The weighted Monge–Ampère energy of quasiplurisubharmonic functions

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Abstract

We study degenerate complex Monge–Ampère equations on a compact Kähler manifold (X, ω). We show that the complex Monge–Ampère operator $(\omega + dd^c \cdot)^n$ is well defined on the class $\mathcal{E}(X, \omega)$ of ω-plurisubharmonic functions with finite weighted Monge–Ampère energy. The class $\mathcal{E}(X, \omega)$ is the largest class of ω-psh functions on which the Monge–Ampère operator is well defined and the comparison principle is valid. It contains several functions whose gradient is not square integrable. We give a complete description of the range of the operator $(\omega + dd^c \cdot)^n$ on $\mathcal{E}(X, \omega)$, as well as on some of its subclasses. We also study uniqueness properties, extending Calabi's result to this unbounded and degenerate situation, and we give applications to complex dynamics and to the existence of singular Kähler–Einstein metrics.
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0. Introduction

Let X be a compact connected Kähler manifold of complex dimension $n \in \mathbb{N}^*$. Let ω be a Kähler form on X. Given μ a positive Radon measure on X such that $\mu(X) = \int_X \omega^n$, we study the complex Monge–Ampère equation

$$\left(\omega + dd^c \varphi\right)^n = \mu, \tag{MA$_\mu$}$$

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where φ, the unknown function, is such that $\omega_\varphi := \omega + dd^c\varphi$ is a positive current. Such functions are called ω-plurisubharmonic (ω-psh for short). We refer the reader to [19] for basic properties of the set $PSH(X, \omega)$ of all such functions. Here $d = \partial + \bar{\partial}$ and $d^c = \frac{1}{2i\pi}(\partial - \bar{\partial})$.

Complex Monge–Ampère equations have been studied by several authors over the last fifty years, in connection with questions from Kähler geometry and complex dynamics (see [1,14,17, 21,22,25,26,28] for references). The first and cornerstone result is due to S.T. Yau who proved [28] that (MA_μ) admits a solution $\varphi \in PSH(X, \omega) \cap \mathcal{C}^\infty(X)$ when $\mu = f\omega^n$ is a smooth volume form.

Motivated by applications towards complex dynamics, we need here to consider measures μ which are quite singular, whence to deal with singular ω-psh functions φ. We introduce and study a class $\mathcal{E}(X, \omega)$ of ω-psh functions for which the complex Monge–Ampère operator $(\omega + dd^c\varphi)^n$ is well defined (see Definition 1.1): following E. Bedford and A. Taylor [6] we show that the operator $(\omega + dd^c\varphi)^n$ is well defined in $X \setminus (\varphi = -\infty)$ for all functions $\varphi \in PSH(X, \omega)$; the class $\mathcal{E}(X, \omega)$ is the set of functions $\varphi \in PSH(X, \omega)$ such that $(\omega + dd^c\varphi)^n$ has full mass $\int_X \omega^n$ in $X \setminus (\varphi = -\infty)$. When $n = \dim_{\mathbb{C}} X = 1$, this is precisely the subclass of functions $\varphi \in PSH(X, \omega)$ whose Laplacian does not charge polar sets. It is striking that the class $\mathcal{E}(X, \omega)$ contains many functions whose gradient is not square integrable, hence several results to follow have no local analogue (compare [8,9]).

One of our main results gives a complete characterization of the range of the complex Monge–Ampère operator on the class $\mathcal{E}(X, \omega)$.

Theorem A. *There exists $\varphi \in \mathcal{E}(X, \omega)$ such that $\mu = (\omega + dd^c\varphi)^n$ if and only if μ does not charge pluripolar sets.*

An important tool we use is the *comparison principle* that we establish in Section 1: we show that $\mathcal{E}(X, \omega)$ is the largest class of ω-psh functions on which the complex Monge–Ampère operator $(\omega + dd^c\cdot)^n$ is well defined and the comparison principle is valid. Another crucial tool for our study is the notion of *weighted Monge–Ampère energy*, defined as

$$E_\chi(\varphi) := \int_X (-\chi) \circ \varphi \left(\omega + dd^c\varphi\right)^n,$$

where $\chi : \mathbb{R}^- \to \mathbb{R}^-$ is an increasing function such that $\chi(-\infty) = -\infty$. The properties of this energy are quite different whether the weight χ is convex ($\chi \in \mathcal{W}^-$) or concave ($\chi \in \mathcal{W}^+$). We show (Proposition 2.2) that

$$\mathcal{E}(X, \omega) = \bigcup_{\chi \in \mathcal{W}^-} \mathcal{E}_\chi(X, \omega),$$

where $\mathcal{E}_\chi(X, \omega)$ denotes the class of functions $\varphi \in \mathcal{E}(X, \omega)$ such that $\chi(\varphi - \sup_X \varphi) \in L^1((\omega + dd^c\varphi)^n)$. At the other extreme, we show (Proposition 3.1) that

$$PSH(X, \omega) \cap L^\infty(X) = \bigcap_{\chi \in \mathcal{W}^+} \mathcal{E}_\chi(X, \omega).$$

The function $\chi(t) = t$ is—up to multiplicative constant—the unique weight in $\mathcal{W}^- \cap \mathcal{W}^+$. We let

$$\mathcal{E}^1(X, \omega) := \left\{ \varphi \in \mathcal{E}(X, \omega) \mid \varphi \in L^1\big((\omega + dd^c\varphi)^n\big) \right\}$$

denote the class $\mathcal{E}_\chi(X, \omega)$ for $\chi(t) = t$. When $n = \dim_{\mathbb{C}} X = 1$, this is the classical class of quasisubharmonic functions of finite energy. It deserves special attention both for the theory and the applications. Indeed all functions $\varphi \in \mathcal{E}^1(X, \omega)$ have gradient in $L^2(\omega^n)$, while the gradient of most functions in $\mathcal{E}_\chi(X, \omega)$, $\chi \in \mathcal{W}^- \setminus \mathcal{W}^+$, does not belong to $L^2(\omega^n)$ (see Example 2.14), in contrast with the local theory [8,9].

