. Observe that the second iterate* f^2 *extends holomorphically to* $\mathbb{P}^2$ *so we get*

$$d_t(f) = 2,\ \lambda_1(f) = \sqrt{2} < 2 = \delta_1(f),\ L_\infty(f) = 1 < \sqrt{2} = DL_\infty(f).$$

See [GN 01] for a detailed study of this mapping.

3) Let f *be a polynomial automorphism of* $\mathbb{C}^2$*. It is known that* f *is conjugate to either an elementary automorphism or a composition of complex Hénon mappings [FM 89]. In the elementary case we get* $d_t(f) = \lambda_1(f) = 1$ *and* $DL_\infty(f) = L_\infty(f) = 1/d$*. In the Hénon case we get* $d_t(f) = 1$*,* $\lambda_1(f) = d$*, so* $DL_\infty(f) \leq 1/d$*. On the other hand* $L_\infty(f) \geq 1/d$ *as follows from lemma 1.3.ii applied to* f *and* f^{-1}*. Therefore* $L_\infty(f) = DL_\infty(f) = 1/d$*.*

2. Classification of quadratic polynomial mappings of $\mathbb{C}^2$

In this section we classify up to conjugacy the quadratic dominating polynomial self mappings of $\mathbb{C}^2$, according to their dynamical degrees. For our purposes, the precise nature of the normal form is not important: the essential point will be to determine their numerical invariants and behavior at infinity. This section is devoted to the proof of the following result.

Theorem

2.1. *Let* $f : \mathbb{C}^2 \to \mathbb{C}^2$ *be a dominating polynomial mapping with* $\delta_1(f) = 2$*.*

Then f is conjugate, by a linear affine automorphism of $\mathbb{C}^2$, to one of the following families.

1) $\underline{d_t(f) < \lambda_1(f)}$:

1.1)
$$f(z,w) = (w+c, zw+c'),$$
where $c, c' \in \mathbb{C}$. We get here $d_t(f) = 1, \lambda_1(f) = (1+\sqrt{5})/2$.

1.2)
$$f(z,w) = (w+c, w[w-az]+bz+c'),$$
where $a,b,c,c' \in \mathbb{C}$ with $(a,b) \neq (0,0)$. We get here $d_t(f) = 1, \lambda_1(f) = 2$.

2) $\underline{d_t(f) = \lambda_1(f)}$:

2.1) $\underline{d_t(f) = \lambda_1(f) = 1}$:

a) $f(z,w) = (az+c, z^2+bw+c')$, where $a,b,c,c' \in \mathbb{C}$ with $ab \neq 0$.

b) $f(z,w) = (az+c, zw+c')$, where $a,c,c' \in \mathbb{C}$ with $a \neq 0$.

2.2) $\underline{d_t(f) = \lambda_1(f) = 2}$:

a) $f(z,w) = (P(z), Q(z,w))$, where $\deg P = \deg Q = 2$ and $\deg_w Q = 1$.

b) $f(z,w) = (P(z), Q(z,w))$, where $\deg P = 1$ and $\deg_w Q = 2 = \deg Q$.

c) $f(z,w) = (w, Q(z,w))$, where $\deg_z Q = \deg_w Q = \deg Q = 2$.

d) $f(z,w) = (zw+c, z[z+aw]+bz+c')$, where $a,b,c,c' \in \mathbb{C}$.

3) $\underline{d_t(f) > \lambda_1(f)}$:

3.1)
$$f(z,w) = (w, z^2+aw+c),$$
where $a,c \in \mathbb{C}$. We get here $d_t(f) = 2, \lambda_1(f) = \sqrt{2} = DL_\infty(f)$.

3.2)
$$f(z,w) = (aw+c, z[z-w]+c'),$$
where $a,c,c' \in \mathbb{C}$, $a \neq 0$. Here $d_t(f) = 2, \lambda_1(f) = (1+\sqrt{5})/2, DL_\infty(f) = 1$.

3.3)
$$f(z,w) = (az^2+bz+c+w, z[w+\alpha z]+c'),$$
where $a,b,c,c',\alpha \in \mathbb{C}$, $a \neq 0$. Here $d_t(f) = 3, \lambda_1(f) = 2, DL_\infty(f) > 1$.

3.4)
$$f(z,w) = (zw+c, z[z+\alpha w]+bz+c'+aw),$$
where $a,b,c,c',\alpha \in \mathbb{C}$ with $a \neq 0$. Here $d_t(f) = 3, \lambda_1(f) = 2, DL_\infty(f) \overset{?}{=} 1$.

3.5) $$f(z,w) = (P(z,w) + L_1(z,w), Q(z,w) + L_2(z,w)),$$

where P, Q *are homogeneous polynomials of degree* 2 *with* $P \wedge Q = 1$ *and* L_1, L_2 *are polynomials of degree* ≤ 1*. Here* $d_t(f) = 4, \lambda_1(f) = 2 = DL_\infty(f)$*.*

Remark

2.2. *We will focus on quadratic mappings with* $d_t(f) \neq \lambda_1(f)$ *in the next sections. For the remaining six families, observe that families 2.1a,2.1b,2.2a,2.2b are skew-products whose dynamics is rather one-dimensional. The remaining two families 2.2c,2.2d may display more intricate dynamical behaviour. A special case of 2.2.d arises in the study of density of states of self-similar diffusion on the interval [0,1] [Sa 01].*

Proof of theorem 2.1 This is a case-by-case analysis.

We first decompose $f(z,w) = (P(z,w) + L_1(z,w); Q(z,w) + L_2(z,w))$, where P, Q are homogeneous polynomials of degree 2 and L_1, L_2 are polynomials of degree ≤ 1. When $P \wedge Q = 1$, f extends holomorphically to $\mathbb{P}^2$ and we obtain the family 3.5. So we only need to consider the cases $P \equiv 0$ or $P = A\tilde{P}$ and $Q = A\tilde{Q}$ with $A, \tilde{P}, \tilde{Q}$ homogeneous of degree 1 and $\tilde{P} \wedge \tilde{Q} = 1$. Indeed the remaining cases $Q \equiv 0$ and $P = \lambda Q$ are both conjugate to the case $P \equiv 0$ respectively by $(z,w) \mapsto (w,z)$ and $(z,w) \mapsto (z + \lambda w, w)$.

1) Case $P \equiv 0$:

We get $f(z,w) = (\alpha z + \beta w + c, Q(z,w) + L_2(z,w))$. When $\beta = 0$, f is a skew-product and a further case by case analysis yields the families 2.1a,2.1b and 2.2b. So let us assume $\beta \neq 0$. Conjugating by $(z,w) \mapsto (z, w/\beta)$ yields $\beta = 1$. Conjugating further by $(z,w) \mapsto (z, w - \alpha z - c)$ yields $\alpha = c = 0$, hence $f(z,w) = (w, Q(z,w) + L_2(z,w))$.

- Subcase $\deg_z Q = 0$: since f is dominating, we get $\deg_z L_2 = 1$ and $Q = Q(w)$ is a degree 2 polynomial. In this case f is a quadratic Hénon mapping, i.e. a mapping in the family 1.2 with $a = 0$ (see [FM 89] for a precise normal form).

• Subcase $\deg_z Q = 1$: then $d_t(f) = 1$, i.e. f is a birational mapping. However its inverse is not polynomial in $\mathbb{C}^2$.

When $\deg_w Q = 2$ we obtain, conjugating by $(z, w) \mapsto (\ z, \lambda w)$,

$$f(z, w) = (w, w[w - az] + bz + b'w + c'), \text{ with } a \neq 0.$$

Further conjugacy by a translation yields the remaining cases of the family 1.2. Observe that f is then algebraically stable in $\mathbb{P}^2$ so $\lambda_1(f) = \delta_1(f) = 2$.

When $\deg_w Q = 1$ we obtain, conjugating by $(z, w) \mapsto (\lambda z, \lambda w)$,

$$f(z, w) = (w, zw + bz + b'w + c').$$

We can further conjugate by a translation to get the normal form of the family 1.1. Observe that f is then algebraically stable in $\mathbb{P}^1 \times \mathbb{P}^1$ so that $\lambda_1(f)$ is the spectral radius of the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ of the degrees of f in $\mathbb{P}^1 \times \mathbb{P}^1$, i.e. $\lambda_1(f) = (1 + \sqrt{5})/2$ [FG 01].

• Last subcase $\deg_z Q = 2$: then $d_t(f) = 2$.

If $\deg_w Q = 2$ then f is algebraically stable in $\mathbb{P}^2$ so $\lambda_1(f) = \delta_1(f) = 2$. We obtain the family 2.2.c.

