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Intrinsic Capacities on Compact Kähler Manifolds

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ABSTRACT. We study fine properties of quasiplurisubharmonic functions on compact Kähler manifolds. We define and study several intrinsic capacities which characterize pluripolar sets and show that locally pluripolar sets are globally "quasi-pluripolar."

1. Introduction

Since the fundamental work of Bedford and Taylor [4, 5], several authors have developed a "Pluripotential theory" in domains of $\mathbb{C}^n$ (or of Stein manifolds). This theory is devoted to the fine study of plurisubharmonic (psh) functions and can be seen as a nonlinear generalization of the classical potential theory (in one complex variable), where subharmonic functions and the Laplace operator Δ are replaced by psh functions and the complex Monge-Ampère operator $(dd^c)^n$. Here d, d^c denote the real differential operators $d := \partial + \overline{\partial}$, $d^c := \frac{i}{2\pi}[\overline{\partial} - \partial]$ so that $dd^c = \frac{i}{\pi}\partial\overline{\partial}$; the normalization being chosen so that the positive measure $\left(dd^c \frac{1}{2}\log[1 + ||z||^2]\right)^n$ has total mass 1 in $\mathbb{C}^n$. We refer the reader to [3, 7, 26] for a survey of this local theory.

Our aim here is to develop a global Pluripotential theory in the context of compact Kähler manifolds. It follows from the maximum principle that there are no psh functions (except constants) on a compact complex manifold X. However, there are usually plenty of positive closed currents of bidegree $(1, 1)$ (we refer the reader to [15, Ch. 3], for basic facts on positive currents). Given ω a real closed smooth form of bidegree $(1, 1)$ on X, we may consider every positive closed current ω' of bidegree $(1, 1)$ on X which is cohomologous to ω. When X is Kähler, it follows from the "dd^c-lemma" that ω' can be written as $\omega' = \omega + dd^c\varphi$, where φ is a function which is integrable with respect to any smooth volume form on X. Such a function φ will be called ω-plurisubharmonic (ω-psh for short). It is globally defined on X and locally given as the sum of a psh and a smooth function. We let $PSH(X, \omega)$ denote the set of ω-psh functions. Such functions were introduced by Demailly, who call them *quasiplurisubharmonic* (qpsh). These are the main objects of study in this article.

There are several motivations to study qpsh functions on compact Kähler manifolds. First of all they arise naturally in complex analytic geometry as positive singular metrics of holomorphic

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line bundles (see Section 5) whose study is central to several questions of complex algebraic geometry. Solving Monge-Ampère equations associated to ω-psh functions has been used to produce metrics with prescribed singularities (see [13]). It is also related to the existence of canonical metrics in Kähler geometry (see [37]). Important contributions have been made by Kolodziej in this direction [28] using techniques from local Pluripotential theory. Quasiplurisubharmonic functions have also been used in [21] to define a notion of ω-polynomial convexity and study the fine approximation of positive currents by rational divisors. Last but not least, such functions are of constant use in complex dynamics in several variables (see [17, 18, 22, 23, 33]).

It seems to us appropriate to develop a theory of qpsh functions of its own rather than view these functions as particular cases of the local theory. Although, the two theories look quite similar, there are important differences which make the "compact theory" both simpler and more difficult than the local one. Here are some examples:

- There is no pluriharmonic functions (except constants) on a compact manifold, hence each ω-psh function φ is canonically associated (up to normalization) to its curvature current $\omega_\varphi := \omega + dd^c\varphi \geq 0$. This yields compactness properties of subsets of $PSH(X, \omega)$ (see Section 2) which are quite useful (e.g., in complex dynamics, see Section 7.2).
- Integration by parts (of constant use in such theories) is quite simple in the compact setting since there is no boundary. As an illustration, we obtain transparent proofs of Chern-Levine-Nirenberg type inequalities (see Example 2.8 and Section 3). A successful application of this simple observation has been made in complex dynamics in [23].
- On the other hand, one loses homogeneity of Monge-Ampère operators in the compact setting. They do have uniformly bounded mass (by Stokes theorem), but there is no performing "comparison principle," which is a key tool in the local theory. This is a source of difficulty when, for example, one wishes to solve Monge-Ampère equations on compact manifolds (see [28, 24]).

We shall develop our study in a series of articles. In the present one we define and study several intrinsic capacities which we shall use in our forthcoming articles.

Let us now describe more precisely the contents of the article.

In Section 2 we define ω-psh functions and gather useful facts about them (especially compacity results such as Proposition 2.7). For locally bounded ω-psh functions φ we define the complex Monge-Ampère operator ω_φ^n in Section 3. We establish Chern-Levine-Nirenberg inequalities (Proposition 3.1) and study the "Monge-Ampère capacity" Cap_ω (Definition 3.4). As in the local theory, ω-psh functions are quasicontinuous with respect to Cap_ω (Corollary 3.8). The capacity Cap_ω is comparable to the local Monge-Ampère capacity of Bedford and Taylor (Proposition 3.10) and moreover, enjoys invariance properties (Proposition 3.5). In Section 4 we define a relative extremal function $h^*_{E,\omega}$ and establish a useful formula (Theorem 4.2)

$$\mathrm{Cap}^*_\omega(E) = \int_X \left(- h^*_{E,\omega}\right) \left(\omega + dd^c h^*_{E,\omega}\right)^n .$$

This is the global version of the fundamental local formula of Bedford-Taylor [5], $\mathrm{Cap}(E, \omega) = \int_\Omega (dd^c u^*_E)^n$.

In Section 5 we study yet another capacity (the Alexander capacity T_ω, Definition 5.7) which is defined by means of a (global) extremal function (Definition 5.1). When ω is a Hodge form, it can be defined as well in terms of Tchebychev constants: These are the contents of Section 6 (Theorem 6.2) where we further give a geometrical interpretation of T_ω when $X = \mathbb{CP}^n$ is the complex projective space and ω is the Fubini-Study Kähler form (Theorem 6.4), following

Alexander's work [1]. In Section 7 we show that locally pluripolar sets can be defined by ω-psh functions when ω is Kähler: This is our version of a result of Josefson (Theorem 7.2). We then give an application in complex dynamics which illustrates how invariance properties of these capacities can be used. Finally, in an Appendix we show how to globally regularize ω-psh functions, following ideas of Demailly.

This article lies at the border of complex analysis and complex geometry. We have tried to make it accessible to mathematicians from both sides. This has of course some consequences for the style of presentation. We have included proofs of some results which may be seen as consequences of results from the local pluripotential theory. We have spent some efforts defining, regularizing and approximating positive singular metrics of holomorphic line bundles, although some of these facts may be considered as classical by complex geometers. Altogether we hope the article is essentially self contained. Our efforts will not be vain if for instance we have convinced specialists of the (local) pluripotential theory that the right point of view in studying the Lelong class $\mathcal{L}(\mathbb{C}^n)$ of psh functions with logarithmic growth in $\mathbb{C}^n$ is to consider qpsh functions on the complex projective space $\mathbb{CP}^n$. We also think this article should be useful to people working in complex dynamics in several variables where pluripotential theory has become an important tool.

Warning. In the whole article *positivity* (like e.g., in *positive* metric and *positive* current) has to be understood in the weak (french, i.e., nonnegativity) sense of currents, except when we talk of a *positive* line bundle L, in which case it means that L admits a smooth metric whose curvature is a Kähler form.

2. Quasiplurisubharmonic functions

In the sequel, unless otherwise specified, L^p-norms will always be computed with respect to a fixed volume form on X, which is a *compact connected Kähler* manifold. Let ω be a closed real current of bidegree $(1, 1)$ on X. We say that a function φ is ω-upper semi-continuous (ω-u.s.c.) if $\varphi + \psi$ is u.s.c. for any local potential ψ of $\omega = dd^c\psi$.

Definition 2.1. Set

$$PSH(X,\omega) := \left\{\varphi \in L^1(X, \mathbb{R} \cup \{-\infty\}) \,/\, dd^c\varphi \geq -\omega \text{ and } \quad \varphi \text{ is } \quad \omega\text{-u.s.c.}\right\}.$$

The set $PSH(X, \omega)$ is the set of "ω-plurisubharmonic" functions.

Observe that $PSH(X, \omega)$ is nonempty if and only if there exists a positive closed current of bidegree $(1, 1)$ on X which is cohomologous to ω. One then says that the cohomology class $\{\omega\}$ is *pseudoeffective.* In the sequel we always assume this property holds. The set $PSH(X, \omega)$ (essentially) only depends on the cohomology class $\{\omega\}$ [see Proposition 2.3 (3)], and its size depends on positivity properties of $\{\omega\}$ (see Remark 2.5). We will usually choose a *smooth* representative ω of this cohomology class. Since ω-psh functions are locally given as the difference of a psh function and a local potential of ω, this will guaranty that ω-psh functions are u.s.c. on X, hence globally bounded from above. We endow $PSH(X, \omega)$ with the L^1-topology. Observe that $PSH(X, \omega)$ is a closed subspace of $L^1(X)$.

Example 2.2. The most fundamental example which may serve as a guideline to everything that follows is the case where $X = \mathbb{CP}^n$ is the complex projective space and $\omega = \omega_{FS}$ is the Fubini-Study Kähler form. There is then a 1-to-1 correspondence between $PSH(\mathbb{CP}^n, \omega_{FS})$ and the Lelong class

$$\mathcal{L}(\mathbb{C}^n) := \left\{ \psi \in PSH(\mathbb{C}^n) \,/\, \psi(z) \leq \frac{1}{2}\log\left[1+|z|^2\right] + C_\psi \right\}$$

which is given by the natural mapping

$$\psi \in \mathcal{L}(\mathbb{C}^n) \mapsto \varphi(x) = \begin{cases} \psi(x) - \frac{1}{2}\log\left[1+|x|^2\right] & \text{if} \quad x \in \mathbb{C}^n \\ \overline{\lim}_{y\in\mathbb{C}^n \to x}\left(\psi(y) - \frac{1}{2}\log\left[1+|y|^2\right]\right) & \text{if} \quad x \in H_\infty , \end{cases}$$

where H_∞ denotes the hyperplane at infinity. One can easily show that this mapping is bicontinuous for the L^1_{loc} topology.

The Lelong class $\mathcal{L}(\mathbb{C}^n)$ of plurisubharmonic functions with logarithmic growth in $\mathbb{C}^n$ has been intensively studied in the last 30 years. It seems to us that the properties of $\mathcal{L}(\mathbb{C}^n)$ are more easily seen when $\mathcal{L}(\mathbb{C}^n)$ is viewed as $PSH(\mathbb{CP}^n, \omega_{FS})$. Furthermore, we shall see hereafter that the class $PSH(X, \omega)$ of ω-psh functions enjoys several properties of $\mathcal{L}(\mathbb{C}^n)$ when ω is Kähler. We start by observing (Propositionx 2.3 (1) and 2.3 (2) below) that $PSH(X, \omega)$ and $PSH(X, \omega')$ are comparable if ω, ω' are both Kähler.

Proposition 2.3.

(1) *If* $\omega_1 \leq \omega_2$ *then* $PSH(X, \omega_1) \subset PSH(X, \omega_2)$.

(2) $\forall A \in \mathbb{R}^*_+$, $PSH(X, A\omega) = A \cdot PSH(X, \omega)$.

(3) *If* ω' *is cohomologous to* ω, $\omega' = \omega + dd^c\chi$, *then*

$$PSH(X, \omega') = PSH(X, \omega) + \chi .$$

(4) *If* $\varphi, \psi \in PSH(X, \omega)$ *then*

$$\max(\varphi, \psi) , \ \frac{\varphi + \psi}{2} , \ \log\left[e^\varphi + e^\psi\right] \in PSH(X, \omega) .$$

Proof. Assertions (1), (2), (3) follow straightforwardly from the definition. Observe that Proposition 2.3 (4) says that $PSH(X, \omega)$ is a convex set which is stable under taking maximum and also under the operation $(\varphi, \psi) \mapsto \log[e^\varphi + e^\psi]$. These are all consequences of the corresponding local properties of psh functions. We nevertheless give a proof, in the spirit of this article. That $(\varphi + \psi)/2 \in PSH(X, \omega)$ follows by linearity. The latter assertion is a consequence of the following computation

$$dd^c \log[e^\varphi + e^\psi] = \frac{e^\varphi\, dd^c\varphi + e^\psi\, dd^c\psi}{e^\varphi + e^\psi} + \frac{e^{\varphi+\psi} d(\varphi - \psi) \wedge d^c(\varphi - \psi)}{\left[e^\varphi + e^\psi\right]^2} ,$$

using that $df \wedge d^c f \geq 0$. This computation makes sense if for instance φ, ψ are smooth. The general case follows then by regularizing φ, ψ (see Appendix). Finally, observe that $\max(\varphi, \psi) = \lim j^{-1} \log[e^{j\varphi} + e^{j\psi}] \in PSH(X, \omega)$. □

It follows from Proposition 2.3 (3) that $PSH(X, \omega)$ essentially depends on the cohomology class $\{\omega\}$. In the same vein we have the following.

Proposition 2.4. *Let $\mathcal{T}_{\{\omega\}}(X)$ denote the set of positive closed currents ω' of bidegree $(1,1)$ on X which are cohomologous to ω. Then*

$$PSH(X,\omega) \simeq \mathcal{T}_{\{\omega\}}(X) \oplus \mathbb{R} .$$

Proof. The mapping

$$\Phi : \varphi \in PSH(X,\omega) \mapsto \omega_\varphi := \omega + dd^c\varphi \in \mathcal{T}_{\{\omega\}}(X)$$

is a continuous affine mapping whose kernel consists of constants mappings: Indeed, $\omega_\varphi = \omega_\psi$ implies that $\varphi - \psi$ is pluriharmonic hence constant by the maximum principle. Moreover, Φ is surjective: If $\omega' \geq 0$ is cohomologous to ω then $\omega' = \omega + dd^c\varphi$ for some $\varphi \in L^1(X,\mathbb{R})$ -this is the celebrated dd^c-lemma on Kähler manifolds (see e.g., Lemma 8.6, Chapter VI in [15]). Thus, φ coincides almost everywhere with a function of $PSH(X,\omega)$ and $\omega' = \Phi(\varphi)$. □

Remark 2.5. The size of $PSH(X,\omega)$ is therefore related to that of $\mathcal{T}_{\{\omega\}}(X)$ hence only depends on the positivity of the cohomology class $\{\omega\}$. The more positive $\{\omega\}$, the bigger $PSH(X,\omega)$.

When $\{\omega\}$ is Kähler then $PSH(X,\omega)$ is large: If e.g., χ is any $\mathcal{C}^2$-function on X then $\varepsilon\chi \in PSH(X,\omega)$ for $\varepsilon > 0$ small enough. We will see (Theorem 7.2) that $PSH(X,\omega)$ characterizes locally pluripolar sets when $\{\omega\}$ is Kähler. It follows from Proposition 2.3 that $PSH(X,\omega)$ and $PSH(X,\omega')$ have the same "size" if $\{\omega\}$ and $\{\omega'\}$ are both Kähler classes.

Note, on the other hand, that $PSH(X,\omega) \simeq \mathbb{R}$ when ω is cohomologous to $[E]$, the current of integration along the exceptional divisor of a smooth blow up. Indeed, let $\pi : X \to \tilde{X}$ be a blow up with smooth center Y, $\text{codim}_{\mathbb{C}} Y \geq 2$ (see e.g., [15, Chapter 2] for the definition of blow-ups). Let $E = \pi^{-1}(Y)$ denote the exceptional divisor and $\omega = [E]$ be the current of integration along E. If $\varphi \in PSH(X,[E])$ then $dd^c(\varphi \circ \pi^{-1}) \geq 0$ in $\tilde{X} \setminus Y$. Since $\text{codim}_{\mathbb{C}} Y \geq 2$, $\varphi \circ \pi^{-1}$ extends trivially through Y has a global psh function on $\tilde{X}$. By the maximum principle $\varphi \circ \pi^{-1}$ is constant hence so is φ. Alternatively there is no positive closed current of bidegree $(1,1)$ on X which is cohomologous to $[E]$ except $[E]$ itself.