We obtain the following extension of Calabi's uniqueness result [10].

Theorem B. *Assume* $(\omega + dd^c\varphi)^n \equiv (\omega + dd^c\psi)^n$ *with* $\varphi \in \mathcal{E}(X, \omega)$ *and* $\psi \in \mathcal{E}^1(X, \omega)$. *Then* $\varphi - \psi$ *is constant.*

It is an interesting open question to prove uniqueness of solutions as above in the larger class $\mathcal{E}(X, \omega)$.

We also study the range of the Monge–Ampère operator on subclasses $\mathcal{E}_\chi(X, \omega)$, when $\chi(t) = -(-t)^p$, $p > 0$, letting

$$\mathcal{E}^p(X, \omega) := \left\{ \varphi \in \mathcal{E}(X, \omega) \mid \varphi \in L^p\big((\omega + dd^c\varphi)^n\big) \right\}$$

denote the corresponding class $\mathcal{E}_\chi(X, \omega)$.

Theorem C. *There exists* $\varphi \in \mathcal{E}^p(X, \omega)$ *such that* $\mu = (\omega + dd^c\varphi)^n$ *if and only if* $\mathcal{E}^p(X, \omega) \subset L^p(\mu)$.

This result also holds for quasihomogeneous weights (see Lemma 3.9). It was obtained in a local context by U. Cegrell [11] when $p \geqslant 1$.

On our way to prove Theorems A–C, we establish several intermediate results (the class $\mathcal{E}(X, \omega)$ is convex, weighted Monge–Ampère operators are continuous under decreasing sequences, etc.) most of which are valid when ω is merely a positive closed (1, 1)-current with bounded potentials. This is motivated by applications towards complex dynamics and complex differential geometry, as we briefly indicate in Section 5.

This article is an expanded version of our previous preprint [20].

1. The class $\mathcal{E}(X, \omega)$

In the sequel X is a compact Kähler manifold of dimension n, and ω is a positive closed (1, 1)-current with bounded potentials, such that $\int_X \omega^n > 0$. We let $PSH(X, \omega) = \{\varphi \in L^1(X) \mid \varphi$ is u.s.c. and $dd^c\varphi \geqslant -\omega\}$ denote the set of ω-plurisubharmonic functions (ω-psh for short) which was introduced and studied in [19].

1.1. Defining the complex Monge–Ampère operator

It follows from their plurifine properties that if u, v are bounded plurisubharmonic functions in some open subset D of $\mathbb{C}^n$, then

$$\mathbf{1}_{\{u>v\}}\big[dd^c u\big]^n = \mathbf{1}_{\{u>v\}}\big[dd^c \max(u, v)\big]^n \tag{1}$$

in the sense of Borel measures in D (see [6]).

Let φ be some unbounded ω-psh function on X and consider $\varphi_j := \max(\varphi, -j) \in PSH(X, \omega)$ the *canonical approximation* of φ by bounded ω-psh functions. This is a decreasing sequence such that, by (1),

$$\mathbf{1}_{\{\varphi_j>-k\}}\big[\omega + dd^c \varphi_j\big]^n = \mathbf{1}_{\{\varphi_j>-k\}}\big[\omega + dd^c \max(\varphi_j, -k)\big]^n.$$

Now if $j \geqslant k$, then $(\varphi_j > -k) = (\varphi > -k)$ and $\max(\varphi_j, -k) = \varphi_k$, thus

$$\mathbf{1}_{\{\varphi>-k\}}\big[\omega + dd^c \varphi_j\big]^n = \mathbf{1}_{\{\varphi>-k\}}\big[\omega + dd^c \varphi_k\big]^n. \tag{2}$$

Observe also that $(\varphi > -k) \subset (\varphi > -j)$, therefore

$$j \geqslant k \quad \Longrightarrow \quad \mathbf{1}_{\{\varphi>-j\}}\big[\omega + dd^c \varphi_j\big]^n \geqslant \mathbf{1}_{\{\varphi>-k\}}\big[\omega + dd^c \varphi_k\big]^n,$$

in the weak sense of Borel measures. Since the total mass of the measures $\mathbf{1}_{\{\varphi>-j\}}[\omega + dd^c \varphi_j]^n$ is uniformly bounded from above by $\int_X \omega^n$, by Stokes theorem, we can define

$$\mu_\varphi := \lim_{j \to +\infty} \mathbf{1}_{\{\varphi>-j\}}\big[\omega + dd^c \varphi_j\big]^n.$$

This is a positive Borel measure which is precisely the non-pluripolar part of $(\omega + dd^c \varphi)^n$, as considered in a local context by E. Bedford and A. Taylor in [6]. Its total mass $\mu_\varphi(X)$ can take any value in $[0, \int_X \omega^n]$.

Definition 1.1. We set

$$\mathcal{E}(X, \omega) := \left\{ \varphi \in PSH(X, \omega) \,\Big|\, \mu_\varphi(X) = \int_X \omega^n \right\}.$$

An alternative definition is given by the following observation.