If $\deg_w Q = 0$ then conjugating by $(z, w) \mapsto (\lambda z + c, \lambda w + c/\lambda)$ yields the family 3.1. These mappings f have the property that the second iterate f^2 is still quadratic and admits an holomorphic extension to $\mathbb{P}^2$ (i.e. f^2 belongs to the family 3.5). The assertion on the dynamical invariants easily follows.

If $\deg_w Q = 1$ then conjugating by $(z, w) \mapsto (\lambda z, \mu w)$ yields

$$f(z, w) = (aw, z[z - w] + \alpha z + \beta w + c').$$

We can further conjugate by a translation to reach the normal form of the family 3.2 Using bihomogeneous coordinates as in [G 02], one can check that these mappings admit an algebraically stable extension to $\mathbb{P}^2$ blown up at the point $[0:1:0]$, with $\lambda_1(f) = (1+\sqrt{5})/2$. We will check in lemma 2.5 below that $DL_\infty(f) = 1$.

2) Case $P = A\tilde{P}, Q = A\tilde{Q}$:

Observe that f is algebraically stable in $\mathbb{P}^2$ so $\lambda_1(f) = \delta_1(f) = 2$. We can write $A(z,w) = \alpha z + \beta w$ with $(\alpha, \beta) \neq (0,0)$. Conjugating by $(z,w) \mapsto (w,z)$ if necessary, we can assume $\alpha = 1$. Further conjugacy by $(z,w) \mapsto (z-\beta, w)$ yields $\beta = 0$.

Similarly we decompose $\tilde{P}(z,w) = az + bw, \tilde{Q}(z,w) = a'z + b'w$ with $(a,b) \neq (0,0) \neq (a',b')$ and $[a:b] \neq [a':b']$ in $\mathbb{P}^1$.

• Subcase $b = 0$: then $ab' \neq 0$. Conjugating by $(z,w) \mapsto (z/b', w)$, we get $b' = 1$ hence

$$f(z,w) = (R(z) + \beta w, z[w + \alpha z] + \delta z + \varepsilon w + c'),$$

where R is a degree 2 polynomial. Either $\beta = 0$ in which case f is a skew-product of the type 2.2.a, or we can assume $\beta = 1$ after conjugating by $(z,w) \mapsto (z, w/\beta)$. A further conjugacy by a translation yields the normal form of the family 3.3. One easily checks that $d_t(f) = 3$ in this case. The dynamical Lojasiewicz exponent at infinity will be estimated in lemma 2.4 below.

• Last subcase $b \neq 0$: conjugating by $(z,w) \mapsto (z, w/b - aw/b)$ yields $a = 0, b = 1$. Thus

$$f(z,w) = (zw + \alpha z + \beta w + c, z[a'z + b'w] + \delta z + c' + aw), \text{ with } a' \neq 0.$$

Further conjugacy by a translation and $(z,w) \mapsto (z/\sqrt{a'}, w)$ yields $\alpha = \beta = 0$ and $a' = 1$. If $a \neq 0$ then we get the normal form of the family 3.4. One easily checks that $d_t(f) = 3$ in this case and the exponent $DL_\infty(f)$ will be considered in lemma 2.6 below. Finally if $a = 0$, then $d_t(f) = 2$ and

f belongs to the family 2.2.d. This ends the proof of the classification. $\square$

Lemma

2.3. *Consider* $f : (z,w) \in \mathbb{C}^2 \mapsto (P(z)+w, z[w+\alpha z]+c') \in \mathbb{C}^2$*, where* P *is a polynomial of degree* 2 *and* $\alpha, c' \in \mathbb{C}$*. Then* $L_\infty(f) = 1, L_\infty(f^2) = 3/2$ *so* $DL_\infty(f) > 1$.

Proof. We leave it to the reader to check that $L_\infty(f) = 1$. Fix $(z,w) \in \mathbb{C}^2$ such that $\max(|z|,|w|) = R >> 1$ and set $(z',w') = f(z,w)$, $(z'',w'') = f(z',w')$. If $|z| = \max(|z|,|w|) = R$, then $|z'| \gtrsim |z|^2 = R^2$ and $|w'| \lesssim R^2$ so $|z''| \gtrsim R^4$.

We assume now $|w| = \max(|z|,|w|) = R$.

• Either $C_1|w|^{1/2} \leq |z| \leq \varepsilon_1|w|$, where C_1 (resp. ε_1) is a fixed large (resp. small) constant. Then $|z'| \gtrsim |z|^2 \geq R^2$, $|w'| \gtrsim |zw| \gtrsim R^{3/2}$ and $|z'| \lesssim |z|^2 \leq \varepsilon_1|zw| \leq \varepsilon_1'|w'|$. Therefore

$$|w'| \gtrsim |z'||w'| \gtrsim R^{5/2}.$$

• Or $|z| \geq \varepsilon_1|w|$. Then $|z'| \gtrsim |z|^2 \gtrsim R^2$ while $|w'| \lesssim |z||w| \lesssim R^2$, so $|z''| \gtrsim R^4$.

• Or $C_1^{-1}|w|^{1/2} \leq |z| \leq C_1|w|^{1/2}$. Then $|w'| \gtrsim |zw| \gtrsim R^{3/2}$ while $|z'| \lesssim R$, thus $|w''| \gtrsim |z'w'| \gtrsim R^{3/2}$ if $|z'| \geq 1$. Now if $|z'| \leq 1$, we get $|z''| \gtrsim |w'| \gtrsim R^{3/2}$.

• Or else $|z| \leq C_1^{-1}|w|^{1/2}$. In this case $|z'| \gtrsim |w| = R$ and $|w'| \lesssim |zw| \lesssim R^{3/2}$. Therefore $|z''| \gtrsim |z'|^2 \gtrsim R^2$.

Altogether this shows $L_\infty(f^2) \geq 3/2$. On the other hand if $(z,w) \in \mathbb{C}^2$ is such that $P(z)+w = 0$, $|w| = R >> |z| >> 1$, we get $w'' = c$ and $|z''| = |P(0)+c'+z[w+\alpha z]| \lesssim |zw| \lesssim R^{3/2}$, so $L_\infty(f^2) = 3/2$. $\square$

Lemma

2.4. *Consider* $f : (z, w) \in \mathbb{C}^2 \mapsto (aw+c, z[z-w]+c') \in \mathbb{C}^2$*, where* $a, c, c' \in \mathbb{C}$ *with* $a \neq 0$*. Then* $L_\infty(f^j) = 1$ *for all* j *so* $DL_\infty(f) = 1$*.*

Proof. It is straightforward to check that $L_\infty(f) = 1$. Observe that $f(w + a, w) = (aw+c, aw+a^2+c')$ hence $f^2(w+a, w) = (a^2w+c_2, a[c-c'-a^2]w+c_2')$ for some constants c_2, c_2'. This shows $L_\infty(f^2) = 1$. Going on this way,

$$f^2\left(w + a + \frac{\varepsilon_1}{w}, w\right) = \left(a^2w + c_2 + O(1/w), a[c - c' - a^2 - \varepsilon_1]w + c_2' + O(1/w)\right),$$

so if we choose $\varepsilon_1 = c - c' - a^2 + a$, we will get

$$f^3\left(w + a + \frac{\varepsilon_1}{w}, w\right) = \left(a^3w + c_3 + O(1/w), a^2\alpha w + c_3' + O(1/w)\right)$$

for some constant α that depends on the next order term in $O(1/w)$. This shows that f^3 grows linearly on the curve $\{zw = w^2 + a + \varepsilon_1\}$ when $|w|$ is large. Therefore $L_\infty(f^3) = 1$. Moreover we can choose the next order term in $O(1/w)$ so that $\alpha = a$. We leave it to the reader to check that there exists constants ε_j, c_j, c_j' such that for all $N \geq 2$,

$$f^N\left(w + a + \sum_{j=1}^{N-2} \frac{\varepsilon_j}{w^j}, w\right) = \left(a^N w + c_N + O(1/w), a^N w + c_N' + O(1/w)\right).$$

This yields $L_\infty(f^N) = 1$ for all N hence $DL_\infty(f) = 1$.

