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Weak solutions to degenerate complex Monge–Ampère flows I

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Abstract Studying the (long-term) behavior of the Kähler-Ricci flow on mildly singular varieties, one is naturally lead to study weak solutions of degenerate parabolic complex Monge–Ampère equations. The purpose of this article, the first of a series on this subject, is to develop a viscosity theory for degenerate complex Monge–Ampère flows in domains of $\mathbb{C}^n$.

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1 Introduction

The study of the (long-term) behavior of the Kähler-Ricci flow on mildly singular varieties in relation to the minimal model program was undertaken by Song and Tian [28,29] and requires a theory of weak solutions for certain degenerate parabolic complex Monge–Ampère equations modelled on:

$$\frac{\partial \phi}{\partial t} + \phi = \log \frac{(dd^c \phi)^n}{V} \tag{1.1}$$

where V is volume form and ϕ a t-dependant Kähler potential on a compact Kähler manifold. The approach in [29] is to regularize the equation and take limits of the solutions of the regularized equation with uniform higher order estimates. But as far as the existence and uniqueness statements in [29] are concerned, we believe that a zeroth order approach would be both simpler and more efficient.

There is a well established pluripotential theory of weak solutions to elliptic complex Monge–Ampère equations, following the pionnering work of Bedford and Taylor [6,7] in the local case (domains in $\mathbb{C}^n$). A complementary viscosity approach has been developed only recently in [14,15,20,30] both in the local and the global case (compact Kähler manifolds).

Suprisingly no similar theory has ever been developed on the parabolic side. The most significant reference for a parabolic flow of plurisubharmonic functions on pseudoconvex domains is [18] but the flow studied there takes the form:

$$\frac{\partial \phi}{\partial t} = \left((dd^c \phi)^n\right)^{1/n} \tag{1.2}$$

which does not make sense in the global case. The purpose of this article, the first of a series on this subject, is to develop a viscosity theory for degenerate complex Monge–Ampère flows of the form (1.1).

This article focuses on solving this problem in domains of $\mathbb{C}^n$, while its companion [16] is concerned with the global case. More precisely we study here the degenerate parabolic complex Monge–Ampère equations

$$e^{\partial_t \varphi + F(t,z,\varphi_t)} \mu(z) - (dd^c \varphi_t)^n = 0 \quad \text{in} \quad \Omega_T \tag{1.3}$$

where

- $\Omega \Subset \mathbb{C}^n$ is a smooth bounded strongly pseudoconvex domain,
- $T \in]0, +\infty]$;

- $F(t, z, r)$ is continuous in $[0, T[\times\Omega \times \mathbb{R}$ and non decreasing in r,
- $\mu(z) \geq 0$ is a bounded continuous volume form on Ω,
- $\varphi : \Omega_T := [0, T[\times\Omega \to \mathbb{R}$ is the unknown function, with $\varphi_t := \varphi(t, \cdot)$.

Our plan is to adapt the viscosity approach developped by Lions et al. (see [12,23]) to the complex case, using the elliptic side of the theory which was developped in [14]. It should be noted that the method used in [29] is a version of the classical PDE method of vanishing viscosity which was superseded by the theory of viscosity solutions.

After developing the appropriate definitions of (viscosity) subsolution, supersolution and solution in the *first section*, we establish in the *second section* an important connection with the elliptic side of the theory:

Theorem A. *If u is a bounded subsolution of the above degenerate parabolic complex Monge–Ampère flow (1.3) in $]0, T[\times\Omega$, then $z \mapsto u(t, z)$ is plurisubharmonic in Ω for all $t > 0$.*

As is often the case in the viscosity theory, one of our main technical tools is the following comparison principle, which we establish in the *third section*:

Theorem B. *If u (resp. v) is a bounded subsolution (resp. supersolution) of the above degenerate parabolic equation then*

$$\max_{\Omega_T}(u - v) \leq \max\{0, \max_{\partial_0\Omega_T}(u - v)\}.$$

Here $\partial_0\Omega_T = \left(\{0\} \times \overline{\Omega}\right) \cup ([0, T[\times\partial\Omega)$ denotes the parabolic boundary of Ω_T. We actually establish several variants of the comparison principle (see Theorem 4.2 and the remarks following its proof).

In the *fourth section* we construct barriers at each point of the parabolic boundary and use the Perron method to eventually show the existence of a viscosity solution to the Cauchy–Dirichlet problem for the Complex Monge–Ampère flow (1.3) (see Sect. 1):

Theorem C. *Let φ_0 be a continuous plurisubharmonic function on $\overline{\Omega}$ such that (φ_0, μ) is admissible in the sense of Definition 5.6.*

The Cauchy–Dirichlet problem for the parabolic complex Monge–Ampère equation with initial data φ_0 admits a unique viscosity solution $\varphi(t, x)$ in infinite time; it is the upper envelope of all subsolutions.

We give simple criteria in Lemma 5.7 to decide whether a data (φ_0, μ) is admissible. This is notably always the case when $\mu > 0$ is positive, while we can not expect the existence of a supersolution if μ vanishes and φ_0 is not a maximal plurisubharmonic function.

We finally study the long term behavior of the flow in *section five*, showing that it asymptotically recovers the solution of the corresponding elliptic Dirichlet problem (see Theorems 6.1 and 6.2):

Theorem D. *Assume (φ_0, μ) is admissible and $F = F(z, r)$ is time independent. The complex Monge–Ampère flow φ_t starting at φ_0 uniformly converges, as $t \to +\infty$,* *to the solution ψ of the Dirichlet problem for the degenerate elliptic Monge–Ampère equation*

$$(dd^c\psi)^n = e^{F(z,\psi)}\mu(z) \text{ in } \Omega, \quad \text{with } \psi_{|\partial\Omega} = \varphi_0.$$

The solution ψ to the above elliptic Dirichlet problem is well known to exist in the pluripotential sense [10], while its existence in the viscosity sense was established in [14,20,30].

Pluripotential theory actually suggests that the solutions to (1.3) should be defined as upper semi continuous t-dependant plurisubharmonic functions which are a.e. derivable w.r.t to the time variable and satisfy the equation almost everywhere where $(dd^c\phi)^n$ is replaced by the Monge–Ampère operator. We did not try and phrase such a definition in a precise and usable way nor determine how it connects to the viscosity concepts developped here.

2 Parabolic viscosity concepts

2.1 A Cauchy–Dirichlet problem

Let $\Omega \Subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain and $T > 0$ a fixed number and define

$$\Omega_T :=]0, T[\times\Omega.$$

We are studying the parabolic complex Monge–Ampère equation (2.1)

$$e^{\partial_t\varphi + F(t,z,\varphi)}\,\mu(t,z) - (dd^c\varphi_t)^n = 0 \quad \text{in} \quad \Omega_T, \tag{2.1}$$

where $F(t,z,r)$ is continuous in $[0,T[\times\Omega\times\mathbb{R}$ and non decreasing in r. The measure $\mu = \mu(t,z) = \mu_t(z) \geq 0$ is assumed to be a bounded continuous non negative volume form depending continuously on the time variable t. It will be often necessary to also impose further that either $\mu > 0$ is positive, or that $\mu = f(z)\nu(t,z)$ with $f(z) \geq 0$ and $\nu = \nu(t,z) > 0$. The positive part of the density can then be absorbed in F. For simplicity, we have therefore stated our main results in the introduction in the case when $\mu = \mu(z) \geq 0$ is time independent but will use a slightly larger framework in the bulk of the article.

We call this equation the parabolic Monge–Ampère equation associated to (F,μ) in Ω_T.

Recall that the parabolic boundary of Ω_T is defined as the set

$$\partial_0\Omega_T := \left(\{0\}\times\overline{\Omega}\right) \cup ([0,T[\times\partial\Omega).$$

We want to study the Cauchy–Dirichlet problem for (2.1) with the following Cauchy–Dirichlet conditions:

$$\begin{cases} \varphi(0,z) = \varphi_0(z), & (0,z) \in \{0\} \times \overline{\Omega}, \\ \varphi(t,\zeta) = h(\zeta), & (t,\zeta) \in [0,T[\times \partial\Omega, \end{cases} \tag{2.2}$$

where $h : \partial\Omega \to \mathbb{R}$ is a continuous function (the Dirichlet boundary data) and φ_0 is a bounded plurisubharmonic function in Ω (the Cauchy data), which extends continuously to $\overline{\Omega}$.

Thus h is actually determined by the boundary values of φ_0. Such a function φ_0 will be called the *Cauchy–Dirichlet data* for the parabolic complex Monge–Ampère equation (2.1) and we will simply write

$$\varphi_{|\partial\Omega_T} = \varphi_0.$$

2.2 Parabolic sub/super-solutions

We assume the reader has some familiarity with the elliptic side of the viscosity theory for complex Monge–Ampère equations which was developed in [14].

The definitions of subsolutions and supersolutions can be extended to the parabolic setting using upper and lower test functions as in the degenerate elliptic case.

We first define what should be a classical solution to our problem. A classical solution to the parabolic complex Monge–Ampère equation (2.1) is a continuous function $\varphi : [0,T[\times\overline{\Omega} \longrightarrow \mathbb{R}$ which is C^1 in t, C^2 in z in $]0,T[\times\Omega$ such that for any $t \in]0,T[$, the function $z \longmapsto \varphi(t,z)$ is a (continous) plurisubharmonic function in Ω that satisfies the following equation

$$(dd^c\varphi_t)^n = e^{\partial_t\varphi(t,z)+F(t,z,\varphi(t,z))}\mu(t,z),$$

for all $z \in \Omega$. The function φ is said to be C^1 in t and C^2 in z (or $C^{(1,2)}$ in short) in $]0,T[\times\Omega$ if $(t,z) \longrightarrow \partial_t\varphi(t,x)$ exists and is continuous in $]0,T[\times\Omega$ and the second partial derivatives of $z \longrightarrow \varphi(t,z)$ with respect to z_j and $\bar{z}_k$ exists and are continuous in all the variables (t,z) in $]0,T[\times\Omega$.

Observe that if we split this equality into two inequalities $\geq$ (resp. $\leq$), we obtain the notion of a classical subsolution (resp. supersolution) to the parabolic Eq. (2.1).

Now let us introduce the general definition.

Definition 2.1 *(Test functions)* Let $w : \Omega_T \longrightarrow \mathbb{R}$ be any function defined in Ω_T and $(t_0,z_0) \in \Omega_T$ a given point. An upper test function (resp. a lower test function) for w at the point (t_0,z_0) is a $C^{(1,2)}$-smooth function q in a neighbourhood of the point (t_0,z_0) such that $w(t_0,z_0) = q(t_0,z_0)$ and $w \leq q$ (resp. $w \geq q$) in a neighbourhood of (t_0,z_0). We will write for short $w \leq_{(t_0,z_0)} q$ (resp. $w \geq_{(t_0,z_0)} q$).

Definition 2.2 1. A function $u : [0,T[\times\overline{\Omega} \longrightarrow \mathbb{R}$ is said to be a (viscosity) subsolution to the parabolic complex Monge–Ampère equation (2.1) in $]0,T[\times\Omega$ if u is upper semi-continuous in $[0,T[\times\overline{\Omega}$ and for any point $(t_0,z_0) \in \Omega_T :=]0,T[\times\Omega$ and any upper test function q for u at (t_0,z_0), we have

$$\left(dd^c q_{t_0}(z_0)\right)^n \geq e^{\partial_t q(t_0,z_0)+F(t_0,z_0,q(t_0,z_0))}\mu(t_0,z_0).$$

In this case we also say that u satisfies the differential inequality $(dd^c\varphi_t)^n \geq e^{\partial_t\varphi(t,z)+F(t,z,\varphi(t,z))}\mu(t,z)$ in the viscosity sense in Ω_T.

2. A function $v : [0,T[\times\overline{\Omega} \longrightarrow \mathbb{R}$ is said to be a (viscosity) supersolution to the parabolic complex Monge–Ampère equation (2.1) in $\Omega_T =]0,T[\times\Omega$ if v is lower semi-continuous in Ω_T and for any point $(t_0,z_0) \in]0,T[\times\Omega$ and any lower test function q for v at (t_0,z_0) such that $dd^c q_{t_0}(z_0) \geq 0$, we have

$$(dd^c q_{t_0})^n(z_0) \leq e^{\partial_t q(t_0,z_0)+F(t_0,z_0,q(t_0,z_0))}\mu(t_0,z_0).$$

In this case we also say that v satisfies the differential inequality $(dd^c\varphi_t)^n \leq e^{\partial_t\varphi(t,z)+F(t,z,\varphi(t,z))}\mu(t,z)$ in the viscosity sense in Ω_T.