It follows from previous proposition that any set of "normalized" ω-psh functions is in 1-to-1 correspondence with $\mathcal{T}_{\{\omega\}}(X)$ which is compact for the weak topology of currents. This is the key to several results to follow: Normalized ω-psh functions form a compact family in $L^1(X)$.

Proposition 2.6. *Assume ω is smooth. Let $(\varphi_j) \in PSH(X,\omega)^{\mathbb{N}}$.*

(1) *If (φ_j) is uniformly bounded from above on X, then either φ_j converges uniformly to $-\infty$ on X or the sequence (φ_j) is relatively compact in $L^1(X)$.*

(2) *If $\varphi_j \to \varphi$ in $L^1(X)$, then φ coincides almost everywhere with a unique function $\varphi^* \in PSH(X,\omega)$. Moreover,*

$$\sup_X \varphi^* = \lim_{j\to+\infty} \sup_X \varphi_j .$$

(3) *In particular, if φ_j is decreasing, then either $\varphi_j \to -\infty$ or $\varphi = \lim \varphi_j \in PSH(X,\omega)$. Similarly, if φ_j is increasing and uniformly bounded from above then $\varphi := (\lim \varphi_j)^* \in PSH(X,\omega)$, where $\cdot^*$ denotes the upper-semi-continuous regularization.*

Proof. This is a straightforward consequence of the analogous local result for sequences of psh functions. We refer the reader to [15, Ch. 1], for a proof. Note, that Proposition 2.6 (2) is a special case of a celebrated lemma attributed to Hartogs. □

The next result is quite useful (see [39, 40] for a systematic use).

Proposition 2.7. *Assume ω is smooth. The family*

$$\mathcal{F}_0 := \{\varphi \in PSH(X,\omega) \,/\, \sup_X \varphi = 0\}$$

is a compact subset of $PSH(X,\omega)$.

If μ is a probability measure such that $PSH(X,\omega) \subset L^1(\mu)$ then

$$\mathcal{F}_\mu := \left\{\varphi \in PSH(X,\omega) \,/\, \int_X \varphi \, d\mu = 0\right\}$$

is a relatively compact subset of $PSH(X,\omega)$. In particular, there exists C_μ such that $\forall \varphi \in PSH(X,\omega)$,

$$-C_\mu + \sup_X \varphi \le \int_X \varphi \, d\mu \le \sup_X \varphi \, .$$

Proof. It follows straightforwardly from Proposition 2.6 (1) that $\mathcal{F}_0$ is a relatively compact subset of $PSH(X,\omega)$. Moreover, $\mathcal{F}_0$ is closed by Hartogs lemma [Proposition 2.6 (2)].

Let $(\varphi_j) \in \mathcal{F}_\mu^{\mathbb{N}}$. Then $\psi_j := \varphi_j - \sup_X \varphi_j \in \mathcal{F}_0$ which is relatively compact. Assume first μ is smooth. Then $(\int_X \psi_j \, d\mu)$ is bounded: This is because if $\psi_{j_k} \to \psi$ in $L^1(X)$ then $\psi_{j_k}\mu \to \psi\mu$ in the weak sense of (negative) measures hence $\int_X \psi_{j_k} d\mu \to \int_X \psi d\mu > -\infty$. Now $\int_X \psi_j \, d\mu = \int_X \varphi_j \, d\mu - \int_X \sup_X \varphi_j \, d\mu = -\sup_X \varphi_j$ thus $(\sup_X \varphi_j)$ is bounded and we can apply the previous proposition to conclude that (φ_j) is relatively compact (it cannot converge uniformly to $-\infty$ since $\int_X \varphi_j \, d\mu = 0$).

When μ is not smooth, it only remains to prove that $(\int_X \psi_j \, d\mu)$ is bounded. Assume on the contrary that $\int_X \psi_j \, d\mu \to -\infty$. Extracting a subsequence if necessary we can assume $\int_X \psi_j \, d\mu \le -2^j$. Set $\psi = \sum_{j \ge 1} 2^{-j} \psi_j$. This is a decreasing sequence of ω-psh functions, hence $\psi \in PSH(X,\omega)$ or $\psi \equiv -\infty$. Now it follows from the previous discussion that $\int_X \psi_j \, dV \ge -C$ if dV denotes some smooth probability measure on X. Thus, $\int_X \psi \, dV > -\infty$ hence $\psi \in PSH(X,\omega)$. We obtain a contradiction since by the Monotone convergence theorem, $\int_X \psi \, d\mu = \sum_{j \ge 1} 2^{-j} \int_X \psi_j \, d\mu = -\infty$. □

Example 2.8. It was part of our Definition 2.1 that ω-psh functions are integrable with respect to a fixed volume form. Therefore $PSH(X,\omega) \subset L^1(\mu)$ for every smooth probability measure μ on X. More generally, if μ is a probability measure on X such that

$$\mu = \Theta + dd^c(S) \, , \tag{2.1}$$

where Θ is smooth and S is a **positive** current of bidimension $(1,1)$ on X, then $PSH(X,\omega) \subset L^1(\mu)$ for any smooth ω. Indeed, let φ in $PSH(X,\omega)$, $\varphi \le 0$. If φ is smooth, it follows from Stokes theorem that

$$\begin{aligned} 0 \le \int_X (-\varphi) \, d\mu &= \int_X (-\varphi)\Theta + \int_X (-\varphi) \, dd^c S \\ &\le C_\Theta ||\varphi||_{L^1} + \int_X S \wedge \big(-dd^c\varphi\big) \\ &\le C_\Theta ||\varphi||_{L^1} + \int_X S \wedge \omega < +\infty \, , \end{aligned}$$

where the last inequality follows from $S \geq 0$ and $-dd^c\varphi \leq \omega$. The general case follows by regularizing φ (see Appendix).

Probability measures satisfying (2.1) naturally arise in complex dynamics (see [23]). Observe also that Monge-Ampère measures arising from the local theory of Bedford and Taylor [5] do satisfy (2.1): If u is psh and locally bounded near e.g., the unit ball B of $\mathbb{C}^n$, we can extend it to $\mathbb{C}^n$ as a global psh function with logarithmic growth considering

$$U(z) := \begin{cases} u(z) & \text{if } z \in B \\ \max(u(z), A\log^+|z| - \sup_B |u| - 1) & \text{if } z \in (1+\varepsilon)B \setminus B \\ A\log^+|z| - \sup_B |u| - 1 & \text{if } z \in \mathbb{C}^n \setminus (1+\varepsilon)B \end{cases}$$

where $\log^+|z| := \max(\log|z|, 0)$ and with A large enough. We assume $A = 1$ for simplicity. Now $\varphi := U - \frac{1}{2}\log[1+|z|^2] + C$ extends as a bounded function in $PSH(\mathbb{CP}^n, \omega)$, where ω is the Fubini-Study Kähler form on $\mathbb{CP}^n$, so $\varphi \geq 0$ if $C > 0$ is large enough. To conclude note that, setting $\omega_\varphi := \omega + dd^c\varphi \geq 0$, we get $\omega_\varphi^n = (dd^c u)^n$ in B and

$$\omega_\varphi^n = \omega^n + dd^c S, \quad \text{where} \quad S = \varphi \sum_{j=0}^{n-1} \omega_\varphi^j \wedge \omega^{n-1-j} \geq 0\,.$$

The Monge-Ampère operator ω_φ^n will be defined in the next section.

Example 2.9. If μ is a probability measure on $X = \mathbb{CP}^n$ and ω denotes as before the Fubini-Study Kähler form, then

$$\varphi_\mu(x) := \int_{\mathbb{CP}^n} \log\left(\frac{||x \wedge y||}{||x|| \cdot ||y||}\right) d\mu(y)$$

defines a ω-psh function on $\mathbb{CP}^n$. Such functions have been considered by Molzon, Shiffman, and Sibony [32, 31] in order to define capacities on $\mathbb{CP}^n$. However, they do not characterize pluripolar sets when $n \geq 2$.

3. Monge-Ampère capacity

In this section we introduce the global Monge-Ampère capacity (see Definition 3.4). The definition only makes sense when the cohomology class $\{\omega\}$ has strong positivity properties. To simplify our exposition we assume throughout this section that ω is a Kähler form.

Let T be a positive closed current of bidegree (p, p) on X, $0 \leq p \leq n = \dim_{\mathbb{C}} X$. It can be thought of as a closed differential form of bidegree (p, p) with measure coefficients whose total variation is controlled by

$$||T|| := \int_X T \wedge \omega^{n-p}\,.$$

We refer the reader to [15, Ch. 3] for basic properties of positive currents. Given $\varphi \in PSH(X, \omega)$ we write $\varphi \in L^1(T)$ if φ is integrable with respect to each (measure) coefficient of T. This is equivalent to φ being integrable with respect to the trace measure $T \wedge \omega^{n-p}$. In this case, the current φT is well defined, hence so is

$$\omega_\varphi \wedge T := \omega \wedge T + dd^c(\varphi T)\,.$$

This is again a *positive* closed current on X, of bidegree $(p+1, p+1)$. Indeed, positivity is a local property which is stable under taking limits. One can locally regularize φ and approximate $\omega_\varphi \wedge T$ by the currents $\omega_{\varphi_\varepsilon} \wedge T$ which are positive since $\omega_{\varphi_\varepsilon}$ are smooth positive forms.

When $\varphi \in PSH(X,\omega) \cap L^\infty(X)$ then $\varphi \in L^1(T)$ for any positive closed current T of bidegree (p,p). One can thus inductively define $\omega_\varphi^j \wedge T$, $1 \le j \le n-p$, for $\varphi \in PSH(X,\omega) \cap L^\infty(X)$. For $T = 0$ and $j = n$ one obtains the *complex Monge-Ampère operator,* ω_φ^n. It follows from the local theory that the operator $\varphi \mapsto \omega_\varphi^n$ is continuous under monotone sequences (see [5]). The proof of these continuity properties is simpler in our compact setting. We refer the reader to [24] where this is proved in a more general global context.

Proposition 3.1 (Chern-Levine-Nirenberg inequalities). *Let T be a positive closed current of bidegree (p,p) on X and $\varphi \in PSH(X,\omega) \cap L^\infty(X)$.*

Then $||\omega_\varphi \wedge T|| = ||T||$. Moreover, if $\psi \in PSH(X,\omega) \cap L^1(T)$, then $\psi \in L^1(T \wedge \omega_\varphi)$ and

$$||\psi||_{L^1(T\wedge\omega_\varphi)} \le ||\psi||_{L^1(T)} + \big[2\sup_X \psi + \sup_X \varphi - \inf_X \varphi\big]\, ||T||\,.$$

Proof. By Stokes theorem, $\int_X dd^c\varphi \wedge T \wedge \omega^{n-p-1} = 0$, hence

$$||\omega_\varphi \wedge T|| := \int_X \omega_\varphi \wedge T \wedge \omega^{n-p-1} = \int_X T \wedge \omega^{n-p} := ||T||\,.$$

Consider now $\psi \in L^1(T)$. Since T has measure coefficients, this simply means that ψ is integrable with respect to the total variation of these measures. Assume first $\psi \le 0$, $\varphi \ge 0$ and φ, ψ are smooth. Then

$$||\psi||_{L^1(T\wedge\omega_\varphi)} := \int_X (-\psi) T \wedge \omega_\varphi \wedge \omega^{n-p-1} = ||\psi||_{L^1(T)} + \int_X (-\psi) T \wedge dd^c\varphi \wedge \omega^{n-p-1}\,.$$

Now it follows from Stokes theorem that

$$\begin{aligned}\int_X (-\psi) T \wedge dd^c\varphi \wedge \omega^{n-p-1} &= \int_X \varphi T \wedge \big(-dd^c\psi\big) \wedge \omega^{n-p-1}\\ &\le \int_X \varphi T \wedge \omega^{n-p} \le \sup_X \varphi \int_X T \wedge \omega^{n-p}\,,\end{aligned}$$

where the next to last inequality follows from $\varphi T \wedge \omega^{n-p} \ge 0$ and $-dd^c\psi \le \omega$. This yields

$$||\psi||_{L^1(T\wedge\omega_\varphi)} \le ||\psi||_{L^1(T)} + \sup_X \varphi ||T||\,.$$

The general case follows by regularizing φ, ψ, observing that $\omega_\varphi = \omega_{\varphi'}$ where $\varphi' = \varphi - \inf_X \varphi \ge 0$, and decomposing $\psi = \psi' + \sup_X \psi$ with $\psi' = \psi - \sup_X \psi \le 0$. □

Remark 3.2. The fact that the L^1-norm of ψ with respect to the probability measure $T \wedge \omega_\varphi \wedge \omega^{n-p-1}$ is controlled by its L^1-norm with respect to $T \wedge \omega^{n-p}$ is similar to the phenomenon already encountered in Example 2.8: One can write

$$T \wedge \omega_\varphi \wedge \omega^{n-p-1} = T \wedge \omega^{n-p} + dd^c S,\ S = \big(\varphi - \inf_X \varphi\big) T \wedge \omega^{n-p-1} \ge 0\,.$$

This type of estimates is usually referred to as "Chern-Levine-Nirenberg inequalities," in reference to [8] where simpler -but fundamental- L^∞-estimates were established (with $\psi = constant$). Estimates involving the L^1-norm of ψ were first proved in the local context by Cegrell [6] and Demailly [12].

A straightforward induction yields the following.

Corollary 3.3. *Let $\psi, \varphi \in PSH(X,\omega)$ with $0 \le \varphi \le 1$. Then*

$$0 \le \int_X |\psi|\omega_\varphi^n \le \int_X |\psi|\omega^n + n\big[1 + 2\sup_X \psi\big]\int_X \omega^n .$$

Following Bedford-Taylor [5] and Kolodziej [29] we introduce the following Monge-Ampère capacity.

Definition 3.4. Let K be a Borel subset of X. We set

$$\mathrm{Cap}_\omega(K) := \sup\left\{\int_K \omega_\varphi^n \,/\, \varphi \in PSH(X,\omega), 0 \le \varphi \le 1\right\} .$$

Note that this definition only makes sense when the cohomology class $\{\omega\}$ is big, i.e., when $\{\omega\}^n > 0$, and when it admits locally bounded potentials φ. This implies, by a regularization result of Demailly [12], that $\{\omega\}$ is big and nef. In order to avoid technicalities, we are assuming throughout this section that $\{\omega\}$ is Kähler.

Proposition 3.5.

(1) *If $K \subset K' \subset X$ are Borel subsets then*

$$\mathrm{Vol}_\omega(K) := \int_K \omega^n \le \mathrm{Cap}_\omega(K) \le \mathrm{Cap}_\omega\left(K'\right) \le \mathrm{Cap}_\omega(X) = \mathrm{Vol}_\omega(X) .$$

(2) *If K_j are Borel subsets of X then $\mathrm{Cap}_\omega(\cup K_j) \le \sum \mathrm{Cap}_\omega(K_j)$. Moreover, $\mathrm{Cap}_\omega(\cup K_j) = \lim \mathrm{Cap}_\omega(K_j)$ if $K_j \subset K_{j+1}$.*

(3) *If $\omega_1 \le \omega_2$ then $\mathrm{Cap}_{\omega_1}(\cdot) \le \mathrm{Cap}_{\omega_2}(\cdot)$. For all $A \ge 1$, $\mathrm{Cap}_\omega(\cdot) \le \mathrm{Cap}_{A\omega}(\cdot) \le A^n\,\mathrm{Cap}_\omega(\cdot)$. In particular, if ω, ω' are two Kähler forms then there exists $C \ge 1$ such that*

$$\frac{1}{C}\,\mathrm{Cap}_\omega(\cdot) \le \mathrm{Cap}_{\omega'}(\cdot) \le C \cdot \mathrm{Cap}_\omega(\cdot) .$$

(4) *If $f : X \to X$ is holomorphic then for all Borel subset K of X,*

$$\mathrm{Cap}_\omega(f(K)) \le \mathrm{Cap}_{f^*\omega}(K) .$$

In particular, $\mathrm{Cap}_\omega(f(K)) = \mathrm{Cap}_\omega(K)$ for every ω-isometry f.