Note that we get $\varepsilon_j = 0$ when $c - c' = a^2 + a$. In this case the line $L = \{z = w + a\}$ is invariant and $f_{|L}(z, w) = (az + c - a^2, aw + c - a^2)$. In particular if $c = a^2$, $c' = -a$ and $a^N = 1$ then $f_{|L}^N = Id_L$ so L is a curve of periodic points. □

Lemma

2.5. *Consider* $f : (z, w) \in \mathbb{C}^2 \mapsto (zw, z[z + \alpha w] + bz + c' + aw) \in \mathbb{C}^2$*, where* $a, b, c', \alpha \in \mathbb{C}$ *with* $a \neq 0$*. Then* $L_\infty(f^j) = 1$ *for all* j *so* $DL_\infty(f) = 1$*.*

Proof. Simple estimates yield $L_\infty(f) = 1$. Observe that $f(0,w) = (0, aw + c')$ so the line $L = (z = 0)$ is invariant and $f_{|L}$ is linear. This shows $L_\infty(f^j) = 1$ for all $j \geq 1$ hence $DL_\infty(f) = 1$. Note moreover that L is a line of periodic points when $c' = 0$ and $a^N = 1$. $\square$

3. Birational quadratic mappings of $\mathbb{C}^2$

In this section we consider the families 1.1 and 1.2. Since $d_t(f) = 1$, they admit an inverse mapping f^{-1} which is rational. There has been intensive work on these birational mappings (see references in [DF 01]). It is difficult in general to analyze the dynamics near the points of indeterminacy. We show that this can be done here at least for open subsets of the parameters.

3.1. **Family 1.1.** It is convenient to consider the meromorphic extension of $f(z,w) = (w + c, zw + c')$ to $\mathbb{P}^1 \times \mathbb{P}^1$, in bihomogeneous coordinates,

$$\begin{array}{rcl} f : \mathbb{P}^1 \times \mathbb{P}^1 & \to & \mathbb{P}^1 \times \mathbb{P}^1 \\ [z_0 : z_1; w_0 : w_1] & \mapsto & [w_0 : w_1 + cw_0; z_0 w_0 : z_1 w_1 + c' z_0 w_0] \end{array}$$

It should be understood that $\mathbb{C}^2$ coïncides with the chart $(z_0 = w_0 = 1)$ and "infinity" consists of the two lines $(z_0 = 0) = (z = \infty)$ and $(w_0 = 0) = (w = \infty)$. Observe that f has two points of indeterminacy, $m = (\infty, 0)$ and $m' = (0, \infty)$, and contracts the line $(w_0 = 0)$ to the superattractive fixed point $q_\infty = (\infty, \infty)$, while it sends $(z_0 = 0)$ to the line $(w_0 = 0)$. This shows f is algebraically stable in $\mathbb{P}^1 \times \mathbb{P}^1$ with $\lambda_1(f) = (1 + \sqrt{5})/2$ (see [FG 01] for further detail). Observe also that

$$I_{f^2} = \{m, m', m''\} = I_{f^n},\ \forall n \geq 2,$$

where $m'' = (\infty, -c)$ is sent by f to the point $m' = (0, \infty)$. The inverse mapping $f^{-1}(z,w) = (\frac{w - c'}{z - c}, z - c)$ is rational in $\mathbb{C}^2$. One easily checks that $I_{f^{-1}} = \{q_\infty, q^-\}$, where $q^- = (c, c') \in \mathbb{C}^2$, and

$$I_{f^{-n}} = \{q_\infty, f^j(q^-),\ 0 \leq j \leq n - 1\}.$$

It is therefore important to get control of the orbit of q^-. Note that

$$f^{-1}(m) = m',\ f^{-1}(m') = m'' \text{ and } f^{-1}(m'') = m',$$

so $\{m', m''\}$ is a $2-$cycle for f^{-1} to which m is strictly preperiodic (if $c \neq 0$).

Lemma

3.1. *The 2-cycle* $\{m', m''\}$ *is* f^{-1}*-attracting iff* $|c| < 1$.

Proof. A simple computation shows $Df^{-2}(m')$ has eigenvalues 0 and $-c$. □

Lemma

3.2. *Assume* $|c|, |c'| < 10^{-3}$. *Let* p_0 *denote the fixed point of* f *which is closest to* $(0,0)$. *Then* p_0 *is attracting and* q^- *belongs to the basin of* p_0.

Remark

3.3. *For such parameters,* f *can be considered as a small perturbation of the case* $c = c' = 0$ *which is the complexification of the Anosov diffeomorphism* $(z,w) \mapsto (w, zw)$ *on the real torus* $\{|z| = |w| = 1\}$.

Proof. Solving $f(z,w) = (z,w)$ yields two fixed points $p_0 = (\alpha, \alpha - c)$ and $p_1 = (1 + c - \alpha, 1 - \alpha)$, where α is the root of $X^2 - (1+c)X + (c + c') = 0$ with smallest modulus. The differential of f at p_0 is

$$Df(p_0) = \begin{bmatrix} 0 & \alpha - c \\ 1 & \alpha \end{bmatrix},$$

so p_0 is an attractive fixed point if $|c|, |c'|$ are small enough. Let us make a local change of coordinates to bring back p_0 to $(0,0)$. Consider

$$g(x,y) = f(\alpha + x, \alpha - c + y) - (\alpha, \alpha - c) = (y, xy + \alpha y + (\alpha - c)x).$$

In these new coordinates, $q^- = (c - \alpha, c' + c - \alpha)$, hence q^- belongs to the basin of $(0,0) = p_0$ if $|c|, |c'|$ are small enough. It is then straightforward to check that $|c|, |c'| < 10^{-3}$ is sufficient. □

Lemma

3.4. *Assume $|c| < 1$ and $|c'| > 4/(1 - |c|)$. Then $q^- = (c, c')$ belongs to the basin of the superattractive fixed point $q_\infty = (\infty, \infty)$.*

Proof. It is more comfortable to work in a local chart near q_∞. Using bihomogeneous coordinates, we work in the chart $(z_1 = w_1 = 1)$. In this chart, f defines a mapping

$$g(x, y) = (\frac{y}{1 + cy}, \frac{xy}{1 + c'xy})$$

and q_∞ has coordinates $(0, 0)$. We get

$$g^2(x, y) = \left(\frac{xy}{1 + (c + c')xy}, \frac{xy^2}{1 + cy + c'xy + (c' + cc')xy^2} \right).$$

Consider

$$\Omega := \left\{ (x, y) \in \mathbb{C}^2 \,/\, |y| < 1/4 \text{ and } |xy| < \frac{1}{4|c + c'|} \right\}.$$

We claim Ω is g^2-invariant and g^2 is contracting in Ω, so Ω is part of the basin of attraction of q_∞. Indeed let $(x, y) \in \Omega$ and set $(x', y') = g^2(x, y)$. Then $|1 + (c + c')xy| > 3/4$ hence $|x'| < 4|xy|/3 < |x|/3$. Moreover

$$\begin{aligned}
|cy| &< |y| < 1/4, \\
|c'xy| &< \frac{|c'|}{4|c + c'|} < 1/3 \quad \text{since} \quad |c'| > 4 > 4|c|, \\
|c'(1 + c)xy^2| &< 2|c'||xy||y| \quad < \quad \frac{1}{2} \cdot \frac{|c'|}{4|c + c'|} < 1/6,
\end{aligned}$$

thus $|1 + cy + c'xy + (c' + cc')xy^2| > 1/4$. This yields $|y'| < 4|xy^2| < |y|/|c + c'| < |y|/3$.

Consider now $\Omega' = \Omega \cap \mathbb{C}^2$. In our original coordinates (z, w), we thus get a portion of the basin (in $\mathbb{C}^2$) of the superattractive fixed point q_∞,

$$\Omega' = \{(z, w) \in \mathbb{C}^2 \,/\, |w| > 1/4 \text{ and } |zw| > 4|c + c'|\}.$$

We claim that $q^- = (c, c')$ belongs to $f^{-1}(\Omega')$ under our assumptions. Indeed $f(q^-) = (c' + c, cc' + c')$ and $|cc' + c'| \geq |c'|(1 - |c|) > 4$, $|cc' + c'||c + c'| > 4|c + c'|$. This shows $f(q^-)$, hence q^-, belongs to the basin of q_∞. □

3.2. **Family 1.2.** We now turn to mappings of family 1.2. When $a \neq 2$, we can further conjugate by a translation and suppose $c = 0$. In order to simplify the exposition we will therefore consider the three parameters family

$$f(z, w) = (w, w[w - az] + bz + c'), \text{ where } a, b, c' \in \mathbb{C} \text{ with } (a, b) \neq (0, 0).$$

We consider their meromorphic extension to $\mathbb{P}^2 = \mathbb{C}^2 \cup (t = 0)$, where $(t = 0)$ denotes the line at infinity. In homogeneous coordinates,

$$f[z : w : t] = [wt : w(w - az) + bzt + c't^2 : t^2].$$

Thus $I_f = \{m, m'\} = I_{f^n}$ for all $n \geq 1$, where $m = [1 : 0 : 0]$, $m' = [1 : a : 0]$ and $f((t = 0)\backslash I_f) = q_\infty := [0 : 1 : 0]$ is a superattractive fixed point for f. So f is algebraically stable in $\mathbb{P}^2$ and $\lambda_1(f) = \delta_1(f) = 2$. The inverse mapping f^{-1} is merely rational in $\mathbb{C}^2$ when $a \neq 0$, $f^{-1}(z, w) = ([w - z^2 - c']/[b - az], z)$. We get $I_{f^{-1}} = \{q_\infty, q^-\}$, where $q^- = (b/a, b^2/a^2) \in \mathbb{C}^2$, except when $a = 0$ in which case $q^- = q_\infty$ and f is then a quadratic Hénon mapping. Therefore

$$I_{f^{-n}} = \{q_\infty, f^j(q^-), \, 0 \leq j \leq n - 1\}, \; \forall n \geq 1.$$

Observe that $f^{-1}(m) = m' = f^{-1}(m')$.