3. A function $\varphi : [0,T[\times\overline{\Omega} \longrightarrow \mathbb{R}$ is said to be a (viscosity) solution to the parabolic complex Monge–Ampère equation (2.1) in $]0,T[\times\Omega$ if it is a subsolution and a supersolution to the parabolic complex Monge–Ampère equation (2.1) in $]0,T[\times\Omega$. Hence φ is continuous in $[0,T[\times\overline{\Omega}$.

We let the reader check that a classical (sub/super) solution of equation (2.1) is a viscosity (sub/super) solution.

Remark 2.3 In order to fit into the framework of viscosity theory, we consider the function $H : [0,T[\times\Omega\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n\times\mathcal{S}_{2n} \longrightarrow \mathbb{R}\cup\{+\infty\}$ defined by

$$\begin{cases} H(t,z,r,\tau,p,Q) := e^{\tau+F(t,z,r)}\mu(t,z) - (dd^c Q)^n, & \text{if } dd^c Q \geq 0, \\ H(t,z,r,\tau,p,Q) := +\infty, & \text{if not,} \end{cases},$$

where $dd^c Q$ is the hermitian $(1,1)$-part of the quadratic form Q in $\mathbb{C}^n \simeq \mathbb{R}^{2n}$.

Observe that the function H is lower semi-continuous in the set $[0,T[\times\Omega\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n\times\mathcal{S}_{2n}$, continuous in its domain

$$Dom H := \{H < +\infty\} = [0,T[\times\Omega\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n\times\{Q\in\mathcal{S}_{2n}; dd^c Q \geq 0\}$$

and is degenerate elliptic in the sense of [12]. Moreover it is non decreasing in the r variable. We call it the *Hamilton function* of the parabolic complex Monge–Ampère equation (2.1).

Observe that if u is a subsolution (resp. a supersolution) of the parabolic equation $H = 0$ then it is a subsolution of the degenerate elliptic equation $H = 0$ in $2n+1$ variables $(t,z) \in]0,T[\times\Omega \subset \mathbb{R}^{2n+1}$ of a special type which does not depend on the gradient w.r.t. z nor on the second derivative w.r.t. t. Actually the two notions are equivalent but we will not use this (see [12]).

The notions of subsolutions and supersolutions for the parabolic equation $H = 0$ as defined in [12] are exactly the ones defined above.

However as far as supersolutions are concerned, it is more useful to work with the finite Hamilton function H_+, where

$$H_+(t,z,r,\tau,p,Q) := e^{\tau+F(t,z,r)}\mu(t,z) - (dd^c Q)_+^n,$$

and $(dd^c Q)_+ = dd^c Q$ if $dd^c Q \geq 0$ and $(dd^c Q)_+ = 0$ if not.

Observe that $H_+ : [0, T[\times\Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}_{2n} \longrightarrow \mathbb{R}$ is an upper semi-continuous and finite Hamilton function such that $H_+ = H$ in $Dom H$, the domain of H .

Therefore most of the general principles of the viscosity method as explained in [12] can be also applied here (at least formally). On the other hand we have to be careful since there is no symmetry between subsolutions and supersolutions.

It follows from [14] that if u is a subsolution to the parabolic equation $H = 0$, any parabolic upper test function q for u at (t_0, z_0) satisfies the condition $dd^c q_{t_0}(z_0) \geq 0$. Hence u is a subsolution to the parabolic equation $H_+ = 0$, but the converse is not true unless $\mu > 0$ (see [14]).

Since the fundamental Jensen–Ishii's maximum principle will be stated in terms of semi-jets, it is convenient to use these notions which we now introduce following [12], in order to characterize as well the notions of sub/super solutions.

Definition 2.4 Let $u : \Omega_T \longrightarrow \mathbb{R}$ be a fixed function. For $(t_0, z_0) \in \Omega_T$, the parabolic second order superjet of u at (t_0, z_0) is the set of $(\tau, p, Q) \in \mathbb{R} \times \mathbb{R}^{2n} \times \mathcal{S}_{2n}$ such that for $(t, z) \in \Omega_T$,

$$\begin{aligned} u(t, z) \leq u(t_0, z_0) + \tau(t - t_0) + o(|t - t_0|) \\ + \langle p, z - z_0\rangle + \frac{1}{2}\langle Q(z - z_0), z - z_0\rangle + o(|z - z_0|^2). \end{aligned}$$

We let $\mathcal{P}^{2,+}u(t_0, z_0)$ denote the set of parabolic second order superjets of u at (t_0, z_0). We define in the same way the set $\mathcal{P}^{2,-}u(t_0, z_0)$ of parabolic second order subjets of u at (t_0, z_0) by

$$\mathcal{P}^{2,-}u(t_0, z_0) = -\mathcal{P}^{2,+}(-u)(t_0, z_0).$$

The set of parabolic second order jets of u at (t_0, z_0) is defined by

$$\mathcal{P}^2 u(t_0, z_0) = \mathcal{P}^{2,+}u(t_0, z_0) \cap \mathcal{P}^{2,-}u(t_0, z_0).$$

We will need a slightly more general notion (see [12]).

Definition 2.5 Let $u : \Omega_T \longrightarrow \mathbb{R}$ be a fixed function and $(t_0, z_0) \in \Omega_T$. The set $\bar{\mathcal{P}}^{2,+}u(t_0, z_0)$ of approximate parabolic second order superjets of u at (t_0, z_0) is defined as the set of $(\tau, p, Q) \in \mathbb{R} \times \mathbb{R}^{2n} \times \mathcal{S}_{2n}$ such that there exists a sequence $(t_j, z_j) \in]0, T[\times\Omega$ converging to (t_0, z_0), such that $u(t_j, z_j) \to u(t_0, z_0)$ and a sequence $(\tau_j, p_j, Q_j) \in \mathcal{P}^{2,+}u(t_j, z_j)$ converging to (τ, p, Q).

In the same way we define the set $\bar{\mathcal{P}}^{2,-}u(t_0, z_0) := -\bar{\mathcal{P}}^{2,+}(-u)(t_0, z_0)$ of approximate parabolic second order subjets of u at (t_0, z_0).

Proposition 2.6 1. *An upper semi-continuous function $u : \Omega_T \longrightarrow \mathbb{R}$ is a subsolution to the parabolic equation (2.1) if and only if for all $(t_0, z_0) \in \Omega_T$ and $(\tau, p, Q) \in \mathcal{P}^{2,+}u(t_0, z_0)$, we have*

$$e^{\tau+F(t_0,z_0,u(t_0,z_0))}\mu(t_0, z_0) \leq (dd^c Q)^n. \tag{2.3}$$

2. *A lower semi-continuous function* $v : \Omega_T \longrightarrow \mathbb{R}$ *is a supersolution to the parabolic equation (2.1) if and only if for all* $(t_0, z_0) \in \Omega_T$ *and* $(\tau, p, Q) \in \mathcal{P}^{2,-}v(t_0, z_0)$ *such that* $dd^c Q \geq 0$, *we have*

$$e^{\tau + F(t_0, z_0, v(t_0, z_0))}\mu(t_0, z_0) \geq (dd^c Q)^n. \tag{2.4}$$

Another way to phrase the definition of supersolutions is to require that, for all $(t_0, z_0) \in \Omega_T$ and all $(\tau, p, Q) \in \mathcal{P}^{2,-}v(t_0, z_0)$, we have

$$e^{\tau + F(t_0, z_0, v(t_0, z_0))}\mu(t_0, z_0) \geq (dd^c Q)^n_+.$$

This statement necessitates some comments:

(1) The reader will easily check that when u is a subsolution (resp. supersolution) to the Eq. (2.1), the inequalities (2.3) (resp. 2.4) are satisfied for all $(\tau, p, Q) \in \overline{\mathcal{P}}^{2,+}u(t_0, z_0)$ (resp. $\overline{\mathcal{P}}^{2,-}v(t_0, z_0)$).
(2) If for a fixed $z_0 \in \Omega$, the function $t \longmapsto u(t, z_0)$ is L-lipschitz in a neighborhood of $t_0 \in]0, T[$. Then for any $(\tau, p, Q) \in \overline{\mathcal{P}}^{2,+}u(t_0, z_0)$, we have $|\tau| \leq L$. Indeed for $|s| << 1$ and $|z - z_0| << 1$,

$$u(t_0 + s, z) \leq u(t_0, z_0) + \tau s + \langle p, z - z_0\rangle + \frac{1}{2}\langle Q(z - z_0), z - z_0\rangle + o(|s| + |z - z_0|^2),$$

hence $-L|s| \leq u(t_0 + s, z_0) - u(t_0, z_0) \leq \tau s + o(|s|)$ for $|s|$ small enough and the conclusion follows.
(3) A discontinuous viscosity solution to the equation (2.1) (in the the sense of [22]) is a function $u : \Omega_T \to [+\infty, -\infty]$ such that
   (i) the usc envelope u^* of u satisfies $\forall x,\ u^*(x) < +\infty$ and is a viscosity subsolution to the Eq. (2.1),
   (ii) the lsc envelope u_* of u satisfies $\forall x,\ u_*(x) > -\infty$ and is a viscosity supersolution to the Eq. (2.1).

If we consider a time independent equation, its static viscosity (sub/super) solutions (i.e.: those who are independent of the time variable) are the time independent extension of the viscosity (sub/super) solutions of the corresponding complex Monge–Ampère equation in the sense of [14] where discontinuous viscosity solutions were not considered.

We introduce discontinuous viscosity solutions here for technical reasons that will be explained later on. Note that the characteristic function u of $\mathbb{C}\backslash\mathbb{Q}^2$ is a discontinuous viscosity solution of $\Delta u = 0$.

2.3 Relaxed semi-limits

Let (h_j) be a sequence of locally uniformly bounded functions on a metric space (Y, d). The upper relaxed semi-limit of (h_j) is

$$\overline{h}(y) = \limsup^*_{j\to+\infty} h_j(y) := \lim_{j\to+\infty} \sup\{h_k(z); k \geq j, d(z, y) \leq 1/j\}.$$

The reader will easily check that $\overline{h}$ is upper semi-continuous on Y.

We define similarly the lower relaxed semi-limit of the sequence (h_j),

$$\underline{h} = \liminf_{*j\to+\infty} h_j.$$

This is a lower semi-continuous function in Y. Observe that

$$\liminf_{*j\to+\infty} h_j \leq (\liminf_{j\to\infty} h_j)_* \leq (\limsup_{j\to+\infty} h_j)^* \leq \limsup^*_{j\to+\infty} h_j.$$

If (h_j) is a non decreasing (resp. non increasing) sequence of continuous functions on Y then $\overline{h} = (\sup h_j)^*$ (resp. $\underline{h} = (\inf h_j)_*$). Moreover if (h_j) converges locally uniformly to a continuous function h on Y then all these limits coincide with h on Y.

The following stability result for viscosity sub/super-solutions is a classical and useful tool (see [12,24]):

Lemma 2.7 *Let $\mu^j(t, x) \geq 0$ be a sequence of continuous volume forms converging uniformly to a volume form μ on Ω_T and let F^j be a sequence of continuous functions in $[0, T[\times\Omega \times \mathbb{R}$ converging locally uniformly to a function F. Let (φ^j) be a locally uniformly bounded sequence of real valued functions defined in Ω_T.*

1. *Assume that for every $j \in \mathbb{N}$, φ^j is a viscosity subsolution to the complex Monge–Ampère flow*

$$e^{\partial_t \varphi^j + F^j(t,z,\varphi^j)} \mu^j(t, z) - (dd^c \varphi_t^j)^n = 0,$$

associated to (F^j, μ^j) in Ω_T. Then its upper relaxed semi-limit

$$\overline{\varphi} = \limsup^*_{j\to+\infty} \varphi^j$$

of the sequence (φ_j) is a subsolution to the parabolic Monge–Ampère equation

$$e^{\partial_t \varphi + F(t,z,\varphi)} \mu - (dd^c \varphi_t)^n = 0,$$

in Ω_T.

2. *Assume that for every $j \in \mathbb{N}$, φ^j is a viscosity supersolution to the complex Monge–Ampère flow associated to (F^j, μ^j) in Ω_T. Then the lower relaxed semi-limit*

$$\underline{\varphi} = \liminf_{*j\to+\infty} \varphi^j$$

of the sequence (φ^j) *is a supersolution to the complex Monge–Ampère flow associated to* (F, μ) *in* Ω_T.

It is a remarkable fact that we do not need any a priori estimate on the time derivatives to pass to the limit in the viscosity differential inequalities.

