J. London Math. Soc. (2) 88 (2013) 737–760
doi:10.1112/jlms/jdt031

Kähler–Einstein fillings

Vincent Guedj, Boris Kolev and Nader Yeganefar

Abstract

We show that on an open bounded smooth strongly pseudoconvex subset of $\mathbb{C}^n$, there exists a Kähler–Einstein metric with positive Einstein constant, such that the metric restricted to the Levi distribution of the boundary is conformal to the Levi form. To achieve this, we solve an associated complex Monge–Ampère equation with Dirichlet boundary condition. We also prove uniqueness of the solution subject to additional restrictions.

1. *Introduction*

The study of Einstein metrics is an important and classical subject in Riemannian geometry; see [**10**]. The most popular framework is that of complete manifolds, either compact (without boundary) or noncompact. However, Einstein metrics on compact manifolds with boundary have also been investigated more recently, mainly in two directions which we now describe.

The first direction is that of conformally compact manifolds. Here, one starts with a compact manifold M with boundary ∂M. A complete Einstein metric on (the interior of) M is called conformally compact if after a suitable conformal transformation, it can be extended smoothly up to the boundary (think of the ball model of real hyperbolic space, or look at [**11**] for the precise definition). This extension is not unique, but different extensions are easily seen to induce Riemannian metrics on the boundary which are in the same conformal class, called the conformal infinity of the conformally compact metric. One of the basic questions is then, a conformal class being fixed on the boundary, is it possible to find a conformally compact Einstein metric on M whose conformal infinity is the given conformal class? One then hopes to get links between the geometric properties of the inner metric and the conformal properties of the boundary; for more on this very active research area, the reader may consult, for example, [**3**, **11**].

We now come to the second direction, which has been explored far less than the first one and is more closely related to our present work. One starts again with a compact manifold M with boundary, and fixes some geometric structure on the boundary (for example, a metric). The problem is then to find an Einstein metric on M which is smooth up to the boundary, and which induces the given geometric structure on ∂M. Assume, for example, that there is an Einstein metric on M with pinched negative curvature such that the boundary is convex and umbilical and let h_0 be the induced metric on ∂M. If h is a metric on ∂M which is sufficiently close to h_0, then it has been shown in [**32**] that there is an Einstein metric on M with negative Einstein constant such that the induced metric on ∂M is h. One of the interesting questions, which has not been fully clarified yet, is to know what 'right' geometric structure has to be fixed on the boundary. Anderson [**2**] considers the Dirichlet problem as in [**32**] (given a metric h on ∂M, can one find an Einstein metric on M inducing h on ∂M?), studies the structure of the space of solutions and observes that this Dirichlet problem is not a well-posed elliptic boundary

Received 30 August 2012; revised 12 February 2013; published online 5 September 2013.

2010 *Mathematics Subject Classification* 32W20, 32Q20 (primary).

Vincent Guedj is partially supported by the French ANR project MACK.

value problem. On the other hand, if one prescribes the metric and the second fundamental form of ∂M, then any Einstein metric on M is essentially unique by Anderson and Herzlich [**3**].

The main purpose of this article is to investigate similar questions in the context of compact Kähler manifolds with boundary. Let M be a compact Kähler manifold with *strongly pseudoconvex* boundary ∂M. The latter is a CR manifold whose geometric properties are encoded by the (conformal class of its) *Levi form*, a positive-definite Hermitian form defined on the *Levi distribution* $T_{\mathbb{C}}(\partial M)$ (the family of maximal complex subspaces within the real tangent bundle). The question we address is the following problem.

PROBLEM. Can one find a Kähler–Einstein metric ω on M such that its restriction to the Levi distribution is conformal to the Levi form on $T_{\mathbb{C}}(\partial M)$?

To simplify, we restrict ourselves in the sequel to studying the case of a strongly pseudoconvex bounded open subset Ω of $\mathbb{C}^n$. One can then always make a conformal change of the Levi form so that the pseudo-Hermitian Ricci tensor (introduced by Webster) is a scalar multiple of the Levi form, that is, $\partial\Omega$ is pseudo-Einstein (see [**25**]). Our problem is thus intimately related to the Riemannian questions recalled above.

It is well known that finding a Kähler–Einstein metric is equivalent to solving a complex Monge–Ampère equation. More specifically, letting μ denote the Lebesgue measure in $\mathbb{C}^n$ normalized such that $\mu(\Omega) = 1$, we will be interested in the following Dirichlet problem: find a smooth strictly plurisubharmonic function φ on Ω which vanishes on the boundary $\partial\Omega$ and satisfies

$$(dd^c\varphi)^n = \frac{e^{-\varepsilon\varphi}\mu}{\int_\Omega e^{-\varepsilon\varphi}\,d\mu} \quad \text{in } \Omega,$$

where $\varepsilon \in \{0, \pm 1\}$ is a fixed constant. If φ is a solution of this problem, then it is easy to see that $dd^c\varphi$ is a Kähler–Einstein metric with the sign of the Einstein constant given by ε, and moreover its restriction to the Levi distribution is conformal to the Levi form on $T_{\mathbb{C}}(\partial\Omega)$ (see Section 2 for more details on this). Actually, if $\varepsilon = 0, -1$, then the Monge–Ampère equation above always has a solution by Caffarelli, Kohn, Nirenberg and Spruck [**14**, Theorem 1.1], so that we will only consider the positive curvature case corresponding to $\varepsilon = 1$. Our main result is the following theorem.

THEOREM 1. *Let $\Omega \subset \mathbb{C}^n$ be a bounded smooth strongly pseudoconvex domain. Then the complex Monge–Ampère problem*

$$\text{(MA)} \qquad (dd^c\varphi)^n = \frac{e^{-\varphi}\mu}{\int_\Omega e^{-\varphi}\,d\mu} \ \text{ in } \Omega \quad \text{and} \quad \varphi_{|\partial\Omega} = 0$$

has a strictly plurisubharmonic solution which is smooth up to the boundary.

By the considerations of Section 2, a consequence of this theorem is that our geometrical problem has a solution.

COROLLARY 2. *Let $\Omega \subset \mathbb{C}^n$ be a bounded smooth strongly pseudoconvex domain. Then there is a smooth (up to the boundary) Kähler–Einstein metric on Ω with positive Einstein constant such that the restriction of the metric to the Levi distribution of $\partial\Omega$ is conformal to the Levi form.*

REMARK 3. It should be noted that if f is an automorphism of the domain Ω and ω is a Kähler–Einstein metric on Ω with positive Einstein constant whose restriction to the Levi distribution is conformal to the Levi form, then the same is true for $f^*\omega$. Therefore, uniqueness can only be expected up to automorphisms [20]. The geometrical problem is actually equivalent to solving a family of Monge–Ampéré equations

$$(dd^c\varphi)^n = e^{-\varphi}\, e^h \mu \ \text{ in } \Omega \quad \text{and} \quad \varphi_{|\partial\Omega} = 0,$$

where h is pluriharmonic in Ω. This invariance property of our geometrical problem under the action of $f \in \mathrm{Aut}(\Omega)$ is equivalent, at the level of (MA), to replacing the pluriharmonic function h by $h \circ f + \log |\mathrm{Jac(f)}|^2$. In the sequel, we shall treat for simplicity the case when h is constant, but our analysis applies to the general case.

Let us now say a few words about the proof of our main theorem. We will use a Ricci inverse iteration procedure, as described first in the compact Kähler setting by Berman, Boucksom, Eyssidieux, Guedj and Zeriahi [7], Keller [23] and Rubinstein [31], whereas related results have recently been obtained in [6, 16] by other interesting approaches. More precisely, fix any smooth strictly plurisubharmonic function φ_0 on Ω which vanishes on the boundary, and for $j \in \mathbb{N}$, let φ_{j+1} be the unique strictly plurisubharmonic solution of the Dirichlet problem

$$(dd^c\varphi_{j+1})^n = \frac{e^{-\varphi_j}\mu}{\int_\Omega e^{-\varphi_j}\, d\mu} \ \text{ in } \Omega \quad \text{and} \quad \varphi_{j+1|\partial\Omega} = 0,$$

whose existence is guaranteed by Caffarelli, Kohn, Nirenberg and Spruck [14]. We will then show that (φ_j) is bounded in $C^\infty(\bar\Omega)$, so that a subsequence converges in $C^\infty(\bar\Omega)$ to a smooth function which is seen to be a solution of (MA). To prove this boundedness in C^∞, we proceed in several steps. First, there is a well-known functional $\mathcal{F}$, defined on the space of plurisubharmonic functions, such that a function φ solves (MA) if and only if φ is a critical point of $\mathcal{F}$ (see Subsection 3.2).

A key result is that this functional is proper in the strong (coercivity) sense of Proposition 11. This properness result is in turn a consequence of a local Moser–Trudinger inequality (Theorem 9, see also the recent independent results of [6, 16]).

Next, we show that the sequence $(\mathcal{F}(\varphi_j))$ is bounded, so that by properness, the sequence (φ_j) has to live in some compact set. Here, compactness is for the L^1-topology in the class of plurisubharmonic functions with finite energy introduced in [5]. Standard results from pluripotential theory then show that (φ_j) is uniformly bounded. To get boundedness in C^∞, we will finally prove higher-order a priori estimates along the lines of [14].

Remark 4. On a closed Fano manifold, the existence of Kähler–Einstein metrics is known to be a difficult problem, and there are in fact closed (Fano) manifolds which do not admit Kähler–Einstein metrics. In contrast, Corollary 2 shows that in the case of domains, there are always Kähler–Einstein metrics with positive Einstein constant (and prescribed behavior at the boundary). The difference lies in the sharp form of the Moser–Trudinger inequality which does not always hold in the compact setting, contrary to Theorem 9. To explain this phenomenon, note that in the local setting, we deal with plurisubharmonic functions which vanish on the boundary and thus have confined singularities. Our result therefore illustrates that the obstruction to the existence of Kähler–Einstein metrics on Fano manifolds is of a global nature.

Now, let us deal with the uniqueness problem. For this, we impose some restrictions on Ω. First, we assume that Ω contains the origin and is circled; this means that Ω is invariant by the natural (diagonal) S^1-action on $\mathbb{C}^n$. Next, if φ is an S^1-invariant solution of the Monge–Ampère equation with Dirichlet boundary condition, then we will say that Ω is (strictly) φ-convex if

Ω is (strictly) convex in the Riemannian sense for the metric $dd^c\varphi$. Note that being φ-convex has a priori nothing to do with being convex in the usual Euclidean sense in $\mathbb{C}^n$. We will prove the following theorem.