Lemma

3.5. *The point m' is attracting for f^{-1} if and only if $|a| < 1$.*

If $|a| < 1$ and $4|a| \leq |b|$, then q^- belongs to the basin of attraction of the point q_∞.

Proof. A simple computation shows $Df^{-1}(m')$ has eigenvalues 0 and a.

Assume $|a| < 1$ and $4|a| \leq |b|$. We work in the chart $(w = 1) \ni q_\infty$. Set $x = z/w, y = t/w$ and

$$g(x, y) = f[x : 1 : y] = \left(\frac{y}{1 - ax + bxy + c'y^2}, \frac{y^2}{1 - ax + bxy + c'y^2} \right).$$

We set $\Omega = \{(x, y) \in \mathbb{C}^2 \, / \, |x| \leq 1/4 \text{ and } |y| \leq \max(1/|b|, 1/'|c'|, 1/16)\}$. Let $(x, y) \in \Omega$ and set $(x', y') = g(x, y)$. Our assumption yields $|1 - ax + bxy + c'y^2| > 1/4$ hence

$$|x'| \leq 4|y| \leq 1/4 \text{ and } |y'| \leq 4|y|^2 \leq |y|/4.$$

This shows Ω is g-invariant and g^n uniformly converges to $q_\infty = (0, 0)$ on Ω. Coming back to the canonical chart $\mathbb{C}^2 = (t = 1)$, this shows

$$\Omega' = \Omega \cap \mathbb{C}^2 = \{(z, w) \in \mathbb{C}^2 \, / \, 4|z| \leq |w| \text{ and } |w| \geq \min(|b|, |4c'|, 16)\}$$

is part of the basin of attraction of the point q_∞. It remains to check that $q^- = (b/a, b^2/a^2) \in \Omega'$, but this readily follows from our assumptions $|b| \geq 4|a|$ and $|a| < 1$. □

3.3. **Ergodic properties.** We mention here some basic questions about ergodic properties of these two families. Let f be one of these mappings. Since f is algebraically stable in X ($\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}^2$), there are two well defined Green currents T_+ and T_- such that $(f^{\pm})^* T_{\pm} = \lambda_1(f) T_{\pm}$. The current T_+ has continuous potentials in $X \setminus I_{f^2}$, so $\mu_f := T_+ \wedge T_-$ is a well defined invariant probability measure (if T_+, T_- are properly normalized) which is mixing [FG 01] and hyperbolic [BDi O2].

When $|c| < 1$ in the family 1.1 (resp. $|a| < 1$ in the family 1.2), then μ_f has maximal entropy$= \log \lambda_1(f)$ [G 02]. If we further assume that q^- belongs to the basin of attraction of some attractive fixed point (see lemmas 3.2,3.4,3.5), then f is a biholomorphism in a neighborhood of $\operatorname{Supp} \mu_f$. In this case one can copy the work of Bedford, Lyubich and Smillie on complex

Hénon mappings to get that μ_f is the unique measure of maximal entropy and that periodic saddle points are equidistributed with respect to μ_f [BLS 93a], [BLS 93b]. It seems that the latter still holds only assuming $|c| < 1$ (resp. $|a| < 1$). It would be interesting however to understand the kind of bifurcation that may occur when e.g. c is fixed, $|c| < 1$, and letting $|c'|$ vary (see lemmas 3.2, 3.4): can q^- belong to $\operatorname{Supp} \mu_f$?

Finally we mention the following:

Question. *Does μ_f always have maximal entropy $= \log \lambda_1(f)$?*

4. Behaviour at infinity when $d_t > \lambda_1$

Let $f : \mathbb{C}^2 \to \mathbb{C}^2$ be a dominating polynomial mapping with $d_t > \lambda_1(f)$. Russakovskii and Shiffman proved in [RS 97] that the sequences of probability measures $d_t^{-n}(f^n)^*\Theta$ converge towards the same limit measure μ_f. Here Θ denotes any smooth probability measure in $\mathbb{C}^2$. Our goal here is to prove that μ_f has compact support in $\mathbb{C}^2$ when f is quadratic. Note that this is obvious when infinity is f-attracting, in particular when $DL_\infty(f) > 1$, i.e. for mappings in the families 3.1,3.3 and 3.5. For the two remaining classes, we will show that infinity is indeed attracting for an open set of parameters and that it is attracting "on the average" for the remaining values of the parameters.

4.1. **A criterion of compactness.** The following proposition was inspired by a result of Douady [Do 01] that concerns the Newton method for solving quadratic equations in $\mathbb{C}^2$.

Proposition

4.1. *Let $f : \mathbb{C}^2 \to \mathbb{C}^2$ be a proper polynomial mapping such that $d_t(f) >$*

$\lambda_1(f)$. *Let g_i denote the inverse branches of f and assume*

$$\log^+ ||g_i(p)|| \leq \alpha_i \log^+ ||p|| + C, \ \forall p \in \mathbb{C}^2,$$

where $C, \alpha_i > 0$ and $\sum_{i=1}^{d_t(f)} \alpha_i < d_t(f)$.

Then the Russakovskii-Shiffman measure μ_f has compact support in $\mathbb{C}^2$.

Proof. Fix ρ such that $d_t^{-1} \sum \alpha_i < \rho < 1$ and $R_0 > 0$ large enough. Let ν be a probability measure in $\mathbb{C}^2$ such that

$$H_\nu(r) := \nu(\log^+ ||p|| > r) \leq C_0 \frac{1}{r}, \text{ for } r \geq R_0. \ (*)$$

Set $\nu_n := d_t^{-n}(f^n)^*(\nu)$. We claim that

$$H_{\nu_n}(r) := \nu_n(\log^+ ||p|| > r) \leq \rho^n C_0 \frac{1}{r}, \text{ for } r \geq R_0.$$

This clearly implies the proposition since every smooth probability measure ν with support in the ball of radius e^{R_0} satisfies $(*)$ and $\nu_n \to \mu_f$, so μ_f will be supported on the ball of radius e^{R_0}.

Let us denote by $h_i(r) := H_{(g_i)_* \nu}(r)$. Observe that

$$\log^+ ||g_i(p)|| > r \Rightarrow \log^+ ||p|| > \frac{r - C}{\alpha_i},$$

hence $h_i(r) \leq H_\nu((r - C)/\alpha_i)$. We infer

$$H_{\nu_1}(r) = \frac{1}{d_t} \sum_{i=1}^{d_t} h_i(r) \leq \frac{C_0}{r} \frac{1}{d_t} \sum_{i=1}^{d_t} \alpha_i \frac{r}{r - C} \leq \rho \frac{C_0}{r},$$

if $r \geq R_0$, R_0 large enough. A straighforward induction yields the claim. $\square$

Remark

4.2. *One may expect that the Russakovskii-Shiffman measure is always compactly supported in $\mathbb{C}^2$ when f is proper. Here is a heuristic argument to support this conjectural fact: let α_i denote the mass of $(g_i)^*\omega$ in $\mathbb{C}^2$. Passing to an iterate we may assume $\delta_1(f) < d_t(f)$, thus we get*

$$\sum \alpha_i = \sum \int_{\mathbb{C}^2} (g_i)^* \omega \wedge \omega = \int_{\mathbb{C}^2} f_* \omega \wedge \omega = \delta_1(f) < d_t(f).$$

On the other hand, it is a well known fact from pluripotential theory that the mass of $(g_i)^*\omega$ *precisely controls the growth of* $\log^+ ||g_i||$.