Remark 2.8 An important example in applications is when $F(t, z, r) = \alpha r$ with $\alpha \geq 0$. In this case a simple change of variables reduces the general case $\alpha > 0$ to the case when $\alpha = 0$. Indeed if $\alpha > 0$ set

$$\psi(s, z) := \alpha(1 + s)\varphi(t, z), \text{ with } t := \alpha^{-1} \log(1 + s),$$

and observe that

$$\partial_s \psi(s, z) = \alpha\varphi(t, z) + \partial_t \varphi(t, z).$$

Thus φ is a (sub/super)solution to the the parabolic Monge– Ampère equation

$$\exp(\partial_t \varphi + \alpha\varphi)\, \mu - (dd^c \varphi_t)^n = 0,$$

if and only ψ is a (sub/super)solution to

$$e^{\partial_t \psi} \tilde{\mu}(s, \cdot) - (dd^c \psi_s)^n = 0,$$

where $\tilde{\mu}(s, z) = \alpha^n (s + 1)^n \mu(z)$.

3 The parabolic Jensen–Ishii's maximum principle

3.1 Maximum principles

Recall that a function $u : U \subset \mathbb{R}^N \longrightarrow \mathbb{R}$ is semi-convex in U if for each small ball $B \Subset U$, there exists a constant $A > 0$ such that the function $x \longmapsto u(x) + A|x|^2$ is convex in B.

We also recall that the upper second order jet $\mathcal{J}^{2,+}u(x_0)$ at $x_0 \in U$ of a function $u : U \longrightarrow \mathbb{R}$ is the set of $(p, Q) \in \mathbb{R}^N \times \mathcal{S}_N$ s.t. for x close to x_0,

$$u(x) \leq u(x_0) + \langle p, x - x_0 \rangle + \frac{1}{2}\langle Q(x - x_0), x - x_0 \rangle + o(|x - x_0|^2).$$

The set $\bar{\mathcal{J}}^{2,+}u(x_0)$ of approximate second order superjets is then defined in the same way as in Definition 2.5.

The following is a consequence of the fundamental Theorem of Jensen on which the Jensen–Ishii maximum principle is based (see [9,12]):

Theorem 3.1 *Let u be a semi-convex function in an open set $U \subset \mathbb{R}^N$, attaining a local maximum at some point $x_0 \in U$. Then there exists $(p, Q) \in \bar{\mathcal{J}}^{2,+}u(x_0)$ such that $p = 0$ and $Q \leq 0$.*

More precisely for any subset $E \subset U$ of Lebesgue measure 0, there exists a sequence (x_k) of points in $U \setminus E$ such that $x_k \to x_0$, $u(x_k) \to u(x_0)$, u is twice differentiable at each point x_k for $k > 1$, $Du(x_k) \to 0$ and $D^2u(x_k) \to Q \leq 0$ as $k \to +\infty$.

A crucial ingredient is Alexandrov's theorem on almost everywhere second order differentiability of convex functions [1]. From this we can derive the following useful result:

Lemma 3.2 *Let $U \subset \mathbb{R}^N$ be an open set and $H : U \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}_N \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a function.*

1. *Assume that H is lower semi-continuous and degenerate elliptic. Let w be a semi-convex function in the open set $U \subset \mathbb{R}^N$ such that for almost all $x_0 \in U$,*

$$H\,(x_0, w(x_0), p, Q) \leq 0, \forall (p, Q) \in \mathcal{J}^{2,+}w(x_0).$$

Then w is a (viscosity) subsolution to the equation $H = 0$ in U.

2. *Assume that $H : U \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}_N \longrightarrow \mathbb{R}$ is finite, upper semi-continuous and degenerate elliptic. Let w be a semi-concave function in the open set $U \subset \mathbb{R}^N$ such that for almost all $x_0 \in U$,*

$$H\,(x_0, w(x_0), p, Q) \geq 0, \forall (p, Q) \in \mathcal{J}^{2,-}w(x_0).$$

Then w is a (viscosity) supersolution to the equation $H = 0$ in U.

In other words, if w is a subsolution (resp. supersolution) of the equation $H = 0$ almost everywhere in U , then it is a subsolution (resp. supersolution) everywhere.

Proof We prove the first statement concerning subsolutions. The second statement concerning supersolutions can be proved in the same way.

We will use the maximum principle of Jensen for semi-convex functions. Let q be a C^2 upper test function for w at a fixed point x_0. Thus $u := w - q$ is semi-convex function in U which takes a local maximum at x_0.

Let us denote by E the exceptional set of points where the viscosity inequality in the lemma is not satisfied. Since E has Lebesgue measure 0, it follows from the local maximum principle of Jensen for the semi-convex function u that there exists a sequence x_k in $U \setminus E$ converging to x_0 such that u is twice differentiable at x_k, $Du(x_k) \to 0$ and $D^2u(x_k) \to A \leq 0$ as $k \to +\infty$ i.e. $(0, A) \in \bar{\mathcal{J}}^{2,+}u(x_0)$ and $A \leq 0$.

By definition $Du(x_k) = Dw(x_k) - Dq(x_k)$ and $D^2u(x_k) = D^2w(x_k) - D^2q(x_k)$ for any k hence $p_k := Dw(x_k) = Du(x_k) + Dq(x_k) \to Dq(x_0)$ and $Q_k := D^2w(x_k) \to A + D^2q(x_0) =: Q$. Therefore $Q \leq D^2q(x_0)$ since $A \leq 0$.

By the choice of x_k, we infer $H(x_k, p_k, Q_k) \leq 0$. By the lower semi-continuity of H we get at the limit $H(x_0, Dq(x_0), Q) \leq 0$. Since $Q \leq D^2q(x_0)$, by the degenerate ellipticity condition, we conclude that

$$H\left(x_0, Dq(x_0), D^2q(x_0)\right) \leq 0.$$

Thus w satisfies the viscosity differential inequality at each point of U. $\square$

We now state the parabolic Jensen–Ishii's maximum principle ([12], [13, p.65]):

Theorem 3.3 *Let $\Omega \subset \mathbb{R}^N$ be a domain, u an upper semi-continuous function and v a lower semi-continuous function in $]0, T[\times\Omega$. Let ϕ be a function defined in $]0, T[\times\Omega^2$ such that $(t, x, y) \longmapsto \phi(t, x, y)$ is continuously differentiable in t and twice continuously differentiable in (x, y).*

Assume that the function $(t, x, y) \longmapsto u(t, x) - v(t, y) - \phi(t, x, y)$ has a local maximum at some point $(\hat{t}, \hat{x}, \hat{y}) \in]0, T[\times\Omega^2$.

Assume furthermore that both $w = u$ and $w = -v$ satisfy:

$$\begin{cases} \forall (s, z) \in \Omega\ \exists r > 0 \text{ such that } \forall M > 0\ \exists C \text{ satisfying} \\ \qquad \left.\begin{array}{l} |(t, x) - (s, z)| \le r, \\ (\tau, p, Q) \in \mathcal{P}^{2,+} w(t, x) \\ |w(t, x)| + |p| + |Q| \le M \end{array}\right\} \Longrightarrow \tau \le C. \end{cases}$$

Then for any $\kappa > 0$, there exists $(\tau_1, p_1, Q^+) \in \bar{\mathcal{P}}^{2,+} u(\hat{t}, \hat{x})$, $(\tau_2, p_2, Q^-) \in \bar{\mathcal{P}}^{2,-} v(\hat{t}, \hat{y})$ such that

$$\tau_1 = \tau_2 + D_t\phi(\hat{t}, \hat{x}, \hat{y}),\ \ p_1 = D_x\phi(\hat{t}, \hat{x}, \hat{y}),\ \ p_2 = -D_y\phi(\hat{t}, \hat{x}, \hat{y})$$

and

$$-\left(\frac{1}{\kappa} + \|A\|\right) I \le \begin{pmatrix} Q^+ & 0 \\ 0 & -Q^- \end{pmatrix} \le A + \kappa A^2,$$

in the sense of quadratic forms on $\mathbb{R}^N$, where $A := D^2_{x,y}\phi(\hat{t}, \hat{x}, \hat{y})$.

Remark 3.4 Condition (2.3) is automatically satisfied for w locally Lipschitz in the time variable or if w is a subsolution of (1.1) with $\mu > 0$. It need not be satisfied for a general supersolution of (1.1) even if $\mu > 0$.

3.2 Regularizing in time

Given a bounded upper semi-continuous function $u : [0, T[\times\Omega \longrightarrow \mathbb{R}$, we consider the upper approximating sequence by Lipschitz functions in t,

$$u^k(t, x) := \sup\{u(s, x) - k|s - t|, s \in [0, T[\},\ \ (t, x) \in [0, T[\times\Omega.$$

If v is a bounded lower semi-continuous function, we consider the lower approximating sequence of Lipschitz functions in t,

$$v_k(t, x) := \inf\{v(s, x) + k|s - t|, s \in [0, T[\},\ \ (t, x) \in [0, T[\times\Omega.$$

Lemma 3.5 *For $k \in \mathbb{R}^+$, u^k is an upper semi-continuous function which satisfies the following properties:*

- *$u(t, z) \le u^k(t, z) \le \sup_{|s-t| \le A/k} u(s, z)$, where $A > 2osc_{X_T} u$.*

- $|u^k(t,x) - u^k(s,x)| \le k|s-t|$, *for* $(s,z) \in [0,T[\times\Omega$, $(t,z) \in [0,T[\times\Omega$.
- *For all* $(t_0, z_0) \in [0, T - A/k] \times \Omega$, *there exists* $t_0^* \in [0, T[$ *such that*

$$|t_0^* - t_0| \le A/k \text{ and } u^k(t_0, z_0) = u(t_0^*, z_0) - k|t_0 - t_0^*|.$$

Moreover if u *satisfies*

$$e^{\partial_t u + F(t, u_t, \cdot)} \mu(t,\cdot) \le (dd^c u_t)^n \text{ in }]0, T[\times\Omega, \tag{3.1}$$

where $\mu = \mu(\cdot,\cdot) \ge 0$ *is a continuous Borel measure in* Ω_T, *then the function* u^k *is a subsolution of*

$$e^{\partial_t w + F_k(t, u_t, \cdot)} \mu_k(t,\cdot) - (dd^c w_t)^n = 0 \text{ in }]A/k, T - A/k[\times\Omega,$$

where $F_k(t,x,z) := \inf_{|s-t|\le A/k} F(s,x,z) + k|s-t|$ *and* $\mu_k(t,z) := \inf_{|s-t|\le A/k} \mu(s,z)$. *The dual statement is true for a lower semi-continuous function* v *which is a supersolution.*

Proof The first statement is elementary. Let us prove the second one in the same spirit as [11]. Let $(t_0, z_0) \in]0, T[\times\Omega$ be fixed and let $q(t,z)$ be an upper test function that touches u^k from above at (t_0, z_0). Consider for k large enough, the following smooth function given by

$$q^*(t,z) := q(t + t_0 - t_0^*, z) + k|t_0 - t_0^*|.$$

Then q^* is an upper test function for u at the point (t_0^*, z_0). Since u satisfies the differential inequality (3.1), we have

$$e^{\partial_t q^*(t_0^*, z_0) + F(t_0^*, q^*(t_0, z_0), z_0)} \mu(t_0^*, z_0) \le (dd^c q^*_{t_0^*}(z_0))^n.$$

Since $\partial_t q^*(t_0^*, z_0) = \partial_t q(t_0, z_0)$, $q^*(t_0^*, z_0) = q(t_0, z_0) + k|t - t_0^*|$ and $dd^c q^*_{t_0^*}(z_0) = dd^c q_{t_0}(x_0)$ and F is non decreasing, we deduce the following inequality

$$e^{\partial_t q(t_0, z_0) + F(t_0^*, q(t_0, z_0), z_0)} \mu(t_0^*, z_0) \le (dd^c q_{t_0}(z_0))^n,$$

which proves the statement of the lemma since $\mu(t_0^*, z_0) \ge \mu_k(t_0, z_0)$ and $F(t_0^*, q(t_0, z_0), z_0) \ge F_k(t_0, q(t_0, z_0), z_0)$.

For a supersolution the same proof works modulo obvious modifications. □

3.3 Spatial plurisubharmonicity of parabolic subsolutions

We first connect sub/super-solutions of certain degenerate elliptic complex Monge–Ampère equations to sub/super-solutions properties of their slices.