Proof. That $\mathrm{Vol}_\omega(\cdot) \le \mathrm{Cap}_\omega(\cdot)$ is a straightforward consequence of the definition (since $\omega_0 = \omega$). It then follows from Stokes theorem that $\int_X \omega_\varphi^n = \int_X \omega^n$ for every $\varphi \in PSH(X,\omega) \cap L^\infty(X)$, thus $\mathrm{Vol}_\omega(X) = \mathrm{Cap}_\omega(X)$.

Property (2) is a straightforward consequence of the definitions.

If $\omega_1 \le \omega_2$ then $PSH(X,\omega_1) \subset PSH(X,\omega_2)$ hence $\mathrm{Cap}_{\omega_1}(\cdot) \le \mathrm{Cap}_{\omega_2}(\cdot)$. Fix $A \ge 1$. If $\psi \in PSH(X, A\omega)$ is such that $0 \le \psi \le 1$ then $\psi/A \in PSH(X,\omega)$ with $0 \le \psi/A \le 1/A \le 1$. Moreover, $(A\omega + dd^c\psi)^n = A^n(\omega + dd^c(\psi/A))^n$. This shows $\mathrm{Cap}_{A\omega}(\cdot) \le A^n\,\mathrm{Cap}_\omega(\cdot)$.

In particular, if ω, ω' are both Kähler then $A^{-1}\omega \le \omega' \le A\omega$ for some constant $A \ge 1$, hence $C^{-1}\,\mathrm{Cap}_\omega(\cdot) \le \mathrm{Cap}_{\omega'}(\cdot) \le C \cdot \mathrm{Cap}_\omega(\cdot)$ with $C = A^n$.

It remains to prove (4). It follows from the change of variables formula that if $\varphi \in PSH(X,\omega)$ with $0 \le \varphi \le 1$ then

$$\int_{f(K)} \omega_\varphi^n \le \int_K f^*\omega_\varphi^n = \int_K \left(f^*\omega + dd^c(\varphi \circ f)\right)^n \le \mathrm{Cap}_{f^*\omega}(K)$$

since $\varphi \circ f \in PSH(X, f^*\omega)$ with $0 \leq \varphi \circ f \leq 1$. We infer $\mathrm{Cap}_\omega(f(K)) \leq \mathrm{Cap}_{f^*\omega}(K)$. When f is a ω-isometry, i.e., $f \in \mathrm{Aut}(X)$ with $f^*\omega = \omega$, then the mapping $\varphi \mapsto \varphi \circ f$ is an isomorphism of $\{u \in PSH(X, \omega) \,/\, 0 \leq u \leq 1\}$, whence $\mathrm{Cap}_\omega(f(K)) = \mathrm{Cap}_{f^*\omega}(K) = \mathrm{Cap}_\omega(K)$. □

Let $PSH^-(X, \omega)$ denote the set of negative ω-psh functions. A set is said to be $PSH(X, \omega)$-*polar* if it is included in the $-\infty$ locus of some function $\psi \in PSH(X, \omega)$, $\psi \not\equiv -\infty$. As we shall soon see, the sets of zero Monge-Ampère capacity are precisely the $PSH(X, \omega)$-polar sets. We start by establishing the following:

Proposition 3.6. *If P is a $PSH(X, \omega)$-polar set, then $\mathrm{Cap}_\omega(P) = 0$. More precisely, if $\psi \in PSH^-(X, \omega)$ then*

$$\mathrm{Cap}_\omega(\psi < -t) \leq \frac{1}{t}\left[\int_X (-\psi)\omega^n + n\,\mathrm{Vol}_\omega(X)\right], \ \forall t > 0\,.$$

Proof. Fix $\varphi \in PSH(X, \omega)$ such that $0 \leq \varphi \leq 1$. Fix $t > 0$ and set $K_t = \{x \in X \,/\, \psi(x) < -t\}$. By Chebyshev's inequality,

$$\int_{K_t} \omega_\varphi^n \leq \int_X (-\psi/t)\omega_\varphi^n \leq \frac{1}{t}\left[\int_X (-\psi)\omega^n + n\,\mathrm{Vol}_\omega(X)\right],$$

where the last inequality follows from Corollary 3.3. Taking supremum over all φ's yields the claim. □

Observe that the previous proposition says that $\mathrm{Cap}^*_\omega(P) = 0$, where

$$\mathrm{Cap}^*_\omega(E) := \inf\{\mathrm{Cap}_\omega(G) \,/\, G \text{ open with } \ E \subset G\},$$

is the outer capacity associate to Cap_ω.

Our aim is now to show that ω-psh functions are quasicontinuous with respect to Cap_ω (Corollary 3.8). We first need to show that decreasing sequences of ω-psh functions converge "in capacity."

Proposition 3.7. *Let $\psi, \psi_j \in PSH(X, \omega) \cap L^\infty(X)$ such that (ψ_j) decreases to ψ. Then for each $\delta > 0$,*

$$\mathrm{Cap}_\omega(\{\psi_j > \psi + \delta\}) \to 0\,.$$

Proof. We can assume w.l.o.g. that $\mathrm{Vol}_\omega(X) = 1$ and $0 \leq \psi_j - \psi \leq 1$. Fix $\delta > 0$ and $\varphi \in PSH(X, \omega)$, $0 \leq \varphi \leq 1$. By Chebyshev inequality, it suffices to control $\int_X (\psi_j - \psi)\omega_\varphi^n$ uniformly in φ. It follows from Stokes theorem that

$$\int_X (\psi_j - \psi)\omega_\varphi^n = \int_X (\psi_j - \psi)\omega \wedge \omega_\varphi^{n-1} - \int_X d(\psi_j - \psi) \wedge d^c\varphi \wedge \omega_\varphi^{n-1}\,.$$

Now by Cauchy-Schwartz inequality,

$$\left|\int_X df_j \wedge d^c\varphi \wedge \omega_\varphi^{n-1}\right| \leq \left(\int_X df_j \wedge d^c f_j \wedge \omega_\varphi^{n-1}\right)^{1/2} \cdot \left(\int_X d\varphi \wedge d^c\varphi \wedge \omega_\varphi^{n-1}\right)^{1/2},$$

where we set $f_j := \psi_j - \psi \geq 0$. Moreover,

$$\int_X d\varphi \wedge d^c\varphi \wedge \omega_\varphi^{n-1} = \int_X \varphi\big(-dd^c\varphi\big) \wedge \omega_\varphi^{n-1} \leq \int_X \varphi\omega \wedge \omega_\varphi^{n-1} \leq 1\,,$$

since $\varphi\omega_\varphi^{n-1} \geq 0$, $-dd^c\varphi \leq \omega$ and $\varphi \leq 1$. Similarly

$$\int_X df_j \wedge d^c f_j \wedge \omega_\varphi^{n-1} = \int_X -f_j\, dd^c f_j \wedge \omega_\varphi^{n-1} \leq \int_X f_j \omega_\psi \wedge \omega_\varphi^{n-1}.$$

Altogether this yields

$$\begin{aligned}\int_X (\psi_j - \psi)\omega_\varphi^n &\leq \int_X (\psi_j - \psi)\omega \wedge \omega_\varphi^{n-1} + \left(\int_X (\psi_j - \psi)\omega_\psi \wedge \omega_\varphi^{n-1}\right)^{1/2} \\ &\leq \sqrt{2}\left(\int_X (\psi_j - \psi)(\omega + \omega_\psi) \wedge \omega_\varphi^{n-1}\right)^{1/2},\end{aligned}$$

where the last inequality follows from the elementary inequalities $0 \leq a \leq \sqrt{a} \leq 1$ and $\sqrt{a} + \sqrt{b} \leq \sqrt{2}\sqrt{a+b}$.

Going on replacing at each step a term ω_φ by $\omega + \omega_\psi$, we end up with

$$\int_X (\psi_j - \psi)\omega_\varphi^n \leq 2\left(\int_X (\psi_j - \psi)(\omega + \omega_\psi)^n\right)^{1/2^n}.$$

The majorant being independent of φ and converging to 0 as $j \to +\infty$ (by dominated convergence theorem), this completes the proof. □

Corollary 3.8 (Quasicontinuity). *Let $\varphi \in PSH(X, \omega)$. For each $\varepsilon > 0$ there exists an open subset O_ε of X such that $\mathrm{Cap}_\omega(O_\varepsilon) < \varepsilon$ and φ is continuous on $X \setminus O_\varepsilon$.*

Proof. For $t > 0$ large enough, the set $O_1 = \{\varphi < -t\}$ has capacity $< \varepsilon/2$ by Proposition 3.6. Working in $X \setminus O_1$ we can thus replace φ by $\varphi_t = \max(\varphi, -t)$ which is bounded on X. Regularizing φ (see Appendix), we can find a sequence ψ_j of smooth $A\omega$-psh functions which decrease to φ_t on X, for some $A \geq 1$. By Proposition 3.7, the set $O_j = \{\psi_{k_j} > \varphi_t + 1/j\}$ has capacity $< \varepsilon 2^{-j-1}$ if k_j is large enough. Now ψ_{k_j} uniformly converges to $\varphi = \varphi_t$ on $X \setminus O_\varepsilon$, $O_\varepsilon = \cup_{j\geq 1} O_j$, so φ is continuous on $X \setminus O_\varepsilon$ and $\mathrm{Cap}_\omega(O_\varepsilon) \leq \varepsilon$. □

Example 3.9. The capacity $\mathrm{Cap}_\omega(\cdot)$ does not distinguish between "big sets." Assume indeed there exists an ample divisor D such that $[D] \sim k\omega$, $k \in \mathbb{N}$. Then there exists $\varphi \in PSH(X, \omega)$ such that $dd^c\varphi = k^{-1}[D] - \omega$. Note that $\varphi \in \mathcal{C}^\infty(X \setminus D)$, $e^\varphi \in \mathcal{C}^0(X)$ and $\{\varphi = -\infty\} = D$. Replacing φ by $\varphi - \sup_X \varphi$ if necessary, we may assume $\sup_X \varphi = 0$. Consider $\varphi_c = \max(\varphi, -c) \in PSH(X, \omega) \cap \mathcal{C}^0(X)$. Then $\varphi_c \equiv \varphi$ outside some neighborhood $V_c = \{\varphi < -c\}$ of D. Since $0 \leq 1 + \varphi_1 \leq 1$ and $\omega_{1+\varphi_1} = \omega_\varphi = 0$ in $X \setminus V_1$, we get

$$\mathrm{Cap}_\omega(X) = \int_X (\omega_{1+\varphi_1})^n = \int_{V_1} (\omega_{1+\varphi_1})^n \leq \mathrm{Cap}_\omega(V_1),$$

hence $\mathrm{Cap}_\omega(V_1) = \mathrm{Cap}_\omega(X)$.

As a concrete example take $X = \mathbb{CP}^n$ and $\omega = \omega_{FS}$, D being some hyperplane H_∞ "at infinity" ($k = 1$). Set $\varphi[z : t] = \log|t| - \frac{1}{2}\log[||z||^2 + |t|^2]$ where z denotes the Euclidean coordinates in $\mathbb{C}^n = \mathbb{CP}^n \setminus H_\infty$ and $H_\infty = (t = 0)$. Observe that $\sup_{\mathbb{CP}^n} \varphi = 0$. One then computes

$$\mathbb{CP}^n \setminus V_1 = \left\{z \in \mathbb{C}^n \,/\, |z| \leq \sqrt{e^2 - 1}\right\}.$$

Thus, the capacity of the complement of any Euclidean ball of radius smaller than $\sqrt{e^2 - 1}$ equals

The definition of Cap_ω mimics the definition of the relative Monge-Ampère capacity introduced by Bedford and Taylor in [5]. Fix $\mathcal{U} = \{\mathcal{U}_\alpha\}$ a finite covering of X by strictly pseudoconvex open subsets of X, $\mathcal{U}_\alpha = \{x \in X \,/\, \varrho_\alpha(x) < 0\}$, where ϱ_α is a strictly psh smooth function defined in a neighborhood of $\overline{\mathcal{U}_\alpha}$. Fix $\delta > 0$ such that $\mathcal{U}^\delta = \{\mathcal{U}_\alpha^\delta\}$ is still a covering of X, where $\mathcal{U}_\alpha^\delta = \{x \in X \,/\, \varrho_\alpha(x) < -\delta\}$. For a Borel subset K of X, we set

$$\mathrm{Cap}_{BT}(K) := \sum_\alpha \mathrm{Cap}_{BT}\left(K \cap \mathcal{U}_\alpha^\delta, \mathcal{U}_\alpha\right),$$

where

$$\mathrm{Cap}_{BT}(E, \Omega) := \sup\left\{\int_E (dd^c u)^n \,/\, u \in PSH(\Omega),\ 0 \le u \le 1\right\}$$

is the capacity studied by Bedford and Taylor. The next proposition is due to Kolodziej [29]. We include a slightly different proof.

Proposition 3.10. *There exists $C \ge 1$ such that*

$$\frac{1}{C}\mathrm{Cap}_\omega(\cdot) \le \mathrm{Cap}_{BT}(\cdot) \le C \cdot \mathrm{Cap}_\omega(\cdot).$$

Proof. Let E be a Borel subset of X. Since $\mathrm{Cap}_\omega(E \cap \mathcal{U}_\alpha^\delta) \le \mathrm{Cap}_\omega(E) \le \sum_\alpha \mathrm{Cap}_\omega(E \cap \mathcal{U}_\alpha^\delta)$, it is sufficient to show that if $\Omega = \{x \in X \,/\, \varrho(x) < 0\}$ is a smooth hyperconvex subset of X, then there exists $C \ge 1$ such that for all $E \subset \Omega_\delta$,

$$\frac{1}{C}\mathrm{Cap}_\omega(E) \le \mathrm{Cap}_{BT}(E, \Omega) \le C \cdot \mathrm{Cap}_\omega(E),$$

where $\Omega_\delta = \{x \in X \,/\, \varrho(x) < -\delta\}$.

It is an easy and well known fact in the local theory that the capacities $\mathrm{Cap}(\cdot, \Omega)$ and $\mathrm{Cap}(\cdot, \Omega')$ are comparable when $\Omega' \subset \Omega$ (see e.g., Theorem 6.5 in [12]). Therefore we can assume (passing to a finer covering if necessary) that $\omega = dd^c\psi$ near $\overline{\Omega}$. Fix $C_1 > 0$ such that $-C_1 \le \psi \le C_1$ on Ω. Fix $\varphi \in PSH(X, \omega)$ such that $0 \le \varphi \le 1$ on X and set $u = (2C_1)^{-1}(\varphi + \psi + C_1)$. Then $u \in PSH(\Omega)$ and $0 \le u \le 1$, hence

$$\int_E \omega_\varphi^n = (2C_1)^n \int_E (dd^c u)^n \le (2C_1)^n \,\mathrm{Cap}_{BT}(E, \Omega),$$

which yields $\mathrm{Cap}_\omega(E) \le (2C_1)^n \,\mathrm{Cap}_{BT}(E, \Omega)$. Observe that we have not used here that ω is Kähler.