THEOREM 5. *Let $\Omega \subset \mathbb{C}^n$ be a bounded smooth strongly pseudoconvex domain which is circled. Let φ be a smooth S^1-invariant strictly plurisubharmonic solution of the complex Monge–Ampère problem* (MA). *If Ω is strictly φ-convex, then φ is the unique S^1-invariant solution of* (MA).

Observe that an S^1-invariant solution always exists, as follows from the proof of Theorem 1: it suffices to start with an initial datum φ_0 which is S^1-invariant, the approximants φ_j will also be S^1-invariant (by the uniqueness part of [**14**]), hence so is any cluster value.

REMARK 6. In the proof of this theorem, we will see that we can replace the φ-convexity hypothesis by a spectral assumption. Namely, if the first eigenvalue of the Laplace operator (of the metric $\omega^\varphi = dd^c\varphi$) with Dirichlet boundary condition is strictly bigger than 1, then (MA) has a unique solution. By Guedj, Kolev and Yeganefar [**21**, Corollary 1.2], the condition on the Ricci curvature of ω^φ and the strict φ-convexity imply this desired spectral estimate. However, [**21**, Proposition 4.1] shows that this estimate may fail if Ω is merely strongly pseudoconvex.

To prove Theorem 5, we follow the approach proposed by Donaldson in the compact (without boundary) setting (see [**8**, **20**]). The heuristic point of view is the following. The space of all plurisubharmonic functions on Ω which vanish on the boundary may be seen as an infinite-dimensional manifold with a natural Riemannian structure. In the S^1-invariant case, we may use a convexity result of Berndtsson [**9**] to show that the functional $\mathcal{F}$ is concave along geodesics of this space. As a consequence, we show that S^1-invariant solutions of (MA) coincide with S^1-invariant maximizers of the functional $\mathcal{F}$. Now, if φ and ψ are two S^1-invariant solutions of (MA), then there exists a geodesic $(\Phi_t)_{0\leqslant t\leqslant 1}$, joining φ to ψ, in the space of Kähler potentials on Ω vanishing on the boundary. Therefore, the function $t \mapsto \mathcal{F}(\Phi_t)$, being concave and attaining its maximum at $t = 0$ and $t = 1$, must be constant. In particular, its derivative vanishes, which implies that $\dot{\Phi}_0$ has to satisfy a partial differential equation (PDE) involving the Laplacian of the metric $dd^c\varphi$ (see equation (5.1)). If Ω is φ-convex, or more generally if the spectral hypothesis alluded to above is satisfied, then the only solution of this PDE is zero, so that $\dot{\Phi}_0$ vanishes identically. From this, we may deduce that (Φ_t) is a constant geodesic, hence $\varphi = \psi$. Note that in the above argument, we have implicitly assumed that (Φ_t) is smooth, which may not be the case. For general continuous geodesics, the proof needs some modifications which will be given in Section 6.

This uniqueness result has the following application. In [**6**, Conjecture 7.5], it is conjectured that if B is a ball in $\mathbb{C}^n$, then any solution of (MA) has to be radial. Theorem 5 shows that this is the case among S^1-invariant solutions if the radius of the ball is not too large. Indeed, let $B \subset \mathbb{C}^n$ be the ball of radius $R > 0$ centered at 0. Consider the radial function

$$\varphi = \frac{n+1}{\pi}[\log\sqrt{1+\|z\|^2} - \log\sqrt{1+R^2}].$$

In an affine chart, φ is the potential of the Fubini–Study metric on complex projective space $\mathbb{P}^n(\mathbb{C})$, normalized to satisfy (MA) on B. Note that B may also be considered as a ball in $\mathbb{P}^n(\mathbb{C})$, whose radius R_{FS} with respect to the Fubini–Study metric is

$$R_{\mathrm{FS}} = \sqrt{\frac{n+1}{\pi}}\arctan R.$$

The diameter of $\mathbb{P}^n(\mathbb{C})$ is then

$$D_{\mathrm{FS}} = \sqrt{\pi(n+1)}/2.$$

If $R_{\mathrm{FS}} < D_{\mathrm{FS}}/2$, then B is strictly convex in $\mathbb{P}^n(\mathbb{C})$, that is, B is strictly φ-convex (this is a well-known result; see, for example, the proof of [**21**, Proposition 4.1]). By Theorem 5, φ is the unique S^1-invariant solution of (MA), so that all such solutions are radial. We have thus proved the following corollary.

Corollary 7. *Let B be a ball in $\mathbb{C}^n$ of radius $0 < R < 1$. Then there is a unique S^1-invariant solution to* (MA) *on B, and this solution is radial.*

The plan of the paper is as follows. In Section 2, we gather some well-known facts on the geometry of pseudoconvex domains and show how our geometrical problem is related to the analytical problem of solving a complex Monge–Ampère equation with Dirichlet boundary condition. In Section 3, we prove a local Moser–Trudinger inequality and use it to prove a properness result for the functional $\mathcal{F}$. In Section 4, we deal with the regularity problem of solutions of (MA), by getting higher-order a priori estimates. This will allow us to prove Theorem 1 in Subsection 4.4. In Section 5, we obtain a variational characterization of solutions of (MA) in the S^1-invariant case. Indeed, we show that S^1-invariant solutions of (MA) are not only critical points of the functional $\mathcal{F}$, but are exactly maximizers of $\mathcal{F}$. Then we proceed to prove Theorem 5. In Section 6, we comment on the difficulty of solving (MA) by the usual continuity method, and finally discuss the optimality of constants in the Moser–Trudinger inequality.

2. *Geometric context*

2.1. *The conformal class of the Levi form*

Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with smooth boundary. Fix a defining function $\rho : \mathbb{C}^n \to \mathbb{R}$ for the boundary $\partial\Omega$, that is, ρ is a smooth function satisfying

$$\Omega = \{\rho < 0\}, \quad \partial\Omega = \{\rho = 0\},$$

and $d\rho$ does not vanish on $\partial\Omega$. Such a function ρ is not unique, but if $\tilde{\rho}$ is another defining function for the boundary, then there is a smooth positive function u such that $\tilde{\rho} = u\rho$.

Let now $x \in \partial\Omega$ be a fixed point, and denote by H_x the maximal complex subspace of the tangent space $T_x\partial\Omega$. If J denotes the complex structure on $\mathbb{C}^n$ (which is just multiplication by $\sqrt{-1}$), then we have

$$H_x = \{v \in T_x\partial\Omega;\ Jv \in T_x\partial\Omega\}.$$

The subspace H_x has real dimension $2n-2$, and as x varies, we get a distribution $H \subset T\partial\Omega$, called the *Levi distribution*. If $(z_1, \ldots, z_n)$ are the coordinates on $\mathbb{C}^n$, then it is easy to see that

$$H_x = \left\{ v = (v_1, \ldots, v_n) \in \mathbb{C}^n;\ \sum_{i=1}^n \frac{\partial \rho}{\partial z_i}(x) v_i = 0 \right\}. \tag{2.1}$$

The *Levi form* is the Hermitian form defined for $v, w \in H_x$ by

$$L_x(v,w) = \sum_{i,j}^n \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j}(x) v_i \bar{w}_j.$$

It is clear from this expression that the Levi form actually depends on ρ, so talking about *the* Levi form is a slight abuse. However, if $\tilde{\rho} = u\rho$ is another defining function for the boundary (with u a smooth positive function), then we have

$$\frac{\partial^2 \tilde{\rho}}{\partial z_i \partial \bar{z}_j} = u \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j} + \frac{\partial u}{\partial z_i} \frac{\partial \rho}{\partial \bar{z}_j} + \frac{\partial u}{\partial \bar{z}_j} \frac{\partial \rho}{\partial z_i} + \rho \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}.$$

Moreover, by using the characterization (2.1) of H and the fact that $\rho = 0$ on $\partial\Omega$, we infer, denoting by $\tilde{L}$ the Levi form corresponding to $\tilde{\rho}$, that

$$\tilde{L} = uL.$$

In other words, the Levi forms corresponding to different defining functions for the boundary differ only by a conformal factor. Thus, the geometrically interesting object on the boundary is the conformal class of the Levi form.

We say that Ω is *strongly pseudoconvex* if the Levi form is a positive-definite Hermitian form at each point of $\partial\Omega$. Our previous discussion shows that this notion does not depend on the choice of a defining function for the boundary. Note also that by changing ρ to $e^{c\rho} - 1$, where $c > 0$ is a large enough positive constant, we may assume the defining function ρ to be plurisubharmonic near the boundary, and not only on the Levi distribution.

2.2. *Kähler metrics*

We give here a brief review of Kähler metrics, mainly to set up some notation and conventions. For more details and proofs, the reader may consult, for example, [**27**]. Although we will be dealing with domains in $\mathbb{C}^n$ in the sequel, we consider a general complex manifold X of complex dimension n, and denote by J its complex structure.

2.2.1. *Kähler form.* A Riemannian metric g on X is called *Hermitian* if it is J-invariant, that is, $g(J\cdot, J\cdot) = g(\cdot, \cdot)$. The $\mathbb{C}$-bilinear extension of g to the complexified tangent bundle $TX \otimes \mathbb{C}$ will also be denoted by the same symbol g. The *fundamental form* associated to g is the real $(1,1)$-form ω defined by

$$\omega(\cdot, \cdot) = g(J\cdot, \cdot).$$

The metric g is called a *Kähler metric* if ω is a closed differential form; ω is then referred to as the *Kähler form* of g. It can be shown that g being a Kähler metric is equivalent to the complex structure J being parallel with respect to the Levi-Civita connection of g.

Let $(z_1, \ldots, z_n)$ be local complex coordinates, and let

$$z_1 = x_1 + \sqrt{-1} y_1, \ldots, z_n = x_n + \sqrt{-1} y_n$$

be the decomposition giving the corresponding real coordinates. As usual, for $i = 1, \ldots, n$, we set

$$\frac{\partial}{\partial z_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} - \sqrt{-1}\frac{\partial}{\partial y_i}\right), \quad \frac{\partial}{\partial \bar{z}_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} + \sqrt{-1}\frac{\partial}{\partial y_i}\right),$$

$$dz_i = dx_i + \sqrt{-1}\, dy_i, \quad d\bar{z}_i = dx_i - \sqrt{-1}\, dy_i,$$

and for $i, j = 1, \ldots, n$,

$$g_{i\bar{j}} = g\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}\right).$$

Then the Kähler form is given locally by

$$\omega = \sqrt{-1} \sum_{i,j=1}^{n} g_{i\bar{j}}\, dz_i \wedge d\bar{z}_j.$$

Note that on $\mathbb{C}^n$, we have $g_{i\bar{j}} = \delta_{ij}/2$ for the canonical Euclidean metric.