Proposition 3.6 *Let* $G :]0, T[\times\mathbb{R}\times\Omega \longrightarrow \mathbb{R}$ *be a continuous function,* $\nu(t, z) = \nu_t(z)$ *a continuous family of volume forms and let* $w :]0, T[\times\Omega \longrightarrow \mathbb{R}$ *be a subsolution (resp. supersolution) to the degenerate elliptic complex Monge–Ampère equation*

$$e^{G(t,z,w)}\nu(t, z) - (dd^c w)^n = 0,$$

in the viscosity sense in $]0, T[\times\Omega$*. Then for all* $t_0 \in [0, T[$*, the function*

$$w_{t_0} : z \longmapsto w(t_0, z)$$

is a subsolution (resp. supersolution) to the degenerate elliptic equation $e^{G(t_0,z,\psi)}\nu_{t_0} - (dd^c w_{t_0})^n = 0$ *in* Ω*.*

Let us stress that these equations do not contain any time derivative $\partial_t w$!

Proof We give the proof for the supersolution case and let the reader deal with the (slightly simpler) case of subsolutions.

Assume that w satisfies the differential inequality

$$(dd^c w)^n \le e^{G(t,z,w)}\nu(t, z)$$

in the sense of viscosity in $U :=]0, T[\times\Omega$. We approximate w by inf-convolution $w_\varepsilon(t, z)$ in all variables, the function w_ε is defined in the open set $U_\varepsilon :=]A\varepsilon, T - A\varepsilon[\times\Omega_\varepsilon \subset U$ for $\varepsilon > 0$ small, where

$$\Omega_\varepsilon = \{z \in \Omega \mid d(z, \partial\Omega) > A\varepsilon\}, \text{ with } A := 4\,\mathrm{osc}_X(u).$$

It is a classical fact (see [11]) that the function $v := w_\varepsilon$ is a supersolution of an approximate parabolic Monge–Ampère equation: it satisfies the differential inequality

$$(dd^c v_\varepsilon)^n \le e^{G^\varepsilon(t,z,v_\varepsilon)}\nu^\varepsilon,$$

in the sense of viscosity, where

$$\nu_\varepsilon(t, z) := \sup\{\nu(t', z'); |t' - t|, |z' - z| \le A\varepsilon\}$$

and G^ε is defined similarly.

Since w_ε is semi-concave in U_ε, it follows from Alexandrov's theorem that it is twice differentiable almost everywhere in U_ε. The above inequality is therefore satisfied pointwise almost everywhere, i.e. at each point (t, z) where w_ε has a second order jet. Observe that for almost all $t_0 \in]A\varepsilon, T - A\varepsilon[$, there exist a set $E^{t_0} \subset \Omega_\varepsilon$ of Lebesgue measure 0 such that for any $z_0 \notin E^{t_0}$, the function w_ε is twice differentiable at (t_0, z_0). By definition we have $\mathcal{J}^2 w_\varepsilon(t_0, z_0) = \{(\tau, p, \kappa, Q)\}$ and $\{(p, Q)\} = \mathcal{J}^2\psi(z_0)$, where $\psi = w_\varepsilon(t_0, \cdot)$. The viscosity inequality satisfied by w at (t_0, z_0) implies

$$(dd^c Q)_+^n \le e^{G^\varepsilon(t_0,z_0,v(t_0,z_0))}\nu^\varepsilon(t_0, z_0).$$

It follows that for almost all fixed $t_0 \in]0, T[$, the function $\psi(z) := w_\varepsilon(t_0, z)$ is pointwise second order differentiable at almost all $z_0 \in \Omega_\varepsilon$ and satisfies

$$\left(dd^c \psi(z_0)\right)_+^n \leq e^{G^\varepsilon(t_0, z_0, \psi(z_0))} \nu^\varepsilon(t_0, z_0).$$

Lemma 3.2 now shows that ψ satisfies the viscosity inequality $(dd^c\psi)^n \leq e^{F^e(t_0,\cdot,\psi)} \nu^\varepsilon(t_0, \cdot)$ at every point of Ω_ε. Since $\nu^\varepsilon \to \nu$ and $G^\varepsilon \to G$ locally uniformly in U, it follows from the stability Lemma 2.7 that $w(t_0, \cdot) = \lim_{\varepsilon \to 0} w_\varepsilon(t_0, \cdot)$ is a supersolution to the degenerate elliptic equation

$$e^{G(t_0,\cdot,\psi)} \nu_{t_0} - (dd^c\psi)^n = 0$$

in the sense of viscosity in Ω. This is true for almost every $t_0 \in]0, T[$. Now given any $t_0 \in]0, T[$, one can find a sequence of points (t^j) converging to t_0 in $]0, T[$ such that for every $j \in \mathbb{N}$, the function $\psi^j := \psi(t^j, \cdot)$ is a supersolution to the degenerate elliptic equation associated to (G^j, ν^j), where $G^j := G(t^j, \cdot)$ and $\nu^j := \nu(t^j, \cdot)$. Since $G^j \to G(t_0, \cdot)$ and $\nu^j \to \nu(t_0, \cdot)$ locally uniformly in Ω, it follows from the stability Lemma in the degenerate elliptic case (see [12]) that $\psi(t_0, \cdot)$ is a supersolution to the degenerate elliptic equation associated to $(G(t_0, \cdot), \nu(t_0, \cdot)$. □

As a consequence we show that subsolutions to parabolic complex Monge–Ampère equations are plurisubharmonic in the space variable:

Corollary 3.7 *Assume that u is a bounded subsolution to the parabolic Monge–Ampère equation (2.1) in $]0, T[\times\Omega$. Then for any fixed $t_0 \in [0, T[$, the function*

$$z \mapsto u(t_0, z) \text{ is plurisubharmonic in } \Omega,$$

Moreover for all $(t_0, z_0) \in \Omega_T$ and $(\tau, p, Q) \in \mathcal{P}^{2,+}u(t_0, z_0)$, we have $dd^c Q \geq 0$ and $dd^c Q > 0$ when $\mu(t_0, z_0) > 0$.

Proof We consider here the parabolic Monge–Ampère equation (2.1) as a degenerate elliptic equation on $]0, T[\times\Omega$ as explained in Remark 2.3.

Since u is a subsolution to the parabolic Monge–Ampère equation (2.1), it is also a subsolution to the degenerate elliptic equation $(dd^c u_t)^n = 0$ in $]0, T[\times\Omega$. Applying Proposition 3.6 with $\mu \equiv 0$, we conclude that for any fixed $t_0 \in]0, T[$, the function $w := u(t_0, \cdot)$ is a subsolution of the degenerate elliptic equation $(dd^c w)^n = 0$ in Ω.

Therefore by [14] the function $\varphi = u(t_0, \cdot)$ is psh in Ω. The last statement follows also from [14]. □

Proposition 3.8 *Assume that $\mu \equiv 0$ vanishes identically in some open set $D \subset \Omega$ and v is a bounded supersolution to the parabolic Monge–Ampère equation (2.1) in* $]0, T, [\times D$.

Then for all $t_0 \in]0, T[$ the function $z \mapsto v(t_0, z)$ is a supersolution to the degenerate elliptic equation $(dd^c w)^n = 0$ in D i.e. $(dd^c v_{t_0})^n \leq 0$ in the viscosity sense in D.

If v is moreover continuous in $]0, T, [\times D$ then the plurisubharmonic envelope $P(v_{t_0}) = \sup\{u \mid u$ psh in D and $u \le v_{t_0}\}$ of the function $z \mapsto v(t_0, z)$ satisfies

$$\left(dd^c P(v_{t_0})\right)^n = 0$$

in the viscosity sense in D, hence it is a maximal psh function in D.

Recall that a psh function u is *maximal* (see [26]) if for every relatively compact open set $U \subset D$ and every psh continuous function h on $\overline{U}$,

$$h \le u \text{ on } \partial U \Rightarrow h \le u \text{ in } U.$$

Proof Since v is a bounded supersolution to the parabolic Monge–Ampère equation (2.1) in $]0, T, [\times D$ and $\mu \equiv 0$ in D, it follows that v is a supersolution to the degenerate elliptic equation $(dd^c v)^n = 0$ in $]0, T[\times D$. Using Lemma 3.6 with $\mu \equiv 0$, we infer that for $t_0 \in]0, T[$, the function $w := v(t_0, \cdot)$ is a supersolution of the degenerate elliptic equation $(dd^c w)^n = 0$ in D.

When w is continuous, it follows from [14, Lemma 4.7] that its plurisubharmonic envelope $\theta := P(w)$ is a (viscosity) supersolution to the equation $(dd^c\theta)^n = 0$ in D. Since θ is also plurisubharmonic, we infer that θ is a viscosity solution to the homogeneous complex Monge–Ampère equation $(dd^c\theta)^n = 0$ in D.

Fix a ball $\mathbb{B} \Subset D$ and observe that the continuous psh function θ is the unique solution to the Dirichlet problem for the homogeneous complex Monge–Ampère equation $(dd^c\psi)^n = 0$ in $\mathbb{B}$ with boundary values $\psi_{|\partial\mathbb{B}} = \theta_{|\partial\mathbb{B}}$.

It is known [14,30] that the viscosity solution to this Dirichlet problem is the upper envelope of all viscosity subsolutions. Since viscosity subsolutions are exactly the pluripotential ones by [14, Theorem 1.9], we infer that θ is the upper envelope of the pluripotential subsolutions to the Dirichlet problem above, hence it coincides with the Perron–Bremermann envelope and is a maximal psh function (see [8,26]). Thus $(dd^c\theta)^n = 0$ in the pluripotential sense and θ is a maximal psh function in the open set D. □

Remark 3.9 Let φ be a continuous plurisubharmonic function and $\mu \ge 0$ an absolutely continuous measure with continuous non-negative density. As the proof of the proposition above shows, the following are equivalent:

(i) $(dd^c\varphi)^n = \mu$ in the pluripotential sense of Bedford–Taylor [7];
(ii) $(dd^c\varphi)^n = \mu$ in the viscosity sense [14].

The dictionary between viscosity and plutipotential theory is quite subtle when μ is allowed to vanish and as far as supersolution are concerned. These notions however coincide for continuous solutions of Dirichlet problems.

The following immediate consequence of the previous proposition shows that one cannot run continuously a parabolic complex Monge–Ampère flow from an arbitrary initial data, if the measure μ is allowed to vanish:

Corollary 3.10 *Assume that φ is a solution to the the parabolic Monge–Ampère equation (2.1) in $]0, T, [\times\Omega$ which extends continuously to $[0, T[\times\Omega$. If μ vanishes in some* *open set D, then for all $t \in [0, T[$, the function φ_t is a maximal psh function in D. In particular φ_0 has to be a maximal plurisubharmonic function in D.*

4 The parabolic comparison principle

In this section we establish a comparison principle for the following parabolic complex Monge–Ampère equation in bounded domains of $\mathbb{C}^n$:

$$e^{\partial_t \varphi + F(t,\cdot,\varphi_t)}\mu_t - (dd^c \varphi_t)^n = 0, \tag{4.1}$$

where $\mu(t, z) = \mu_t(z) \geq 0$ is a continuous family of Borel measure on Ω.

We begin with a technical lemma.

Lemma 4.1 *Let $\mu(t, z) \geq 0$ and $\nu(t, x) \geq 0$ be two continuous Borel measures on Ω depending on the variables (t, z) and $F, G : [0, T[\times\overline{\Omega} \times \mathbb{R} \longrightarrow \mathbb{R}$ two continuous functions.*

Assume that $u : \bar{\Omega}_T \to \mathbb{R}$ is upper semicontinuous and $v : \bar{\Omega}_T \to \mathbb{R}$ is lower semicontinuous. Assume that the restriction of u to Ω_T is a bounded subsolution to the parabolic complex Monge–Ampère equation (4.1) associated to (F, μ) in Ω_T and that the restriction of v to Ω_T is a bounded supersolution to the parabolic complex Monge–Ampère equation (4.1) associated to (G, ν) in Ω_T.

Assume (†) that v is locally Lipschitz in the time variable and either u is locally Lipschitz in the time variable or $\mu > 0$.