It should be noted that examples of polynomial mappings of $\mathbb{C}^2$ *with non-compactly supported Russakovskii-Shiffman measure are given in [FG 01], however these are non proper mappings.*

4.2. **Family 3.2.** We consider here mappings

$$f(z,w) = (aw + c, z[z-w] + c'), \text{ where } a \neq 0.$$

Lemma

4.3. *If* $|a| > 1$ *then infinity is* f*-attracting.*

Proof. Assume $|a| = 1 + 2t$, $t > 0$. Set $V_\varepsilon = \{(z,w) \in \mathbb{C}^2 \,/\, \max(|z|,|w|) > 1/\varepsilon\}$. The lemma will follow from the existence of $\varepsilon_0 > 0$ such that

$$0 < \varepsilon < \varepsilon_0 \Rightarrow f(V_\varepsilon) \subset V_{\varepsilon/(1+t)}.$$

Fix $(z,w) \in V_\varepsilon$ and set $(z',w') = f(z,w)$. If $|w| = \max(|z|,|w|) > \varepsilon^{-1}$, then

$$|z'| = |aw + c| \geq (1+2t)|w| - |c| > \frac{1 + 3t/2}{\varepsilon} \text{ if } 0 < \varepsilon < \varepsilon_1.$$

So assume $|z| = \max(|z|,|w|) > \varepsilon^{-1}$. Either $|z - w| \geq 1 + 2t$ in which case $|w'| \geq (1+2t)|z| - |c'| > (1+t)/\varepsilon$ for $0 < \varepsilon < \varepsilon_2$. Or $|z - w| < 1 + 2t$, then $|w| > (1+t)/(1+3t/2)\varepsilon^{-1}$ yields $|z'| > (1+t)/\varepsilon$ for $0 < \varepsilon < \varepsilon_3$. We get the desired inclusion choosing $\varepsilon_0 = \min(\varepsilon_1, \varepsilon_2, \varepsilon_3)$. $\square$

We now consider the remaining cases $0 < |a| \leq 1$. Recall that $d_t(f) = 2$. Since f is proper, there are two well defined inverse branches of f in $\mathbb{C}^2$ which we denote by g^+, g^-, ordered so that if $g^\pm(x,y) = (z^\pm, w^\pm)$ then $|z^+| \geq |z^-|$.

Lemma

4.4. *There exists $C > 0$ such that for all $(x, y) \in \mathbb{C}^2$,*

$$\begin{aligned}
\log^+ ||g^+ \circ g^+(x,y)|| &\leq \log^+ ||(x,y)|| + C \\
\log^+ ||g^+ \circ g^-(x,y)|| &\leq \frac{1}{2} \log^+ ||(x,y)|| + C \\
\log^+ ||g^- \circ g^+(x,y)|| &\leq \frac{1}{2} \log^+ ||(x,y)|| + C. \\
\log^+ ||g^- \circ g^-(x,y)|| &\leq \frac{1}{4} \log^+ ||(x,y)|| + C.
\end{aligned}$$

Therefore μ_f has compact support in $\mathbb{C}^2$.

Proof. Fix $(x, y) \in \mathbb{C}^2$. The two preimages of (x, y) satisfy $w = (x - c)/a$ and $z^2 - (x - c)z/a + (c' - y) = 0$. From $|z^+ z^-| = |c' - y|$ we get $|z^-| \leq |c' - y|^{1/2}$ hence $|z^-| \leq C_1 \max(|y|^{1/2}, 1)$. Since $|z^+ + z^-| = |x - c|/|a|$ we get $|z^+| \leq C_2 \max(|x|, |y|^{1/2}, 1)$. Finally $|w^\pm| = |x - c|/|a| \leq C_3 \max(|x|, 1)$. Iterating these inequalities yields the lemma.

It follows from proposition 4.1 that μ_f has compact support in $\mathbb{C}^2$ since here $\sum \alpha_i = 9/4 < 4 = d_t(f^2)$. □

4.3. **Family 3.4.** We consider here mappings of the form

$$f(z, w) = (zw + c, z[z + \alpha w] + bz + c' + aw), \text{ where } a \neq 0.$$

Lemma

4.5. *If $|a| > 1$ then infinity is f-attracting.*

Proof. Define $t > 0$ by $|a| = 1 + 3t$ and fix $\lambda > 0$ small enough so that $|\alpha\lambda| < t$. For technical reason we first conjugate f by $(z, w) \mapsto (\lambda z, w)$. Thus we will show that infinity is attracting for g, where

$$g(z, w) = (zw + c_1, z[\lambda^2 z + \alpha\lambda w] + \alpha\lambda z + c_2 + aw).$$

Set $V_\varepsilon := \{(z,w) \in \mathbb{C}^2 \,/\, \max(|z|,|w|) > 1/\varepsilon\}$. It is clearly sufficient to show the existence of $\varepsilon_0 > 0$ such that $g(V_\varepsilon) \subset V_{\varepsilon/(1+t)}$ for $0 < \varepsilon < \varepsilon_0$. Pick $(z,w) \in V_\varepsilon$ and set $(z',w') = g(z,w)$.

Assume first $|z| = \max(|z|,|w|) > 1/\varepsilon$. Then $|z'| \geq |w||z| - |c_1| > (1+t)/\varepsilon$ if $|w| \geq 1+2t$ and $0 < \varepsilon < \varepsilon_1$. Now if $|w| \leq 1+2t$, then $|w'| \geq \lambda^2|z|^2/2 > (1+t)/\varepsilon$ for $0 < \varepsilon < \varepsilon_2$, so $(z',w') \in V_{\varepsilon/(1+t)}$ in both cases.

Assume now $|w| = \max(|z|,|w|) > 1/\varepsilon$. Then $|z'| \geq (1+t)/\varepsilon$ if $|z| \geq 1+2t$ and $0 < \varepsilon < \varepsilon_3$. Now if $|z| \leq 1+2t$, we get

$$|w'| \geq (|a| - |\alpha\lambda|)|w| - C \geq (1+2t)|w| - C > \frac{1+t}{\varepsilon} \text{ if } 0 < \varepsilon < \varepsilon_4.$$

The desired inclusion follows with $\varepsilon_0 = \min(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$. □

We now consider the case $0 < |a| \leq 1$.

Lemma

4.6. *Let f be as above. Denote by g_1, g_2, g_3 the three inverse branches of f ordered so that if $g_i(x,y) = (z_i, w_i)$, then $|z_1| \leq |z_2| \leq |z_3|$. Then there exists $C > 0$ such that for all $(x,y) \in \mathbb{C}^2$,*

$$\begin{aligned}
\log^+ ||g_1(x,y)|| &\leq \log^+ ||(x,y)|| + C \\
\log^+ ||g_2(x,y)|| &\leq \frac{2}{3}\log^+ ||(x,y)|| + C \\
\log^+ ||g_3(x,y)|| &\leq \frac{2}{3}\log^+ ||(x,y)|| + C.
\end{aligned}$$

Therefore μ_f has compact support in $\mathbb{C}^2$.

Proof. We fix $R_0 = R_0(a,b,c,c',\alpha) >> 1$. In order to simplify notations, we will denote by $\lesssim$ an inequality $\leq$ that holds true up to a constant that only depends on the parameters a,b,c,c',α. Without loss of generality we may assume $(x,y) \in \mathbb{C}^2$ are such that $\max(|x|,|y|) > R_0$.

Let (z_i, w_i), $1 \leq i \leq 3$ be the solutions of $f(z,w) = (x,y)$ ordered so that $|z_3| \geq |z_2| \geq |z_1|$. Observe that $zw = x - c$ hence

$$z^3 + bz^2 + [\alpha(x-c) + c' - y]z + a(x-c) = 0 = (z - z_1)(z - z_2)(z - z_3).$$

From $|z_1 z_2 z_3| = |a(x-c)|$ we get

$$|z_1| \leq |a(x-c)|^{1/3} \leq |z_3|. \quad (1)$$

Assume $|x| > R_0$. Using $|z_1 + z_2 + z_3| = |b|$ this yields, if R_0 is chosen large enough,

$$\frac{1}{2}|a(x-c)|^{1/3} \leq |z_2|. \quad (2)$$

Indeed otherwise $|z_1| \leq |z_2| \leq |a(x-c)|^{1/3}/2$ yields $|z_3| \geq 4|a(x-c)|^{1/3}$ hence $|b| = |z_1 + z_2 + z_3| \geq 3|a(x-c)|^{1/3}$ contradicting $|x| > R_0$. From $w_i = (x-c)/z_i$, we infer

$$|w_3| \leq \frac{|x-c|}{|a(x-c)|^{1/3}} \lesssim \max(|x|^{2/3}, 1) \quad (3)$$

and

$$|w_2| \leq 2\frac{|x-c|}{|a(x-c)|^{1/3}} \lesssim \max(|x|^{2/3}, 1). \quad (4)$$