Lemma 1.2. *Fix* $\varphi \in PSH(X, \omega)$. *Let* (s_j) *be any sequence of real numbers converging to* $+\infty$, *such that* $s_j \leqslant j$ *for all* $j \in \mathbb{N}$. *The following conditions are equivalent*:

(a) $\varphi \in \mathcal{E}(X, \omega)$;
(b) $(\omega + dd^c \varphi_j)^n (\varphi \leqslant -j) \to 0$;
(c) $(\omega + dd^c \varphi_j)^n (\varphi \leqslant -s_j) \to 0$,

where $\varphi_j := \max(\varphi, -j)$ *denotes the canonical approximation.*

Proof. By definition of μ_φ, we have

$$\mu_\varphi(X) = \int_X \omega^n - \lim_{j \to +\infty} \left(\omega + dd^c\varphi_j\right)^n (\varphi \leqslant -j),$$

hence (a) is equivalent to (b). It follows from (2) (with $k = s_j$) that

$$\left(\omega + dd^c\varphi_j\right)^n (\varphi \leqslant -s_j) = \left(\omega + dd^c\varphi_{s_j}\right)^n (\varphi \leqslant -s_j),$$

since $s_j \leqslant j$. This shows that *(b)* is equivalent to (c). $\quad\square$

When X is a compact Riemann surface ($n = \dim_{\mathbb{C}} X = 1$), the set $\mathcal{E}(X, \omega)$ is the set of ω-subharmonic functions whose Laplacian does not charge polar sets. It follows from the above discussion that any ω-psh function has a well-defined complex Monge–Ampère operator $(\omega + dd^c\varphi)^n$ in $X \setminus (\varphi = -\infty)$, and μ_φ is the trivial extension of $(\omega + dd^c\varphi)^n$ through $(\varphi = -\infty)$. A function φ belongs to $\mathcal{E}(X, \omega)$ precisely when its complex Monge–Ampère has total mass $\int_X \omega^n$ in $X \setminus (\varphi = -\infty)$, hence it is natural to use the notation

$$\left(\omega + dd^c\varphi\right)^n := \mu_\varphi = \lim_{j \to +\infty} \mathbf{1}_{\{\varphi > -j\}}\left[\omega + dd^c\varphi_j\right]^n,$$

for $\varphi \in \mathcal{E}(X, \omega)$.

Theorem 1.3. *Let $\varphi \in \mathcal{E}(X, \omega)$. Then for all bounded Borel function b,*

$$\left\langle \left(\omega + dd^c\varphi\right)^n, b\right\rangle = \lim_{j \to +\infty} \left\langle \left(\omega + dd^c\varphi_j\right)^n, b\right\rangle, \tag{3}$$

where $\varphi_j := \max(\varphi, -j)$ is the canonical approximation of φ.
   In particular $(\omega + dd^c\varphi)^n$ puts no mass on pluripolar sets, and

$$\mathbf{1}_{\{\varphi > -j\}}\left(\omega + dd^c\varphi\right)^n (B) = \mathbf{1}_{\{\varphi > -j\}}\left(\omega + dd^c\varphi_j\right)^n (B) \tag{4}$$

for all Borel subsets $B \subset X$.

Let us emphasize that the convergence in (3) implies—but is much stronger than—the convergence in the weak sense of positive Borel measures,

$$\left(\omega + dd^c\varphi_j\right)^n \to \left(\omega + dd^c\varphi\right)^n.$$

Several results to follow are a consequence of fact (3): the complex Monge–Ampère measure $(\omega + dd^c\varphi)^n$ of a function $\varphi \in \mathcal{E}(X, \omega)$ is very well approximated by the Monge–Ampère measures $(\omega + dd^c\varphi_j)^n$.
   Note also that the complex Monge–Ampère operator $(\omega + dd^c\varphi)^n$ is thus well defined for functions $\varphi \in \mathcal{E}(X, \omega)$, although these functions need not have gradient in $L^2(X)$ (see Theorem 1.9 and Example 2.14). This is in contrast with the local theory [8,9].

Proof. Recall that $\varphi \in \mathcal{E}(X, \omega)$ if and only if $(\omega + dd^c\varphi_j)^n(\varphi \leqslant -j) \to 0$, where $\varphi_j := \max(\varphi, -j)$. We infer that for all Borel subset $B \subset X$,

$$\left(\omega + dd^c\varphi\right)^n(B) := \lim_{j \to +\infty} \int_{B \cap (\varphi > -j)} \left(\omega + dd^c\varphi_j\right)^n = \lim_{j \to +\infty} \int_B \left(\omega + dd^c\varphi_j\right)^n.$$

This yields (3), by using Lebesgue dominated convergence theorem.

Since $(\omega + dd^c\varphi_j)^n$ does not charge any pluripolar set B, the same property holds for $(\omega + dd^c\varphi)^n$. Equality (4) now follows from (2) and (3). $\quad\square$

Since $(\omega + dd^c\varphi)^n$ does not charge the pluripolar set $(\varphi = -\infty)$ when $\varphi \in \mathcal{E}(X, \omega)$, one can construct a continuous increasing function $h : \mathbb{R}^+ \to \mathbb{R}^+$ such that $h(+\infty) = +\infty$ and $h \circ |\varphi| \in L^1((\omega + dd^c\varphi)^n)$. This motivates the following result.

Proposition 1.4. *Fix* $\varphi \in PSH(X, \omega)$ *and let* $h : \mathbb{R}^+ \to \mathbb{R}^+$ *be a continuous increasing function such that* $h(+\infty) = +\infty$. *Then*

$$\varphi \in \mathcal{E}(X, \omega) \text{ and } \int_X h \circ |\varphi|\left(\omega + dd^c\varphi\right)^n < +\infty \quad \Longleftrightarrow \quad \sup_{j \geqslant 0} \int_X h \circ |\varphi_j|\left(\omega + dd^c\varphi_j\right)^n < +\infty,$$

where $\varphi_j := \max(\varphi, -j)$.

Moreover if this condition holds, then for all Borel subset $B \subset X$,

$$\int_B h \circ |\varphi_j|\left(\omega + dd^c\varphi_j\right)^n \to \int_B h \circ |\varphi|\left(\omega + dd^c\varphi\right)^n. \tag{5}$$

Proof. We can assume without loss of generality that $\varphi, \varphi_j \leqslant 0$.