Then, for every $\delta > 0$ either $\sup_{\bar{\Omega}_T} \left(u(t, x) - v(t, x) - \frac{\delta}{T-t}\right)$ is attained on $\partial_0\Omega_T$ or there exists $(\hat{t}, \hat{x}) \in]0, T'] \times \Omega$ where

$$T' = T - 2\delta(\|u\|_\infty + \|v\|_\infty) \tag{4.2}$$

such that $\sup \left(u(t, x) - v(t, x) - \frac{\delta}{T-t}\right)$ *is attained at $(\hat{t}, \hat{x})$ and*

$$e^{\frac{\delta}{(T-\hat{t})^2} + F(\hat{t},\hat{x},u(\hat{t},\hat{x})) - G(\hat{t},\hat{x},v(\hat{t},\hat{x}))}\mu(\hat{t}, \hat{x}) \leq \nu(\hat{t}, \hat{x}). \tag{4.3}$$

Proof Consider

$$w(t, x) := u(t, x) - v(t, x) - \frac{\delta}{T - t}.$$

This function is upper semi-continuous and bounded from above on the compact set $\overline{\Omega}_T$ and it is locally Lipschitz in the time variable. Since $w(t, z)$ tends to $-\infty$ as $t \to T^-$, there exists a point $(t_0, z_0) \in [0, T[\times\bar{\Omega}$, such that

$$\overline{M} := \sup_{\overline{\Omega}_T} w = w(t_0, z_0).$$

By construction this maximum cannot be achieved on $]T', T[\times\bar{\Omega}$. We can assume that this maximum of w on $\overline{\Omega}_T$ is not attained on $\partial_0\Omega_T$. The set $\{(t,x) \in \bar{\Omega}_T,\ w(t,x) = \overline{M}\}$ is then a compact subset contained in $]0, T'] \times \Omega$. Consider for small $\varepsilon > 0$, the function defined on $]0, T[\times\Omega^2$ by

$$w_\varepsilon(t,x,y) := u(t,x) - v(t,y) - \frac{\delta}{T-t} - \frac{1}{2\varepsilon}|x-y|^2.$$

This function is upper semi-continuous and bounded from above in $[0, T[\times\overline{\Omega}^2$ by a uniform constant C, and it tends to $-\infty$ as $t \to T^-$, so it reaches its maximum on $[0, T[\times\overline{\Omega}^2$ at some point $(t_\varepsilon, x_\varepsilon, y_\varepsilon) \in [0, T[\times\overline{\Omega}^2$ i.e.

$$\overline{M}_\varepsilon := \sup_{t\in[0,T[\times\overline{\Omega}^2} w_\varepsilon = u(t_\varepsilon, x_\varepsilon) - v(t_\varepsilon, y_\varepsilon) - \frac{\delta}{T-t_\varepsilon} - \frac{1}{2\varepsilon}|x_\varepsilon - y_\varepsilon|^2.$$

Observe that $\overline{M} \le \overline{M}_\varepsilon \le C$, which implies that any limit point of (t_ε) is in $[0, T[$. It follows from [12, Proposition 3.7] that $|x_\varepsilon - y_\varepsilon|^2 = o(\varepsilon)$ and that there is a subsequence $(t_{\varepsilon_j}, x_{\varepsilon_j}, y_{\varepsilon_j})$ converging to $(\hat{t}, \hat{x}, \hat{x}) \in [0, T[\times\overline{\Omega}^2$ where $(\hat{t}, \hat{x})$ is a maximum point of w on $\overline{\Omega}_T$ and

$$\lim_{j\to\infty} \overline{M}_{\varepsilon_j} = \overline{M}. \tag{4.4}$$

To simplify notation we set for any $j \in \mathbb{N}$, $(t_j, x_j, y_j) = (t_{\varepsilon_j}, x_{\varepsilon_j}, y_{\varepsilon_j})$. Extracting and relabelling we may assume that $\big(u(t_j, x_j, y_j)\big)_j$ and $\big(v(t_j, x_j, y_j)\big)_j$ converge. By the semicontinuity of u and v,

$$\lim_{j\to\infty} u(t_j, x_j, y_j) \le u(\hat{t}, \hat{x}), \quad \lim_{j\to\infty} v(t_j, x_j, y_j) \ge v(\hat{t}, \hat{x}). \tag{4.5}$$

On the other hand (4.4) implies that:

$$\lim_{j\to\infty} u(t_j, x_j, y_j) - \lim_{j\to\infty} v(t_j, x_j, y_j) - \frac{\delta}{T-\hat{t}} = u(\hat{t}, \hat{x}) - v(\hat{t}, \hat{x}) - \frac{\delta}{T-\hat{t}},$$
$$\lim_{j\to\infty} u(t_j, x_j, y_j) - \lim_{j\to\infty} v(t_j, x_j, y_j) = u(\hat{t}, \hat{x}) - v(\hat{t}, \hat{x}).$$

Together with (4.5), this yields

$$\lim_{j\to\infty} u(t_j, x_j) = u(\hat{t}, \hat{x}), \quad \lim_{j\to\infty} v(t_j, y_j) = v(\hat{t}, \hat{x}). \tag{4.6}$$

From our assumption we conclude that $(\hat{t}, \hat{x}) \in]0, T[\times\Omega$ and

$$\overline{M} = u(\hat{t}, \hat{x}) - v(\hat{t}, \hat{x}) - \frac{\delta}{T-\hat{t}}. \tag{4.7}$$

Applying the parabolic Jensen–Ishii's maximum principle Theorem 3.3 [the technical assumption (2.3) being satisfied thanks to (†)] to the functions $U(t,x) :=$

$u(t,x)-\frac{\delta}{T-t}$, v and the penality function $\phi(t,x,y) := \frac{1}{2\varepsilon}|x-y|^2$ for any fixed $\varepsilon = \varepsilon_j$, we find approximate parabolic second order jets $(\tau_j, p_j^{\pm}, Q_j^{\pm}) \in \mathbb{R}\times\mathbb{R}^{2n}\times\mathcal{S}_{2n}$ such that

$$\left(\tau_j + \frac{\delta}{(T-t_j)^2}, p_j^+, Q_j^+\right) \in \bar{\mathcal{P}}^{2,+}u(t_j,x_j),\quad \left(\tau_j, p_j^-, Q_j^-\right) \in \bar{\mathcal{P}}^{2,-}v(t_j,y_j)$$

with $p_j^+ = -p_j^- = \frac{1}{\varepsilon_j}(x_j-y_j)$ and $Q_j^+ \le Q_j^-$ (see [12, p.17] for the classical deduction of this inequality from Theorem 3.3 using the form of the flat space penalization function and compare [3] for the difficulties in curved space).

Applying the parabolic viscosity differential inequalities, we obtain for all $j\in\mathbb{N}$,

$$\begin{aligned} e^{\tau_j + \frac{\delta}{(T-t_j)^2} + F(t_j,x_j,u(t_j,x_j))}\mu(t_j,x_j) &\le (dd^c Q_j^+)^n \\ &\le (dd^c Q_j^-)^n \\ &\le e^{\tau_j + G(t_j,y_j,v(t_j,y_j))}\nu(t_j,y_j), \end{aligned}$$

which implies that

$$e^{\frac{\delta}{(T-t_j)^2} + F(t_j,x_j,u(t_j,x_j)) - G(t_j,y_j,v(t_j,y_j))}\mu(t_j,x_j) \le \nu(t_j,y_j),$$

Letting $j\to+\infty$ and using (4.6) (4.7) , we obtain the inequality (4.3). □

Theorem 4.2 *Assume that $\mu(z)\ge 0$ is a continuous non negative volume form on $\overline{\Omega}$. Let u be a bounded subsolution to the parabolic complex Monge–Ampère equation (4.1) and v a bounded supersolution to the parabolic complex Monge–Ampère equation (4.1) in Ω_T. Then*

$$\max_{\bar{\Omega}_T}(u-v) \le \max\{\max_{\partial_0\Omega_T}(u-v), 0\},$$

where u (resp. v) has been extended as an upper (resp. a lower) semicontinuous function to $\bar{\Omega}_T$.

Proof **Step 1.** Assume that $\mu > 0$ in Ω_T and that v is locally Lipschitz in the time variable. Fix $\delta > 0$. Apply Lemma 4.1 with $\mu=\nu$, $F=G$. It follows that either

$$\sup_{\bar{\Omega}_T}\left\{u(t,x)-v(t,x)-\frac{\delta}{T-t}\right\} = \sup_{\partial_0\bar{\Omega}_T}\left\{u(t,x)-v(t,x)-\frac{\delta}{T-t}\right\}$$

or

$$e^{\frac{\delta}{(T-\hat{t})^2} + F(\hat{t},\hat{x},u(\hat{t},\hat{x})) - F(\hat{t},\hat{x},v(\hat{t},\hat{x}))} \le 1$$

which implies $F\left(\hat{t},\hat{x},u(\hat{t},\hat{x})\right) - F\left(\hat{t},\hat{x},v(\hat{t},\hat{x})\right) < 0$ hence $u(\hat{t},\hat{x}) < v(\hat{t},\hat{x})$. In either case, every $(t,x)\in\bar{\Omega}_T$ satisfies

$$u(t,x) - v(t,x) - \frac{\delta}{T-t} < \max\{0, \sup_{\partial_0 \bar{\Omega}_T} (u-v)\}.$$

Since δ can be choosen arbitrary small, we infer:

$$u - v \le \max\{0, \sup_{\partial_0 \bar{\Omega}_T} (u-v)\}.$$

Step 2. Still assuming that $\mu > 0$ in Ω_T, let us remove the assumption that v is locally Lipschitz in the time variable. Fix $\delta > 0$. Either

$$\sup_{\bar{\Omega}_T} \left\{ u(t,x) - v(t,x) - \frac{\delta}{T-t} \right\} = \sup_{\partial_0 \bar{\Omega}_T} \left\{ u(t,x) - v(t,x) - \frac{\delta}{T-t} \right\}$$

or

$$\sup_{\bar{\Omega}_T} \left\{ u(t,x) - v(t,x) - \frac{\delta}{T-t} \right\} > \sup_{\partial_0 \bar{\Omega}_T} \left\{ u(t,x) - v(t,x) - \frac{\delta}{T-t} \right\}$$

Suppose we are in the second case. Fix $\bar{s} \in \mathbb{R}$ such that

$$\sup_{\bar{\Omega}_T} \left\{ u(t,x) - v(t,x) - \frac{\delta}{T-t} \right\} > \bar{s} > \sup_{\partial_0 \bar{\Omega}_T} \left\{ u(t,x) - v(t,x) - \frac{\delta}{T-t}) \right\}.$$

Since $\bar{s} > \sup_{\partial_0 \bar{\Omega}_T} (u(t,x) - v(t,x) - \frac{\delta}{T-t})$, we have $\partial_0 \bar{\Omega}_T \subset \{w(t,x) < \bar{s}\}$. Since $\{w(t,x) < \bar{s}\}$ is open and contains $\{0\} \times \bar{\Omega}$, we can find $\eta > 0$ such $[0,\eta] \times \bar{\Omega} \subset \{w(t,x) < \bar{s}\}$ so that every $(t,x) \in [0,\eta] \times \bar{\Omega} \cup \partial_0 \bar{\Omega}_T$ satisfies

$$u(t,x) - v(t,x) - \frac{\delta}{T-t} < \bar{s}.$$

We now apply Lemma 3.5 to v. Then, by Dini-Cartan's lemma we have

$$\lim_{k \to \infty} \sup_{\bar{\Omega}_T} \left\{ u(t,x) - v_k(t,x) - \left(\frac{\delta}{T-t} \right) = \sup_{\bar{\Omega}_T} (u(t,x) - v(t,x)) - \frac{\delta}{T-t} \right\}$$

and similarly

$$\lim_{k \to \infty} \sup_{[0,\eta] \times \bar{\Omega} \cup \partial_0 \bar{\Omega}_T} \left\{ u - v_k - \frac{\delta}{T-t} \right\} = \sup_{[0,\eta] \times \bar{\Omega} \cup \partial_0 \bar{\Omega}_T} \left\{ u - v - \frac{\delta}{T-t} \right\}.$$

Hence we can assume that for k large enough the maximum of $w_k(t,x) := u(t,x) - v_k(t,x) - \frac{\delta}{T-t}$ is not attained on $[0,\eta] \times \bar{\Omega} \cup \partial_0 \Omega_T$. Choose k large enough so that the supersolution property of v_k is valid for $\eta/2 < t < T'$. Lemma 4.1 applied to $\tilde{u}(t,x) = u(t+\eta, x)$, $\tilde{v}(t,x) = v_k(t+\eta, x)$, yields

$$F\left(\hat{t}, \hat{x}, u(\hat{t}, \hat{x})\right) - F^k\left(\hat{t}, \hat{x}, v_k(\hat{t}, \hat{x})\right) + \frac{\delta}{T^2} \leq \log(\mu^k/\mu)(\hat{t}, \hat{x}), \tag{4.8}$$

where $(\hat{t}, \hat{x}) = (\hat{t}_k, \hat{x}_k) \in]0, T[\times\Omega$ is a point where the function $w_k(t, x)$ takes its maximum in $\overline{\Omega}_T$.