We now give a bound from above for $|w_1|$. Recall that $z_1 + \alpha(x-c) + bz_1 + c' + aw_1 = y$. Thus

$$|w_1| \leq \frac{1}{|a|}\left(|y| + |\alpha(x-c)| + |c'| + |bz_1| + |z_1|^2\right) \lesssim \max(||(x,y)||, 1), \quad (5)$$

where the last inequality follows from (1). Note finally that z_3 is one of the solutions of $z^2 + bz + [c' + aw_3 - y + \alpha(x-c)] = 0$. Therefore $|z_3| \lesssim \max(|b|, |c' + aw_3 - y + \alpha(x-c)|^{1/2})$. Together with (3) this yields

$$|z_2| \leq |z_3| \lesssim \max(||(x,y)||^{1/2}, 1). \quad (6)$$

This yields the lemma when $|x| > R_0$. Assume now $|y| > R_0 \geq |x|$. Without loss of generality we may actually assume $|y| > R_0^2 >> R_0 \geq |x|$. There only remains to show $|z_2| \geq \frac{1}{2}|a(x-c)|^{1/3}$. Assume the contrary, then

$$|y| \sim |\alpha(x-c)+c'-y| = |z_1z_2+z_1z_3+z_2z_3| \lesssim |y|^{\frac{1}{2}+\frac{1}{3}},$$

by (6), a contradiction.

Using the notations of proposition 4.1, we get $\sum \alpha_i = 7/3 < 3 = d_t(f)$, hence μ_f has compact support in $\mathbb{C}^2$. □

5. The Russakovskii-Shiffman measure

Let $f : \mathbb{C}^2 \to \mathbb{C}^2$ be a dominating polynomial mapping such that $d_t > \lambda_1(f)$. Following [G 02b] we give in this section an elementary construction of the Russakovskii-Shiffman measure μ_f. When infinity is f-attracting, we then show that every plurisubharmonic function is in $L^1(\mu_f)$. This is stronger than the general result proved in [G 02] that every quasiplurisubharmonic function on $\mathbb{P}^2$ is in $L^1(\mu_f)$.

• **Construction of μ_f.** Let $a \in \mathbb{C}^2$ be a non critical value of f and Θ a smooth probability measure supported near a. Then $d_t^{-1} f^*\Theta$ is again a smooth probability measure with compact support in $\mathbb{C}^2$. Thus Θ and $d_t^{-1} f^*\Theta$ are cohomologous, when viewed as global smooth forms of maximal bidegree on $\mathbb{P}^2$. Hence there exists a smooth form $\mathcal{T}$ of bidegree $(1,1)$ on $\mathbb{P}^2$ such that

$$\frac{1}{d_t} f^*\Theta = \Theta + dd^c\mathcal{T}. \quad (\dagger)$$

Adding some multiple of the Fubini-Study form ω, we can further assume $0 \leq \mathcal{T} \leq C\omega$, for some constant $C > 0$. Pulling back $(\dagger)$ by f^n yields

$$\frac{1}{d_t^n}(f^n)^*\Theta = \Theta + dd^c\mathcal{T}_n, \text{ where } \mathcal{T}_n = \sum_{j=0}^{n-1} \frac{1}{d_t^j}(f^j)^*(\mathcal{T}). \quad (\dagger\dagger)$$

The sequence $(\mathcal{T}_n)$ is an increasing sequence of positive currents of bidegree $(1,1)$ on $\mathbb{P}^2$ such that

$$0 \leq \mathcal{T}_n \leq C \sum_{j=0}^{n-1} \frac{1}{d_t^j}(f^j)^*\omega.$$

The latter series is convergent since $(f^j)^*\omega$ has mass $\delta_1(f^j) \leq [\lambda_1(f)+\varepsilon]^j$ for $j \geq j_\varepsilon$ and $\varepsilon > 0$ small enough that $d_t(f) > \lambda_1(f)+\varepsilon$. Therefore $\mathcal{T}_n$ converges towards some positive current $\mathcal{T}_\infty$. This yields

$$\frac{1}{d_t^n}(f^n)^*\Theta = \Theta + dd^c\mathcal{T}_n \to \mu_f := \Theta + dd^c\mathcal{T}_\infty.$$

Observe that if Θ' is any other smooth probability measure, then $\Theta' = \Theta + dd^cS$ for some smooth $(1,1)$ form S on $\mathbb{P}^2$, so

$$\frac{1}{d_t^n}(f^n)^*\Theta' = \frac{1}{d_t^n}(f^n)^*\Theta + dd^c\left(\frac{1}{d_t^n}(f^n)^*S\right) \to \mu_f$$

because $||(f^n)^*S|| = \delta_1(f^n) = o(d_t^n)$. In particular $d_t^{-n}(f^n)^*\omega^2 \to \mu_f$.

Remark

5.1. *Assume infinity is an attracting set for f in the following sense: there exists a neighborhood V of infinity in $\mathbb{C}^2$ such that $\cap_{j\geq 1} f^j(V) = \emptyset$. In this case we get $\mathbb{C}^2 = K^+ \cup \mathcal{B}^+(\infty)$, where $K^+ = \{a \in \mathbb{C}^2 \,/\, (f^n(a))_{n\geq 0}$ is bounded$\}$ is a compact subset of $\mathbb{C}^2$ and $\mathcal{B}^+(\infty)$ denotes the basin of attraction of infinity, $\mathcal{B}^+(\infty) = \cup_{n\geq 0} f^{-n}(V)$. The measure μ_f is supported on the compact set ∂K^+ in this case. Infinity is always an attracting set for f when $DL_\infty(f) > 1$, but it may also be attracting when $DL_\infty(f) = 1$ as we have seen in section 4.*

An alternative construction of μ_f was given in [G 02] under the more restrictive assumption that $DL_\infty(f) = d_t(f)/\lambda_1(f)$.

Theorem

5.2. *Let $f : \mathbb{C}^2 \to \mathbb{C}^2$ be a dominating polynomial mapping such that $d_t(f) >$*

$\lambda_1(f)$. Assume μ_f has compact support and either infinity is f-attracting or $d_t(f) > \lambda_1(f)^{3/2}$.

Then every plurisubharmonic function is in $L^1(\mu_f)$.

Proof. Let B be a ball in $\mathbb{C}^2$ containing $\operatorname{Supp}\mu_f$ and φ a plurisubharmonic function near $\overline{B}$. Without loss of generality $\varphi \leq 0$ on B. Let $\chi \geq 0$ be a test function in B such that $\chi \equiv 1$ near B_1, $\operatorname{Supp}\mu_f \subset B_1 \subset\subset B$. We get

$$0 \leq \int_{B_1}(-\varphi)d\mu_f = \int_{B_1}(-\varphi)\Theta + \int_{B_1}(-\varphi)dd^c(\chi\mathcal{T}_\infty).$$

Since Θ is smooth, we only need to get an upper bound on the second integral. By Stokes theorem this latter reads

$$I = -\int_B \chi\mathcal{T}_\infty \wedge dd^c\varphi + \int_{B\setminus B_1}\varphi dd^c(\chi\mathcal{T}_\infty) = I' + I''.$$

Note that $I' \leq 0$ because φ is plurisubharmonic, hence we only need to get an upper bound on I''. Observe that $dd^c(\chi\mathcal{T}_\infty) = dd^c\chi\wedge\mathcal{T}_\infty + 2d\chi\wedge d^c\mathcal{T}_\infty + \chi dd^c\mathcal{T}_\infty$. Since $\mu_f = 0$ in $B\setminus B_1$, we get that $\chi dd^c\mathcal{T}_\infty = -\chi\Theta$ is smooth in $B\setminus B_1$. It is therefore sufficient to get control on

$$I_1 = \int_{B\setminus B_1}\varphi d\chi\wedge d^c\mathcal{T}_\infty \text{ and } I_2 = \int_{B\setminus B_1}\varphi dd^c\chi\wedge\mathcal{T}_\infty.$$

Since $\mathcal{T}_\infty$ is positive, we get

$$|I_2| \leq ||\chi||_{\mathcal{C}^2}\int_B(-\varphi)\omega\wedge\mathcal{T}_\infty \leq C_1\sum_{j\geq 0}\int_B(-\varphi)\omega\wedge\frac{1}{d_t^j}(f^j)^*\omega.$$

Since $d_t > \lambda_1(f)$, we have $d_t^l > \delta_1(f^l)$ for l large enough. We assume for simplicity $l = 1$ and set $d = \delta_1(f) < d_t(f)$. Now $(f^j)^*\omega = d^j dd^c G_j^+$ in $\mathbb{C}^2$, where G_j^+ is locally uniformly bounded in $\mathbb{C}^2$. It follows therefore from Chern-Levine-Nirenberg inequalities [S 99] that

$$|I_2| \leq C_2||\varphi||_{L^1(B_2)}\sum_{j\geq 0}\left(\frac{d}{d_t}\right)^j < +\infty,$$

where B_2 is a slightly larger ball than B.