2.2.2. *Ricci curvature form.* We denote by r the Ricci tensor of X as a Riemannian manifold. The *Ricci form* of X, to be denoted by $\mathrm{Ric}\,(\omega)$ or simply Ric, is the $(1,1)$-form associated to r, that is,

$$\mathrm{Ric}\,(\omega)(\cdot,\cdot) = r(J\cdot,\cdot).$$

In local holomorphic coordinates, it can be shown that

$$\mathrm{Ric}\,(\omega) = -\sqrt{-1}\partial\bar{\partial}\log\det g_{i\bar{j}}.$$

It follows that the Ricci form is a closed form. Moreover, its cohomology class is equal to $2\pi c_1(X)$, where $c_1(X)$ is the first Chern class of X. A Kähler metric ω on X is called *Kähler–Einstein* if for some constant $\lambda \in \mathbb{R}$ we have

$$\mathrm{Ric}\,(\omega) = \lambda\omega.$$

2.2.3. *Normalization of d^c.* We set

$$d^c = \frac{1}{2\pi\sqrt{-1}}(\partial - \bar{\partial}),$$

so that

$$\sqrt{-1}\partial\bar{\partial} = \pi\, dd^c.$$

This normalization is in common use in complex analytic geometry, having the following advantages: the positive current $T = dd^c \log\|z\|$ then has Lelong number 1 at the origin in $\mathbb{C}^n$; moreover, the Fubini–Study form ω_{FS}, in some affine chart $\mathbb{C}^n$, is written

$$\omega_{\mathrm{FS}} = dd^c \log\sqrt{1+\|z\|^2}.$$

Its cohomology class thus coincides with that of a hyperplane (as it should), having total volume

$$\int_{\mathbb{P}^n} \omega_{\mathrm{FS}}^n = \int_{\mathbb{C}^n} (dd^c \log\sqrt{1+\|z\|^2})^n = 1.$$

Note finally that $\mathrm{Ric}\,(\omega_{\mathrm{FS}}) = (n+1)\pi\omega_{\mathrm{FS}}$.

Likewise, the Laplacian Δ associated to a Kähler metric ω is defined as

$$\Delta = \mathrm{tr}\,(dd^c),$$

where tr denotes the trace with respect to ω. Hence, we have

$$\Delta = -\frac{1}{\pi}\bar{\partial}^*\bar{\partial}.$$

2.3. *Kähler–Einstein metrics on strongly pseudoconvex domains*

Fix $\Omega \subset \mathbb{C}^n$ a bounded strongly pseudoconvex domain.

2.3.1. *Associated complex Monge–Ampère equations.* In this section, we show that finding Kähler–Einstein metrics is equivalent to solving a complex Monge–Ampère equation.

We assume first that Ω is endowed with a Kähler metric ω which is smooth up to the boundary, and which satisfies the following normalized Einstein condition:

$$\mathrm{Ric}\,(\omega) = \varepsilon\pi\omega,$$

where $\varepsilon \in \{0, \pm 1\}$ (the somewhat unusual π factor is due to our normalization convention for the d^c operator). We choose a smooth potential φ for ω, so that

$$\omega = dd^c\varphi.$$

Such a potential is unique up to the addition of a pluriharmonic function on Ω. We are going to see that φ satisfies a complex Monge–Ampère equation. As recalled in the previous section, if we denote by $(g_{i\bar{j}})$ the components of the metric in coordinates, then the Ricci form is given by

$$\mathrm{Ric}\,(\omega) = -\pi\, dd^c \log\left(\det g_{i\bar{j}}\right).$$

Letting V_0 be the canonical volume form on $\mathbb{C}^n$, it is easily checked that ω^n is equal to $\det(g_{i\bar{j}})V_0$, up to a multiplicative constant. Therefore, we have the following intrinsic formula for the Ricci form:

$$\mathrm{Ric}\,(\omega) = -\pi\, dd^c \log \frac{\omega^n}{V_0}.$$

The Einstein condition on ω can then be written

$$dd^c\left[\log\frac{(dd^c\varphi)^n}{V_0} + \varepsilon\varphi\right] = 0.$$

Thus, there is a pluriharmonic function h such that

$$\log\frac{(dd^c\varphi)^n}{V_0} + \varepsilon\varphi = h,$$

which we may write as a complex Monge–Ampère equation

$$(dd^c\varphi)^n = e^{-\varepsilon\varphi}\, e^h V_0. \tag{2.2}$$

Conversely, if φ is a smooth function satisfying the previous equation for some given pluriharmonic function h, and if $\omega = dd^c\varphi$ is positive definite, then we let the reader verify that ω is a Kähler–Einstein metric with Einstein constant $\varepsilon\pi$.

2.3.2. *Boundary conditions.* Let ρ be a boundary-defining function for Ω, as described in Subsection 2.1. Recall that L is the Levi form associated to ρ. The $(1,1)$-form associated to L, that is $L(J\cdot,\cdot)$, is equal to $\pi\, dd^c\rho_{|H}$ with our normalization conventions. Let now φ be a smooth real-valued function defined on $\bar{\Omega}$. On a collar neighborhood $[-\delta,0]\times\partial\Omega$ of $\partial\Omega$ (where $\delta>0$ is fixed), we can write the expansion of φ in powers of ρ as follows: for all $N\in\mathbb{N}$,

$$\varphi = \varphi_0 + \rho\varphi_1 + \rho^2\varphi_2 + \cdots + \rho^N\varphi_n + o(\rho^N). \tag{2.3}$$

Here, the functions φ_i are initially defined on $\{0\}\times\partial\Omega \simeq \partial\Omega$, but we can view them as functions defined on the collar neighborhood $[-\delta,0]\times\partial\Omega$ by setting, with obvious notation, $\varphi_i(\rho,x) = \varphi_i(0,x)$. Thus, we have, for example, $\varphi_0 = 0$ if $\varphi|_{\partial\Omega} = 0$. From the expansion (2.3), we get

$$dd^c\varphi = dd^c\varphi_0 + \varphi_1\, dd^c\rho + d\rho\wedge d^c\varphi_1 + (d\varphi_1 + 2\varphi_2\, d\rho)\wedge d^c\rho + O(\rho).$$

Using the fact that $d\rho_{|H} = d^c\rho_{|H} = 0$ (see the characterization (2.1) of H), the previous expansion implies

$$dd^c\varphi|_H = dd^c\varphi_0|_H + \varphi_1\, dd^c\rho.$$

In particular, if $\varphi_0 = 0$, or more generally if $dd^c\varphi_0 = 0$, then $dd^c\varphi|_H$ is conformal to the Levi form.

Consider now the following geometrical problem: find a Kähler–Einstein metric ω on Ω such that its restriction to the Levi distribution is conformal to the Levi form. Our previous discussion shows that in order to solve this problem, it is enough to solve the following analytical problem: find a function φ such that the following conditions hold:

(1) $dd^c\varphi$ is positive definite;
(2) φ satisfies the Monge–Ampère equation (2.2);
(3) φ satisfies the Dirichlet boundary condition on $\partial\Omega$, that is, $\varphi|_{\partial\Omega} = 0$.

Indeed, the form $\omega = dd^c\varphi$ is then a solution to the geometrical problem. Note that in the case of nonpositive Ricci curvature, which corresponds to $\varepsilon = 0$ or -1 in equation (2.2), the geometrical problem always has a solution by Caffarelli, Kohn, Nirenberg and Spruck [14, Theorem 1.1]. We will therefore consider only the positive curvature case ($\varepsilon = 1$).

2.4. *The strategy*

In the sequel, we let $\Omega = \{\rho < 0\} \subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain and μ denote the Euclidean Lebesgue volume form in $\mathbb{C}^n$, normalized so that

$$\mu(\Omega) = 1.$$

We consider the following Dirichlet problem:

$$\text{(MA)} \qquad (dd^c\varphi)^n = \frac{e^{-\varphi}\mu}{\int_\Omega e^{-\varphi} d\mu} \text{ in } \Omega \quad \text{with } \varphi_{|\partial\Omega} = 0,$$

where φ is strictly plurisubharmonic and $\mathcal{C}^\infty$-smooth up to the boundary of Ω.

Following [7] (where such a technique is used in a compact setting), we are going to solve (MA) by *an iterative process*, solving for each $j \in \mathbb{N}$ the Dirichlet problem

$$\text{(MA)}_j \qquad (dd^c\varphi_{j+1})^n = \frac{e^{-\varphi_j}\mu}{\int_\Omega e^{-\varphi_j}\, d\mu} \text{ in } \Omega \quad \text{with } \varphi_{j+1|\partial\Omega} = 0,$$

where $\varphi_0 = \rho$ (we could actually start from any smooth plurisubharmonic initial data φ_0 with zero boundary values).

It follows from the work of Cafarelli–Kohn–Nirenberg–Spruck [14] that the Dirichlet problem $(\text{MA})_j$ admits a unique plurisubharmonic solution φ_{j+1} which is smooth up to the boundary. We are going to show that a subsequence of the sequence (φ_j) converges in $\mathcal{C}^\infty(\bar\Omega)$ toward a solution φ of (MA).

In a compact setting, this approach coincides with the time-one discretization of the Kähler–Ricci flow and was first considered by Keller [23] and Rubinstein [31] (see also [7]).

REMARK 8. As the proof will show, our result actually holds for any (normalized) volume form μ and with more general boundary values.

3. *Energy estimates*

We now move on to showing that the sequence (φ_j) is relatively compact in $\mathcal{C}^\infty(\bar\Omega)$. The proof reduces to establishing a priori *estimates*. We first show that one has a uniform a priori control on the *energy* of the solutions.

3.1. *Local Moser–Trudinger inequality*

The following local Moser–Trudinger-type inequality is of independent interest. (While we were finishing the writing of this paper, two preprints appeared [6, 16] which propose similar inequalities with different and interesting proofs.)

THEOREM 9. *There exist $0 < \beta_n < 1$ and $C > 0$ such that for all smooth plurisubharmonic functions φ in Ω with $\varphi_{|\partial\Omega} = 0$,*

$$\int_\Omega e^{-\varphi} d\mu \leqslant C \exp(\beta_n |\mathcal{E}(\varphi)|),$$

where $\mathcal{E}(\varphi) = (1/(n+1)) \int_\Omega \varphi (dd^c\varphi)^n$.

We refer the reader to [19, 28–30, 35, 36] for related results both in a local and global context. The proof we propose is new and relies on pluripotential techniques, as developed in [4, 5, 15, 22, 24, 38].