For the reverse inequality we consider $\chi \in C^\infty(X)$ such that $\chi \equiv 0$ in $X \setminus \Omega$ and $\chi < 0$ in Ω. Replacing χ by $\varepsilon\chi$ if necessary, we can assume $\chi \in PSH(X, \omega)$. This is because ω is Kähler (and this is the only place where we shall use this crucial assumption). Fix $\varepsilon > 0$ so small that $\chi \le -\varepsilon$ on Ω_δ. Let now $u \in PSH(\Omega)$ be such that $0 \le u \le 1$ on Ω. Consider

$$\varphi(x) = \begin{cases} \frac{u-\psi+C_1}{2+2C_1} & \text{in } \Omega_\delta \\ \max\left(\frac{u-\psi+C_1}{2+2C_1}, \frac{2}{\varepsilon}\chi(x) + 1\right) & \text{in } \Omega \setminus \Omega_\delta \\ 1 & \text{in } X \setminus \Omega \end{cases}.$$

Observe that $0 \le u' := (u - \psi + C_1)/(2 + 2C_1) \le (1 + 2C_1)/(2 + 2C_1) < 1$ in Ω. Therefore $\varphi \in PSH(X, \frac{2}{\varepsilon}\omega)$ since $\frac{2}{\varepsilon}\chi(x) + 1 \le -1 < u'$ in Ω_δ, while $\frac{2}{\varepsilon}\chi(x) + 1 \equiv 1 > u'$ on $\partial\Omega$. Note, also that $0 \leq \varphi \leq 1$ thus for $E \subset \Omega_\delta$,

$$\frac{1}{(2+2C_1)^n}\int_E (dd^c u)^n = \int_E \left(\frac{\omega}{2+2C_1}+dd^c\varphi\right)^n \leq \int_E \left(\frac{2}{\varepsilon}\omega+dd^c\varphi\right)^n$$
$$\leq \mathrm{Cap}_{2\omega/\varepsilon}(E) \leq \left(\frac{2}{\varepsilon}\right)^n \mathrm{Cap}_\omega(E)$$

hence $\mathrm{Cap}_{BT}(E,\Omega) \leq 4^n(1+C_1)^n\varepsilon^{-n}\,\mathrm{Cap}_\omega(E)$. □

Since locally pluripolar sets are precisely the sets of zero relative capacity [5], we obtain the following:

Corollary 3.11. $\mathrm{Cap}_\omega(P)=0 \Leftrightarrow \mathrm{Cap}_{BT}(P)=0 \Leftrightarrow$ *P is locally pluripolar.*

We shall show later on that locally pluripolar sets are $PSH(X,\omega)$-polar when ω is Kähler (see Theorem 7.2).

The following two results are direct consequences of the corresponding results of Bedford and Taylor [4, 5].

Theorem 3.12 (Dirichlet Problem). *Let $\varphi \in PSH(X,\omega)\cap L^\infty(X)$. Let B be a small ball in X. Then there exists $\hat{\varphi} \in PSH(X,\omega)$ such that $\hat{\varphi}=\varphi$ in $X\setminus B$, $\hat{\varphi}\geq\varphi$ and $(\omega_{\hat{\varphi}})^n=0$ in B. Moreover, if $\varphi_1\leq\varphi_2$ then $\hat{\varphi}_1\leq\hat{\varphi}_2$.*

Theorem 3.13 (Comparison principle). *Let $\varphi,\psi \in PSH(X,\omega)\cap L^\infty(X)$. Then*

$$\int_{\{\varphi<\psi\}}\omega_\psi^n \leq \int_{\{\varphi<\psi\}}\omega_\varphi^n .$$

Many results in this section hold when ω is merely a smooth semi-positive form such that $\{\omega\}^n>0$, as the following example shows.

Example 3.14. Let $\pi : X \to \tilde{X}$ be the blow up of $\tilde{X}$ along a smooth center Y of codimension ≥ 2. Let $\tilde{\omega}$ be a Kähler form on $\tilde{X}$ and set $\omega=\pi^*\tilde{\omega}$. Then ω is a smooth semi-positive form on X such that $\omega_{|E}\equiv 0$, where E denotes the exceptional divisor. Clearly

$$PSH(X,\omega)=\pi^*PSH(\tilde{X},\tilde{\omega}),$$

hence ω-psh functions do not separate points of E. However, the Monge-Ampère capacity $\mathrm{Cap}_\omega(\cdot)$ is well-defined and enjoys all previous properties.

4. The relative extremal function

We assume in this section, as in Sections 3 and 7, that ω is a Kähler form. We now introduce a substitute for the relative extremal function which has revealed so useful in the local theory [5]: If E is a Borel subset of X, we set

$$h_{E,\omega}(x) := \sup\left\{\varphi(x)\,/\,\varphi\in PSH(X,\omega),\ \varphi\leq 0 \text{ and } \varphi_{|E}\leq -1\right\}.$$

We let $h^*_{E,\omega}$ denote its upper-semi-continuous regularization, which we call the relative ω-plurisubharmonic extremal function of the subset $E\subset X$. It enjoys several natural properties; we list some of them below. The proofs follow from standard arguments together with Theorem 7.2:

- The function $h^*_{E,\omega}$ is ω-psh. It satisfies $-1 \leq h^*_{E,\omega} \leq 0$ on X and $h^*_{E,\omega} = -1$ on $E \setminus P$, where P is pluripolar.
- If $E \subset X$ and $P \subset X$ is pluripolar, then $h^*_{E \setminus P} \equiv h^*_E$.
- If (E_j) increases towards $E \subset X$, then $h^*_{E_j}$ decreases towards h^*_E.
- If (K_j) is a sequence of compact subsets decreasing towards K, then $h^*_{K_j}$ increases (a.e.) towards h^*_K.

As in the local theory, the complex Monge-Ampère of the relative extremal function of a subset $E \subset X$ vanishes outside $\overline{E}$, except perhaps on the set $\{h^*_E = 0\}$ which, in the local theory, lies in the boundary of the domain.

Proposition 4.1. *When the open set* $\Omega_E := \{x \in X \,/\, h^*_{E,\omega}(x) < 0\}$ *is nonempty, then*

$$\left(\omega_{h^*_{E,\omega}}\right)^n = \left(\omega + dd^c h^*_{E,\omega}\right)^n = 0 \ \ in \quad \Omega_E \setminus \overline{E}\,.$$

Proof. Assume that $\Omega_E := \{h^*_{E,\omega} < 0\}$ is nonempty. It follows from Choquet's lemma that there exists an increasing sequence φ_j of ω-psh functions such that $\varphi_j = -1$ on E, $\varphi_j \leq 0$ on X, and $h^*_{E,\omega} = (\lim \varphi_j)^*$. Let $a \in \Omega_E \setminus \overline{E}$ and fix a small ball $B \subset \Omega_E$ centered at point a. Let $\widehat{\varphi_j} = \widehat{(\varphi_j)}_B$ denote the functions obtained by applying Theorem 3.12, so that $(\omega_{\widehat{\varphi_j}})^n = 0$ in B. If B is chosen small enough, then $\widehat{\varphi_j} < 0$ in B, hence $\widehat{\varphi_j} \leq 0$ on X, while $\widehat{\varphi_j} = \varphi_j = -1$ on E. This can be seen by showing that $\widehat{0}_B \to 0$ as the radius of the ball B shrinks to 0. Therefore $\lim \nearrow \widehat{\varphi_j} = h^*_{E,\omega}$, hence $(\omega_{h^*_{E,\omega}})^n \equiv 0$ in a neighborhood of a, which prove our claim. □

We now establish an important result which expresses the capacity in terms of the relative extremal function for any subset. It will show in particular that the set function Cap^*_ω is a capacity in the sense of Choquet which is outer regular. For simplicity, we write h_E for $h_{E,\omega}$.

Theorem 4.2. *Let* $E \subset X$ *be any Borel subset, then*

$$\mathrm{Cap}^*_\omega(E) = \int_X \left(-h^*_E\right)\omega^n_{h^*_E}\,. \tag{†}$$

The Monge-Ampère capacity satisfies the following continuity properties:

(1) *If* $(E_j)_{j\geq 0}$ *is an increasing sequence of arbitrary subsets of* X *and* $E := \cup_{j\geq 0} E_j$ *then*

$$\mathrm{Cap}^*_\omega(E) = \lim_{j\to+\infty} \mathrm{Cap}^*_\omega(E_j)\,.$$

(2) *If* $(K_j)_{j\geq 0}$ *is a decreasing sequence of compact subsets of* X *and* $K := \cap_{j\geq 0} K_j$ *then*

$$\mathrm{Cap}_\omega(K) = \mathrm{Cap}^*_\omega(K) = \lim_{j\to+\infty} \mathrm{Cap}_\omega(K_j)\,.$$

In particular, $\mathrm{Cap}^*_\omega(\cdot)$ *is an outer regular Choquet capacity on* X.

Proof. We first establish (†) when $E = K \subset X$ is compact. Observe that in the definition of the capacity, it is enough to restrict ourselves to ω-psh functions φ such that $-1 < \varphi < 0$. Let φ be such a function. The pluripolar set $N := \{h_K < h^*_K\}$ is of measure 0 for the measure ω^n_φ. Since $-1 < \varphi$, we have $K \subset N \cup \{h^*_K < \varphi\}$, hence the comparison principle yields

$$\int_K \omega^n_\varphi \leq \int_{\{h^*_K < \varphi\}} \omega^n_\varphi \leq \int_{\{h^*_K < \varphi\}} \omega^n_{h^*_E}\,.$$

This shows $\mathrm{Cap}_\omega(K) \le \int_{\Omega_K} \omega_{h_K^*}^n$ since $\{h_K^* < \varphi\} \subset \Omega_K$. It follows from Proposition 4.1 that

$$\mathrm{Cap}_\omega(K) \le \int_{\Omega_K} (\omega_{h_K}^*)^n = \int_K (\omega_{h_K^*})^n ,$$

whence equality.

Recall that $K \cap \{h_K^* > -1\} \subset \{h_K < h_K^*\}$ is of measure 0 for $\omega_{h_K^*}^n$, so

$$\mathrm{Cap}_\omega(K) = \int_K (\omega_{h_K}^*)^n = \int_K (-h_K^*)(\omega_{h_K}^*)^n = \int_X (-h_K^*)(\omega_{h_K}^*)^n ,$$

where the last equality follows from the fact that the equilibrium measure $(\omega_{h_K^*})^n$ is supported on $K \cup \{h_K^* = 0\}$.

We assume now that $E = G \subset X$ is an *open* subset. Let (K_j) be an exhaustive sequence of compact subsets of G which increases to G. Since $h_{K_j}^* \downarrow h_G$ on X, it follows from classical convergence results that $(-h_{K_j}^*)\omega_{h_{K_j}^*}^n \to (-h_G^*)\omega_{h_G}^n$ in the weak sense of measures on X, therefore

$$\int_X (-h_G)\omega_{h_G}^n = \lim_{j\to+\infty} \int_X (-h_{K_j}^*)\omega_{h_{K_j}^*}^n = \lim_{j\to+\infty} \mathrm{Cap}_\omega(K_j) = \mathrm{Cap}_\omega(G) .$$

This proves (†) when $E \subset X$ is an open subset.

Finally, let $E \subset X$ be any subset. By definition of the outer capacity, there is a sequence of open subsets $(O_j)_{j\ge 1}$ of X containing E such that $\mathrm{Cap}_\omega^*(E) = \lim_{j\to+\infty} \mathrm{Cap}_\omega(O_j)$. We can assume w.l.o.g. that the sequence $(O_j)_{j\ge1}$ is decreasing.

By a classical topological lemma of Choquet, there exists an increasing sequence $(u_j)_{j\ge1}$ negative ω-psh functions on X s.t. $u_j = -1$ on E with $u_j \uparrow h_E^*$ almost everywhere on X. We set for each $j \in \mathbb{N}$, $G_j := O_j \cap \{u_j < -1 + 1/j\}$. Then (G_j) is a decreasing sequence of open subsets of X such that $E \subset G_j \subset O_j$ and $u_j - 1/j \le h_{G_j} \le h_E$, so $h_{G_j} \uparrow h_E^*$ almost everywhere on X. We infer $(-h_{G_j})\omega_{h_{G_j}}^n \to (-h_E^*)\omega_{h_E}^n$ in the weak sense of measures on X, thus

$$\int_X (-h_E^*)\omega_{h_E^*}^n = \lim_{j\to+\infty} \int_X (-h_{G_j}^*)\omega_{h_{G_j}}^n .$$

On the other hand, we have by construction $\mathrm{Cap}_\omega^*(E) \le \lim_{j\to+\infty} \mathrm{Cap}_\omega^*(G_j) \le \lim_{j\to+\infty} \mathrm{Cap}_\omega^*(O_j) = \mathrm{Cap}_\omega^*(E)$. Therefore using (†) for open subsets, we get

$$\int_X (-h_E^*)\omega_{h_E^*}^n = \mathrm{Cap}_\omega^*(E) .$$

Observe that (1) follows straightforwardly from this formula. Indeed, if (E_j) increases towards E, then $h_{E_j}^*$ decreases towards h_E^*, hence $(-h_{E_j}^*)(\omega_{h_{E_j}^*})^n \to (-h_E^*)(\omega_{h_E^*})^n$, so that

$$\mathrm{Cap}_\omega^*(E) = \int_X (-h_E^*)\omega_{h_E^*}^n = \lim \int_X (-h_{E_j}^*)(\omega_{h_{E_j}^*})^n = \lim \mathrm{Cap}_\omega^*(E_j) .$$

It remains to prove (2). Let (K_j) be a decreasing sequence of compact subsets of X which converges to K. We claim that $h_{K_j}^* \uparrow h_K^*$ almost everywhere on X. Indeed, the extremal function $h_{K_j}^*$ increases almost everywhere to a ω-psh function h such that $h \le h_K^*$ on X. We want to prove that $h_K \leq h$ on X. Let $u \in PSH(X, \omega)$ such that $u \leq 0$ on X and $u_{|K} \leq -1$. Fix $\varepsilon > 0$ and consider the open subset $G_\varepsilon := \{u < -1 + \varepsilon\}$. Then $K \subset G_\varepsilon$, thus $K_j \subset G_\varepsilon$ for j large enough. This yields $u - \varepsilon \leq h_{K_j}$ for j large enough, hence $u \leq h$ on X. Therefore $h_K \leq h$ on X as claimed.

Since $(-h_{K_j})\omega^n_{h^*_{K_j}}$ converges weakly to $(-h^*_K)\omega^n_{h^*_K}$ on X, we infer

$$\mathrm{Cap}_\omega(K) = \int_X (-h^*_K)\omega^n_{h^*_K} = \lim_{j\to+\infty} \int_X (-h^*_{K_j})\omega^n_{h^*_{K_j}} = \lim_{j\to+\infty} \mathrm{Cap}_\omega(K_j)\,.$$

From this last property, taking a decreasing sequence $(K_j)_{j\geq 0}$ of compact subsets such that $K = \cap_{j\geq 0} K_j$ and $K_{j+1} \subset K_j^\circ$ for any $j \in \mathbb{N}$ we obtain $\mathrm{Cap}^*_\omega(K) = \lim_{j\to+\infty} \mathrm{Cap}_\omega(K_j) = \mathrm{Cap}_\omega(K)$. □

5. Alexander capacity

We now introduce another capacity which is defined by means of a global extremal function. It is closely related to the projective capacity introduced by Alexander in [1].

5.1. Global extremal functions

Definition 5.1. Let K be a Borel subset of X. We set

$$V_{K,\omega} := \sup\{\varphi(x) \,/\, \varphi \in PSH(X,\omega),\ \varphi \leq 0 \text{ on } K\}\ .$$

This definition mimics the definition of the so-called "Siciak's extremal function" usually defined for Borel subset of $X = \mathbb{CP}^n$ that are bounded in $\mathbb{C}^n = X \setminus H_\infty$, where H_∞ denotes some hyperplane at infinity. This function was introduced and studied by Siciak in [35, 36] (see also [38]). One can indeed check that this definition coincides with the classical one if one chooses $\omega = [H_\infty]$ to be the current of integration along the hyperplane H_∞. Similarly one could consider the case where $\omega = [D]$ is the current of integration along a positive divisor D on X and let D play the role of infinity. This approach has been used by some authors working in Arakelov geometry to define capacities on projective varieties (see [30, 9] and references therein). However, this forces them to consider only compact subsets of $X \setminus D$ and leads to less intrinsic notions of capacities.

In the sequel we assume that ω is smooth (but not necessarily positive).