Assume first that $\sup_{j \geqslant 0} \int_X h \circ |\varphi_j|(\omega + dd^c\varphi_j)^n < +\infty$. Then $\varphi \in \mathcal{E}(X, \omega)$ by Lemma 1.2. Since the Borel measures $h(-\varphi_j)(\omega + dd^c\varphi_j)^n$ have uniformly bounded masses, they form a weakly compact sequence. Let ν be a cluster point. Since the functions $h(-\varphi_j)$ increase towards $h(-\varphi)$ and $(\omega + dd^c\varphi_j)^n$ converges towards $(\omega + dd^c\varphi)^n$, it follows from semi-continuity that $h(-\varphi)(\omega + dd^c\varphi)^n \leqslant \nu$, hence $\int_X h(-\varphi)(\omega + dd^c\varphi)^n \leqslant \nu(X) < +\infty$.

Conversely assume that $\varphi \in \mathcal{E}(X, \omega)$ and $h(-\varphi) \in L^1((\omega + dd^c\varphi)^n)$. It follows from (4) that

$$\int_{(\varphi \leqslant -j)} \omega_{\varphi_j}^n = \int_X \omega_{\varphi_j}^n - \int_{(\varphi > -j)} \omega_{\varphi_j}^n = \int_{(\varphi \leqslant -j)} \omega_\varphi^n.$$

Here—and in the sequel—we use the notation $\omega_\varphi := \omega + dd^c\varphi$, $\omega_{\varphi_j} = \omega + dd^c\varphi_j$. Thus by (4) again,

$$\int_X h(-\varphi_j)\omega_{\varphi_j}^n = h(j) \int_{(\varphi \leqslant -j)} \omega_{\varphi_j}^n + \int_{(\varphi > -j)} h(-\varphi)\omega_{\varphi_j}^n$$

$$= \int_{(\varphi \leqslant -j)} h(j)\omega_\varphi^n + \int_{(\varphi > -j)} h(-\varphi)\omega_\varphi^n \leqslant \int_X h(-\varphi)\omega_\varphi^n.$$

Moreover if $B \subset X$ is a Borel subset, then

$$\left| \int_B h(-\varphi_j)\omega_{\varphi_j}^n - \int_B h(-\varphi)\omega_{\varphi}^n \right| \leqslant \int_{B \cap (\varphi \leqslant -j)} h(-\varphi_j)\omega_{\varphi_j}^n + \int_{B \cap (\varphi \leqslant -j)} h(-\varphi)\omega_{\varphi}^n$$

$$\leqslant 2 \int_{(\varphi \leqslant -j)} h(-\varphi)\omega_{\varphi}^n \to 0. \qquad \square$$

1.2. The comparison principle

We now establish the *comparison principle* which will be an important tool in the sequel.

Theorem 1.5. *Let* $\varphi, \psi \in \mathcal{E}(X, \omega)$, *then*

$$\int_{(\varphi < \psi)} \left(\omega + dd^c\psi\right)^n \leqslant \int_{(\varphi < \psi)} \left(\omega + dd^c\varphi\right)^n.$$

The class $\mathcal{E}(X, \omega)$ *is the largest subclass of* $PSH(X, \omega)$ *on which the operator* $(\omega + dd^c \cdot)^n$ *is well defined and the comparison principle is valid.*

Proof. Assume first that φ, ψ are bounded. It follows from (1) that

$$\int_{(\varphi < \psi)} \omega_{\psi}^n = \int_{(\varphi < \psi)} \left[\omega + \max(\varphi, \psi)\right]^n = \int_X \omega^n - \int_{(\varphi \geqslant \psi)} \left[\omega + \max(\varphi, \psi)\right]^n$$

$$\leqslant \int_X \omega^n - \int_{(\varphi > \psi)} \omega_{\varphi}^n = \int_{(\varphi \leqslant \psi)} \omega_{\varphi}^n.$$

Replacing ψ by $\psi - \varepsilon$, $\varepsilon > 0$, yields when $\varepsilon \searrow 0$,

$$\int_{(\varphi < \psi)} \omega_{\psi}^n = \lim \nearrow \int_{(\varphi < \psi - \varepsilon)} \omega_{\psi}^n \leqslant \lim \nearrow \int_{(\varphi \leqslant \psi - \varepsilon)} \omega_{\varphi}^n = \int_{(\varphi < \psi)} \omega_{\varphi}^n.$$

When φ, ψ are unbounded, we set $\varphi_j = \max(\varphi, -j)$ and $\psi_k = \max(\psi, -k)$. The comparison principle for bounded ω-psh functions yields

$$\int_{(\varphi_j < \psi_k)} \omega_{\psi_k}^n \leqslant \int_{(\varphi_j < \psi_k)} \omega_{\varphi_j}^n.$$

Observe that $(\varphi_j < \psi) \subset (\varphi_j < \psi_k) \subset (\varphi < \psi_k)$. Letting $k \to +\infty$ in the corresponding inequality yields, by using monotone convergence theorem together with (3),

$$\int_{(\varphi_j < \psi)} \omega_{\psi}^n \leqslant \int_{(\varphi \leqslant \psi)} \omega_{\varphi_j}^n.$$

Letting now $j \to +\infty$, we infer $\int_{(\varphi < \psi)} \omega_\psi^n \leqslant \int_{(\varphi \leqslant \psi)} \omega_\varphi^n$. The desired inequality now follows by replacing ψ by $\psi - \varepsilon$, $\varepsilon > 0$, and letting $\varepsilon \to 0^+$.