Proof. Recall that the Monge–Ampère capacity was introduced by Bedford and Taylor [4]. By definition, the capacity of a compact subset $K \subset \Omega$ is

$$\operatorname{Cap}(K) := \sup\left\{\int_K (dd^c u)^n ; u \text{ plurisubharmonic in } \Omega \text{ with } 0 \leqslant u \leqslant 1\right\}.$$

We will use the following useful inequalities: for any $\gamma < 2$, there exists $C_\gamma > 0$ such that for all $K \subset \Omega$,

$$\mu(K) \leqslant C_\gamma \exp\left[-\frac{\gamma}{\operatorname{Cap}(K)^{1/n}}\right] \tag{3.1}$$

(see, for example, [38]). For all smooth plurisubharmonic functions φ in Ω with zero boundary values, for all $t > 0$,

$$\operatorname{Cap}(\varphi < -t) \leqslant \frac{(n+1)|\mathcal{E}(\varphi)|}{t^{n+1}},$$

where

$$\mathcal{E}(\varphi) := \frac{1}{n+1}\int_\Omega \varphi (dd^c\varphi)^n.$$

For the latter inequality, we refer the reader to [1, Lemma 2.2]. We infer

$$\int_\Omega e^{-\varphi}\, d\mu = -1 + \int_0^{+\infty} e^t \mu(\varphi < -t)\, dt \leqslant C\int_0^{+\infty} \exp(t - \lambda t^{1+1/n})\, dt,$$

where

$$\lambda := \frac{\gamma}{(n+1)^{1/n}|\mathcal{E}(\varphi)|^{1/n}}.$$

We let the reader check that the function $h(t) = t - \lambda t^{1+1/n}$ attains its maximum value at point $t_c = \lambda^{-n}(1 + 1/n)^{-n}$. Moreover, $h(t) \leqslant -t$ for $t \geqslant 4^n t_c$. This shows

$$\int_0^{+\infty} \exp(t - \lambda t^{1+1/n}) dt \leqslant 4^n t_c \exp(h(t_c)) + \int_{4^n t_c}^{+\infty} \exp(-t)\, dt \tag{3.2}$$

$$\leqslant 4^n t_c \exp\left(\frac{t_c}{n+1}\right) + 1. \tag{3.3}$$

Using the definition of λ and the formula defining t_c, we arrive at

$$\int_0^{+\infty} \exp(t - \lambda t^{1+1/n})\, dt \leqslant c_n |\mathcal{E}(\varphi)| \exp(\beta_n' |\mathcal{E}(\varphi)|) + 1,$$

where

$$\beta_n' = \frac{1}{\gamma^n (1 + 1/n)^n}.$$

We can fix, for example, $\gamma = 1$ so that $\beta_n' < 1$ for all $n \geqslant 1$. Moreover, the desired inequality is obtained by choosing β_n so that $\beta_n' < \beta_n < 1$ and enlarging the constant C. $\square$

Remark 10. Note for later use that the same proof yields an inequality

$$\int_\Omega e^{-A\varphi}\, d\mu \leqslant C_A \exp(\beta_A |\mathcal{E}(\varphi)|), \tag{3.4}$$

where

$$\beta_A := \frac{A^{n+1}}{\gamma^n (1 + 1/n)^n}$$

is smaller than 1 only if $A = A_n$ is not too large. When $n = 1$, the critical value is $A = 2$. This is related to a theorem of Bishop as we shall see in Subsection 4.4.

It follows from the recent work [**1**] that the optimal exponent γ is actually $2n$, improving the bound 2 obtained in [**38**], hence also enlarging the allowed constant A_n above, when $n > 1$.

3.2. *Properness*

We let

$$\mathcal{E}(\varphi) := \frac{1}{n+1}\int_\Omega \varphi(dd^c\varphi)^n$$

denote the *energy* of a plurisubharmonic function φ and set

$$\mathcal{F}(\varphi) := \mathcal{E}(\varphi) + \log\left[\int_\Omega e^{-\varphi}\, d\mu\right].$$

Recall that the energy functional is a primitive of the complex Monge–Ampère operator, namely if ψ_s is a curve of plurisubharmonic functions with zero boundary values, then

$$\frac{d\mathcal{E}(\psi_s)}{ds} = \int_\Omega \dot{\psi}_s(dd^c\psi_s)^n,$$

as follows from Stokes theorem. A similar computation shows that a function φ solves (MA) if and only if it is a critical point of the functional $\mathcal{F}$ (in other words (MA) is the Euler–Lagrange equation for $\mathcal{F}$).

Inspired by techniques from the calculus of variations, it is thus natural to try and maximize the functional $\mathcal{F}$ so as to build a critical point. This usually requires the functional to be *proper* in order to be able to restrict to compact subsets of the space of functions involved. It follows from the Moser–Trudinger inequality (Theorem 9) that the functional $\mathcal{F}$ is indeed *proper*, in the following strong sense.

Proposition 11. *There exist $a > 0$, $b \in \mathbb{R}$ such that for all smooth plurisubharmonic functions ψ in Ω, with zero boundary values,*

$$\mathcal{F}(\psi) \leqslant a\mathcal{E}(\psi) + b.$$

Proof. This is an immediate consequence of Theorem 9 with $a = 1 - \beta_n$ and $b = \log C$. □

3.3. *Ricci inverse iteration*

We let $PSH(\Omega)$ denote the set of plurisubharmonic functions in Ω. Given $\varphi \in \mathrm{PSH}(\Omega) \cap \mathcal{C}^\infty(\bar{\Omega})$ with zero boundary values, it follows from the work of Cafarelli, Kohn, Nirenberg and Spruck [**14**] that there exists a unique function $\psi \in \mathrm{PSH}(\Omega) \cap \mathcal{C}^\infty(\bar{\Omega})$ with zero boundary values such that

$$(dd^c\psi)^n = \frac{e^{-\varphi}\mu}{\int_\Omega e^{-\varphi} d\mu} \quad \text{in } \Omega.$$

We let

$$\mathcal{T} := \{\varphi \in \mathrm{PSH}(\Omega) \cap \mathcal{C}^\infty(\bar{\Omega}) \mid \varphi_{|\partial\Omega} = 0\}$$

denote the space of test functions and

$$T : \varphi \in \mathcal{T} \longmapsto \psi \in \mathcal{T}$$

denote the operator such that $\psi = T(\varphi)$ is the unique solution of $(*)$. Observe that solving (MA) is equivalent to finding a fixed point of T.

The key to the dynamical construction of solutions to (MA) lies in the following monotonicity property.

Proposition 12. *For all $\varphi \in \mathcal{T}$,*

$$\mathcal{F}(T\varphi) \geqslant \mathcal{F}(\varphi),$$

with strict inequality unless $T\varphi = \varphi$.

Proof. Fix $\varphi \in \mathcal{T}$ and set $\psi := T\varphi$. Recall that

$$\mathcal{F}(\varphi) = \mathcal{E}(\varphi) + \log\left[\int_\Omega e^{-\varphi}\, d\mu\right]$$

and

$$\mathcal{E}(\psi) - \mathcal{E}(\varphi) = \frac{1}{n+1}\sum_{j=0}^{n}\int_\Omega (\psi - \varphi)(dd^c\psi)^j \wedge (dd^c\varphi)^{n-j}.$$

It follows from Stokes theorem that for all j,

$$\int (\psi-\varphi)(dd^c\psi)^j \wedge (dd^c\varphi)^{n-j} = \int (\psi - \varphi)(dd^c\psi)^n + \int d(\psi-\varphi)\wedge d^c(\psi-\varphi)\wedge S,$$

where S is a positive closed form of bidegree $(n-1,n-1)$. Thus,

$$\mathcal{E}(\psi) - \mathcal{E}(\varphi) \geqslant \int_\Omega (\psi-\varphi)(dd^c\psi)^n.$$

We now set

$$\tilde{\varphi} := \varphi + \log\left[\int e^{-\varphi}\, d\mu\right], \quad \tilde{\psi} := \psi + \log\left[\int e^{-\psi}\, d\mu\right]$$

and

$$\mu_\varphi := e^{-\tilde{\varphi}}\mu, \quad \mu_\psi := e^{-\tilde{\psi}}\mu.$$

Note that the latter are probability measures in Ω with $(dd^c\psi)^n = \mu_\varphi$.

It follows from the definition of $\mathcal{F}$ and our last inequality that

$$\mathcal{F}(\psi) - \mathcal{F}(\varphi) \geqslant \int_\Omega (\tilde{\psi} - \tilde{\varphi})d\mu_\varphi = \int_\Omega F\log F\, d\mu_\psi,$$

where $F = e^{\tilde{\psi}-\tilde{\varphi}}$, hence the latter quantity denotes the relative entropy of the probability measures μ_φ, μ_ψ. It follows from the convexity of $-\log$ and Jensen's inequality (note that $Fd\mu_\psi = d\mu_\varphi$ is a probability measure) that

$$\int_\Omega -\log[F^{-1}]\, Fd\mu_\psi \geqslant -\log\left[\int_\Omega F^{-1}F\, d\mu_\psi\right] = 0.$$

Since $-\log(x)$ is a strict convex function for $x > 0$, it follows that equality holds only when $F = 1$ almost everywhere, that is, $\tilde{\varphi} = \tilde{\psi}$.

Observe finally that since ψ and φ both have zero boundary values, the equality $\tilde{\varphi} = \tilde{\psi}$ can only occur when $\varphi \equiv \psi$, that is, when $\varphi = T\varphi$ is a fixed point of T, as claimed. □

We infer that the energies $\mathcal{E}(\varphi_j)$ of the solutions φ_j of $(\mathrm{MA})_{j-1}$ are uniformly bounded.

Corollary 13. *The sequence $(\mathcal{F}(T^j\varphi_0))_j$ is bounded, hence so is $(\mathcal{E}(T^j\varphi_0))_j$.*

Proof. Fix $\varphi_0 \in \mathcal{T}$ (for example, $\varphi_0 = \rho$) and set $\varphi_j = T^j \varphi_0$. Observe that $\mathcal{E}(\varphi_j) \leqslant 0$ since $\varphi_j \leqslant 0$, hence it suffices to establish a bound from below. The previous proposition ensures that the sequence $(\mathcal{F}(T^j \varphi_0))_j$ is increasing. It follows from Proposition 11 that

$$\mathcal{F}(\varphi_0) \leqslant \mathcal{F}(T^j \varphi_0) \leqslant a\mathcal{E}(T^j \varphi_0) + b \leqslant b,$$

so that the energies $\mathcal{E}(T^j \varphi_0)$ are uniformly bounded. □

4. *Higher-order estimates*

4.1. *Uniform a priori estimates*

Recall that φ_j is a smooth plurisubharmonic solution of $(\mathrm{MA})_{j-1}$. Its Monge–Ampère measure thus satisfies

$$(dd^c \varphi_j)^n = f_j \mu \quad \text{with } f_j = \frac{e^{-\varphi_{j-1}} \mu}{\int_\Omega e^{-\varphi_{j-1}} d\mu}.$$

It follows from the previous section that the φ_j's have uniformly bounded energy. Thus, they form a relatively compact family (for the L^1-topology) in the class $\mathcal{E}^1(\Omega)$ of plurisubharmonic functions with finite energy (see [**5**]). When the complex dimension is $n = 1$, the latter is the class of negative plurisubharmonic functions with zero boundary values and whose gradient is in L^2; since (normalized) plurisubharmonic functions are uniformly L^2, the family (φ_j) is thus included in a finite ball of the Sobolev space $W^{1,2}$. In higher dimensions, the class $\mathcal{E}^1(\Omega)$ is a convenient substitute for the Sobolev spaces; we refer the reader to [**5**] for more details.