Since F and μ are uniformly continuous in $[0, T'] \times \overline{\Omega} \times [-K, K]$ with $K = \max(\|u\|_\infty, \|v\|_\infty)$, it follows that, for k large enough, we have

$$F\left(\hat{t}, \hat{x}, u(\hat{t}, \hat{x})\right) - F\left(\hat{t}, \hat{x}, v_k(\hat{t}, \hat{x})\right) \leq -\frac{\delta}{2T^2}. \tag{4.9}$$

From this we get $u(\hat{t}, \hat{x}) < v_k(\hat{t}, \hat{x})$. Hence $\sup_{\bar{\Omega}_T} w_k(t, x) < 0$ and $\sup_{\bar{\Omega}_T} (u(t, x) - v(t, x) - \frac{\delta}{T-t}) \leq 0$.

In particular, whether we are in the first or the second case, we infer

$$\begin{aligned}\sup_{\bar{\Omega}_T}\left(u(t, x) - v(t, x) - \frac{\delta}{T-t}\right) &\leq \max\left(0, \sup_{\partial_0\bar{\Omega}_T} u(t, x) - v(t, x) - \frac{\delta}{T-t}\right)\\ &\leq \max\left(0, \sup_{\partial_0\bar{\Omega}_T} u(t, x) - v(t, x)\right),\end{aligned}$$

and for every $(t, x) \in \Omega_T$ we have

$$u(t, x) - v(t, x) - \frac{\delta}{T-t} \leq \max\left(0, \sup_{\partial_0\bar{\Omega}_T}(u(t, x) - v(t, x))\right).$$

Since δ can be chosen arbitrary small, we conclude once again that

$$u - v \leq \max\left\{0, \sup_{\partial_0\bar{\Omega}_T}(u - v)\right\}.$$

Step 3. Assume that $\mu(t, x) = \nu(t, x) \geq 0$ and the subsolution u is locally uniformly Lipschitz in t.

More precisely we assume that $t \longmapsto u(t, z)$ is C-Lipschitz in t in some subset $[0, T'] \subset [0, T[$ uniformly in $z \in \Omega$. The idea is to perturb μ by adding an arbitrary small positive term.

Consider for $\eta > 0$ small enough, the positive volume form $\tilde{\mu} := \mu + \eta\beta^n$, where $\beta = dd^c\rho > 0$ is the standard Kähler form on Ω i.e. $\rho(z) := |z|^2 - R^2$, where $R > 1$ is large enough so that $\rho < 0$ in Ω. Then fix $\varepsilon > 0$ and consider the function $\psi(t, z) := u(t, z) + \varepsilon\rho(z)$. This is an upper semi-continuous function in Ω_T. We claim that ψ is a subsolution to the equation

$$e^{\partial_t\psi(t,\cdot)+F(t,\psi_t,\cdot)}\tilde{\mu}(t, \cdot) \leq (dd^c\psi_t)^n, \tag{4.10}$$

in $[0, T'] \times \Omega$ for an appropriate choice of η in terms of ε.

Indeed since ρ is C^2, any parabolic upper test function θ for ψ at any point (t_0, z_0) can be written as $\theta(t,z) := \tilde{\theta}(t,z) + \varepsilon\rho(z)$, where $\tilde{\theta}$ is a parabolic upper test function for u at the point (t_0, z_0). From the viscosity inequality for u we know that $dd^c\tilde{\theta}_{t_0} \geq 0$ and

$$(dd^c\tilde{\theta}_{t_0})^n_{z_0} \geq e^{\partial_t\tilde{\theta}(t_0,z_0)+F(t_0,z_0,\tilde{\theta}(t_0,z_0))}\mu(t_0,z_0).$$

Therefore $dd^c\theta_{t_0}(z_0) = dd^c\tilde{\theta}_{t_0}(z_0) + \varepsilon\beta \geq 0$ and then

$$(dd^c\theta_{t_0})^n_{z_0} \geq (dd^c\tilde{\theta}_{t_0})^n + \varepsilon^n\beta^n \geq e^{\partial_t\theta(t_0,z_0)+F(t_0,\theta(t_0,z_0),z_0)}\mu(t_0,z_0) + \varepsilon^n\beta^n,$$

since $\theta \leq \tilde{\theta}$ and F in non decreasing in the second variable.

Now set $M := \sup_{\Omega_{T'}} u$ and $A := \sup\{F(t,z,M); 0 \leq t \leq T', z \in \Omega\}$. Since u is C-Lipschitz in t uniformly in z and $u \leq_{(t_0,z_0)} \theta$, it follows from Taylor's formula that $\partial_t\theta(t_0,z_0) \leq C$. Then

$$e^{\partial_t\theta(t_0,z_0)+F(t_0,z_0,\theta(t_0,z_0))} \leq e^{C+A}.$$

Therefore if we choose $\eta := \varepsilon^n e^{-A-C}$, we obtain the inequality

$$(dd^c\theta_{t_0})^n_{z_0} \geq e^{\partial_t\theta(t_0,z_0)+F(t_0,z_0,\theta(t_0,z_0))}(\mu(t_0,z_0) + \eta\beta^n),$$

which proves our claim.

Since $\tilde{\mu} \geq \mu$, the function v is also a supersolution to the parabolic equation associated to $(F, \tilde{\mu})$. We can apply the comparison principle of the first part and conclude that $u(t,z) + \varepsilon\rho(z) - v(t,z) \leq \max_{\partial_0\Omega_T}(u-v)^+ + O(\varepsilon)$. Letting $\varepsilon \to 0$ we obtain the conclusion of the theorem.

Step 4. Finally assume that $\mu(z) = \nu(z) \geq 0$ does not depend on t. Regularizing u in the time variable only according to Lemma 3.5, we obtain a decreasing sequence u^k of k-Lipschitz functions in t converging to u. We know by (3.1) that $U = u^k$ is a subsolution to the parabolic Monge–Ampère equation associated to (F_k, μ) i.e.

$$e^{\partial_t U(t,\cdot)+F_k(t,\cdot,U_t)}\mu \leq (dd^c U_t)^n,$$

where $F_k(t,z,r) := \inf_{|s-t|\leq 1/k} F(s,z,r)$ on $[A/k, T - A/k] \times \Omega$. Observe that μ does not change after regularization in the time variable since it does not depend on t.

Using the perturbation argument of Step 3, we see that the function $\psi^k := u^k(t,z) + \varepsilon\rho(z)$ satisfies the differential inequality

$$e^{\partial_t\psi(t,\cdot)+F_k(t,\cdot,\psi_t)}(\mu + \eta\beta^n) \leq (dd^c\psi_t)^n,$$

where $\eta := \varepsilon^n e^{-A-k}$ on $[A/k, T - A/k] \times \Omega$.

We now regularise v in the time variable only according to Lemma 3.5 and argue as in Step 2 to conclude that, in the second case,

$$F_k\left(\hat{t},\hat{x},u_k(\hat{t},\hat{x})+\varepsilon\rho(\hat{t},\hat{x})\right) - F^k\left(\hat{t},\hat{x},v^k(\hat{t},\hat{x})\right) + \frac{\delta}{T^2} \leq \log(\mu/\tilde{\mu})(\hat{t},\hat{x}), \quad (4.11)$$

where $(\hat{t}, \hat{x}) = (\hat{t}_k, \hat{x}_k) \in]0, T[\times\Omega$ is a point where the function $u_k(t, x) + \varepsilon\rho(t, x) - v^k(t, x)) - \frac{\delta}{T-t}$ achieves its maximum. Since F is uniformly continuous in $[0, T'] \times \overline{\Omega} \times [-K - 2\epsilon R^2, K]$ and $\mu \le \tilde{\mu}$ with $K = \max(\|u\|_\infty, \|v\|_\infty)$, it follows that for k large enough,

$$F(\hat{t}, \hat{x}, u_k(\hat{t}, \hat{x}) + \varepsilon\rho(\hat{t}, \hat{x})) - F(\hat{t}, \hat{x}, v^k(\hat{t}, \hat{x})) \le -\frac{\delta}{2T^2}. \tag{4.12}$$

This yields:

$$u_k(\hat{t}, \hat{x}) + \varepsilon\rho(\hat{t}, \hat{x}) < v^k(\hat{t}, \hat{x})$$

and for all $(t, x) \in \bar{\Omega}_T$

$$u_k(t, x) + \varepsilon\rho(t, x) - v^k(\hat{t}, \hat{x}) - \frac{\delta}{T-t} < 0.$$

We can then let ε decrease to 0, then let k go to $+\infty$ arguing as in the last part of Step 2, to conclude that

$$\begin{aligned}\sup_{\bar{\Omega}_T}\left(u(t, x) - v(t, x) - \frac{\delta}{T-t}\right) &\le \max\left(0, \sup_{\partial_0\bar{\Omega}_T} u(t, x) - v(t, x) - \frac{\delta}{T-t}\right)\\ &\le \max\left(0, \sup_{\partial_0\bar{\Omega}_T} u(t, x) - v(t, x)\right).\end{aligned}$$

Letting δ decrease to 0, we conclude the proof. □

Remark 4.3 As the proof shows, the comparison principle is valid under more general conditions than those stated in the theorem, in particular: when the volume form $\mu(t, z) > 0$ depends on (t, z) and does not vanish on $\overline{\Omega_T}$.

Remark 4.4 An important case for applications is when F is strongly increasing in the last variable, meaning that there exists $\alpha > 0$ such that for any (t, z), the function $r \longmapsto F(t, z, r) - \alpha r$ is non decreasing in $\mathbb{R}$. Then we can prove a more precise comparison principle. Namely assume that $\mu(t, z) > 0$ and $\nu(t, z) \ge 0$ are two continuous volume forms in Ω_T, u is a subsolution to the parabolic complex Monge–Ampère equation associated to (F, μ) and v is a supersolution to the parabolic Monge–Ampère equation associated to (F, ν), then

$$\max_{\Omega_T}(u - v) \le \max\{M_0, (1/\alpha)\log\gamma\}$$

where $M_0 := \max_{\partial_0\Omega_T}(u - v)^+$ and $\gamma := \max_{\Omega_T} \nu/\mu$. This follows from the fundamental inequality (4.3).

Remark 4.5 A change of variables in time leads to the more general twisted parabolic complex Monge–Ampère equation

$$e^{h(t)\partial_t\varphi + F(t,z,\varphi)}\mu(t, z) - (dd^c\varphi_t)^n = 0, \tag{4.13}$$

where $h > 0$ is positive continuous function in $[0, T[$.

The comparison principle holds for the more general parabolic complex Monge–Ampère equation (4.13). This follows from the change of variables

$$u(s,z) := \varphi(t,z), \text{ with, } t = \gamma(s)$$

where γ is a positive increasing function in $[0, S[$ with values in $[0, T[$ such that $\gamma(0) = 0$.

Indeed observe that $\partial_s u = \gamma'(s)\partial_t\varphi(t,z)$ for $(s,z) \in [0, S[\times\Omega$. Thus if we set $\gamma'(s) = h(t)$, then φ is a solution to the twisted parabolic complex Monge–Ampère equation (4.13) if and only if u is a solution to the parabolic complex Monge–Ampère equation

$$e^{\partial_s u + G(s,z,u)}\nu(s,z) - (dd^c u_s)^n = 0,$$

where $G(s,z,r) := F(\gamma(s), z, r)$ and $\nu(s,z) := \mu(\gamma(s), z)$.

Since $r \longmapsto G(t,z,r)$ in non decreasing, we can apply the comparison principle proved above and obtain the claim. Observe that the equation $\gamma'(s) = h(t)$ means that the inverse function $g(t) = \gamma^{-1}(t) = s$ satisfies $g'(t) = 1/h(t)$ and $g(0) = 0$, thus γ is uniquely determined by h.

5 Existence of solutions

We now study the Cauchy–Dirichlet problem for the parabolic complex Monge–Ampère equation

$$e^{\partial_t\varphi + F(t,\cdot,\varphi)}\mu - (dd^c\varphi_t)^n = 0, \text{ in } \Omega_T, \tag{5.1}$$

with Cauchy–Dirichlet conditions,

$$\varphi(t,z) = \varphi_0(z), \quad (t,\zeta) \in \partial_0\Omega, \tag{5.2}$$

where $\mu(z) \geq 0$ is a continuous volume form in Ω, $\varphi_0 : \Omega \longrightarrow \mathbb{R}$ is continuous in $\bar{\Omega}$ and plurisubharmonic in Ω and $F : [0, T[\times\Omega\times\mathbb{R} \longmapsto \mathbb{R}$ is a continuous function non decreasing in the last variable.

We assume that Ω is a strictly pseudoconvex domain and let ρ be a defining function for Ω which is strictly plurisubharmonic in a neighborhood of $\overline{\Omega}$, with $-1 \leq \rho < 0$ in Ω.

5.1 Existence of sub/super-solutions

We first introduce the notions of sub/super-solution for the Cauchy–Dirichlet problem:

Definition 5.1 Let φ_0 be a Cauchy–Dirichlet data function for the parabolic Monge–Ampère equation (5.2).