Theorem 5.2. *Let K be a Borel subset of X.*

(1) *K is $PSH(X,\omega)$-polar iff $\sup_X V^*_{K,\omega} = +\infty$ iff $V^*_{K,\omega} \equiv +\infty$.*

(2) *If K is not $PSH(X,\omega)$-polar then $V^*_{K,\omega} \in PSH(X,\omega)$ and satisfies $V^*_{K,\omega} \equiv 0$ in the interior of K, $(\omega_{V^*_{K,\omega}})^n = 0$ in $X \setminus \overline{K}$ and*

$$\int_{\overline{K}} (\omega_{V^*_{K,\omega}})^n = \int_X \omega^n = \mathrm{Vol}_\omega(X)\,.$$

Proof. Assume $\sup_X V^*_{K,\omega} = +\infty$. By a lemma of Choquet (see Lemma 4.23 in [15, Ch. 1]), we can find an increasing sequence of functions $\varphi_j \in PSH(X,\omega)$ such that $\varphi_j = 0$ on K and

$V^*_{K,\omega} = (\lim \nearrow \varphi_j)^*$. Extracting a subsequence if necessary, we can assume $\sup_X \varphi_j \geq 2^j$. Set $\psi_j = \varphi_j - \sup_X \varphi_j$. These functions belong to $\mathcal{F}_0$ which is a compact subfamily of $PSH(X, \omega)$ (Corollary 2.7). Recall that if μ is a smooth volume form on X then there exists C_μ such that $\int \psi_j \, d\mu \geq -C_\mu$ for all j. Set $\psi := \sum_{j \geq 1} 2^{-j} \psi_j$. Then $\psi \in PSH(X, \omega)$ as a decreasing limit of functions in $PSH(X, \omega)$ with $\int_X \psi \, d\mu \geq -C_\mu > -\infty$. Now for every $x \in K$ we get $\psi(x) = -\sum_{j \geq 1} 2^{-j} \sup_X \varphi_j = -\infty$ hence $K \subset \{\psi = -\infty\}$, i.e., K is $PSH(X, \omega)$-polar.

Conversely assume K is $PSH(X, \omega)$-polar, $K \subset \{\psi = -\infty\}$ for some $\psi \in PSH(X, \omega)$. Then for all $c \in \mathbb{R}$, $\psi + c \in PSH(X, \omega)$ and $\psi + c \leq 0$ on K. Therefore $V_{K,\omega} \geq \psi + c, \forall c \in \mathbb{R}$. This yields $V_{K,\omega} = +\infty$ on $X \setminus \{\psi = -\infty\}$ hence $V^*_{K,\omega} \equiv +\infty$ on X since $\{\psi = -\infty\}$ has zero volume. We have thus shown the following circle of implications: K is $PSH(X, \omega)-$polar $\Rightarrow V^*_{K,\omega} \equiv +\infty \Rightarrow \sup_X V^*_{K,\omega} = +\infty \Rightarrow K$ is $PSH(X, \omega)-$polar.

Assume now that K is not $PSH(X, \omega)$-polar. Then $V^*_{K,\omega} \in PSH(X, \omega)$ [see Proposition 2.6 (2)] and clearly satisfies $V^*_{K,\omega} = 0$ in the interior of K. If we show that $(\omega_{V^*_{K,\omega}})^n = 0$ in $X \setminus \overline{K}$ then

$$\int_{\overline{K}} (\omega_{V^*_{K,\omega}})^n = \int_X (\omega_{V^*_{K,\omega}})^n = \int_X \omega^n ,$$

as follows from Stokes theorem. Let $\varphi_j \in PSH(X, \omega)$ be an increasing sequence such that $\varphi_j = 0$ on K and $V^*_{K,\omega} = (\lim \nearrow \varphi_j)^*$. Fix B a small ball in $X \setminus \overline{K}$. Let $\hat{\varphi}_j$ be the solution of the Dirichlet problem with boundary values φ_j. Then $\hat{\varphi}_j \in PSH(X, \omega)$, $\hat{\varphi}_j = \varphi_j$ in $X \setminus B$ (in particular, $\hat{\varphi}_j = 0$ on K hence $\hat{\varphi}_j \leq V_{K,\omega}$) and the sequence $(\hat{\varphi}_j)$ is again increasing (Theorem 3.12). Since $(\omega_{\hat{\varphi}_j})^n = 0$ in B and $(\lim \nearrow \hat{\varphi}_j) = V^*_{K,\omega}$, it follows from the continuity of the complex Monge-Ampère on increasing sequences that $(\omega_{V^*_{K,\omega}})^n = 0$ in B. As B was an arbitrarily small ball in $X \setminus \overline{K}$ we infer $(\omega_{V^*_{K,\omega}})^n = 0$ in $X \setminus \overline{K}$. □

The following corollary has to be related to Proposition 2.7.

Corollary 5.3. *Let K be a Borel subset of X and set*

$$\mathcal{F}_K := \{\varphi \in PSH(X, \omega) \,/\, \sup_K \varphi = 0\} .$$

Then $\mathcal{F}_K$ is relatively compact iff K is not $PSH(X, \omega)$-polar.

Proof. Observe that $V_{K,\omega}(x) = \sup\{\varphi(x) \,/\, \varphi \in \mathcal{F}_K\}$. Thus, if $\mathcal{F}_K$ is relatively compact then it is uniformly bounded from above, hence $\sup_X V_{K,\omega} < +\infty$, i.e., K is not $PSH(X, \omega)$-polar.

Assume conversely that K is not $PSH(X, \omega)$-polar. Let $(\varphi_j) \in \mathcal{F}_K^{\mathbb{N}}$. Then $\varphi_j \leq V_{K,\omega} \leq \sup_X V_{K,\omega} < +\infty$ hence (φ_j) is uniformly bounded from above. It follows from Proposition 2.6 that (φ_j) is relatively compact. Indeed, it can not converge uniformly to $-\infty$ since $\sup_K \varphi_j = 0$ (see Proposition 2.6). □

Proposition 5.4. *Let K be a Borel subset of X.*

(1) *If $K' \subset K$ then $V_{X,\omega} \leq V_{K,\omega} \leq V_{K',\omega}$ and $\sup_X V_{X,\omega} = 0$. Furthermore $V_{X,\omega} \equiv 0$ when $\omega \geq 0$.*

(2) *If $\omega_1 \leq \omega_2$ then $V_{K,\omega_1} \leq V_{K,\omega_2}$.*

(3) *For all $A > 0$, $V_{K,A\omega} = A \cdot V_{K,\omega}$.*

(4) *If* $\omega' = \omega + dd^c\chi$ *then*

$$-\chi + \inf_X \chi + V_{K,\omega} \le V_{K,\omega'} \le V_{K,\omega} + \sup_X \chi - \chi .$$

(5) *If* $f : X \to X$ *is holomorphic then*

$$V_{f(K),\omega} \circ f \le V_{K,f^*\omega} .$$

In particular, if f *is a* ω*-isometry then* $V_{f(K),\omega} = V_{K,\omega}$.

Proof. That $K \mapsto V_{K,\omega}$ is decreasing follows straightforwardly from the definition. Observe that $0 \in PSH(X,\omega)$ when $\omega \ge 0$, hence $V_{X,\omega} \equiv 0$ in this case. When ω is smooth (but not positive), considering $V^*_{X,\omega}$ will be a useful way of constructing a positive closed current $\omega_{V^*_{X,\omega}} \sim \omega$ with minimal singularities (see Section 6).

Assertions (2), (3), (4) are simple consequences of Proposition 2.3. The last assertion results from the following observation: If $\varphi \in PSH(X,\omega)$ is such that $\varphi \le 0$ on $f(K)$, then $\varphi \circ f$ belongs to $PSH(X, f^*\omega)$ and satisfies $\varphi \circ f \le 0$ on K. □

Example 5.5. Assume $X = \mathbb{CP}^n$, ω is the Fubini-Study Kähler form and let B_R denote the Euclidean ball centered at the origin and of radius R in $\mathbb{C}^n \subset \mathbb{CP}^n$. Then for $x \in \mathbb{C}^n$,

$$V_{B_R,\omega}(x) = \max\left(\log\frac{||x||}{R} + \frac{1}{2}\log\left[1+R^2\right] - \frac{1}{2}\log\left[1+||x||^2\right]; 0\right) .$$

Indeed, set $\psi_R := \max(\frac{1}{2}\log[1+||x||^2], \varphi_R)$, where $\varphi_R = \frac{1}{2}\log[1+R^2] + \log\frac{||x||}{R}$. Recall that the usual Siciak's extremal function of B_R is $\log^+\frac{||x||}{R}$. Therefore $\frac{1}{2}\log[1+||x||^2] \le \varphi_R = \psi_R$ for $||x|| \ge R$. On the other hand, if $||x|| < R$ then $1 + ||x||^2 > (1+R^2)\frac{||x||^2}{R^2}$ hence $\frac{1}{2}\log[1+||x||^2] > \varphi_R$ in B_R.

Now let $u \in PSH(\mathbb{CP}^n,\omega)$ such that $u \le 0$ in B_R. Then $v = u + \frac{1}{2}\log[1+||x||^2] \in \mathcal{L}(\mathbb{C}^n)$. Since $v \le \frac{1}{2}\log[1+R^2]$ in B_R we infer $v \le \frac{1}{2}\log[1+R^2] + \log^+\frac{||x||}{R} = \psi_R$ in $\mathbb{C}^n \setminus B_R$. Moreover, $v \le \frac{1}{2}\log[1+||x||^2] = \psi_R$ in B_R hence $v \le \psi_R$ in $\mathbb{C}^n$. This shows $V_{B_R,\omega} = \psi_R - \frac{1}{2}\log[1+||x||^2]$ on $\mathbb{CP}^n$.

Proposition 5.6.

(1) *If* E *is an open subset, then* $V_E = V_E^*$.

(2) *Let* E *be a Borel subset and* P *a* $PSH(X,\omega)$*-polar set. Then*

$$V^*_{E\cup P} \equiv V^*_E .$$

(3) *Let* (E_j) *be an increasing sequence of Borel subsets and set* $E = \cup E_j$. *Then* $V^*_{E,\omega} = \lim \searrow V^*_{E_j,\omega}$ *if* ω *is Kähler.*

(4) *Let* K_j *be a decreasing sequence of compact subsets of* X *and set* $K = \cap K_j$. *Then* $V_{K_j,\omega} \nearrow V_{K,\omega}$, *hence* $V^*_{K_j,\omega} \nearrow V^*_{K,\omega}$ *a.e.*

(5) *Fix* $E \subset X$ *a nonpluripolar set. Then there exists* G_j *a decreasing sequence of open subsets,* $E \subset G_j$, *such that* $V^*_E = \lim V^*_{G_j}$.

Proof. We write here V_E for $V_{E,\omega}$ since ω is fixed and no confusion can arise.

Let E be an open subset of X. Observe that $V_E \leq 0$ on E, hence $V_E^* \leq 0$ on E which is open. Therefore $V_E^* \leq V_E$, whence equality. This proves (1).

Let $w \in PSH(X, \omega)$, $w \leq 0$, and fix $P \subset \{w = -\infty\}$. Fix E a Borel subset of X. Clearly $V_{E\cup P} \leq V_E$ hence $V_{E\cup P}^* \leq V_E^*$. Conversely let $\varphi \in PSH(X, \omega)$ be such that $\varphi \leq 0$ on E. Then $\forall \varepsilon > 0$, $\psi_\varepsilon := (1-\varepsilon)\varphi + \varepsilon w \in PSH(X, \omega)$ satisfies $\psi_\varepsilon \leq 0$ on $E \cup P$, hence $\psi_\varepsilon \leq V_{E\cup P}$. Letting $\varepsilon \to 0$ we infer $\varphi \leq V_{E\cup P}$ on $X \setminus P$, hence $\varphi \leq V_{E\cup P}^*$ on X. Thus, $V_E^* \leq V_{E\cup P}^*$.

Let E_j be an increasing sequence of subsets of X and set $E = \cup_{j\geq 1} E_j$. Let $v := \lim \searrow V_{E_j}^*$ [the limit is decreasing by Proposition 5.4 (1)]. If E is $PSH(X, \omega)$-polar then so are all the E_j's, hence $V_E^* \equiv +\infty = \lim V_{E_j}^*$. So let us assume E is not $PSH(X, \omega)$-polar. Then $v \in PSH(X, \omega)$ since $v \geq V_{E,\omega}^* \not\equiv -\infty$ [see Proposition 2.6 (3)]. Observe that $v = 0$ on the set $E \setminus N$, where $N = \cup_{j\geq 1}\{V_{E_j} < V_{E_j}^*\}$. The latter is called a *negligible set.* It follows from the local theory [5] together with Theorem 7.2 that N is $PSH(X, \omega)$-polar. Therefore $V_E^* \leq v \leq V_{E\setminus N}^* = V_E^*$ by (2).

Let K_j be a decreasing sequence of compact subsets and set $K = \cap_j K_j$. Clearly $\lim \nearrow V_{K_j} \leq V_K$. Fix $\varepsilon > 0$ and let $\varphi \in PSH(X, \omega)$ be such that $\varphi \leq 0$ on K. Then $\{\varphi < \varepsilon\}$ is an open set which contains all $K_j's$, for $j \geq j_\varepsilon$ large enough. Thus, $\varphi - \varepsilon \leq 0$ on K_j, hence $\varphi - \varepsilon \leq \lim \nearrow V_{K_j}$. Taking the supremum over all such φ's and letting $\varepsilon \to 0$ yields the reverse inequality $V_K \leq \lim \nearrow V_{K_j}$. The conclusion on the convergence of the upper semi-continuous regularizations follows now from Proposition 2.6.

It remains to prove (5). By Choquet's lemma, there exists an increasing sequence $\varphi_j \in PSH(X, \omega)$ such that $\varphi_j \leq 0$ on E and $V_E^* = (\sup_j \varphi_j)^*$. Set $G_j := \{\varphi_j < 1/j\}$. This defines a decreasing sequence of open subsets containing E. Observe that $\varphi_j - 1/j \leq V_{G_j} \leq V_E$, hence $\lim \varphi_j \leq \lim V_{G_j} \leq V_E$. Therefore $V_E^* = \lim V_{G_j}^*$. □

5.2. Alexander capacity

Definition 5.7. Let K be a Borel subset of X. We set

$$T_\omega(K) := \exp\left(-\sup_X V_{K,\omega}^*\right).$$

This capacity characterizes again $PSH(X, \omega)$-polar sets:

Proposition 5.8. *Let P be a Borel subset. Then $T_\omega(P) = 0$ iff P is $PSH(X, \omega)$-polar. Moreover, if $\varphi \in PSH(X, \omega)$ then*

$$T_\omega(\varphi < -t) \leq C_\varphi \exp(-t), \ \forall t \in \mathbb{R},$$

where $C_\varphi = \exp(-\sup_X \varphi)$.

Proof. The first assertion follows from Theorem 5.2. Let $\varphi \in PSH(X, \omega)$, $t \in \mathbb{R}$ and set $K_t = \{\varphi < -t\}$. Then $\varphi + t \leq 0$ on K_t hence $\varphi + t \leq V_{K_t,\omega}^*$. We infer $\sup_X \varphi + t \leq \sup_X V_{K_t,\omega}^*$ which yields $T_\omega(K_t) \leq \exp(-\sup_X \varphi)\exp(-t)$. □

The following proposition is an immediate consequence of Proposition 5.4. It shows that capacities T_ω, $T_{\omega'}$ are comparable if ω, ω' are both Kähler. Further they enjoy nice invariance properties.

Proposition 5.9.

(1) *For all Borel subsets* $K' \subset K \subset X$, $T_\omega(K') \le T_\omega(K) \le T_\omega(X) = 1$.

(2) *If* $\omega_1 \le \omega_2$ *then* $T_{\omega_1}(\cdot) \ge T_{\omega_2}(\cdot)$. *For all* $A > 0$, $T_{A\omega}(\cdot) = [T_\omega(\cdot)]^A$. *In particular, if* ω *and* ω' *are both Kähler then there exists* $C \ge 1$ *such that*

$$[T_\omega(\cdot)]^C \le T_{\omega'}(\cdot) \le [T_\omega(\cdot)]^{1/C} .$$

(3) *If* $\omega' = \omega + dd^c\chi$ *then*

$$\frac{1}{C} T_\omega(\cdot) \le T_{\omega'}(\cdot) \le C \cdot T_\omega(\cdot) ,$$

where $C = \exp(\sup_X \chi - \inf_X \chi) \ge 1$.

(4) *If* $f : X \to X$ *is a holomorphic map then* $T_{f^*\omega}(\cdot) \le T_\omega \circ f(\cdot)$. *In particular, if* f *is a* ω*-isometry then* $T_\omega \circ f = T_\omega$.