We simply recall here that functions in $\mathcal{E}^1(\Omega)$ have zero Lelong numbers. For such a function ψ, Skoda's integrability theorem [**34**] ensures that $e^{-\psi}$ is in L^q for all $q > 1$. Since the family (φ_j) is, moreover, relatively compact, Skoda's uniform integrability theorem [**38**] ensures that the densities f_j satisfy

$$\int_\Omega f_j^2 \, d\mu \leqslant C,$$

for some uniform constant $C > 0$. This can also be seen as a consequence of Theorem 9.

Recall now the following fundamental result due to Kolodziej [**24**] (see also [**17**] for the case of L^2 densities): if ψ is a smooth plurisubharmonic function in Ω with zero boundary values and such that

$$(dd^c \psi)^n = f \, d\mu,$$

where $f \in L^2(\mu)$, then

$$\|\psi\|_{L^\infty(\Omega)} \leqslant C_f,$$

where the constant C_f only depends on Ω and $\|f\|_{L^2}$. Applying this to $\psi = \varphi_j$ yields the following lemma.

LEMMA 14. *For all $j \in \mathbb{N}$,*

$$-C_0 \leqslant \varphi_j \leqslant 0, \tag{4.1}$$

for some uniform constant $C_0 > 0$.

4.2. $\mathcal{C}^2$ *a priori estimates*

The goal of this section is to establish the following a priori estimates on the Laplacian of the solutions to $(\mathrm{MA})_{j-1}$.

THEOREM 15. *There exists $C > 0$ such that for all $j \in \mathbb{N}$,*

$$\sup_{\bar{\Omega}} |\Delta \varphi_j| \leqslant C.$$

These estimates are 'almost' contained in [**14**], however, hypothesis (1.3) on p. 213 is not satisfied, hence neither [**14**, Theorem 1.1] nor [**14**, Theorem 1.2] can be applied to our situation.

We nevertheless follow their proof as organized by Boucksom [**12**], explaining some of the necessary adjustments. It will be a consequence of the following series of lemmas.

LEMMA 16. *There exists $C_1 > 0$ such that*

$$\sup_{\partial\Omega} |\nabla \varphi_j| \leqslant C_1.$$

Proof. It follows from the order-zero uniform estimates (4.1) that

$$(dd^c \varphi_j)^n \leqslant e^{C_0} \mu \quad \text{in } \Omega.$$

Let u denote the unique smooth plurisubharmonic function in $\bar{\Omega}$ such that

$$(dd^c u)^n = e^{C_0} \mu \ \text{ in } \Omega \quad \text{with } u_{|\partial\Omega} \equiv 0.$$

The latter exists by Caffarelli, Kohn, Nirenberg and Spruck [**14**, Theorem 1.1]. It follows from the comparison principle that

$$u \leqslant \varphi_j \leqslant 0 \quad \text{in } \Omega.$$

This yields the desired control of $\nabla \varphi_j$ on $\partial\Omega$. □

LEMMA 17. *There exists $C_2 > 0$ such that*

$$\sup_{\Omega} |\Delta \varphi_j| \leqslant C_2 (1 + \sup_{\partial\Omega} |\Delta \varphi_j|).$$

Proof. We let Δ_j denote the Laplace operator with respect to the Kähler form $\omega_j = dd^c \varphi_j$, while Δ denotes the Euclidean Laplace operator. We claim that for all $j \geqslant 1$,

$$\Delta_j \{\log \Delta \varphi_j + \varphi_{j-1}\} \geqslant 0. \tag{4.2}$$

Assuming this for the moment, we show how to derive the desired control on $\Delta \varphi_j$. Let $z_j \in \bar{\Omega}$ be a point which realizes the maximum of the function

$$h_j := \varphi_j + \varphi_{j-1} + \log \Delta \varphi_j.$$

It follows from (4.2) that $z_j \in \partial\Omega$, otherwise $\Delta_j h_j(z_j) \leqslant 0$ contradicting

$$\Delta_j h_j \geqslant \Delta_j \varphi_j > 0.$$

We infer from Lemma 14 that for all $w \in \Omega$,

$$\log \Delta \varphi_j(w) \leqslant 2C_0 + h_j(z_j) \leqslant 2C_0 + \log \sup_{\partial\Omega} \Delta \varphi_j,$$

which yields the desired upper bound.

It remains to establish (4.2). We shall need the following local differential inequality which goes back to the works of Aubin and Yau: if ω is an arbitrary Kähler form and $\beta = dd^c \|z\|^2$ denotes the Euclidean Kähler form, then

$$\Delta_\omega \log \operatorname{tr}_\beta(\omega) \geqslant - \frac{\operatorname{tr}_\beta(\operatorname{Ric} \omega)}{\operatorname{tr}_\beta(\omega)}. \tag{4.3}$$

We apply this inequality to $\omega = \omega_j = dd^c\varphi_j$. Observe that $\mathrm{Ric}(\omega_j) = \omega_{j-1}$ since

$$(dd^c\varphi_j)^n = e^{-\varphi_{j-1}}\, e^{c_j} dV.$$

Observe that

$$\frac{\mathrm{tr}_\beta(\omega_{j-1})}{\mathrm{tr}_\beta(\omega_j)} = \frac{\Delta_\beta(\varphi_{j-1})}{\mathrm{tr}_\beta(\omega_j)} \leqslant \Delta_j(\varphi_{j-1}).$$

Combined with (4.3), this yields

$$\Delta_j \log \mathrm{tr}_\beta(\omega_j) \geqslant -\Delta_j(\varphi_{j-1}),$$

whence (4.2). □

LEMMA 18. *There exists $C_3 > 0$ such that*

$$\sup_{\partial\Omega} |D^2\varphi_j| \leqslant C_3(1 + \sup_{\Omega} |\nabla\varphi_j|^2).$$

Proof. This follows from a long series of estimates which are the same as those of [**14**], up to minor modifications. We only sketch these out, following the proof of [**12**, Lemma 7.17]. To fit in with the notation of [**12**], we set $\psi = \varphi_j - \rho$ and $\eta = dd^c\rho$ so that ψ is a η-psh function (still) with zero boundary values on $\partial\Omega$ such that

$$(\eta + dd^c\psi)^n = e^{-\psi}\, e^F \eta^n,$$

where F is some smooth density. Our problem is thus equivalent to showing an a priori estimate

$$\sup_{\partial\Omega} |D^2\psi| \leqslant C_3(1 + \sup_{\Omega} |\nabla\psi|^2),$$

where C_3 is under control.

Fix $p \in \Omega$. It is classical that one can choose complex coordinates $(z_j)_{1\leqslant j\leqslant n}$ so that $p = 0$ and

$$\rho = -x_n + \Re\left(\sum_{j,k=1}^{n} a_{jk} z_j \bar{z}_k\right) + O(|z|^3),$$

where $z_j = x_j + iy_j$. For convenience, we set

$$t_1 = x_1,\ t_2 = x_1, \ldots, t_{2n-1} = y_n,\ t_{2n} = x_{2n}.$$

Let (D_j) be the dual basis of $dt_1, \ldots, dt_{2n-1}, -d\rho$ so that for $j < 2n$,

$$D_j = \frac{\partial}{\partial t_j} - \frac{\rho_{t_j}}{\rho_{x_n}} \frac{\partial}{\partial x_n} \quad \text{and} \quad D_{2n} = -\frac{1}{\rho_{x_n}} \frac{\partial}{\partial x_n}.$$

Step 0: *Bounding the tangent–tangent derivatives.* Observe that the D_j's commute and are tangent to $\partial\Omega$ for $j < 2n$; we thus have a trivial control on the tangent–tangent derivatives at $p = 0$,

$$D_i D_j \psi(0) = 0 \quad \text{for } 1 \leqslant i, j < 2n.$$

Step 1: *Bounding the normal–tangent derivatives.* Set $K = \sup_{\partial\Omega} |\nabla\psi|$. We claim that for all $1 \leqslant i < 2n$,

$$|D_i D_{2n} \psi(0)| \leqslant C(1 + K),$$

for some uniform constant $C > 0$.

Let h be the smooth function in Ω with zero boundary values such that

$$\Delta_\eta h := n \frac{dd^c h \wedge \eta^{n-1}}{\eta^n} = -n \quad \text{in } \Omega.$$

The proof requires the construction of a barrier $b = \psi + \varepsilon h - \mu\rho^2$ such that

$$0 \leqslant b \quad \text{and} \quad \Delta_\psi b := n\frac{dd^c b \wedge (\eta + dd^c\psi)^{n-1}}{(\eta + dd^c\psi)^n} \leqslant -\frac{1}{2}\operatorname{tr}_\psi(\eta) \text{ in } B,$$

where B is a half ball centered at $p = 0$ of positive radius and $\varepsilon, \mu > 0$ are under control. This can be done exactly as in [**12**, Lemma 7.17, Step 1] since the only information needed is that $(\eta + dd^c\psi)^n$ is uniformly bounded from above by $C\eta^n$, which follows here from our $\mathcal{C}^0$-estimate.

One then shows the existence of uniform constants $\mu_1, \mu_2 > 0$ such that the functions $v_\pm := K(\mu_1 + \mu_2|z|^2) \pm D_j\psi$ both satisfy

$$0 \leqslant v_\pm \text{ on } B \quad \text{and} \quad \Delta_\psi v_\pm \leqslant 0 \text{ in } B.$$

It follows then from the maximum principle that $v_\pm \geqslant 0$ in B so that $D_{2n}v_\pm(0) \geqslant 0$ since $v_\pm(0) = 0$. Thus,

$$|D_{2nj}\psi(0)| \leqslant CK(1 + D_{2n}b(0)) \leqslant C'(1 + K),$$

as claimed.