It turns out that the class $\mathcal{E}(X, \omega)$ is the largest class of ω-plurisubharmonic functions on which the complex Monge–Ampère operator is well defined and the comparison principle is valid. Indeed let $\mathcal{F}$ be the largest class with these properties, so that $\mathcal{E}(X, \omega) \subset \mathcal{F} \subset PSH(X, \omega)$ and

$$\int\limits_{(u \leqslant \psi)} \left(\omega + dd^c \psi\right)^n \leqslant \int\limits_{(u \leqslant \psi)} \left(\omega + dd^c u\right)^n,$$

for all $u, \psi \in \mathcal{F}$. Note that this inequality is equivalent to the comparison principle Theorem 1.5 since the measures $(\omega + dd^c u)^n$, $(\omega + dd^c \psi)^n$ have the same total mass $\int_X \omega^n$. Fix $\varphi \in \mathcal{E}(X, \omega)$ and $\psi \in \mathcal{F}$ and apply previous inequality with $u = \varphi + c$, $c \in \mathbb{R}$, to obtain

$$\left(\omega + dd^c \psi\right)^n (\varphi = -\infty) = \lim_{c \to +\infty} \int\limits_{(\varphi + c \leqslant \psi)} \left(\omega + dd^c \psi\right)^n \leqslant \omega_\varphi^n (\varphi = -\infty) = 0.$$

Since $\mathcal{E}(X, \omega)$ characterizes pluripolar sets (Example 2.14), we infer $\omega_\psi^n (\psi = -\infty) = 0$, hence $\psi \in \mathcal{E}(X, \omega)$, i.e. $\mathcal{F} = \mathcal{E}(X, \omega)$.   □

This principle allows us to derive important properties of the class $\mathcal{E}(X, \omega)$.

Proposition 1.6. *The class $\mathcal{E}(X, \omega)$ is convex. Moreover if $\varphi \in \mathcal{E}(X, \omega)$ and $\psi \in PSH(X, \omega)$ are such that $\varphi \leqslant \psi$, then $\psi \in \mathcal{E}(X, \omega)$.*

Proof. The proof follows from the comparison principle together with the following elementary observation: *if $\varphi \in PSH(X, \omega)$ and $\varphi/2 \in \mathcal{E}(X, \omega)$, then $\varphi \in \mathcal{E}(X, \omega)$.* Indeed set $u = \varphi/2$, $u_j := \max(u, -j)$, and $\varphi_j := \max(\varphi, -j)$. Observe that $u_j = \varphi_{2j}/2$ and

$$\omega_{u_j} := \omega + dd^c(\varphi_{2j}/2) = \frac{1}{2}(\omega + \omega_{\varphi_{2j}}) \geqslant \frac{1}{2}\omega_{\varphi_{2j}},$$

therefore

$$\int\limits_{(\varphi \leqslant -2j)} \left(\omega + dd^c \varphi_{2j}\right)^n = \int\limits_{(u \leqslant -j)} \left(\omega + dd^c \varphi_{2j}\right)^n \leqslant 2^n \int\limits_{(u \leqslant -j)} \left(\omega + dd^c u_j\right)^n \to 0.$$

We now use the comparison principle to show that if $\varphi \in \mathcal{E}(X, \omega)$, $\psi \in PSH(X, \omega)$ are such that $\varphi \leqslant \psi$, then $\psi/2 \in \mathcal{E}(X, \omega)$, so that $\psi \in \mathcal{E}(X, \omega)$ by previous observation. Set $v := \psi/2$ and $v_j := \max(v, -j)$, $\varphi_j := \max(\varphi, -j)$. We can assume without loss of generality that $\varphi \leqslant \psi \leqslant -2$, hence $v \leqslant -1$. It follows from $\varphi \leqslant \psi$ that

$$(v \leqslant -j) \subset (\varphi_{2j} < v_j - j + 1) \subset (\varphi \leqslant -j),$$

where the last inclusion simply uses $v_j \leqslant -1$. We infer

$$\omega_{v_j}^n (v \leqslant -j) \leqslant \int_{(\varphi_{2j} < v_j - j + 1)} \omega_{v_j}^n \leqslant \int_{(\varphi_{2j} < v_j - j + 1)} \omega_{\varphi_{2j}}^n \leqslant \omega_{\varphi_{2j}}^n (\varphi \leqslant -j) \to 0,$$

as follows from Lemma 1.2. Thus $v = \psi/2 \in \mathcal{E}(X, \omega)$.

Using the comparison principle again, we now show that if $\varphi, \psi \in \mathcal{E}(X, \omega)$, then $(\varphi + \psi)/4 \in \mathcal{E}(X, \omega)$. It follows then from our first observation that $(\varphi + \psi)/2 \in \mathcal{E}(X, \omega)$, thus $\mathcal{E}(X, \omega)$ is convex. Set $w := (\varphi + \psi)/4$, $w_j := \max(w, -j)$, $\varphi_j := \max(\varphi, -j)$ and $\psi_j := \max(\psi, -j)$. Observe that $(v \leqslant -j) \subset (\varphi \leqslant -2j) \cup (\psi \leqslant -2j)$, thus it suffices to show that $\omega_{v_j}^n (\varphi \leqslant -2j) \to 0$. Assuming as above that $\varphi, \psi \leqslant -2$ yields $(\varphi \leqslant -2j) \subset (\varphi_{2j} < w_j - j + 1) \subset (\varphi \leqslant -j)$, hence

$$\omega_{v_j}^n (\varphi \leqslant -2j) \leqslant \omega_{\varphi_{2j}}^n (\varphi \leqslant -j) = \omega_{\varphi_j}^n (\varphi \leqslant -j) \to 0,$$

as follows from Lemma 1.2. $\quad\square$

Corollary 1.7. *If $\varphi \in \mathcal{E}(X, \omega)$, $\psi \in PSH(X, \omega)$, then $\max(\varphi, \psi) \in \mathcal{E}(X, \omega)$ and*

$$\mathbf{1}_{\{\varphi > \psi\}} \big[\omega + dd^c \varphi\big]^n = \mathbf{1}_{\{\varphi > \psi\}} \big[\omega + dd^c \max(\varphi, \psi)\big]^n.$$