Remark 5.10. Following Zeriahi [40] one can prove that for all $\alpha < 2/\nu(X,\omega)$ there exists $C_\alpha > 0$ such that

$$\mathrm{Vol}_\omega(\cdot) \le C_\alpha T_\omega(\cdot)^\alpha ,$$

where $\nu(X,\omega) = \sup\{\nu(\varphi, x) \,/\, \varphi \in PSH(X,\omega), x \in X\}$ and $\nu(\varphi, x)$ denotes the Lelong number of φ at point x. In particular, it follows from Proposition 5.8 that $\forall \varphi \in PSH(X,\omega)$ with $\sup_X \varphi = 0$,

$$\mathrm{Vol}_\omega(\varphi < -t) \le C_\alpha \exp(-\alpha t), \ \forall t \in \mathbb{R} .$$

Such inequalities are quite useful in complex dynamics [20, 22] and in the study of the complex Monge-Ampère operator [28].

Example 5.11. Assume $X = \mathbb{CP}^n$, ω is the Fubini-Study Kähler form and B_R is the Euclidean ball centered at the origin and of radius R in a chart $\mathbb{C}^n \subset \mathbb{CP}^n$. We have explicitly computed the extremal function in this case (Example 5.5). This yields

$$T_\omega(B_R) = \frac{R}{\sqrt{1+R^2}} .$$

Observe that $T_\omega(B_R) \sim R$ as $R \to 0$. This shows the optimality of the rate of decreasing in Proposition 5.8.

The capacity T_ω in Example 5.11 has to be related to the capacity $T_{\mathbb{B}^n}$ which measures compact subsets of the unit ball $\mathbb{B}^n$ of $\mathbb{C}^n$. It is defined as follows: Given K a Borel subset of $\mathbb{C}^n$, $T_{\mathbb{B}^n}(K) := \exp(-\sup_{\mathbb{B}^n} L_K)$, where

$$L_K(z) = \sup\left\{v(z) \,/\, v \in \mathcal{L}(\mathbb{C}^n), \ \sup_K v \le 0\right\}$$

is the Siciak's extremal function of K and $\mathcal{L}(\mathbb{C}^n)$ denotes the Lelong class of psh functions with logarithmic growth in $\mathbb{C}^n$ (see Example 2.2). Let $\omega = \omega_{FS}$ denote the Fubini-Study Kähler form on $\mathbb{CP}^n$. One easily checks that

$$V_{K,\omega} - \log\sqrt{2} \le L_K - \frac{1}{2}\log\left[1 + |z|^2\right] \le V_{K,\omega} \text{ in } \mathbb{C}^n .$$

We infer straightforwardly $\sup_{\mathbb{B}^n} L_K \leq \log\sqrt{2} + \sup_{\mathbb{CP}^n} V_{K,\omega}$ hence $T_{\mathbb{B}^n}(K) \geq 2^{-1/2}T_\omega(K)$. We also have a reverse inequality. Indeed, $\forall\varphi \in PSH(\mathbb{CP}^n,\omega)$, $\sup_{\mathbb{CP}^n}\varphi \leq \sup_{\mathbb{B}^n}\varphi + C_1$, where $C_1 = \sup_{\mathbb{CP}^n} V_{\mathbb{B}^n,\omega} = \log\sqrt{2}$. Therefore

$$\sup_{\mathbb{CP}^n} V_{K,\omega} \leq \sup_{\mathbb{B}^n} V_{K,\omega} + \log\sqrt{2} \leq \sup_{\mathbb{B}^n} L_K + \log 2\ ,$$

which yields

$$\frac{1}{\sqrt{2}}T_\omega(K) \leq T_{\mathbb{B}^n}(K) \leq 2T_\omega(K)\ .$$

Example 5.12. Assume again $X = \mathbb{CP}^n$ and ω is the Fubini-Study Kähler form. Consider the totally real subspace $\mathbb{RP}^n$ of points with real coordinates (the closure of $\mathbb{R}^n \subset \mathbb{C}^n$ in $\mathbb{CP}^n$). Then

$$\frac{1}{2(1+\sqrt{2})} \leq T_\omega(\mathbb{RP}^n) \leq 1\ .$$

Indeed, set $B_{\mathbb{R}^n} := \mathbb{R}^n \cap \mathbb{B}^n$. It follows from the discussion above that

$$T_\omega(\mathbb{RP}^n) \geq \frac{1}{2}T_{\mathbb{B}^n}(\mathbb{B}_{\mathbb{R}^n})\ .$$

Now there is an explicit formula for $L^*_{B_{\mathbb{R}^n}}$ (Lundin's formula, see [27]),

$$L^*_{B_{\mathbb{R}^n}}(z) = \sup\left\{\log^+|h(<z,\xi>)|\,/\,||\xi|| = 1\right\},\ z \in \mathbb{C}^n\ ,$$

where $h(\zeta) = \zeta + \sqrt{\zeta^2 - 1}$. A simple computation yields $|h(\zeta)| \leq \log[|z| + \sqrt{|z|^2+1}]$ for $\zeta = <z,\xi>$ with $||\xi|| = 1$. We infer

$$L_{\mathbb{B}_{\mathbb{R}^n}}(z) \leq \log\left[|z| + \sqrt{|z|^2+1}\right] \leq \log\left[1+\sqrt{2}\right] \quad \text{in } \mathbb{B}^n\ ,$$

which yields the desired inequality.

Observe that the minorant is independent of the dimension n. This has been used recently in complex dynamics by Dinh and Sibony [18], who also considered this capacity.

Remark 5.13. It follows from Proposition 5.6 that T_ω is a generalized capacity in the sense of Choquet which is outer regular.

6. Tchebychev constants

In this section we consider the case where $\{\omega\} = c_1(L)$ is the first Chern class of a holomorphic line bundle L on X.

Recall that a holomorphic line bundle L on X is a family of complex lines $\{L_x\}_{x\in X}$ together with a structure of complex manifold of dimension $1 + \dim_{\mathbb{C}} X$ such that the projection map $\pi : L \to X$ taking L_x on x is holomorphic. Moreover, one can always locally trivialize L: There exists an open covering $\{\mathcal{U}_\alpha\}$ of X and biholomorphisms $\Phi_\alpha : \pi^{-1}(\mathcal{U}_\alpha) \to \mathcal{U}_\alpha \times \mathbb{C}$ which take $L_x = \pi^{-1}(x)$ isomorphically onto $\{x\} \times \mathbb{C}$. The line bundle L is then uniquely (i.e., up to isomorphism) determined by its transition functions $g_{\alpha\beta} \in \mathcal{O}^*(\mathcal{U}_{\alpha\beta})$, $\mathcal{U}_{\alpha\beta} := \mathcal{U}_\alpha \cap \mathcal{U}_\beta$, where

$$g_{\alpha\beta} := \left(\Phi_\alpha \circ \Phi_\beta^{-1}\right)_{|\{x\}\times\mathbb{C}}\ .$$

Note that the $g_{\alpha\beta}$'s satisfy the cocycle condition $g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} \equiv 1$, hence define a class $[\{g_{\alpha\beta}\}] \in H^1(X, \mathcal{O}^*)$. The first Chern class of L is the image $c_1(L) \in H^2(X, \mathbb{Z})$ of $[\{g_{\alpha\beta}\}]$ under the mapping $c_1 : H^1(X, \mathcal{O}^*) \to H^2(X, \mathbb{Z})$ induced by the exponential short exact sequence $0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$.

We let $\Gamma(X, L)$ denote the set of holomorphic sections of L on X: $s \in \Gamma(X, L)$ is a collection $s = \{s_\alpha\}$ of holomorphic functions s_α on $\mathcal{U}_\alpha$ satisfying the compatibility condition $s_\alpha = g_{\alpha\beta} s_\beta$ on $\mathcal{U}_{\alpha\beta}$. Similarly a (singular) metric ψ of L on X is a collection $\psi = \{\psi_\alpha\}$ of functions $\psi_\alpha \in L^1(\mathcal{U}_\alpha)$ satisfying $\psi_\alpha = \psi_\beta + \log|g_{\alpha\beta}|$ in $\mathcal{U}_{\alpha\beta}$. The metric is said to be *smooth* if the $\psi_\alpha' s$ are $\mathcal{C}^\infty$-smooth functions. A smooth metric always exists. The metric ψ is said to be *positive* if the ψ_α's are psh functions. In particular, if $s = \{s_\alpha\}$ is a holomorphic section of L on X, then $\psi = \{\psi_\alpha := \log|s_\alpha|\}$ is a positive (singular) metric of L on X. Note that we make here a slight abuse of terminology: Differential geometers usually call "metric' the nonnegative (usually smooth and nonvanishing) quantities $e^{-\psi} = \{e^{-\psi_\alpha}\}$.

Given a (singular) metric $\psi = \{\psi_\alpha\}$ of L on X, we consider its curvature $\Theta_\psi := dd^c \psi_\alpha$ in $\mathcal{U}_\alpha$. This yields a globally well-defined, real closed current on X since $dd^c \log|g_{\alpha\beta}| = 0$ in $\mathcal{U}_{\alpha\beta}$. It is a standard consequence of de Rham's isomorphism that this current represents the image of the first Chern class of L under the mapping $i : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{R})$ (induced by the inclusion $i : \mathbb{Z} \to \mathbb{R}$). The line bundle L is said to be *pseudoeffective* (resp. *positive*) if it admits a (singular) positive metric (resp. a smooth metric whose curvature is a Kähler form).

Fix $h = \{h_\alpha\}$ a smooth metric of L on X and set $\omega := \Theta_h$. Then $PSH(X, \omega)$ is in 1-to-1 correspondence with the set of positive singular metrics of L on X. Indeed, if ψ is such a metric then $\varphi := \psi - h$ is globally well defined on X and such that $dd^c\varphi \geq -\omega$. Conversely if $\varphi \in PSH(X, \omega)$ then $\psi = \{\psi_\alpha := \varphi + h_\alpha\}$ defines a positive singular metric of L on X. We can thus rephrase the pseudoeffectivity property as follows:

$$L \text{ is pseudoeffective} \iff PSH(X, \omega) \neq \emptyset\,.$$

Given L a pseudoeffective line bundle, it is interesting to know whether L admits a positive metric which is less singular than any another. This notion has been introduced in [16] and happens to be related to very special extremal functions.

Proposition 6.1. *Let (L, h) be a pseudoeffective line bundle on X equipped with a smooth metric h. Set $\omega := \Theta_h$. Then*

$$h_{\min} := h + V_{X,\omega}^*$$

is a positive singular metric of L on X with "minimal singularities." More precisely, if ψ is a positive singular metric of L on X, then there exists a constant C_ψ such that $\psi \leq h_{\min} + C_\psi$.

Proof. Let ψ be a positive singular metric of L on X. Then $\psi - h$ is a globally well defined ω-psh function. It is u.s.c. hence bounded from above on X: We let C_ψ denotes its maximum. Then $\psi - h - C_\psi \leq 0$ on X, hence $\psi - h \leq V_{X,\omega}^* + C_\psi$, which yields $\psi \leq h_{\min} + C_\psi$. □

In the sequel we assume L is positive and h has been chosen so that $\omega := \Theta_h$ is a Kähler form. For $s \in \Gamma(X, L^N)$, we let $||s||_{Nh}$ denote the norm of s computed with respect to the metric Nh: It is defined in $\mathcal{U}_\alpha$ by $||s||_{Nh} := |s_\alpha| e^{-Nh_\alpha}$. The definition is independent of α thanks to the compatibility conditions.

For a given Borel subset K of X, we define its *Tchebychev constants*

$$M_{N\omega}(K) := \inf\left\{ \sup_K ||s||_{Nh} \,/\, s \in \Gamma\big(X, L^N\big),\ \sup_X ||s||_{Nh} = 1 \right\}.$$

Note that an obvious rescaling argument shows that $M_{d\omega}$ remains unchanged if we replace h by $h + C$ so that it really depends on $\omega = \Theta_h$ rather than on h. Consider

$$T'_\omega(K) := \inf_{N \geq 1} [M_{N\omega}(K)]^{1/N} .$$

Theorem 6.2. *Let K be a compact subset of X. Then*

$$T_\omega(K) = T'_\omega(K) .$$

Proof. The core of the proof consists in showing that

$$V_{K,\omega}(x) = \sup \left\{ \frac{1}{N} \log ||s||_{Nh}(x) \,/\, N \geq 1, s \in \Gamma\big(X, L^N\big) \text{ and } \sup_K ||s||_{Nh} \leq 1 \right\} .$$

Note that for any of the sections s involved in the supremum, $\varphi := N^{-1} \log ||s||_{Nh}$ belongs to $PSH(X, \omega)$ and satisfies $\varphi \leq 0$ on K. Therefore $\varphi \leq V_{K,\omega}$.

Conversely, fix $x_0 \in X$ and $a < V_{K,\omega}(x_0)$. Fix $\varphi \in PSH(X, \omega)$ such that $\sup_K \varphi \leq 0$ and $\varphi(x_0) > a$. Regularizing φ (see Appendix) and translating, we can assume $\varphi \in PSH(X, \omega) \cap C^\infty(X)$, $\sup_K \varphi < 0$ and $\varphi(x_0) > a$. Fix $\varepsilon > 0$. Let $B = B(x_0, r)$ be a small ball on which $\varphi > a$. We choose B so small that the oscillation of h is smaller than ε on B. Let χ be a test function with compact support in B and such that $\chi \equiv 1$ in $B(x_0, r/2)$. We can assume w.l.o.g. that $B \subset \mathcal{U}_{\alpha_0}$ for some α_0 but $B \cap \mathcal{U}_\beta = \emptyset$ for all $\beta \neq \alpha_0$. This insures that χ is a smooth section of L^N for all $N \geq 1$.

Let ψ_1 be a smooth positive metric of $L^{N_1} \otimes K_X^*$ on X (this is possible if N_1 is chosen large enough since L is positive). Let ψ_2 be a positive metric of L^{N_2} on X which is smooth in $X \setminus \{x_0\}$ and with Lelong number $\nu(\psi_2, x_0) \geq n = \dim_{\mathbb{C}} X$ (this is again possible if N_2 is large enough, since L is ample). Observe that $\bar\partial\chi$ is a smooth $\bar\partial$-closed (0, 1)-form with values in L^N (for all $N \geq 1$). Alternatively it is a smooth $\bar\partial$-closed $(n, 1)$-form with values in $L^N \otimes K_X^*$. Applying Hörmander's L^2-estimates (see e.g., [15, Ch. VIII]) with weight $\psi_N := (N - N_1 - N_2)(\varphi + h) + \psi_1 + \psi_2$, we find a smooth section f of L^N such that $\bar\partial f = \bar\partial\chi$ and

$$\int_X |f|^2 e^{-2(N-N_1-N_2)(\varphi+h)-2\psi_1-2\psi_2} \, dV_\omega \leq C_1 \int_X |\bar\partial\chi|^2 e^{-2\psi_N} \, dV_\omega .$$

Note that $\bar\partial\chi$ has support in $B \setminus B(x_0, r/2)$ where ψ_N is smooth so that both integrals are finite. Since $\nu(\psi_2, x_0) \geq n$, this forces $f(x_0) = 0$. The second integral is actually bounded from above by $C_2 e^{-2N(a-\varepsilon)}$, where C_2 is independent of N, since $-\varphi < -a$ on B and the oscillation of h is smaller than ε on B. Therefore $s := \chi - f \in \Gamma(X, L^N)$ satisfies $s(x_0) = 1$ and

$$\int_X |s|^2 e^{-2N(\varphi+h)} \, dV_\omega \leq C_3 e^{-2N(a-\varepsilon)} ,$$

where C_3 is independent of N. Now $\varphi < 0$ in a neighborhood of K, so the mean-value inequality applied to the subharmonic functions $|s_\alpha|^2$ yields for all x in K,

$$\begin{aligned} |s|^2 e^{-2Nh}(x) &\leq C_\delta \int_{B(x,\delta)} |s|^2(y) e^{-2N[\varphi+h](y)} e^{2N[h(y)-h(x)+\varphi(y)]} \, d\lambda(y) \\ &\leq C_4 e^{-2N(a-\varepsilon)} \end{aligned}$$

if δ is so small that $|\sup_{B(x,\delta)} \varphi| > 0$ is bigger than the oscillation of h on $B(x, \delta)$. Therefore $S := C_4^{-1/2} e^{N(a-\varepsilon)} s \in \Gamma(X, L^N)$ satisfies $\sup_K ||S||_{Nh} \leq 1$ and $N^{-1} \log ||S||_{Nh}(x_0) \geq a - \varepsilon - \frac{\log C_4}{2N}$. Letting $N \to +\infty$, $\varepsilon \to 0$ and $a \to V_{K,\omega}(x_0)$ completes the proof of the equality.