Step 2: *Bounding the normal–normal derivatives.* This is somehow the most delicate estimate. Set again $K = \sup_{\partial\Omega}|\nabla\psi|$. We want to show $|D^2_{2n}\psi(0)| \leqslant C(1 + K^2)$ for some uniform constant $C > 0$. Using previous estimates on $D_iD_j\psi(0)$, it suffices to show

$$|\psi_{z_n\bar{z}_n}(0)| \leqslant C(1 + K^2).$$

Recall that

$$\det(\rho_{z_i\bar{z}_j}(0) + \psi_{z_i\bar{z}_j}(0))_{1\leqslant i,j\leqslant n} = e^{-\psi(0)+F(0)}$$

is bounded from above, and for $i < n$,

$$|\psi_{z_i\bar{z}_n}(0)| \leqslant C(1 + K).$$

Expanding the determinant with respect to the last row thus yields the expected upper bound, provided we can bound from below the $(n-1, n-1)$-minor

$$\det(\rho_{z_i\bar{z}_j}(0) + \psi_{z_i\bar{z}_j}(0))_{1\leqslant i,j\leqslant n-1}.$$

A (by now) classical barrier argument shows $dd^c\varphi = \eta + dd^c\psi$ is uniformly bounded from below by $\varepsilon\eta$ on the complex tangent space to $\partial\Omega$ (see [**12**, Lemma 7.16] which can be used since φ_j is uniformly bounded). □

The following blowup argument was used by Chen [**18**] for constructing geodesics in the space of Kähler metrics.

Lemma 19. *There exists $C_4 > 0$ such that*

$$\sup_\Omega |\nabla\varphi_j| \leqslant C_4.$$

Proof. It follows from previous estimates that

$$\sup_\Omega \Delta\varphi_j \leqslant C(1 + \sup_\Omega |\nabla\varphi_j|^2).$$

Assume that $\sup_\Omega|\nabla\varphi_j|$ is unbounded. Up to extracting and relabeling, this means that

$$M_j := |\nabla\varphi_j(x_j)| = \sup_\Omega |\nabla\varphi_j| \to +\infty,$$

where $x_j \in \bar{\Omega}$ converges to $a \in \bar{\Omega}$. We set

$$\psi_j(z) := \varphi_j(x_j + M_j^{-1} z).$$

This is a sequence of uniformly bounded plurisubharmonic functions which are well defined (at least) in a half ball B around zero and satisfy

$$|\nabla \psi_j(0)| = 1 \quad \text{and} \quad \sup_B \Delta \psi_j \leqslant C.$$

We infer that the sequence (ψ_j) is relatively compact in $\mathcal{C}^1$, hence we can assume (up to relabeling) $\psi_j \to \psi \in \mathcal{C}^1(B)$, where ψ is plurisubharmonic and satisfies $\nabla \psi(0) = 1$.

If $a \in \partial\Omega$, then it follows from the proof of Lemma 16 that $\psi \equiv 0$, contradicting $\nabla\psi(0) = 1$. Therefore, $a \in \Omega$, so we can actually assume that B is a ball of arbitrary size, hence ψ can be extended as a plurisubharmonic function on the whole of $\mathbb{C}^n$. Since φ_j is uniformly bounded, so are ψ_j and ψ. Thus, ψ has to be constant, contradicting $\nabla\psi(0) = 1$. $\square$

4.3. *Evans–Krylov theory*

It follows from Schauder's theory for linear elliptic equations with variable coefficients that it suffices to obtain a priori estimates

$$\|\varphi_j\|_{2,\alpha} \leqslant C, \tag{4.4}$$

for some positive exponent $\alpha > 0$, in order to obtain a priori estimates

$$\|\varphi_j\|_{k+2,\alpha} \leqslant C_k, \tag{4.5}$$

at all orders $k \in \mathbb{N}$. Here

$$\|h\|_{k,\alpha} := \sum_{j=0}^{k} \sup_\Omega |D^j h| + \sup_{z,w \in \Omega, z \neq w} \frac{|D^k h(z) - D^k h(w)|}{|z - w|^\alpha}$$

denotes the norm associated to the Hölder space of functions h which are k-times differentiable on $\bar{\Omega}$ with Hölder-continuous of exponent $\alpha > 0$ kth-derivative.

The a priori estimates (4.4) follow from Theorem 15, as is shown in [**14**, Theorem 1].

4.4. *Conclusion*

It follows from the previous sections that the sequence (φ_j) is relatively compact in $\mathcal{C}^\infty(\bar{\Omega})$. We let $\mathcal{K}$ denote the set of its cluster values. We infer from Proposition 12 that the functional $\mathcal{F}$ is constant on $\mathcal{K}$: for all $\psi \in \mathcal{K}$,

$$\mathcal{F}(\psi) = \lim_{j \to +\infty} \nearrow \mathcal{F}(T^j \varphi_0).$$

Now $\mathcal{K}$ is clearly T-invariant, hence $\mathcal{F}(T\psi) = \mathcal{F}(\psi)$ for all $\psi \in \mathcal{K}$. Thus, Proposition 12 again ensures that $T\psi = \psi$, that is, ψ is a solution of (MA).

As explained earlier, this is equivalent to saying that there exists a Kähler–Einstein metric $\omega = dd^c\varphi$ with $\mathrm{Ric}(\omega) = \pi\omega$ and prescribed values on the boundary of Ω, hence we have solved our geometrical problem.

5. *Uniqueness*

Recall that (MA) is the Euler–Lagrange equation of the functional

$$\mathcal{F}(\varphi) := \mathcal{E}(\varphi) + \log\left[\int_\Omega e^{-\varphi}\, d\mu\right].$$

If a smooth strictly plurisubharmonic function φ with zero boundary values maximizes $\mathcal{F}$, then it is a critical point of $\mathcal{F}$, hence φ is a solution of (MA). Indeed for any smooth function v,

$$\frac{d}{dt}\mathcal{F}(\varphi+tv)_{|t=0} = \int_\Omega v(dd^c\varphi)^n - \frac{\int_\Omega v\, e^{-\varphi}\, d\mu}{\int_\Omega e^{-\varphi}\, d\mu} = 0,$$

thus $(dd^c\varphi)^n = e^{-\varphi}\mu/(\int_\Omega e^{-\varphi}\, d\mu)$.

Our purpose here is to show that the converse holds true when Ω satisfies an additional symmetry property.

5.1. *Continuous geodesics*

In the setting of compact Kähler manifolds, Mabuchi [26], Semmes [33] and Donaldson [20] have shown that the set of all Kähler metrics in a fixed cohomology class has the structure of an infinite Riemannian manifold with nonnegative curvature. The notion of a geodesic joining two Kähler metrics plays an important role there and we refer the reader to [18] for more information on this.

Our purpose here is to consider similar objects for pseudoconvex domains in order to study the uniqueness of solutions to (MA). Let A denote the annulus $A = \{\zeta \in \mathbb{C}/1 < |\zeta| < e\}$ and fix two functions ϕ_0, ϕ_1 which are plurisubharmonic in Ω, continuous up to the boundary, with zero boundary values. We let $\mathcal{G}$ denote the set of all plurisubharmonic functions Ψ on $\Omega \times A$ which are continuous on $\bar\Omega \times \bar A$ and such that

$$\Psi_{|\partial\Omega\times A} \equiv 0 \quad \text{and} \quad \Psi_{|\Omega\times\partial A} \leqslant \phi,$$

where $\phi(z,\zeta) = \phi_0(z)$ for $|\zeta| = 1$ and $\phi(z,\zeta) = \phi_1(z)$ for $|\zeta| = e$. We set

$$\Phi(z,\zeta) := \sup\{\Psi(z,\zeta)/\Psi \in \mathcal{G}\}.$$

Proposition 20. *The function Φ is plurisubharmonic in $\Omega \times A$, continuous on $\bar\Omega \times \bar A$ and satisfies the following conditions:*

(i) $\Phi(z, e^{i\theta}\zeta) = \Phi(z,\zeta)$ *for all* $(z,\zeta,\theta) \in \Omega \times A \times \mathbb{R}$;
(ii) $\Phi(z,1) = \phi_0(z)$ *and* $\Phi(z,e) = \phi_1(z)$ *for all* $z \in \Omega$;
(iii) $(dd^c_{z,\zeta}\Phi)^{n+1} \equiv 0$ *in* $\Omega \times A$.

Proof. The invariance by rotations (i) follows from the corresponding invariance property of the family $\mathcal{G}$. The continuity and boundary properties (ii) follow standard arguments which go back to Bremermann [13] and Walsh [37].

The maximality property (iii) is a consequence of Bedford–Taylor's solution to the homogeneous complex Monge–Ampère equation on balls, through a balayage procedure: by Choquet's lemma, the sup can be achieved along an increasing sequence which is maximal on an arbitrary ball $B \subset \Omega \times A$; one then concludes by using the continuity property of the complex Monge–Ampère operator along increasing sequences [4]. □

Definition 21. Set $\Phi_t(z) = \Phi(z, e^t)$. The continuous family $(\Phi_t)_{0\leqslant t\leqslant 1}$ is called the geodesic joining ϕ_0 to ϕ_1.

Recall that

$$\mathcal{E}(\varphi) := \frac{1}{n+1}\int_\Omega \varphi(dd^c\varphi)^n$$

denotes the *energy* of a plurisubharmonic function φ.

LEMMA 22. *Let $(\Phi_t)_{0\leqslant t\leqslant 1}$ be a continuous geodesic. Then $t\mapsto \mathcal{E}(\Phi_t)$ is affine.*

Proof. We let the reader verify that if $(z,\zeta)\mapsto\Phi(z,\zeta)$ is a continuous plurisubharmonic function in $\Omega\times A$, then

$$dd^c_\zeta\mathcal{E}\circ\Phi=\pi_*((dd^c_{z,\zeta}\Phi)^{n+1}),$$

where $\pi:\Omega\times A\to A$ denotes the projection onto the second factor.

It thus follows from Proposition 20 that $\zeta\in A\mapsto\mathcal{E}\circ\Phi(\zeta)\in\mathbb{R}$ is harmonic in ζ. The same proposition ensures that it is also invariant by rotation, hence it is affine in $t=\log|\zeta|$. □

5.2. *Variational characterization*

We now make an additional hypothesis of S^1-invariance in order to use an important result by Berndtsson [9]. Namely, we assume here below that Ω is *circled*, that is,

$$\Omega \textit{ contains the origin and is invariant under the rotations } z\longmapsto e^{i\theta}z$$

and

$$\phi_0,\phi_1 \text{ are } S^1\text{-invariant, that is, } \phi_i(e^{i\theta}z)=\phi_i(z).$$

Under this assumption, it follows from [9, Theorem 1.2] that

$$t\longmapsto -\log\left(\int_\Omega e^{-\Phi_t}\,d\mu\right)$$

is a convex function of t if (Φ_t) is a continuous geodesic.