Proof. It follows from the previous proposition that $u := \max(\varphi, \psi)$ belongs to $\mathcal{E}(X, \omega)$. Set $\varphi_j = \max(\varphi, -j)$, $\psi_{j+1} = \max(\psi, -j - 1)$ and $u_j := \max(u, -j)$. Observe that $\max(\varphi_j, \psi_{j+1}) = \max(\varphi, \psi, -j) = u_j$. Applying (1) yields

$$\mathbf{1}_{\{\varphi_j > \psi_{j+1}\}} \big(\omega + dd^c \varphi_j\big)^n = \mathbf{1}_{\{\varphi_j > \psi_{j+1}\}} \big(\omega + dd^c u_j\big)^n.$$

Recall from (3) that $\mathbf{1}_{(\varphi > \psi)} \omega_{\varphi_j}^n \to \mathbf{1}_{(\varphi > \psi)} \omega_\varphi^n$. Now $(\varphi > \psi) \subset (\varphi_j > \psi_{j+1})$ and $(\varphi_j > \psi_{j+1}) \setminus (\varphi > \psi) \subset (\varphi \leqslant -j)$, hence

$$0 \leqslant [\mathbf{1}_{(\varphi_j > \psi_{j+1})} - \mathbf{1}_{(\varphi > \psi)}] \omega_{\varphi_j}^n \leqslant \mathbf{1}_{(\varphi \leqslant -j)} \omega_{\varphi_j}^n \to 0,$$

since $\varphi \in \mathcal{E}(X, \omega)$. This shows that

$$\mathbf{1}_{\{\varphi_j > \psi_{j+1}\}} \big(\omega + dd^c \varphi_j\big)^n \to \mathbf{1}_{\{\varphi > \psi\}} \big(\omega + dd^c \varphi\big)^n.$$

One proves similarly that $\mathbf{1}_{\{\varphi_j > \psi_{j+1}\}} (\omega + dd^c u_j)^n \to \mathbf{1}_{\{\varphi > \psi\}} (\omega + dd^c u)^n$, observing that $(\varphi_j > \psi_{j+1}) \setminus (\varphi > \psi) \subset (\max[\varphi, \psi] \leqslant -j)$. $\quad\square$

The previous proposition also shows that functions which belong to the class $\mathcal{E}(X, \omega)$, although possibly unbounded, have mild singularities.

Corollary 1.8. *Assume ω is Kähler. A function $\varphi \in \mathcal{E}(X, \omega)$ has zero Lelong number at every point $x \in X$.*

Note that this is not a sufficient condition to belong to $\mathcal{E}(X, \omega)$. It is for instance well known that, when $n = 1$, there are subharmonic functions whose Laplacian has no Dirac mass but nevertheless charges a polar set.

Proof. For any point $x \in X$, one can construct a function $\varrho \in PSH(X, \omega)$ which is smooth but at point x, and such that $\varrho \sim c \log \mathrm{dist}(\cdot, x)$ near x, for some constant $c > 0$. Such a function ϱ has well-defined Monge–Ampère measure $(\omega + dd^c \varphi)^n$ and the latter has mass $\geqslant c^n$ at point x (see [13]). Therefore $\rho \notin \mathcal{E}(X, \omega)$.

Now if $\varphi \in PSH(X, \omega)$ has a positive Lelong number at point x, then $\varphi \leqslant \gamma \varrho + C$ on X, for some constants $\gamma, C > 0$, so it follows from the previous result that φ does not belong to $\mathcal{E}(X, \omega)$. $\quad\square$

We will see in the next section (Theorem 2.6) that the complex Monge–Ampère operator $\varphi \in \mathcal{E}(X, \omega) \mapsto (\omega + dd^c \varphi)^n$ is continuous under *any* decreasing sequences.

Theorem 1.9. *Let $\varphi_j \in \mathcal{E}(X, \omega)$ be any sequence decreasing towards $\varphi \in \mathcal{E}(X, \omega)$. Then $(\omega + dd^c \varphi_j)^n \to (\omega + dd^c \varphi)^n$.*

The proof of this result uses some properties of the weighted Monge–Ampère energy that we introduce in Section 2. The following consequence will be quite useful when solving Monge–Ampère equations in Section 4. It is due to E. Bedford and A. Taylor [3,5] when φ, ψ are bounded.

Corollary 1.10. *Assume $\varphi, \psi \in \mathcal{E}(X, \omega)$ are such that $(\omega + dd^c \varphi)^n \geqslant \mu$ and $(\omega + dd^c \psi)^n \geqslant \mu$ for some positive Borel measure μ on X. Then*

$$\left[\omega + dd^c \max(\varphi, \psi)\right]^n \geqslant \mu.$$

Proof. Observe that $\max(\varphi, \psi) \in \mathcal{E}(X, \omega)$ by Proposition 1.6. It follows from Corollary 1.7 that

$$\left[\omega + dd^c \max(\varphi, \psi)\right]^n \geqslant \mathbf{1}_{\{\varphi > \psi\}} \left(\omega + dd^c \varphi\right)^n + \mathbf{1}_{\{\varphi < \psi\}} \left(\omega + dd^c \psi\right)^n \geqslant \mathbf{1}_{\{\varphi \neq \psi\}} \mu.$$

Thus we are done if $\mu(\varphi = \psi) = 0$.

Assume now that $\mu(\varphi = \psi) > 0$. We show hereafter that $\mu(\varphi = \psi + t_0) = 0$ for all $t_0 \in \mathbb{R} \setminus I_\mu$ where I_μ is at most countable. Assuming this, we can find a decreasing sequence $\varepsilon_j \searrow 0$ such that $\mu(\varphi = \psi + \varepsilon_j) = 0$. Replacing ψ by $\psi + \varepsilon_j$ above yields

$$\left[\omega + dd^c \max(\varphi, \psi + \varepsilon_j)\right]^n \geqslant \mu.$$

The desired inequality therefore follows from Theorem 1.9.