1. We say that an upper semi-continuous function $u : [0, T[\times\overline{\Omega} \longrightarrow \mathbb{R}$ is a subsolution to the Cauchy–Dirichlet problem (5.2) if u is a subsolution to the parabolic equation (5.1) in Ω_T which satisfies $u \leq \varphi_0$ on the parabolic boundary $\partial_0\Omega_T$.

2. We say that a lower semi-continuous function $v : [0, T[\times\overline{\Omega} \longrightarrow \mathbb{R}$ is a supersolution to the Cauchy–Dirichlet problem (5.2) if v is a supersolution to the parabolic equation (5.1) in Ω_T which satisfies $v \geq \varphi_0$ on the parabolic boundary $\partial_0\Omega_T$.

Observe that sub/supersolutions to the parabolic complex Monge–Ampère equation we are interested in always exist:

Lemma 5.2 1. *The constant function $\overline{v} \equiv \sup_{\overline{\Omega}} \varphi_0$ is a supersolution to the Cauchy–Dirichlet problem (5.2) with Cauchy–Dirichlet data φ_0.*
2. *The Cauchy–Dirichlet problem for the parabolic equation (5.1) with initial data φ_0 admits a subsolution $\underline{u}$ in $]0, T[\times\Omega$, which is continuous in $[0, T[\times\overline{\Omega}$ and satisfies $\underline{u} \leq \overline{v}$ in $[0, T[\times\overline{\Omega}$.*

Proof The first statement is obvious. To prove the second one, we consider the function defined for $t \in [0, T[$ by

$$B(t) := \int_0^t b_+(s)ds, \quad \text{where } b(t) := \sup\{F(t, z, \varphi_0(z)); z \in \Omega\}. \tag{5.3}$$

Choose $A > 0$ large enough so that $A(dd^c\rho)^n \geq \mu$ in Ω. The function

$$(t, z) \mapsto \underline{u}(t, z) := A\rho(z) + \varphi_0(z) - B(t)$$

is a subsolution to Cauchy–Dirichlet problem for the parabolic complex Monge–Ampère equation (5.2) which clearly satisfies $\underline{u} \leq \overline{v}$. □

Remark 5.3 Observe that the supersolution given above is bounded, while the subsolution $\underline{u}$ is continuous in $[0, T[\times\overline{\Omega}$, hence locally bounded. When $(t, z) \longmapsto F(t, z, \varphi_0(z))$ is bounded from above on $[0, T[\times\overline{\Omega}$, there exists a globally bounded subsolution. Indeed set

$$B := \sup\{F(t, z, \varphi_0(z)); 0 \leq t < T, z \in \Omega\},$$

and let $A > 1$ be so large that $A^n(dd^c\rho)^n \geq e^B\mu$. Then

$$\underline{u}(t, z) := A\rho(z) + \varphi_0(z),$$

does the job.

Consider the upper envelope

$$\varphi := \sup\{u\ ; u \in \mathcal{S}, \underline{u} \leq u \leq \overline{v}\}, \tag{5.4}$$

where $\mathcal{S}$ is the family of all subsolutions to the Cauchy–Dirichlet problem for the parabolic equation (5.1) with the Cauchy–Dirichlet condition (5.2), and $\underline{u}, \overline{v}$ are the sub/super-solutions from Lemma 5.2.

Lemma 5.4 *Given any non empty family $\mathcal{S}_0$ of bounded subsolutions to the parabolic equation (5.1) which is bounded above by a continuous function, the usc regularization of the upper envelope $\phi_{\mathcal{S}_0} := \sup_{\phi \in \mathcal{S}_0} \phi$ is a subsolution to (5.1) in Ω_T.*

If $\mathcal{S}$ is the family of all subsolutions to the Cauchy–Dirichlet problem (5.2), its envelope $\phi_{\mathcal{S}}$ coincides with the upper envelope φ given by the formula (5.4) and is a discontinuous viscosity solution to (5.1) in Ω_T.

Moreover for any $(t, z) \in [0, T[\times\Omega$,

$$\varphi^*(t, z) - \varphi_*(t, z) \leq \sup_{\partial_0\Omega_T} (\varphi^* - \varphi_*)_+. \tag{5.5}$$

Proof The first statement follows from the standard method of Perron (see [12,24]). Observe that the family $\mathcal{S}$ of all subsolutions to the Cauchy–Dirichlet problem (5.2) is not empty since $\underline{u} \in \mathcal{S}$ and bounded from above by $\overline{v}$, thanks to Lemma 5.2 and Theorem 4.2.

The fact that φ is a discontinuous viscosity solution to (5.1) in Ω_T follows by the general argument of Perron as in the degenerate elliptic case (see [12,14]). □

5.2 Barriers

In order to prove that the above (a priori discontinuous) viscosity solution is a continuous viscosity solution and satisfies the Cauchy–Dirichlet condition, we need to construct appropriate barriers.

Definition 5.5 Fix $(t_0, x_0) \in \partial_0\Omega_T$ and $\varepsilon > 0$.

1. We say that an upper semi-continuous function $u : \Omega_T \longrightarrow \mathbb{R}$ is an ε-subbarrier for the Cauchy problem (5.2) at the point (t_0, x_0), if u is a subsolution to the parabolic complex Monge–Ampère equation (5.1) such that $u \leq \varphi_0$ in $\partial\Omega_T$ and $u_*(t_0, x_0) \geq \varphi_0(x_0) - \varepsilon$.
2. We say that a lower semi-continuous function $v : \Omega_T \longrightarrow \mathbb{R}$ is an $\varepsilon-$superbarrier to the Cauchy problem (5.2) at the boundary point (t_0, x_0) if v is a supersolution to the parabolic equation (5.1) such that $v \geq \varphi_0$ in $\partial_0\Omega_T$ and $v^*(t_0, x_0) \leq \varphi(x_0) + \varepsilon$.

Definition 5.6 We say (φ_0, μ) is admissible whenever for all $\epsilon > 0$ we can find $\psi_0 \in C^0(\bar{\Omega}) \cap PSH(\Omega)$ such that $\varphi_0 \leq \psi_0 \leq \varphi_0 + \epsilon$ and $C = C_\epsilon \in \mathbb{R}$ such that $(dd^c\psi_0)^n \leq e^C\mu$ in the viscosity sense.

In other words a Cauchy data φ_0 is admissible with respect to μ if it is the uniform limit on Ω of continuous psh functions whose Monge–Ampère measure is controlled by μ. In particular if $(dd^c\varphi_0)^n \leq e^C\mu$ in the viscosity sense then (φ_0, μ) is admissible. We also note the following useful criterion:

Lemma 5.7 *If $\mu > 0$ then (φ_0, μ) is admissible.*

Proof This follows from classical results on approximation of plurisubharmonic functions. Indeed, any psh function in Ω, continuous up to the boundary can be approximated uniformly in $\overline{\Omega}$ by psh functions in Ω that are smooth up to the boundary (see [2,17,27]).

Therefore given $\varepsilon > 0$, we can find a function ψ_0 psh in Ω, smooth up to the boundary such that $\varphi_0 \leq \psi_0 \leq \varphi_0 + \epsilon$ in Ω. If $\mu > 0$ on $\{0\} \times \bar{\Omega}$, there is a constant $C > 0$ such that $(dd^c \psi_0)^n \leq e^C \mu$ pointwise in Ω , hence in the sense of viscosity in Ω [14]. □

Example 5.8 If $\mu(z) \equiv 0$ vanishes identically on some open set $D \subset \Omega$ where φ_0 is not a maximal psh function (i.e. where the Monge–Ampère measure $(dd^c \varphi_0)^n$ is not zero) then (φ_0, μ) is not admissible.

Indeed (φ_0, μ) is admissible if and only if φ_0 is the uniform limit (in Ω hence in particular on D) of a sequence of continuous psh functions ψ_j such that

$$(dd^c \psi_j)^n \leq C_j \mu$$

for some $C_j > 0$. In particular ψ_j has to be maximal in D, hence so is φ_0.

Proposition 5.9 *Assume that (φ_0, μ) is admissible. For all $\varepsilon > 0$ and $(t_0, x_0) \in \partial_0 \Omega_T$, there exists a continuous function U (resp. V) in $[0, T[\times \overline{\Omega}$, which is an $\varepsilon-$subbarrier (resp. ε-superbarrier) to the Cauchy–Dirichlet problem (5.2) at (t_0, x_0).*

Proof Fix $\varepsilon > 0$ and $(t_0, z_0) \in \partial_0 \Omega_T$.

1. We first construct ε-subbarriers. There are two cases:

1.1. Assume $t_0 = 0$ and $z_0 \in \overline{\Omega}$. Fix $\varepsilon > 0$ and define the following function

$$U(t, z) := \varphi_0(z) + \varepsilon \rho(z) - B(t) - Mt, \ \ (t, z) \in [0, T[\times \Omega,$$

where $B(t)$ is the C^1 positive function defined by the formula (5.3) and $M > 0$ is a large constant to be chosen later. Recall that $B'(t) \geq F(t, z, \varphi_0(z))$ in $[0, T[\times \Omega$.

The function U is continuous in $[0, T[\times \overline{\Omega}$, it is plurisubharmonic in the space variable $z \in \Omega$ and C^1 in the time variable $t \in [0, T[$. Moreover it satisfies the inequality $(dd^c U_t)^n \geq \varepsilon^n (dd^c \rho)^n$ in the pluripotential sense in Ω for any fixed $t \in [0, T[$. Observe that

$$\partial_t U(t, z) + F(t, z, U(t, z)) \leq F(t, z, \varphi_0(z)) - C'(t) - M \leq -M,$$

pointwise in Ω_T.

If we choose $M = M(\varepsilon) > 1$ large enough so that $\varepsilon^n (dd^c \rho)^n \geq e^{-M} \mu$, then U satisfies the inequality

$$(dd^c U_t)^n \geq e^{\partial_t U(t,\cdot) + F(t,\cdot,U_t)} \mu,$$

in the pluripotential sense in Ω, for each t. Moreover it follows from [14] that the function U satisfies the differential inequality

$$(dd^c U_t)^n \geq e^{\partial_t U(t,\cdot) + F(t,\cdot,U_t)}$$

in the viscosity sense in Ω_T. Therefore the function U is a viscosity subsolution to the Cauchy–Dirichlet problem (5.2).

Since $U(0,\cdot)=\varphi_0+\varepsilon\rho\leq\varphi_0$ in Ω, $U(0,z_0)=\varphi_0(z_0)+\varepsilon\rho(z_0)$ and $\rho\geq -1$, we see that $U(0,z_0)\geq\varphi_0(z_0)-\varepsilon$. Hence U is an $\varepsilon-$subbarrier at any point $(0,z_0)\in\{0\}\times\overline{\Omega}$.

1.2. If $t_0>0$ and $x_0\in\partial\Omega$, we argue as in the proof of Lemma 5.2. We consider, for $t\in[0,T[$,

$$B(t):=\int_{t_0}^t b_+(s)ds,\ \text{ where } b(t):=\sup\{F(t,z,\varphi_0(z));z\in\overline{\Omega}\}.$$

This is C^1 function in $[0,T[$ satisfying $B(t_0)=0$ and $B'(t)\geq F(t,z,\varphi_0(z))$ for any $(t,z)\in[0,T[\times\overline{\Omega}$. Choosing $A>1$ large enough so that $\mu\leq A^n(dd^c\rho)^n$, the function

$$(t,x)\in[0,T[\times\Omega\mapsto U(t,x):=\varphi_0(x)+A\rho(x)-B(t)\in\mathbb{R}$$

is a subbarrier at any point $(t_0,x_0)\in[0,T[\times\partial\Omega$.

We have not used the admissibility of the Cauchy–Dirichlet data to construct subbarriers.

2. Constructing superbarriers is a more delicate task that requires besides the admissibily some pluripotential tools. We also consider two cases:

2.1. Fix $\varepsilon>0$ and use that (φ_0,μ) is admissible to obtain a psh function ψ_0 in Ω continuous up to the boundary such that $\varphi_0-\varepsilon\leq\psi_0\leq\varphi_0$ in $\overline{\Omega}$. The maximal psh function $\bar{\psi}_0$ solving the Dirichlet problem

$$(dd^c\bar{\psi}_0)^n=0\ \text{ and }\ \bar{\psi}_0|_{\partial\Omega}=\psi_0|_{\partial\Omega}$$

is continuous and plurisubharmonic [6]. It can be used as a subbarrier at any $(t_0,z_0)\in\partial_0\Omega$ such that $t_0\geq 0$ and $z_0\in\partial\Omega$. The fact that it is a viscosity supersolution follows from [14,30].