It remains to get control on I_1. We decompose $\mathcal{T} = \sum \mathcal{T}_{ij} dz_i \wedge d\overline{z}_j$ in $\mathbb{C}^2$, where the $\mathcal{T}'_{ij}$s are smooth functions. By Cauchy-Schwarz inequality we get

$$
\begin{aligned}
&\left| \int_{B\setminus B_1} (-\varphi) d\chi \wedge (f^n)^* d^c \mathcal{T} \right| \\
&\quad \leq \sum_{i,j} \left| \int_{B\setminus B_1} (-\varphi) d\chi \wedge (f^n)^* (d^c \mathcal{T}_{ij} \wedge dz_i \wedge d\overline{z}_j) \right| \\
&\quad \leq \sum_{i,j} \left| \int_{B\setminus B_1} (-\varphi) d\chi \wedge d^c\chi \wedge (f^n)^* (dz_i \wedge d\overline{z}_j) \right|^{1/2} \\
&\quad \cdot \left| \int_{B\setminus B_1} (-\varphi) (f^n)^* (d\mathcal{T}_{ij} \wedge d^c \mathcal{T}_{ij} \wedge dz_i \wedge d\overline{z}_j) \right|^{1/2} \\
&\quad \leq C_3 \left[\int_{B\setminus B_1} (-\varphi) \omega \wedge (f^n)^* \omega \right]^{1/2} \cdot \left[\int_{B\setminus B_1} (-\varphi) (f^n)^* \omega^2 \right]^{1/2}.
\end{aligned}
$$

When infinity is f-attracting, we can assume $B \setminus B_1$ is a relatively compact subset of the basin of attraction of infinity. Therefore $\frac{1}{2} \log[1 + ||f^n||^2] = \log ||f^n|| + u_n$, where u_n is uniformly bounded on $B \setminus B_1$. Thus $(f^n)^*(\omega^2) = (dd^c u_n)^2 + 2dd^c \log ||f^n|| \wedge dd^c u_n$ yields, by Chern-Levine-Nirenberg inequalities again

$$
0 \leq \int_{B\setminus B_1} (-\varphi)(f^n)^* \omega^2 \leq C_4 d^n,
$$

for some constant C_4 independent of n. On the other hand $(f^n)^*\omega = d^n dd^c G_n^+$ with G_n^+ uniformly bounded on $B \setminus B_1$. This shows

$$
\left| \int_{B\setminus B_1} (-\varphi) d\chi \wedge (f^n)^* (d^c \mathcal{T}) \right| \leq C_5 d^n.
$$

Therefore $|I_1| \leq C_5 \sum_{j\geq 0} (d/d_t)^j < +\infty$.

When infinity is not f-attracting we can still get an upper bound

$$
0 \leq \int_{B\setminus B_1} (-\varphi)(f^n)^* \omega^2 \leq C_4 d^{2n},
$$

so $|I_1| \leq C_5 \sum_{j\geq 0} (d^{3/2}/d_t)^j < +\infty$ if $d_t(f) > \lambda_1(f)^{3/2}$. $\square$

Remark

5.3. *The main ergodic properties of μ_f are established in [G 02b]. It is mixing with positive Lyapunov exponents, repelling periodic points are equidistributed with respect to μ_f, and μ_f is the unique measure of maximal entropy $h_{\mu_f}(f) = h_{top}(f) = \log d_t(f)$.*

6. Algebraicity of $\mathcal{E}_f$

Let $f : \mathbb{C}^2 \to \mathbb{C}^2$ be a dominating polynomial mapping such that $d_t > \lambda_1(f)$. Russakovskii and Shiffman have shown [RS 97] the existence of a pluripolar set $\mathcal{E}_f \subset \mathbb{C}^2$ such that

$$\frac{1}{d_t^n}(f^n)^*\varepsilon_a \longrightarrow \mu_f, \quad \forall a \in \mathbb{C}^2 \setminus \mathcal{E}_f.$$

Here ε_a denotes the Dirac mass at point a. Following Briend and Duval [BD 01], we show here that $\mathcal{E}_f$ is actually algebraic when f is quadratic.

We denote by $\deg_p f$ the local topological degree of f at p, i.e. the number of points in $f^{-1}(q)$ which are close to p when q is close to $f(p)$. So $\deg_p f > 1$ iff p belongs to the critical set $\mathcal{C}_f$ of f. For an irreducible algebraic curve A of $\mathbb{C}^2$, we set $\deg_A f = \min_{p\in A} \deg_p f = \deg_p f$ for a generic point $p \in A$. When $A = \cup A_i$ is not irreducible, we set $\deg_A f = \max_i \deg_{A_i} f$.

Lemma

6.1. *Let f, g be two proper polynomial self-mappings of $\mathbb{C}^2$. Then*

1) $\deg_p(f \circ g) = \deg_p g \cdot \deg_{g(p)} f$ *hence* $\deg_A(f \circ g) = \deg_A g \cdot \deg_{g(A)} f$.

2) $\deg_{\mathcal{C}_{f\circ g}}(f \circ g) \leq \deg_{\mathcal{C}_g} g \cdot \deg_{\mathcal{C}_f} f$.

3) $1 \leq \deg_p f \leq d_t(f)$.

4) $1 \leq \deg_A f \leq \deg_{\mathcal{C}_f} f \leq \delta_1(f)$.

5) Assume $d_t(f) > \delta_1(f)$. If $\deg_{f^j(p)} f = d_t(f)$ for all $j \geq 0$, then p is periodic and the corresponding cycle is totally invariant.

Proof. Assertion 1) is a straightforward consequence of the definition. We refer the interested reader to [GH 78] chapter 5.1,5.2 for further details on local topological degree. The chain rule yields $\mathcal{C}_{f\circ g} = \mathcal{C}_g \cup g^{-1}(\mathcal{C}_f)$. Therefore

$$\begin{aligned}\deg_{\mathcal{C}_{f\circ g}}(f\circ g) &= \max\Big(\deg_{\mathcal{C}_g}(f\circ g), \deg_{g^{-1}(\mathcal{C}_f)}(f\circ g)\Big)\\ &= \max\Big(\deg_{\mathcal{C}_g} g \cdot \deg_{g(\mathcal{C}_g)} f, \deg_{g^{-1}(\mathcal{C}_f)} g \cdot \deg_{\mathcal{C}_f} f\Big)\\ &\leq \deg_{\mathcal{C}_g} g \cdot \deg_{\mathcal{C}_f} f.\end{aligned}$$

Assertion 3) is clear and 4) easily follows from Bezout theorem [BD 01].

It follows from 4) that the set $E = \{p \in \mathbb{C}^2 \,/\, \deg_p f = d_t(f)\}$ is finite when $d_t(f) > \delta_1(f)$. So if $\deg_{f^j(p)} f = d_t(f)$ for all $j \geq 0$ then p is preperiodic to a cycle in E. To simplify we assume $f^n(p) = q$ with $q = f(q) \in E$. Now $f^{-1}(q)$ contains q with multiplicity d_t, so we get $f^*\varepsilon_q = d_t\varepsilon_q$, hence q is totally invariant. This shows $p = q$ so p is periodic and the corresponding cycle is totally invariant. ☐

Note in particular that $\deg_{\mathcal{C}_{f^j}} f^j$ is submultiplicative. Therefore we can define the asymptotic critical degree $\mathcal{T}(f)$ of f by

$$\mathcal{T}(f) := \lim_{j\to+\infty}\Big(\deg_{\mathcal{C}_{f^j}} f^j\Big)^{1/j}.$$

Observe that $\mathcal{T}(f) > 1$ implies strong recurrence of the critical set so $\mathcal{T}(f) = 1$ "generically". This motivates the following proposition which is a weak version of a result of Briend and Duval [BD 01] on holomorphic endomorphisms.

Proposition

6.2. *Let $f : \mathbb{C}^2 \to \mathbb{C}^2$ be a proper polynomial mapping. Assume $\lambda_1(f)\mathcal{T}(f) <$*

$d_t(f)$, then the exceptional set $\mathcal{E}_f$, if non empty, is finite and consists of totally invariant cycles.