To conclude observe that by rescaling one gets

$$\begin{aligned}-\log T_\omega(K) &= \sup_X V_{K,\omega}\\ &= \sup\left\{\frac{1}{N}\sup_X \log ||S||_{Nh} \,/\, N \geq 1, S \in \Gamma\big(X, L^N\big) \text{ and } \sup_K ||S||_{Nh} = 1\right\}\\ &= \sup\left\{-\frac{1}{N}\sup_K \log ||S||_{Nh} \,/\, N \geq 1, S \in \Gamma\big(X, L^N\big) \text{ and } \sup_X ||S||_{Nh} = 1\right\}\\ &= -\log T'_\omega(K)\,.\end{aligned}$$

□

Projective capacity. We assume here that $X = \mathbb{CP}^n$ is the complex projective space and $\omega = \omega_{FS}$ is the Fubini-Study Kähler form. We give in this context a geometrical interpretation of the capacity T_ω. This will shed some light on the notion of projective capacity introduced by Alexander [1].

Let $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ denote the canonical projection map. We let $\mathbb{B}^{n+1}$ denote the unit ball in $\mathbb{C}^{n+1}$. Recall that the polynomially convex hull $\widehat{F}$ of a compact set F of $\mathbb{C}^{n+1}$ is defined as $\widehat{F} := \{x \in \mathbb{C}^{n+1} \,/\, |P(x)| \leq \sup_F |P|,\ \forall P \text{polynomial}\}$.

The following result gives an interesting interpretation of the capacity T_ω.

Theorem 6.3. *Let K be a compact subset of $\mathbb{CP}^n$. Then*

$$T_\omega(K) = \sup\left\{r > 0 \,/\, r\mathbb{B}^{n+1} \subset \widehat{K_0}\right\},$$

where $K_0 = \pi^{-1}(K) \cap \partial\mathbb{B}^{n+1}$.

Proof. Let K, K_0 be as in the theorem. Observe that K_0 is a circled subset of $\partial\mathbb{B}^{n+1}$: If $z \in K_0$ then $e^{i\theta} z \in K_0, \forall\theta \in [0, 2\pi]$. For such compacts, the polynomial hull $\widehat{K_0}$ coincides with the "homogeneous polynomial hull,"

$$\widehat{K_0}^h := \left\{x \in \mathbb{C}^{n+1} \,/\, |P(x)| \leq \sup_F |P|,\ \forall P \text{ homogeneous polynomial}\right\}.$$

Indeed, one inclusion $\widehat{K_0} \subset \widehat{K_0}^h$ is clear, so assume $z_0 \in \widehat{K_0}^h$. Let $P = \sum_{j=0}^d P_j$ be a polynomial of degree d decomposed into its homogeneous components. Observe that $P_j(x) = (2\pi)^{-1}\int_0^{2\pi} P(e^{i\theta}x)e^{-ij\theta}d\theta$. Therefore $\sup_{K_0}|P_j| \leq \sup_{K_0}|P|$ since K_0 is circled. Fix $t \in]0, 1[$. Then

$$|P(tz_0)| \leq \sum_{j=0}^d t^j |P_j(z_0)| \leq \frac{1}{1-t}\sup_{K_0}|P|\,.$$

We infer $tz_0 \in \widehat{K_0}$. Letting $t \to 1^-$ and using that K_0 is closed we get $z_0 \in \widehat{K_0}$, whence $\widehat{K_0} = \widehat{K_0}^h$.

Fix now $z \in \mathbb{C}^{n+1}$ such that $||z|| \leq T_\omega(K)$. Let P be a homogeneous polynomial of degree d. Then

$$|P(z)| = ||z||^d \left|P\left(\frac{z}{||z||}\right)\right| \leq T_\omega(K)^d \sup_{\partial\mathbb{B}^{n+1}} |P|\,. \tag{6.1}$$

Now set $\psi(z) = d^{-1}\log|P(z)| - \log||z||$ and $\varphi = \psi - \sup_K \psi$. Then $\varphi \in PSH(X,\omega)$ with $\sup_K \varphi \leq 0$ hence $\varphi \leq V_{K,\omega}$. Therefore

$$T_\omega(K)^d \leq \exp\Big(-d\sup_{\mathbb{CP}^n}\varphi\Big) = \frac{\sup_{K_0}|P|}{\sup_{\partial\mathbb{B}^{n+1}}|P|}\,.$$

Together with (6.1) this yields $|P(z)| \leq \sup_{K_0}|P|$ hence $z \in \widehat{K_0}^h = \widehat{K_0}$. Thus, K_0 contains the ball centered at the origin of radius $T_\omega(K)$.

Conversely since $T_\omega(K) = T'_\omega(K)$ (Theorem 5.1), one can find homogeneous polynomials P_j of degree d_j such that $\sup_{\partial\mathbb{B}^{n+1}}|P_j|^{-1/d_j}\cdot\sup_{K_0}|P_j|^{1/d_j} \to T_\omega(K)$. Assume $r\mathbb{B}^{n+1} \subset \widehat{K_0}$. Then

$$r^{d_j}\sup_{\partial\mathbb{B}^{n+1}}|P_j| = \sup_{r\mathbb{B}^{n+1}}|P_j| \leq \sup_{K_0}|P_j|$$

yields $r \leq T_\omega(K)$. □

Remark 6.4. Sibony and Wong [34] have been first in showing that if a compact subset K of $\mathbb{CP}^n$ is large enough then the polynomial hull of K_0 contains a full neighborhood of the origin in $\mathbb{C}^{n+1}$. They used the (complicated) notion of Γ-capacity. Their approach has been simplified by Alexander [1] who introduced a projective capacity which is comparable to T_ω (see Theorem 4.4 in [1]). The proof given above is essentially Alexander's (see also Theorem 4.3 in [36]).

This result has been used recently in complex dynamics (see [17, 23]).

Further capacities. In our definition of Chebyshev constants we have normalized holomorphic sections $s \in \Gamma(X, L^N)$ by requiring $\sup_X ||s||_{Nh} = 1$. Given μ a probability measure such that $PSH(X,\omega) \subset L^1(\mu)$ and $A \in \mathbb{R}$, we could as well consider

$$M^{\mu,A}_{N,\omega}(K) := \left\{\sup_K ||s||_{Nh} \,/\, s \in \Gamma\big(X, L^N\big),\ \int_X \log||s||_{Nh}\,d\mu = A\right\}.$$

This normalization has the following pleasant property: If $s \in \Gamma(X, L^N)$ and $s' \in \Gamma(X, L^{N'})$ are so normalized then $s\cdot s' \in \Gamma(X, L^{N+N'})$ again satisfies $\int_X \log||ss'||_{(N+N')h}\,d\mu = A$. We infer $M^{\mu,A}_{N+N',\omega} \leq M^{\mu,A}_{N,\omega}\cdot M^{\mu,A}_{N',\omega}$ so that

$$T^{\mu,A}_\omega(K) := \inf_{N\geq 1}\big[M^{\mu,A}_{N,\omega}(K)\big]^{1/N} = \lim_{N\to+\infty}\big[M^{\mu,A}_{N,\omega}(K)\big]^{1/N}\,.$$

This yields a whole family of capacities which are all comparable to T_ω thanks to Proposition 2.7: There exists $C = C(\mu, A) \geq 1$ such that

$$\frac{1}{C}T_\omega(\cdot) \leq T^{\mu,A}_\omega(\cdot) \leq CT_\omega(\cdot)\,.$$

The *projective capacity* of Alexander [1] is precisely $T^{\mu,A}_\omega$ for $X = \mathbb{CP}^n$, $\omega = \omega_{FS}$, $\mu = \omega^n$ and $A = \int_{\mathbb{CP}^n}(\log|z_n| - \log||(z_0,\dots,z_n)||)\,\omega^n([z])$.

7. Comparison of capacities and applications

7.1. Josefson's theorem

In this section we assume that ω is Kähler and normalized by $\mathrm{Vol}_\omega(X) = 1$. We first prove inequalities relating T_ω and Cap_ω. Then we prove (Theorem 7.2) a quantitative version of Josefson's theorem that every locally pluripolar set is actually $PSH(X,\omega)$-polar. In the local theory this result is due to El Mir [19]. We follow the approach of Alexander-Taylor [2].

Proposition 7.1. *There exists $A>0$ s.t. for all compact subsets K of X,*

$$\exp\left[-\frac{A}{\mathrm{Cap}_\omega(K)}\right] \leq T_\omega(K) \leq e\cdot\exp\left[-\frac{1}{\mathrm{Cap}_\omega(K)^{1/n}}\right].$$

Proof. Set $M_K=\sup_X V_{K,\omega}$. If $M_K=+\infty$ then K is $PSH(X,\omega)$-polar (Theorem 5.2) and there is nothing to prove: $T_\omega(K)=\mathrm{Cap}_\omega(K)=0$. So we assume in the sequel $M_K<+\infty$ hence $V^*_{K,\omega}\in PSH(X,\omega)$. If $M_K\geq 1$ then $u_K:=M_K^{-1}V^*_{K,\omega}\in PSH(X,\omega)$ with $0\leq u_K\leq 1$ on X. Since $\omega_{V^*_{K,\omega}}\leq M_K\omega_{u_K}$, we get

$$\frac{1}{M_K^n}=\frac{1}{M_K^n}\int_K\left(\omega_{V^*_{K,\omega}}\right)^n\leq\int_K(\omega_{u_K})^n\leq\mathrm{Cap}_\omega(K)$$

whence $T_\omega(K)\leq\exp(-\mathrm{Cap}_\omega(K)^{-1/n})$.

If $0\leq M_K\leq 1$ then $0\leq V^*_{K,\omega}\leq 1$ hence $V_{K,\omega}-1$ coincides with the relative extremal function $h_{K,\omega}$ (see Proposition 3.14). We infer

$$1=\int_K\left(\omega_{V^*_{K,\omega}}\right)^n\leq\mathrm{Cap}_\omega(K)\leq\mathrm{Cap}_\omega(X)=1\,,$$

while $T_\omega(K)\leq T_\omega(X)=1$. Thus, in both cases $T_\omega(K)\leq e\cdot\exp(\mathrm{Cap}_\omega(K)^{-1/n})$.

We now prove the reverse inequality. We can assume $M_K\geq 1$, otherwise it is sufficient to adjust the value of A. Let $\varphi\in PSH(X,\omega)$ be such that $\varphi\leq 0$ on K. Then $\varphi\leq M_K$ on X, hence $w:=M_K^{-1}(\varphi-M_K)\in PSH(X,\omega)$ satisfies $\sup_X w\leq 0$ and $w\leq -1$ on K. We infer $w\leq h^*_{K,\omega}$, hence

$$w_K:=\frac{V^*_{K,\omega}-M_K}{M_K}\leq h^*_{K,\omega}\leq 0\,.$$

Now $\sup_X(V^*_{K,\omega}-M_K)=0$, so it follows from Proposition 2.7 that $\int_X|V^*_{K,\omega}-M_K|\omega^n\leq C_1$ for some constant $C_1>0$ independent of K. We infer

$$\begin{aligned}\mathrm{Cap}_\omega(K)=\int_K\left(\omega_{h^*_{K,\omega}}\right)^n &\leq \int_X\left[-h^*_{K,\omega}\right]\left(\omega_{h^*_{K,\omega}}\right)^n\\ &\leq \frac{1}{M_K}\int_X-\left(V^*_{K,\omega}-M_K\right)\left(\omega_{h^*_{K,\omega}}\right)^n\leq\frac{C_2}{M_K}\,,\end{aligned}$$

using Corollary 3.3 and the fact that $h^*_{K,\omega}=-1$ on K, except perhaps on a pluripolar set which has zero $(\omega_{h^*_{K,\omega}})^n$-measure. This yields the desired inequality. □

It follows from the previous proposition and Corollary 3.8 that ω-psh functions are quasi-continuous with respect to the capacity T_ω.

Theorem 7.2. *Locally pluripolar sets are $PSH(X,\omega)$-polar.*

Proof. More precisely, we are going to show the following: Consider Ω an open subset of X, $v\in PSH^-(\Omega)$ and $P\subset\{v=-\infty\}$. Fix $0<\varepsilon<1/n$ and $V_t:=V_{G_t,\omega}$ where $G_t=\{x\in\Omega\,/\,v(x)<-t\}$. Then

$$\varphi_\varepsilon(x):=\frac{1}{\varepsilon}\int_1^{+\infty}\frac{1}{t^{1+\varepsilon}}\left[V_t(x)-\sup_X V_t\right]dt$$

is a ω-psh function such that $P \subset \{\varphi_\varepsilon = -\infty\}$.

Indeed, since G_t is open, we have $V_t \in PSH(X, \omega)$ and $V_t = 0$ on G_t (see Proposition 5.6). Observe that φ_ε is a sum of negative ω-psh functions hence it is either identically $-\infty$ or a well defined $A\omega$-psh function with $A = \varepsilon^{-1}\int_1^{+\infty} t^{-(1+\varepsilon)}\,dt = 1$. Recall that $-C + \sup_X V_t \leq \int_X V_t\omega^n \leq \sup_X V_t$ (Proposition 2.7). Therefore $\int_X \varphi_\varepsilon\omega^n \geq -C$ hence $\varphi_\varepsilon \in PSH(X, \omega)$.

Fix $x \in \Omega$ such that $v(x) < -1$. Observe that $V_t - \sup_X V_t \leq 0$ with $V_t(x) = 0$ if $x \in G_t$, i.e., when $|v(x)| > t$. Therefore

$$\varphi_\varepsilon(x) \leq -\frac{1}{\varepsilon}\int_1^{|v(x)|} \frac{\sup_X V_t}{t^{1+\varepsilon}}\,dt\,.$$

Recall now that Cap_ω is always dominated by Cap_{BT} hence $\mathrm{Cap}_\omega(G_t) \leq C_1/t < 1$ if t is large enough. We infer from the previous proposition that

$$-\sup_X V_t \leq -[\mathrm{Cap}_\omega(G_t)]^{-1/n} \leq -C_2 t^{1/n}\,,$$

which yields

$$\varphi_\varepsilon(x) \leq \frac{-C_2}{\varepsilon}\int_1^{|v(x)|} \frac{dt}{t^{1+\varepsilon-1/n}} \leq -C_3|v(x)|^{1/n-\varepsilon} + C_4\,.$$

Note that $\varphi_\varepsilon(x) = -\infty$ whenever $v(x) = -\infty$ hence $P \subset \{\varphi_\varepsilon = -\infty\}$. □

7.2. Dynamical capacity estimates

Let $f : \mathbb{CP}^n \to \mathbb{CP}^n$ be an holomorphic endomorphism. We let ω denote again the Fubini-Study Kähler form. Then $f^*\omega$ is a smooth positive closed $(1, 1)$-form of mass $\lambda = \int_{\mathbb{CP}^n} f^*\omega \wedge \omega^{n-1} =:$ The *first algebraic degree of f*. Thus, $\lambda^{-1} f^*\omega = \omega + dd^c\varphi$, where φ is a smooth ω-psh function on $\mathbb{CP}^n$. Iterating this functional equation yields

$$\frac{1}{\lambda^j}(f^j)^*\omega = \omega + dd^c g_j,\ g_j = \sum_{l=0}^{j-1}\frac{1}{\lambda^l}\varphi \circ f^l\,.$$

We assume $\lambda \geq 2$. Thus, the sequence (g_j) uniformly converges on $\mathbb{CP}^n$ towards a continuous function $g_f \in PSH(X, \omega)$ called the *Green function* of f. We refer the interested reader to [33] for a detailed study of the properties of the *Green current* $T_f = \omega + dd^c g_f$.

Dynamical volume estimates have revealed quite useful in establishing ergodic properties of the Green current T_f (see [20, 22] and references therein). We establish herebelow very simple dynamical capacity estimates and show how to derive from them dynamical volume estimates.