PROPOSITION 23. *Assume that Ω is circled and let φ be an S^1-invariant solution of* (MA). *Then*

$$\mathcal{F}(\varphi)\geqslant\mathcal{F}(\psi)$$

for all S^1-invariant plurisubharmonic functions ψ in Ω which are continuous up to the boundary, with zero boundary values.

Proof. Let $(\Phi_t)_{0\leqslant t\leqslant 1}$ denote the *geodesic* joining $\phi_0:=\varphi$ to $\phi_1:=\psi$. It follows from the above-mentioned work of Berndtsson [9] that

$$t\longmapsto -\log\left(\int e^{-\Phi_t}\,d\mu\right)$$

is convex, while we have just observed that

$$t\longmapsto\mathcal{E}(\Phi_t)$$

is affine, thus

$$t\longmapsto\mathcal{F}(\Phi_t)\text{ is concave.}$$

It therefore suffices to show that the derivative of $\mathcal{F}(\Phi_t)$ at $t=0$ is nonpositive to conclude $\mathcal{F}(\varphi)=\mathcal{F}(\Phi_0)\geqslant\mathcal{F}(\Phi_t)$ for all t, in particular at $t=1$ where it yields $\mathcal{F}(\varphi)\geqslant\mathcal{F}(\psi)$. When $t\mapsto\Phi_t$ is smooth, a direct computation yields, for $t=0$,

$$\frac{d}{dt}(\mathcal{F}(\Phi_t))=\int_\Omega\dot{\Phi}_t\left[(dd^c\Phi_t)^n-\frac{e^{-\Phi_t}\mu}{\left(\int e^{-\Phi_t}d\mu\right)}\right]=0,$$

since $\Phi_0=\varphi$ is a solution of (MA). For the general case, one can argue as in the proof of [8, Theorem 6.6]. □

Corollary 24. *A smooth S^1-invariant plurisubharmonic function $\varphi : \bar{\Omega} \to \mathbb{R}$ with zero boundary values is a solution of* (MA), *that is, satisfies*

$$(dd^c\varphi)^n = \frac{e^{-\varphi}\mu}{\int_\Omega e^{-\varphi} d\mu} \quad \text{in } \Omega,$$

if and only if it maximizes the functional $\mathcal{F}$.

5.3. *Uniqueness of solutions*

The purpose of this section is to establish a uniqueness result for (MA). Recall that if φ is a solution of (MA), then we say that Ω is strictly φ-convex if Ω is strictly convex for the metric $dd^c\varphi$.

Theorem 25. *Assume that Ω is circled and strictly φ-convex, where φ is an S^1-invariant solution of* (MA)*. Then φ is the only S^1-invariant solution to* (MA).

Proof. Assume that we are given φ, ψ, which are two S^1-invariant solutions of (MA). Let $(\Phi_t)_{0\leqslant t\leqslant 1}$ denote the continuous geodesic joining $\phi_0 = \varphi$ to $\phi_1 = \psi$. Since the functional $\mathcal{F}$ is concave along this geodesic and attains its maximum both at ϕ_0 and ϕ_1, it is actually constant, hence each Φ_t is an S^1-invariant solution to (MA) by Corollary 24, so that

$$(dd^c\Phi_t)^n = \frac{e^{-\Phi_t}\mu}{\int_\Omega e^{-\Phi_t}\, d\mu} \quad \text{in } \Omega.$$

Assume that the mapping $(z, t) \in \Omega \times A \mapsto \Phi_t(z) \in \mathbb{R}$ is smooth. Taking derivatives with respect to t, we infer

$$n\, dd^c\dot{\Phi}_t \wedge (dd^c\Phi_t)^{n-1} = \left[-\dot{\Phi}_t + \int_\Omega \dot{\Phi}_t (dd^c\Phi_t)^n\right](dd^c\Phi_t)^n,$$

so that 1 is an eigenvalue with eigenvector $\dot{\Phi}_t - \int_\Omega \dot{\Phi}_t(dd^c\Phi_t)^n$ for the Laplacian Δ_t associated to the Kähler form $dd^c\Phi_t$. Without the regularity assumption, we can take derivatives in the sense of distributions to ensure that at $t = 0$,

$$n\, dd^c\dot{\Phi}_0 \wedge (dd^c\Phi_0)^{n-1} = \left[-\dot{\Phi}_0 + \int_\Omega \dot{\Phi}_0 (dd^c\Phi_0)^n\right](dd^c\Phi_0)^n,$$

as in the proof of [**8**, Theorem 6.8]. Note that $\Phi_0 = \varphi$ is smooth. In particular, $\dot{\Phi}_0$ is a solution of

$$-\triangle\psi = \psi - c(\psi) \;\text{ in } \Omega \quad \text{with } \psi_{|\partial\Omega} = 0, \tag{5.1}$$

where

$$c(\psi) = \int_\Omega \psi (dd^c\varphi)^n.$$

We are going to show that any solution of equation (5.1) has to vanish identically if Ω is strictly φ-convex. Namely, assume first that $c(\psi) \geqslant 0$. Write $\psi = \psi^+ - \psi^-$, where $\psi^+ = \max\{\psi, 0\}$ and $\psi^- = \max\{-\psi, 0\}$. Multiplying equation (5.1) by ψ^+ and integrating by parts, we get

$$\begin{aligned}\int_\Omega |d\psi^+|^2 (dd^c\varphi)^n &= \int_\Omega (\psi^+)^2 (dd^c\varphi)^n - c(\psi)\int_\Omega \psi^+(dd^c\varphi)^n \\ &\leqslant \int_\Omega (\psi^+)^2 (dd^c\varphi)^n.\end{aligned}$$

By the variational characterization of the first eigenvalue of the Laplacian, if ψ^+ does not vanish identically, then the last inequality means that the first eigenvalue of Δ with Dirichlet boundary condition is at most 1. However, by Guedj, Kolev and Yeganefar [21, Corollary 1.2], we know that this eigenvalue is strictly bigger than 1 because of the strict convexity condition. (Due to our normalization convention for d^c, there is a π factor difference between the definition of Δ in our present work and the one in [21].) This shows $\psi^+ = 0$ and therefore $\psi = 0$ because $c(\psi) \geqslant 0$. If $c(\psi) \leqslant 0$, then the reasoning is similar and $\psi = 0$ as well.

As a conclusion, we see that $\dot{\Phi}_0 = 0$ on Ω. Therefore, since the energy

$$t \longmapsto \mathcal{E}(\Phi_t)$$

is affine along the geodesic, and its derivative at $t = 0$ vanishes, it is constant on the interval $[0, 1]$. Now, along the geodesic, the derivative of $\mathcal{F}$ vanishes and since

$$\mathcal{F}(\Phi_t) = \mathcal{E}(\Phi_t) + \log\left(\int e^{-\Phi_t}\, d\mu\right),$$

we obtain finally that

$$\int \dot{\Phi}_t\, e^{-\Phi_t}\, d\mu = 0.$$

But $\dot{\Phi}_t \geqslant 0$ since $t \mapsto \Phi_t$ is convex (by subharmonicity and S^1-invariance) and therefore $\dot{\Phi}_t = 0$ almost everywhere. This leads to $\Phi_0 = \Phi_1$. □

6. *Concluding remarks*

6.1. *The continuity method*

A classical strategy to solve (MA) is to use the *continuity method*, looking at a continuous family of similar Dirichlet problems,

$$(\mathrm{MA})_t \qquad (dd^c\varphi_t)^n = \frac{e^{-t\varphi_t}\mu}{\int_\Omega e^{-t\varphi_t}\, d\mu} \text{ in } \Omega \quad \text{with } \varphi_{t|\partial\Omega} = 0,$$

where the parameter t runs from 0 to 1. One sets

$$I := \{t \in [0, 1]/(\mathrm{MA})_t \text{ admits a (smooth plurisubharmonic) solution}\}$$

and then tries to show that I is nonempty, open and closed, so that $I = [0, 1]$. Observe that $1 \in I$ is equivalent to solving the Dirichlet problem $(\mathrm{MA}) = (\mathrm{MA})_1$.

It follows from the work of Cafarelli–Kohn–Nirenberg–Spruck [14] that $0 \in I$, hence the latter is nonempty (see the discussion in Paragraph 2.3.2).

The a priori estimates derived in Section 4 can be adapted to show that I is closed. This is in general the most difficult part of the method. However, it turns out here that proving the openness is a delicate issue. Indeed, to do so, we need to show that the linearized $(\mathrm{MA})_t$ equation has a trivial kernel. More precisely, we have to prove that if φ_t is a solution of $(\mathrm{MA})_t$, then every solution of

$$-\triangle\psi - t\psi + tc(\psi) = 0 \text{ in } \Omega \quad \text{with } \psi_{|\partial\Omega} = 0, \tag{6.1}$$

where

$$c(\psi) := \int \psi (dd^c\varphi_t)^n,$$

must vanish. Here and in the following, covariant derivative, Ricci tensor and Laplacian refer to the metric defined by φ_t. Let us introduce the differential operator

$$D : C^\infty(\Lambda^{0,1}\Omega) \longrightarrow C^\infty(\Lambda^{0,1}\Omega \otimes \Lambda^{0,1}\Omega),$$

defined by

$$D\alpha := \nabla^{0,1}\alpha.$$

We have then a *Bochner formula* (up to an inessential multiplicative π factor which we omit for brevity)

$$-\triangle\alpha = D^*D\alpha + \mathrm{Ric}(\alpha), \quad \alpha \in C^\infty(\Lambda^{0,1}\Omega). \tag{6.2}$$

Here and in the following computation, $\triangle$ is the $\bar\partial$-Laplacian. Applying (6.2) to $\bar\partial\psi$, where ψ is a solution of (6.1), we get

$$-\triangle\bar\partial\psi = t\bar\partial\psi = D^*D\bar\partial\psi + t\bar\partial\psi,$$

because $\triangle$ and $\bar\partial$ commute and $\mathrm{Ric}(\alpha) = t\alpha$. Therefore,

$$D^*D\bar\partial\psi = 0. \tag{6.3}$$

Then, taking the L^2 inner product of $D^*D\bar\partial\psi$ and $\bar\partial\psi$ and integrating by parts, without neglecting boundary terms (see [**21**] for details) and using the fact that on the boundary we have

$$\triangle\psi = tc(\psi),$$

we obtain

$$\|D\bar\partial\psi\|^2_{L^2} = -\frac{1}{2}\int_{\partial\Omega}(n\cdot\psi)^2[\mathrm{tr}\, L_\rho + \mathrm{Hess}\,\rho(Jn, Jn)]\sigma, \tag{6.4}$$

where ρ is a boundary-defining function for $\partial\Omega$, n is the outward unit normal vector field on $\partial\Omega$ and L_ρ is the *Levi form* corresponding to ρ (see Subsection 2.1).