Proof. Replacing f by f^l if necessary, we can assume $\delta_1(f)\deg_{\mathcal{C}_f} f < d_t(f)$. Set $E = \{p \in \mathbb{C}^2 \,/\, \deg_p f = d_t(f)\}$. It follows from lemma 6.1.4 that E is a finite set. Passing to an iterate if necessary, we can further assume E is totally invariant using lemma 6.1.5. We claim then $\mathcal{E}_f = E$. It is sufficient to prove $\mu_{n,p}(\mathcal{C}_f) \to 0$, for all $p \notin E$, where $\mu_{n,p} = d_t^{-n}(f^n)^*\varepsilon_p$.

Set $F = \{p \in \mathbb{C}^2 \,/\, \deg_p f > \deg_{\mathcal{C}_f} f\}$. Then $F \setminus E$ consists of finitely many points with degree $\leq d_t - 1$. Let $\rho < 1$ be close to 1 (to be chosen later) and fix $p \in \mathbb{C}^2 \setminus E$. Since E is totally invariant, $f^{-n}(p) \cap E = \emptyset$ for all n. Therefore $\mu_{n,p}(F) = \mu_{n,p}(F \setminus E) \leq \sharp F (d_t - 1)^n / d_t^n$. Similarly

$$\mu_{n,p}(F \cup f^{-1}(F) \cup \ldots \cup f^{-n\rho}(F)) \leq \sum_{j=0}^{n\rho} \mu_{n-j,p}(F \setminus E) \leq C \left(\frac{d_t - 1}{d_t}\right)^{n(1-\rho)}.$$

Following [BD 01] we now count the number of points in $f^{-n}(p) \cap \mathcal{C}_f$. It follows from Bezout theorem that there are no more than $\delta_1(f^n)$ points (forgetting multiplicities). Points in $f^{-n}(p) \cap \mathcal{C}_f \setminus F \cup f^{-1}(F) \cup \ldots \cup f^{-n\rho}(F)$ have multiplicity bounded from above by $(\deg_{\mathcal{C}_f} f)^{n\rho}(d_t - 1)^{n(1-\rho)}$. Therefore we get

$$\begin{aligned} \mu_{n,p}(\mathcal{C}_f) &\leq \mu_{n,p}(F \cup \ldots \cup f^{-n\rho}(F)) + \delta_1(f^n)\frac{(\deg_{\mathcal{C}_f} f)^{n\rho}(d_t - 1)^{n(1-\rho)}}{d_t^n} \\ &\leq C\left(\frac{d_t - 1}{d_t}\right)^{n(1-\rho)} + \left(\frac{\delta_1(f)\deg_{\mathcal{C}_f}}{d_t^{\rho}}\right)^n. \end{aligned}$$

Choosing $\rho < 1$ close enough to 1 yields $\mu_{n,p}(\mathcal{C}_f) \to 0$ hence $\mu_{n,p} \to \mu_f$. $\square$

We now check that the condition $\lambda_1(f)\mathcal{T}(f) < d_t(f)$ is satisfied for quadratic families.

Lemma

6.3.

1) Let f be a mapping from family 3.1. Then $\deg_{\mathcal{C}_{f^4}} f^4 = 2$.

2) Let f be a mapping from family 3.2. Then $\deg_{\mathcal{C}_{f^5}} f^5 = 2$.

3) Let f be a mapping from family 3.3. Then $\deg_{\mathcal{C}_{f^2}} f^2 = 2$.

4) Let f be a mapping from family 3.4. Then $\deg_{\mathcal{C}_{f^2}} f^2 = 2$.

So in all cases $\lambda_1(f)\mathcal{T}(f) < d_t(f)$.

Remark

6.4. *Mappings from family 3.5 extend as holomorphic endomorphisms of $\mathbb{P}^2$. It follows from [BD 01] that $\mathcal{E}_f$ is algebraic in this case. The condition $\lambda_1(f)\mathcal{T}(f) < d_t(f)$ is not necessarily satisfied and the set $\mathcal{E}_f$ may well be infinite. In the latter case, f (or f^2) is conjugate to $(z,w) \mapsto (z^d, Q(z,w))$ so $(z=0) \subset \mathcal{E}_f$.*

Proof.

1) Consider $f(z,w) = (w, z^2 + aw + c')$. Then $\mathcal{C}_f = (z=0)$ and $\deg_{\mathcal{C}_f} = 2 = \delta_1(f)$. One easily checks that $f(\mathcal{C}_f), f^2(\mathcal{C}_f), f^3(\mathcal{C}_f)$ and $f^{-1}(\mathcal{C}_f), f^{-2}(\mathcal{C}_f)$, $f^{-3}(\mathcal{C}_f)$ are all different from $\mathcal{C}_f$. It follows therefore from lemma 6.1 that $\deg_{\mathcal{C}_{f^4}} f^4 = 2$, while $\delta_1(f^4) = 4$ and $d_t(f^4) = 16$.

Observe that $E = \{p \in \mathbb{C}^2 \,/\, \deg_p f^2 = 4 = d_t(f^2)\}$ is empty except when $a = 0$. When $a = 0$ then $E = \{(0,0)\}$ is totally invariant only when $c' = 0$. Therefore $\mathcal{E}_f = \emptyset$ except when $a = c' = 0$ in which case $\mathcal{E}_f = \{(0,0)\}$.

2) Consider $f(z,w) = (aw + c, z[z-w] + c')$, $a \neq 0$. The critical set is $\mathcal{C}_f = \{w = 2z\}$. By induction, we easily get that

$$f^j(\mathcal{C}_f) = \left\{(A_j(\zeta), B_j(\zeta)) \in \mathbb{C}^2 \,/\, \zeta \in \mathbb{C}^2\right\},$$

where A_j, B_j are polynomials of degree $\deg A_j = d_{j-1}, \deg B_j = d_j$ with $d_{j+2} = d_{j+1} + d_j$. This shows $f^j(\mathcal{C}_f) \neq \mathcal{C}_f$ for all $j \geq 1$. Similarly

$$f^{-j}(\mathcal{C}_f) = \left\{w^{d_{j-1}} z^{d_j} (z-w)^{d_j} = R_j(z,w)\right\},$$

where R_j is a polynomial of degree $\deg R_j < 2d_j + d_{j-1}$. So $f^{-j}(\mathcal{C}_f) \neq \mathcal{C}_f$ for $j \geq 1$. In particular we get $\delta_1(f^5) \cdot \deg_{\mathcal{C}_{f^5}} f^5 = 13 \cdot 2 < 32 = d_t(f^5)$.

3) Consider $f(z,w) = (az^2 + bz + c + w, z[w + \alpha z] + c')$, $a \neq 0$. The critical set $\mathcal{C}_f = \{w = 2az^2 + (b - 2\alpha)z\}$ is irreducible. We get $f^{-1}(\mathcal{C}_f) = \{w(1-z) = (\alpha - a)z^2 - bz + c' - c\} \neq \mathcal{C}_f$ and

$$f(\mathcal{C}_f) = \left\{(3a\zeta^2 + (2b - 2\alpha)\zeta, 2a\zeta^3 + (b - \alpha)\zeta^2 + c') \in \mathbb{C}^2 \,/\, \zeta \in \mathbb{C}\right\} \neq \mathcal{C}_f.$$

Therefore $\deg_{\mathcal{C}_{f^2}} f^2 = 2$, hence $\delta_1(f^2) \cdot \deg_{\mathcal{C}_{f^2}} f^2 = 8 < 9 = d_t(f^2)$.

4) Consider $f(z,w) = (zw + c, z[z + \alpha w] + bz + c' + aw)$, $a \neq 0$. The critical set $\mathcal{C}_f = \{aw = 2z^2 + bz\}$ is irreducible and straightforward computations yield again $f(\mathcal{C}_f) \neq \mathcal{C}_f \neq f^{-1}(\mathcal{C}_f)$, so $\deg_{\mathcal{C}_{f^2}} f^2 = 2$. $\square$

Remark

6.5. *It is perhaps worth mentioning that pull-backs of Dirac masses are not everywhere well defined when f is not proper. Consider e.g. $f(z,w) = (P(z), zw^2)$, where P is a polynomial of degree* $\deg P = d \geq 3$. *Then $d_t(f) = 2d > d = \lambda_1(f)$. The line $(z = 0)$ is contracted to the point $a = (P(0), 0)$. So pull-backs of Dirac masses at points $f^j(a)$ by f^n are not well defined. This shows that the orbit of point a as to be included in the exceptional set $\mathcal{E}_f$, hence we can not expect the latter to be algebraic in general.*

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