Proposition 7.3. *There exists $0 < \alpha < 1$ such that for all Borel subsets K of X, for all $j \in \mathbb{N}$,*

$$[\alpha T_\omega(K)]^{\lambda^j} \leq T_\omega\big(f^j(K)\big)\,.$$

Proof. This follows straightforwardly from Proposition 5.9:

$$T_\omega\big(f^j(K)\big) \geq T_{(f^j)^*\omega}(K) = \left[T_{\lambda^{-j}(f^j)^*\omega}(K)\right]^{\lambda^j} \geq [\alpha T_\omega(K)]^{\lambda^j}\,,$$

where the first two inequalities follow from Proposition 5.9 (4) and 5.9 (2) and last one follows from Proposition 5.9 (3) and the fact that $\lambda^{-j}(f^j)^*\omega = \omega + dd^c g_j$, where g_j is uniformly bounded. □

Corollary 7.4. *Let $\varphi \in PSH(X, \omega)$. Then the sequence $(\lambda^{-j}\varphi \circ f^j)$ is relatively compact in $L^1(\mathbb{CP}^n)$.*

Proof. Set $\varphi_j = \lambda^{-j}\varphi \circ f^j$. Observe that φ_j is uniformly bounded from above and that $\varphi_j + g_j \in PSH(\mathbb{CP}^n, \omega)$. It follows from Proposition 2.6 that either φ_j converges uniformly towards $-\infty$ or it is relatively compact in $L^1(\mathbb{CP}^n)$. It is sufficient to show that for $A > 0$ large enough, $\overline{\lim}_{j\to+\infty} T_\omega(\varphi_j < -A) < T_\omega(X) = 1$. Observe that $f^j(\varphi_j < -A) = \{\varphi < -A\lambda^j\}$. Therefore

$$\left[\alpha T_\omega(\varphi_j < -A)\right]^{\lambda^j} \leq T_\omega\big(\varphi < -A\lambda^j\big) \leq C \exp\big(-A\lambda^j\big),$$

where the last inequality follows from Proposition 5.8. We infer

$$\overline{\lim}_{j\to+\infty} T_\omega(\varphi_j < -A) \leq \frac{1}{\alpha}\exp(-A) < 1$$

for $A > -\log\alpha$ large enough. □

Corollary 7.5. *There exists $C > 0$ such that for all Borel subset K of $\mathbb{CP}^n$ and for all $j \in \mathbb{N}$,*

$$\mathrm{Vol}_\omega\left(f^j(K)\right) \geq \exp\left(-\frac{C\lambda^j}{\mathrm{Vol}_\omega(K)}\right).$$

In other words the volume of a given set can not decrease too fast under iteration. Such volume estimates are used in complex dynamics to prove fine convergence results towards the Green current T_f (see [20, 22]). One may hope that dynamical capacity estimates will allow to establish convergence results in higher codimension.

Proof. By the change of variables formula one gets

$$\mathrm{Vol}_\omega\left(f^j K\right) = \int_{f^j K} \omega^n \geq \frac{1}{d_t^j}\int_K (f^j)^*\omega^n = \frac{1}{d_t^j}\int_K \left|J_{FS}(f^j)\right|^2\omega^n,$$

where $d_t = \lambda^n$ denotes the topological degree of f and $J_{FS}(f)$ stands for the jacobian of f with respect to the Fubini-Study volume form. Observe that $\log|J_{FS}(f)| = u - v$ is a difference of two qpsh functions $u, v \in PSH(X, A\omega)$ for some $A = A(\lambda, f)$. Moreover, by the chain rule,

$$\frac{1}{\lambda^j}\log\left|J_{FS}(f^j)\right| = \sum_{l=0}^{j-1}\frac{1}{\lambda^j}\log\left|J_{FS}(f)\circ f^l\right|.$$

Since $\lambda^{-l}\log|J_{FS}(f)\circ f^l|$ is relatively compact in $L^1(\mathbb{CP}^n)$ (previous corollary), the concavity of the log yields

$$\begin{aligned}\frac{1}{\mathrm{Vol}_\omega(K)}\int_K \left|J_{FS}(f^j)\right|^2\omega^n &\geq \exp\left(\frac{2\lambda^j}{\mathrm{Vol}_\omega(K)}\int_K \frac{1}{\lambda^j}\log\left|J_{FS}(f^j)\right|\omega^n\right)\\ &\geq \exp\left(-\frac{C_1\lambda^j}{\mathrm{Vol}_\omega(K)}\right).\end{aligned}$$

The conclusion follows by observing that $\alpha\exp(-x/\alpha) \geq \exp(-2x/\alpha)$, for all $\alpha > 0$ and all $x \geq 1/e$. □

8. Appendix: Regularization of qpsh functions

It is well-known that every psh function φ can be *locally regularized,* i.e., one can find *locally* a sequence φ_j of smooth psh functions which decrease towards φ (see e.g., [15, Ch. 1]). Similarly one can always locally regularize ω-psh functions. It is interesting to know whether one can also *globally regularize* ω-psh functions.

When X is a complex homogeneous manifold (i.e., when $\mathrm{Aut}(X)$ acts transitively on X), it is possible to approximate any ω-psh function by a decreasing sequence of smooth ω-psh functions (see [21, 25]). In general however there is a loss of positivity: It will be possible to approximate $\varphi \in PSH(X, \omega)$ by a decreasing sequence of smooth functions φ_j but the curvature forms $dd^c\varphi_j$ will have to be more negative than $-\omega$. How negative depends on the positivity of the cohomology class $\{\omega\}$.

Consider e.g., $\pi : X \to \mathbb{P}^2$ the blow up of $\mathbb{P}^2$ at point p, $E = \pi^{-1}(p)$ the exceptional divisor and let $\omega = [E]$ be the current of integration along E. Then $PSH(X, \omega) \simeq \mathbb{R}$ (see Remark 2.5) so every psh function has logarithmic singularities along E, hence is not smooth. Alternatively E has self-intersection -1 so its cohomology class cannot be represented by smooth nonnegative forms, not even by smooth forms with (very) small negativity.

Following Demailly's fundamental work [10, 12, 16] (to cite a few) we show herebelow that regularization with no loss of positivity is possible when ω is a Hodge form (i.e., a Kähler form with integer class). This yields a "simple" regularization process when X is projective. We would like to mention that Demailly has produced over the last twenty years much finer regularization results. We nevertheless think it is worth including a proof, since it is far less technical than Demailly's more general results (although our proof heavily relies on his ideas). We thank P. Eyssidieux for his helpful contribution regarding that matter.

Theorem 8.1. *Let $L \to X$ be a positive holomorphic line bundle equipped with a smooth strictly positive metric h, and set $\omega := \Theta_h > 0$.*

Then for every $\varphi \in PSH(X, \omega)$, there exists a sequence $\varphi_j \in PSH(X, \omega) \cap C^\infty(X)$ such that φ_j decreases towards φ.

Proof. Let $\varphi \in PSH(X, \omega)$. We can assume w.l.o.g. that $\varphi \leq 0$ on X. Let $\psi = \{\psi_\alpha := \varphi + h_\alpha \in PSH(\mathcal{U}_\alpha)\}$ denote the associated (singular) positive metric of L on X, where $\{\mathcal{U}_\alpha\}$ denotes an open over of X trivializing L (see Section 5).

Step 1. We consider the following Bergman spaces

$$\mathcal{H}_{j,j_0} := \left\{ s \in \Gamma\left(X, L^j\right) / \int_X |s|^2 e^{-2h_{j,j_0}}\, dV_\omega < +\infty \right\} ,$$

where $h_{j,j_0} = (j - j_0)\psi + j_0 h$, j_0 a fixed large integer (to be specified later). Let $\sigma_1^{(j,j_0)}, \dots,$ $\sigma_{s_j}^{(j,j_0)}$ be an orthonormal basis of $\mathcal{H}_{j,j_0}$ and set

$$\psi_{j,j_0} := \frac{1}{2j} \log \left[\sum_{l=1}^{s_j} |\sigma_l^{(j,j_0)}|^2 \right] = \frac{1}{2j} \sup_{s \in B_{j,j_0}} \log |s|^2 ,$$

where B_{j,j_0} denotes the unit ball of radius 1 centered at 0 in $\mathcal{H}_{j,j_0}$. Clearly ψ_{j,j_0} defines a positive (singular) metric of L on X, equivalently $\varphi_{j,j_0} := \psi_{j,j_0} - h \in PSH(X, \omega)$. If $x \in \mathcal{U}_\alpha$

and $s = \{s_\alpha\} \in \mathcal{H}_{j,j_0}$, then $|s_\alpha|^2$ is subharmonic in $\mathcal{U}_\alpha$ hence

$$|s_\alpha(x)|^2 \leq \frac{C_1}{r^{2n}} \int_{B(x,r)} |s_\alpha(x)|^2 \leq \frac{C_2}{r^{2n}} e^{2\sup_{B(x,r)} h_{j,j_0}} \int_X |s|^2 e^{-2h_{j,j_0}} \, dV_\omega ,$$

where $r > 0$ is so small that $B(x,r) \subset \mathcal{U}_\alpha$. We infer

$$\varphi_{j,j_0}(x) \leq (1 - j_0/j) \sup_{B(x,r)} \varphi + \frac{C_3 - n\log r}{j} . \tag{8.1}$$

There is also a reverse inequality which uses a deep extension result of Ohsawa-Takegoshi-Manivel (see [14]): There exists $j_0 \in \mathbb{N}$ and $C_4 > 0$ large enough so that $\forall x \in X, \forall j \in \mathbb{N}$, there exists $s \in \Gamma(X, L^j)$ with

$$\int_X |s|^2 e^{-2h_{j,j_0}} \, dV_\omega \leq C_4 |s(x)|^2 e^{-2h_{j,j_0}(x)} .$$

Choose s so that the right-hand side is equal to 1, hence $s \in B_{j,j_0}$. Then

$$\psi_{j,j_0}(x) \geq \frac{1}{2j} \log |s(x)|^2 = \left(1 - \frac{j_0}{j}\right) \psi(x) + \frac{j}{j_0} h(x) - \frac{\log C_4}{2j} .$$

We infer

$$\varphi_{j,j_0}(x) \geq \left(1 - \frac{j_0}{j}\right) \varphi(x) - \frac{\log C_4}{2j} \geq \varphi(x) - \frac{\log C_4}{2j} \tag{8.2}$$

since $\varphi \leq 0$ on X. It follows from (8.1) and (8.2) that $\varphi_j \to \varphi$ in $L^1(X)$.

Step 2. We now show, following [16], that $(\varphi_{j,j_0})_j$ is almost subadditive. Let $s \in \Gamma(X, L^{j_1+j_2})$ with

$$\int_X |s|^2 e^{-2h_{j_1+j_2,j_0}} \, dV_\omega \leq 1 .$$

We may view s as the restriction to the diagonal Δ of $X \times X$ of a section $S \in \Gamma(X \times X, L_1^{j_1} \otimes L_2^{j_2})$, where $L_i = \pi_i^* L$ and $\pi : X \times X \to X$ denotes the projection onto the i^{th} factor, $i = 1, 2$. Consider the Bergman spaces

$$\mathcal{H}_{j_1,j_2,j_0} := \Big\{ S \in \Gamma\big(X \times X, L_1^{j_1} \otimes L_2^{j_2}\big) \, / \\ \int_{X\times X} |S|^2 e^{-2h_{j_1,j_0/2}(x) - 2h_{j_2,j_0/2}(y)} \, dV_{\omega_1}(x) \, dV_{\omega_2}(y) < +\infty \Big\} ,$$

where $\omega_i = \pi^* \omega$. It follows from the Ohsawa-Takegoshi-Manivel L^2-extension theorem [16] that there exists $S \in \Gamma(X \times X, L_1^{j_1} \otimes L_2^{j_2})$ such that $S_{|\Delta} = s$ and

$$\int_{X\times X} |S|^2 e^{-2h_{j_1,j_0/2} - 2h_{j_2,j_0/2}} \, dV_{\omega_1} \, dV_{\omega_2} \leq C_5 \int_X |s|^2 e^{-2h_{j_1+j_2,j_0}} \, dV_\omega \leq C_5 ,$$

where C_5 only depends on the dimension $n = \dim_{\mathbb{C}} X$. Observe that $\{\sigma_{l_1}^{(j_1,j_0/2)}(x) \cdot \sigma_{l_1}^{(j_1,j_0/2)}(y)\}_{l_1,l_2}$ forms an orthonormal basis of $\mathcal{H}_{j_1,j_2,j_0}$, thus

$$S(x,y) = \sum_{l_1,l_2} c_{l_1,l_2} \sigma_{l_1}^{(j_1,j_0/2)}(x) \sigma_{l_2}^{(j_2,j_0/2)}(y)$$

with $\sum |c_{l_1,l_2}|^2 \le C_5$. It follows therefore from Cauchy-Schwarz inequality that

$$|s(x)|^2 = |S(x,x)|^2 \le C_5 \sum_{l_1} |\sigma_{l_1}^{(j_1,j_0/2)}(x)|^2 \sum_{l_2} |\sigma_{l_2}^{(j_2,j_0/2)}(y)|^2 ,$$

which yields

$$\varphi_{j_1+j_2,j_0} \le \frac{\log C_5}{2(j_1+j_2)} + \frac{j_1}{j_1+j_2}\varphi_{j_1,j_0/2} + \frac{j_2}{j_1+j_2}\varphi_{j_2,j_0/2} .$$

Note, finally, that $\varphi_{j,j_0/2} \le \varphi_{j,j_0}$ since $\varphi = \psi - h \le 0$, therefore $\hat{\varphi}_j := \varphi_{2^j,j_0} + 2^{-j-2}\log C_5$ is decreasing.

Step 3. It remains to make $\hat{\varphi}_j$ smooth. Indeed, it has all the other required properties: It is decreasing and by Step 1 we have for all $x \in X$,

$$\varphi(x) \le \hat{\varphi}_j(x) \le \left(1 - j_0 2^{-j}\right) \sup_{B(x,r)} \varphi + \frac{C_6 - n\log r}{2^j} , \tag{8.3}$$

so that $\hat{\varphi}_j \to \varphi$. Let $\sigma_{1+s_{2^j}}^{(2^j)}, \dots, \sigma_{N_j}^{(2^j)} \in \Gamma(X, L^{2^j})$ be such that $(\sigma_l^{(2^j)})_l$ is a basis of $\Gamma(X, L^{2^j})$ and set

$$\varphi_j := \frac{1}{2^{j+1}} \log\left[\sum_{l=1}^{s_{2^j}} |\sigma_l^{(2^j)}|^2 + \varepsilon_j \sum_{l=1+s_{2^j}}^{N_j} |\sigma_l^{(2^j)}|^2\right] + \frac{\log C_5}{2^{j+2}} - h .$$

Clearly $\varphi_j \in PSH(X,\omega)$. Moreover, $\varphi_j \in \mathcal{C}^\infty(X)$ because L^{2^j} is very ample if j is large enough (hence we can find, for every $x \in X$, a holomorphic section of L^{2^j} on X which does not vanish at x). Finally, we can choose $\varepsilon_j > 0$ that decrease so fast to zero that (φ_j) is still decreasing and converges to φ. □

Corollary 8.2. *Let ω be a Kähler form on a **projective** algebraic manifold X. Then there exists $A \ge 1$ such that for every $\varphi \in PSH(X,\omega)$, we can find $\varphi_j \in PSH(X, A\omega) \cap \mathcal{C}^\infty(X)$ which decrease towards φ.*

Proof. Let $\varphi \in PSH(X,\omega)$. Since X is projective, we can find a Hodge form ω'. Then $C^{-1}\omega' \le \omega \le C\omega'$ for some constant $C \ge 1$. Since $PSH(X,\omega) \subset PSH(X, C\omega')$, it follows from the previous theorem that we can find $\varphi_j \in PSH(X, C\omega') \cap \mathcal{C}^\infty(X)$ that decrease towards φ. Now the result follows from $PSH(X, C\omega') \subset PSH(X, A\omega)$ with $A = C^2$. □

Remark 8.3. When X is merely Kähler, the above result still holds but the proof is far more intricate. We refer the reader to Demailly's articles for a proof.

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