It remains to show that $I_\mu := \{t_0 \in \mathbb{R} \mid \mu(\varphi = \psi + t_0) > 0\}$ is at most countable. Consider $f : t \in \mathbb{R} \mapsto \mu(\{\varphi < \psi + t\}) \in \mathbb{R}^+$. This is an increasing function which is left continuous since μ is a Borel measure. Moreover

$$\lim_{\substack{t > t_0 \\ t \to t_0}} f(t) = \mu\left(\{\varphi \leqslant \psi + t_0\} \setminus \{\psi = -\infty\}\right) = \mu\left(\{\varphi \leqslant \psi + t_0\}\right)$$

since $\mu(\psi = -\infty) \leqslant \omega_\psi^n(\psi = -\infty) = 0$. Then f is continuous at t_0 unless $\mu(\varphi = \psi + t_0) > 0$. Thus the set I_μ coincides with the set of discontinuity of f which is at most countable. $\quad\square$

2. Low-energy classes

Let $\chi : \mathbb{R}^- \to \mathbb{R}^-$ be a convex increasing function such that $\chi(-\infty) = -\infty$. It follows from the convexity assumption that

$$0 \leqslant (-t)\chi'(t) \leqslant (-\chi)(t) + \chi(0), \quad \text{for all } t \in \mathbb{R}^-.$$

A straightforward computation shows that $\chi \circ \varphi \in PSH(X, \omega)$ whenever $\varphi \in PSH(X, \omega)$ is non-positive and such that $\chi' \circ \varphi \leqslant 1$. Indeed

$$dd^c \chi \circ \varphi = \chi'' \circ \varphi \, d\varphi \wedge d^c\varphi + \chi' \circ \varphi \, dd^c\varphi \geqslant -\omega.$$

Our aim here is to study the class of ω-psh functions with finite χ-energy.

Definition 2.1. We let $\mathcal{E}_\chi(X, \omega)$ denote the set of ω-plurisubharmonic functions with finite χ-energy, i.e.

$$\mathcal{E}_\chi(X, \omega) := \left\{ \varphi \in \mathcal{E}(X, \omega) \mid \chi(-|\varphi|) \in L^1\big((\omega + dd^c\varphi)^n\big) \right\}.$$

Observe that this definition is invariant under translation both of the function $\varphi \mapsto \varphi + c$, $c \in \mathbb{R}$, and of the weight $\chi \mapsto \chi - \chi(0)$. We shall therefore often assume that the functions we are dealing with are non-negative, and we will always normalize the weight by requiring $\chi(0) = 0$. We let

$$\mathcal{W}^- := \left\{ \chi : \mathbb{R}^- \to \mathbb{R}^- \mid \chi \text{ convex increasing}, \chi(0) = 0, \chi(-\infty) = -\infty \right\}$$

denote the set of admissible weights.

Proposition 2.2.

$$\mathcal{E}(X, \omega) = \bigcup_{\chi \in \mathcal{W}^-} \mathcal{E}_\chi(X, \omega).$$

Proof. Fix $\varphi \in \mathcal{E}(X, \omega)$. Its complex Monge–Ampère measure $(\omega + dd^c\varphi)^n$ is well defined and does not charge the pluripolar set $(\varphi = -\infty)$. One can construct a continuous increasing function $h : \mathbb{R}^+ \to \mathbb{R}^+$ such that $h(+\infty) = +\infty$ and $h \circ |\varphi| \in L^1((\omega + dd^c\varphi)^n)$. Note that we can assume without loss of generality that h is concave (replacing if necessary h by a concave increasing minorant $\tilde{h} \leqslant h$ such that $\tilde{h}(+\infty) = +\infty$). Therefore $\varphi \in \mathcal{E}_\chi(X, \omega)$, where $\chi(t) := -h(-t)$. $\quad\square$

Observe that the union is increasing in the sense that $\mathcal{E}_\chi(X, \omega) \subset \mathcal{E}_{\tilde{\chi}}(X, \omega)$ whenever $\tilde{\chi} = O(\chi)$ at infinity. Observe also that $\chi = O(Id)$, as follows from the convexity assumption, hence for all $\chi \in \mathcal{W}^-$,

$$\mathcal{E}_\chi(X, \omega) \supset \mathcal{E}^1(X, \omega) := \left\{ \varphi \in \mathcal{E}(X, \omega) \mid \varphi \in L^1(\omega_\varphi^n) \right\}.$$

When $n = \dim_{\mathbb{C}} X = 1$, $\mathcal{E}^1(X, \omega)$ is the classical class of quasisubharmonic functions of finite (unweighted) energy.

In the rest of Section 2, $\chi \in \mathcal{W}^-$ denotes a fixed admissible weight and

$$E_\chi(\varphi) := \int\limits_X (-\chi) \circ \varphi \big(\omega + dd^c\varphi\big)^n$$

denotes the χ-energy of a function $\varphi \in \mathcal{E}_\chi(X, \omega)$, $\varphi \leqslant 0$.

2.1. Useful inequalities

Many properties of the class $\mathcal{E}_\chi(X, \omega)$ follow from simple integration by parts, as shown by the following result.