2.2. Assume $t_0=0$ and $z_0\in\overline{\Omega}$. Set for $t\in[0,T[$

$$\Gamma(t):=\int_0^t\gamma_+(s)ds,\ \text{ where } \gamma(t):=-\inf\{F(t,z,\psi_0(z));z\in\overline{\Omega}\}.$$

Observe that Γ is C^1 in $[0,T[$ and satisfies $\Gamma'(t)+F(t,z,\psi_0(z))\geq 0$ for all $(t,z)\in[0,T[\times\Omega$. Thus

$$V(t,z):=\psi_0(z)+Ct+\Gamma(t),$$

is a continuous function in $[0,T[\times\overline{\Omega}$, C^1 in t and psh in z. Moreover for any $t\in[0,T[$, it satisfies

$$(dd^cV_t)^n=(dd^c\psi_0)\leq e^C\mu\leq e^{\partial_tV+F(t,\cdot,V_t)}\mu,$$

in the pluripotential sense in Ω. As above we infer that V is a subsolution to the parabolic equation (5.2).

Since $V(0,z) = \psi_0(z) \geq \varphi_0(z) - \varepsilon$, it follows that V is an ε-superbarrier to the Cauchy problem (5.2) at any parabolic boundary point $(0, z_0) \in \Omega_T$. □

Note that one cannot expect the existence of superbarriers when $\mu = 0$ and φ_0 is not maximal.

5.3 The Perron envelope

We are now ready to show the existence of solutions to the Cauchy–Dirichlet problem for degenerate complex Monge–Ampère flows:

Theorem 5.10 *Assume $\mu > 0$ or $\mu = \mu(z)$ is independent of t and (φ_0, μ) is admissible. Then the Cauchy-Dirichlet problem for the parabolic complex Monge–Ampère equation (5.1) with Cauchy–Dirichlet condition (5.2) admits a unique viscosity solution $\varphi(t,z)$ in infinite time.*

Proof It follows from Proposition 5.9 that there is at least a subsolution $\underline{u}$ and a supersolution $\overline{v} = 0$ to the Cauchy problem for the parabolic complex Monge–Ampère equation (5.1) with Cauchy–Dirichlet condition (5.2), which satisfy the inequality $\underline{u} \leq \overline{v}$ in $\mathbb{R}^+ \times \Omega$. We can thus consider the upper envelope φ of those subsolutions u that satisfy $\underline{u} \leq u \leq \overline{v}$ in $\mathbb{R}^+ \times \Omega$ as defined in 5.4.

Fix $T > 0$ large and observe that the restriction of φ^* to Ω_T is a subsolution to the parabolic complex Monge–Ampère equation (5.1), while the restriction of φ_* to Ω_T is a supersolution to the same parabolic complex Monge–Ampère equation. By Lemma 5.4, they satisfy the inequalitiy (5.5) and then by semi-continuity there exists $(t_0, x_0) \in (\{0\} \times \overline{\Omega}) \cup ([0,T] \times \partial\Omega)$ such that

$$\max_{(t,x)\in\overline{\Omega}_T} \{\varphi^*(t,x) - \varphi_*(t,x)\} = \varphi^*(t_0,x_0) - \varphi_*(t_0,x_0).$$

Fix $\varepsilon > 0$ arbitrary small. By Proposition 5.9, there exists a continuous $\varepsilon-$subbarrier U and an $\varepsilon-$superbarrier V to the Cauchy–Dirichlet problem (5.2) in Ω_T at the parabolic boundary point $(t_0, x_0) \in \partial_0\Omega_T$ such that $U(t_0,x_0) \geq \varphi_0(x_0) - \varepsilon$ and $V(t_0,x_0) \leq \varphi_0(x_0) + \varepsilon$. Since $U_0 \leq \varphi_0 \leq V_0$ in Ω, it follows from the comparison principle that $U \leq \varphi_* \leq \varphi \leq \varphi^* \leq V$ in $[0,T] \times \Omega$. Hence $U = U_* \leq \varphi_*$ and $\varphi \leq V^* = V$ is $[0,T] \times \overline{\Omega}$. At the boundary point (t_0, x_0) we have

$$\varphi_0(x_0) - \varepsilon \leq U(t_0,x_0) \leq \varphi_*(t_0,x_0) \leq \varphi^*(t_0,x_0) \leq V(t_0,x_0) \leq \varphi_0(x_0) + \varepsilon.$$

We infer that for all $(t,x) \in [0, T[\times\Omega$,

$$\varphi^*(t,x) - \varphi_*(t,x) \leq \varphi^*(t_0,x_0) - \varphi_*(t_0,x_0) \leq 2\varepsilon.$$

Since $T > 0$ was arbitrary large, this implies that $\varphi^* \leq \varphi_*$ in $\mathbb{R}^+ \times \Omega$, hence $\varphi^* = \varphi_*$ in $\mathbb{R}^+ \times \Omega$.

The same reasoning as above shows that $\varphi(0,\cdot) = \varphi_0$ in $\overline{\Omega}$. This proves that $\varphi = \varphi^*$ is a continuous solution to the Cauchy–Dirichlet problem (5.2) in $\mathbb{R}^+ \times \Omega$ with initial data φ_0. □

Remark 5.11 When μ vanishes identically in a non empty open set $D \subset \Omega$ where φ_0 is not maximal (in particular (φ_0, μ) is not admissible), then there is no viscosity solution to the above Cauchy–Dirichlet problem by Corollary 3.10.

6 Long term behavior of the flows

We assume in this last section that $F = F(z, r)$ is time independent. It follows from Theorem 5.10 that the complex Monge–Ampère flow

$$e^{\partial_t \varphi + F(\cdot, \varphi)} \mu(z) - (dd^c \varphi_t)^n = 0 \tag{6.1}$$

admits a unique solution for all times (i.e. makes sense in $\mathbb{R}^+ \times \Omega$) and for every Cauchy–Dirichlet data $\varphi_0 \in \mathcal{C}^0(\partial\Omega) \cap PSH(\Omega)$ such that (φ_0, μ) is admissible: we always assume such is the case in the sequel.

Our aim in this final section is to analyze, the asymptotic behavior of this flow when $t \to +\infty$. By analogy with the Kähler-Ricci flow, the model case is when

$$F(z, r) = h(z) + \alpha r, \quad (t, z) \in \Omega \times \mathbb{R}.$$

The situation is simple when $\alpha > 0$ (negative curvature), more involved when $\alpha = 0$ (Ricci flat case), often intractable when $\alpha < 0$ (positive curvature).

6.1 Negative curvature

We first make a strong assumption on F [corresponding to the model case $F(z, x) = \alpha x + h(z)$ with $\alpha > 0$] so as to obtain a good control on the speed of convergence of the flow, starting from any admissible initial data φ_0:

Theorem 6.1 *Assume that the function $r \mapsto F(\cdot, r) - \alpha r$ is increasing for some $\alpha > 0$. Then the complex Monge–Ampère flow φ_t starting at φ_0 uniformly converges, as $t \to +\infty$, to the solution ψ of the Dirichlet problem for the degenerate elliptic Monge–Ampère equation*

$$(dd^c \psi)^n = e^{F(z,\psi)} \mu(z) \text{ in } \Omega, \quad \text{with } \psi_{|\partial\Omega} = \varphi_0.$$

More precisely

$$||\varphi_t - \psi||_{L^\infty(\Omega)} \leq e^{-\alpha t} ||\varphi_0 - \psi||_{L^\infty(\Omega)}$$

The existence of the solution ψ is well known in this case (see [10]).

Proof Consider

$$u(t, z) := e^{\alpha t} \varphi(t, z).$$

Then u is a solution to the parabolic complex Monge–Ampère equation

$$e^{h(t) \partial_t u + G(t, \cdot, u_t)} \mu = (dd^c u_t)^n,$$

where $h(t) := e^{-\alpha t}$ and

$$G(t, z, r) := F(z, e^{-\alpha t} r) - \alpha r e^{-\alpha t} + n\alpha t.$$

We let the reader check that $v(t, z) := e^{\alpha t}\psi(z)$ is a solution to the same parabolic complex Monge–Ampère equation. Our hypothesis on F implies that $r \longmapsto G(t, z, r)$ in non decreasing. We can thus apply the comparison principle (see Remark 4.5), which yields the desired bound. □

6.2 The general case

We now show that the convergence holds in full generality, without any control on the speed of convergence:

Theorem 6.2 *The complex Monge–Ampère flow φ_t starting at φ_0 uniformly converges, as $t \to +\infty$, to the solution ψ of the Dirichlet problem for the degenerate elliptic Monge–Ampère equation*

$$(dd^c\psi)^n = e^{F(z,\psi)}\mu(z) \text{ in } \Omega, \quad \text{with } \psi_{|\partial\Omega} = \varphi_0.$$

Proof We are going to use Theorem 6.1 by considering the perturbed Monge–Ampère flows associated to the functions $F(z, r) + \varepsilon(r - c)$, where $\varepsilon > 0$ is small and c is a carefully chosen constant.

We first establish an upper bound. Set $M_0 := \sup_{\bar{\Omega}} \varphi_0$. Since the constant M_0 is a supersolution to the Monge–Ampère flow associated to (F, μ) with boundary value M_0, it follows from the comparison principle that

$$\varphi(t, z) \leq M_0 \text{ in } \mathbb{R}^+ \times \Omega.$$

Fix $\varepsilon > 0$ and set $F^\varepsilon(z, r) := F(z, r) + \varepsilon(r - M_0)$. Let $\varphi^\varepsilon(t, z)$ be the solution of the complex Monge–Ampère flow associated to (F^ε, μ) with Cauchy–Dirichlet data $\varphi_0^\varepsilon = \varphi_0$ i.e.

$$(dd^c\varphi_t^\varepsilon)^n = e^{\partial_t\varphi^\varepsilon + F(z,\varphi^\varepsilon) + \varepsilon(\varphi^\varepsilon - M_0)}\mu(z). \qquad (\star)_\varepsilon$$

Observe that φ is a subsolution to the flow $(\star)_\varepsilon$ since $\varphi \leq M_0$. The comparison principle therefore implies $\varphi \leq \varphi^\varepsilon$ in $\mathbb{R}^+ \times \Omega$.

Let u^ε be the solution of the degenerate elliptic Monge–Ampère equation $(dd^c u^\varepsilon)^n = e^{F(z,u^\varepsilon)+\varepsilon(u^\varepsilon - M_0)}\mu(z)$ with Dirichlet data $u^\varepsilon|_{\partial\Omega} = \varphi_0|_{\partial\Omega}$ ([10]). It follows from the stability of the solutions to the Dirichlet problem for the complex Monge–Ampère operator that u^ε uniformly converges to u in Ω as $\varepsilon \to 0$ (see [19]).

Fix $\delta > 0$ and choose ε such that $u - \delta \leq u^\varepsilon \leq u + \delta$. It follows from Theorem 6.1 that $\lim_{t\to\infty} \varphi_t^\varepsilon(z) = u^\epsilon(z)$ uniformly in Ω. Therefore there exists $T_\delta > 1$ so that for $t \geq T_\delta$ and $z \in \Omega$, $\varphi_t(z) \leq u(z) + 2\delta$. This is the desired upper bound.

We now establish a lower bound. Observe first that the family (φ_t) is uniformly bounded from below. Indeed let ρ be a strongly psh defining function for Ω and choose

$B > 1$ such that

$$B^n (dd^c \rho)^n \geq e^{F(z,0)} \mu(z)$$

pointwise in Ω. Since $\rho \leq 0$, the function $\psi(t, z) := B\rho(z)$ is a subsolution to the parabolic Monge–Ampère equation $(dd^c \psi_t)^n = e^{\partial_t \psi + F(z,\psi)} \mu(z)$. It therefore follows from the comparison principle that

$$B\rho(z) - \varphi(t, z) \leq \max_{\bar{\Omega}} (B\rho - \varphi_0)_+ \text{ in } \mathbb{R}^+ \times \Omega.$$

Thus φ is uniformly bounded from below by a constant m_0 in $\mathbb{R}^+ \times \Omega$.

We now consider the perturbed Monge–Ampère flow associated to (F_ε, μ) with Cauchy–Dirichlet data $\varphi_0^\varepsilon = \varphi_0$, where $F_\varepsilon(z, r) := F(z, r) + \varepsilon(r - m_0)$. Observe that φ is a supersolution of this new perturbed flow since $\varphi \geq m_0$. Arguing as above shows the existence of $T'_\delta > 1$ such that

$$\varphi_t(z) \geq u(z) - \delta \text{ for } t \geq T'_\delta$$

and $z \in \Omega$. This proves that $\varphi_t \to u$ uniformly in Ω. □

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