If Ω is a strictly pseudoconvex domain, then $\mathrm{tr}\, L_\rho$ is positive at each point of $\partial\Omega$, however, we do not have a priori any control on $\mathrm{Hess}\,\rho(Jn, Jn)$. So, contrary to what happens on a closed manifold where we do not have to deal with this disturbing boundary term, we cannot conclude here.

REMARK 26. In the same spirit, we have shown in [**21**] that a ball of sufficiently large radius in complex projective space provides an example of a strongly pseudoconvex domain which is not convex, and for which the *Lichnerowicz estimate* fails.

6.2. *Optimal constants*

It is natural to wonder whether it is possible to solve

$$(\mathrm{MA})_t \qquad (dd^c\varphi_t)^n = \frac{e^{-t\varphi_t}\mu}{\int_\Omega e^{-t\varphi_t}d\mu} \ \text{ in } \Omega \quad \text{with } \varphi_{t|\partial\Omega} = 0,$$

for bigger values of $t > 1$. As noted in Remark 10, our Moser–Trudinger inequality allows us to get control for slightly larger values of t, with a maximal value depending on n, namely

$$t < (2n)^{1+1/n}(1+1/n)^{(1+1/n)}.$$

It should be noted that one cannot expect to solve $(\mathrm{MA})_t$ for big values of t, as follows from Bishop's volume comparison theorem. Indeed, let $\mathbb{B}$ denote the unit ball in $\mathbb{C}^n$. If we can find a solution φ of $(\mathrm{MA})_t$ on $\mathbb{B}$, then this means that we can find a Kähler–Einstein metric $\omega = dd^c\varphi$ on $\mathbb{B}$ satisfying $\mathrm{Ric}\,(\omega) = t\pi\omega$. Moreover, the volume V of this metric is

$$V = \int_{\mathbb{B}} \frac{(dd^c\varphi)^n}{n!} = \frac{1}{n!}.$$

But by the Bishop volume comparison theorem, the volume has to be less than or equal to the volume of the $2n$-real-dimensional sphere endowed with a metric of constant curvature k, with $k = (t\pi)/(2n-1)$. This implies

$$\frac{1}{n!} \leqslant \frac{(4\pi)^n(n-1)!}{k^n(2n-1)!},$$

so that

$$t \leqslant 4(2n-1)\left[\frac{(n-1)!n!}{(2n-1)!}\right]^{1/n}.$$

The interested reader will find in [6] further motivation and references for $(\mathrm{MA})_t$ for large (critical) values of t.

Acknowledgements. It is a pleasure to thank Robert Berman and Bo Berndtsson for stimulating discussions related to their joint work [6]. We would also like to thank Sébastien Boucksom, Thierry Gallouët and Marc Herzlich for helpful discussions during the preparation of this paper. Finally, we are grateful to the referee for several useful remarks and comments.

References

1. P. Ahag, U. Cegrell, S. Kołodziej, H. H. Phạm and A. Zeriahi, 'Partial pluricomplex energy and integrability exponents of plurisubharmonic functions', *Adv. Math.* 222 (2009) 2036–2058.
2. M. T. Anderson, 'On boundary value problems for Einstein metrics', *Geom. Topol.* 12 (2008) 2009–2045.
3. M. T. Anderson and M. Herzlich, 'Unique continuation results for Ricci curvature and applications', *J. Geom. Phys.* 58 (2008) 179–207.
4. E. Bedford and B. A. Taylor, 'A new capacity for plurisubharmonic functions', *Acta Math.* 149 (1982) 1–40.
5. S. Benelkourchi, V. Guedj and A. Zeriahi, 'Plurisubharmonic functions with weak singularities', *Complex analysis and digital geometry*, Acta Universitatis Upsaliensis. Skrifter rörande Uppsala Universitet. C. Organisation och Historia 86 (Uppsala Universitet, Uppsala, 2009) 57–74.
6. R. J. Berman and B. Berndtsson, 'Moser–Trudinger type inequalities for complex Monge–Ampère operators and Aubin's "hypothèse fondamentale"', ArXiv e-prints, 2011, http://adsabs.harvard.edu/abs/2011arXiv1109.1263B.
7. R. Berman, S. Boucksom, P. Eyssidieux, V. Guedj and A. Zeriahi, *Ricci iteration and Kähler–Ricci flow on log-Fano varieties* ArXiv e-prints, 2011, http://adsabs.harvard.edu/abs/2011arXiv1111.7158B.
8. R. J. Berman, S. Boucksom, V. Guedj and A. Zeriahi, 'A variational approach to complex Monge–Ampère equations', *Publ. Math. I.H.E.S.* 117 (2013), 179–245.
9. B. Berndtsson, 'Subharmonicity properties of the Bergman kernel and some other functions associated to pseudoconvex domains', *Ann. Inst. Fourier (Grenoble)* 56 (2006) 1633–1662.
10. A. L. Besse, *Einstein manifolds*, Classics in Mathematics (Springer, Berlin, 2008), Reprint of the 1987 edition.
11. O. Biquard, 'Métriques d'Einstein asymptotiquement symétriques', *Astérisque* 265 (2000) vi+109.
12. S. Boucksom, 'Monge–Ampère equations on complex manifolds with boundary', *Complex Monge–Ampère equations and geodesics in the space of Kähler metrics* (ed. V. Guedj; Springer, Berlin, 2011) 253–278.
13. H. J. Bremermann, 'On a generalized Dirichlet problem for plurisubharmonic functions and pseudo-convex domains. Characterization of Šilov boundaries', *Trans. Amer. Math. Soc.* 91 (1959) 246–276.
14. L. Caffarelli, J. J. Kohn, L. Nirenberg and J. Spruck, 'The Dirichlet problem for nonlinear second-order elliptic equations. II. Complex Monge–Ampère, and uniformly elliptic, equations', *Comm. Pure Appl. Math.* 38 (1985) 209–252.
15. U. Cegrell, 'Pluricomplex energy', *Acta Math.* 180 (1998) 187–217.
16. U. Cegrell, 'Measures of finite pluricomplex energy', ArXiv e-prints, 2011, http://adsabs.harvard.edu/abs/2011arXiv1107.1899C.
17. U. Cegrell and L. Persson, 'The Dirichlet problem for the complex Monge–Ampère operator: stability in L^2', *Michigan Math. J.* 39 (1992) 145–151.
18. X. Chen, 'The space of Kähler metrics', *J. Differential Geom.* 56 (2000) 189–234.
19. W. S. Cohn and G. Lu, 'Sharp constants for Moser–Trudinger inequalities on spheres in complex space $\mathbb{C}^n$', *Comm. Pure Appl. Math.* 57 (2004) 1458–1493.
20. S. K. Donaldson, 'Symmetric spaces, Kähler geometry and Hamiltonian dynamics', *Northern California Symplectic Geometry Seminar*, American Mathematical Society Translations, Series 2 196 (American Mathematical Society, Providence, RI, 1999) 13–33.
21. V. Guedj, B. Kolev and N. Yeganefar, 'A Lichnerowicz estimate for the first eigenvalue of convex domains in Kähler manifolds', *Anal. Partial Differential Equations*, to appear. http://msp.org/scripts/coming.php?jpath=apde.
22. V. Guedj and A. Zeriahi, 'Intrinsic capacities on compact Kähler manifolds', *J. Geom. Anal.* 15 (2005) 607–639.
23. J. Keller, 'Ricci iterations on Kähler classes', *J. Inst. Math. Jussieu* 8 (2009) 743–768.
24. S. Kolodziej, 'The complex Monge–Ampère equation', *Acta Math.* 180 (1998) 69–117.
25. J. M. Lee, 'Pseudo-Einstein structures on CR manifolds', *Amer. J. Math.* 110 (1988) 157–178.

26. T. Mabuchi, 'Some symplectic geometry on compact Kähler manifolds. I', *Osaka J. Math.* 24 (1987) 227–252.
27. A. Moroianu, *Lectures on Kähler geometry*, London Mathematical Society Student Texts 69 (Cambridge University Press, Cambridge, 2007).
28. J. Moser, 'A sharp form of an inequality by N. Trudinger', *Indiana Univ. Math. J.* 20 (1970/71) 1077–1092.
29. E. Onofri, 'On the positivity of the effective action in a theory of random surfaces', *Comm. Math. Phys.* 86 (1982) 321–326.
30. D. H. Phong, J. Song, J. Sturm and B. Weinkove, 'The Moser–Trudinger inequality on Kähler–Einstein manifolds', *Amer. J. Math.* 130 (2008) 1067–1085.
31. Y. A. Rubinstein, 'Some discretizations of geometric evolution equations and the Ricci iteration on the space of Kähler metrics', *Adv. Math.* 218 (2008) 1526–1565.
32. J.-M. Schlenker, 'Einstein manifolds with convex boundaries', *Comment. Math. Helv.* 76 (2001) 1–28.
33. S. Semmes, 'Complex Monge–Ampère and symplectic manifolds', *Amer. J. Math.* 114 (1992) 495–550.
34. H. Skoda, 'Sous-ensembles analytiques d'ordre fini ou infini dans C^n', *Bull. Soc. Math. France* 100 (1972) 353–408.
35. G. Tian, 'Kähler–Einstein metrics with positive scalar curvature', *Invent. Math.* 130 (1997) 1–37.
36. G. Tian, 'Notes taken by Meike Akveld', *Canonical metrics in Kähler geometry*, Lectures in Mathematics ETH Zürich (Birkhäuser, Basel, 2000).
37. J. B. Walsh, 'Continuity of envelopes of plurisubharmonic functions', *J. Math. Mech.* 18 (1968/69) 143–148.
38. A. Zeriahi, 'Volume and capacity of sublevel sets of a Lelong class of plurisubharmonic functions', *Indiana Univ. Math. J.* 50 (2001) 671–703.

Vincent Guedj
Institut Universitaire de France et Institut de Mathématiques de Toulouse
Université Paul Sabatier
31062 Toulouse cedex 09
France

vincent.guedj@math.univ-toulouse.fr

Boris Kolev and Nader Yeganefar
LATP, CNRS & Université d'Aix-Marseille
39 Rue F. Joliot-Curie
13453 Marseille cedex 13
France

kolev@cmi.univ-mrs.fr
nader.yeganefar@cmi.univ-mrs.fr