Lemma 2.3 (*The fundamental inequality*). *Let* $\varphi, \psi \in PSH(X, \omega) \cap L^\infty(X)$ *be such that* $\varphi \leqslant \psi \leqslant 0$. *Then*

$$0 \leqslant \int\limits_X (-\chi) \circ \psi \big(\omega + dd^c\psi\big)^n \leqslant 2^n \int\limits_X (-\chi) \circ \varphi \big(\omega + dd^c\varphi\big)^n.$$

Proof. The proof follows from a repeated application of the following inequality, of independent interest: let T be any positive closed current of bidimension $(1, 1)$ on X, then

$$0 \leqslant \int\limits_X (-\chi) \circ \varphi \omega_\psi \wedge T \leqslant 2 \int\limits_X (-\chi) \circ \varphi \omega_\varphi \wedge T. \tag{6}$$

Indeed observe that $\int_X (-\chi) \circ \psi \omega_\psi^n \leqslant \int_X (-\chi) \circ \varphi \omega_\psi^n$, hence it suffices to apply (6) with $T = \omega_\varphi^j \wedge \omega_\psi^{n-1-j}$, $0 \leqslant j \leqslant n-1$, to conclude.

It remains to prove (6). Here—and quite often in the sequel—we are going to use Stokes theorem which yields $\int_X u\, dd^c v \wedge T = \int v\, dd^c u \wedge T$, whenever u, v are bounded ω-psh functions. Let us stress that there is no need for a *global* regularization of ω-psh functions to justify this integration by parts. We simply use the fact that $u\, dd^c v \wedge T$ and $v\, dd^c u \wedge T$ are well defined and cohomologous currents on X, since

$$u\, dd^c v \wedge T - v\, dd^c u \wedge T = d\big[u\, d^c v \wedge T - v\, d^c u \wedge T\big].$$

The latter computation can be justified by using local regularizations together with continuity results of [5]. We can also assume the weight χ is smooth and then approximate it by using convolutions. Observe that $\chi' \circ \varphi \omega + dd^c(\chi \circ \varphi) \geqslant 0$, hence

$$0 \leqslant \int\limits_X (-\chi) \circ \varphi \omega_\psi \wedge T = \int\limits_X (-\chi) \circ \varphi \omega \wedge T + \int\limits_X (-\psi)\, dd^c \chi \circ \varphi \wedge T$$

$$\leqslant \int\limits_X (-\chi) \circ \varphi \omega \wedge T + \int\limits_X (-\varphi)\big[\chi' \circ \varphi \omega + dd^c(\chi \circ \varphi)\big] \wedge T$$

$$= \int\limits_X (-\varphi)\chi' \circ \varphi \omega \wedge T + \int\limits_X (-\chi) \circ \varphi \omega_\varphi \wedge T.$$

Observe now that $0 \leqslant (-\varphi)\chi' \circ \varphi \leqslant (-\chi) \circ \varphi$ and

$$\int\limits_X (-\chi) \circ \varphi \omega_\varphi \wedge T = \int\limits_X (-\chi) \circ \varphi \omega \wedge T + \int\limits_X \chi' \circ \varphi \, d\varphi \wedge d^c\varphi \wedge T$$

$$\geqslant \int\limits_X (-\chi) \circ \varphi \omega \wedge T,$$

therefore $\int_X (-\varphi)\chi' \circ \varphi \omega \wedge T \leqslant \int_X (-\chi) \circ \varphi \omega_\varphi \wedge T$, which yields (6).   □

It follows from Lemma 2.3 that the class $\mathcal{E}_\chi(X, \omega)$ is stable under taking maximum. Another consequence of Lemma 2.3 is that a given ω-psh function belongs to $\mathcal{E}_\chi(X, \omega)$ if and only if *any* sequence $\varphi_j \in PSH(X, \omega) \cap L^\infty(X)$ decreasing to φ has uniformly bounded χ-energy.

Corollary 2.4. *Fix $\varphi \in PSH(X, \omega)$. The following conditions are equivalent*:

(1) $\varphi \in \mathcal{E}_\chi(X, \omega)$;
(2) *for any sequence $\varphi_j \in PSH(X, \omega) \cap L^\infty(X)$ decreasing towards φ,*

$$\sup_{j \geqslant 0} \int\limits_X (-\chi)(|\varphi_j|)(\omega + dd^c\varphi_j)^n < +\infty;$$

(3) *there exists one sequence as in (2).*

Proof. It follows from Proposition 1.4 that $\varphi \in \mathcal{E}_\chi(X, \omega)$ if and only if

$$\sup_{j \geqslant 0} \int\limits_X (-\chi)(-|\varphi_j|)(\omega + dd^c\varphi_j)^n < +\infty,$$

where $\varphi_j := \max(\varphi, -j)$ is the canonical approximating sequence. Thus (1) is equivalent to (3).
Assume (3) holds. If $\psi_j \in PSH(X, \omega) \cap L^\infty(X)$ is any other sequence decreasing towards φ, then $\psi_j \geqslant \varphi_{k_j}$ for some (possibly large) $k_j \in \mathbb{N}$, hence by Lemma 2.3, the sequence $\int_X (-\chi)(-|\psi_j|)(\omega + dd^c\psi_j)^n$ is still uniformly bounded, showing (2). The reverse implication $(2) \Rightarrow (3)$ is obvious.   □

We now establish an inequality similar to (6), without any assumption on the relative localization of φ, ψ.

Proposition 2.5. *Let T be a positive closed current of bidimension (j, j) on X, $0 \leqslant j \leqslant n$, and let $\varphi, \psi \leqslant 0$ be bounded ω-psh functions. Then*

$$0 \leqslant \int\limits_X (-\chi) \circ \varphi \omega_\psi^j \wedge T \leqslant 2 \int\limits_X (-\chi) \circ \varphi \omega_\varphi^j \wedge T + 2 \int\limits_X (-\chi) \circ \psi \omega_\psi^j \wedge T.$$

Observe that when T has bidimension $(0, 0)$, $T \equiv 1$, this shows that

$$\chi \circ \mathcal{E}_\chi(X, \omega) \subset L^1(\mu),$$

for any Monge–Ampère measure $\mu = (\omega + dd^c \psi)^n$, $\psi \in \mathcal{E}_\chi(X, \omega)$. We shall use this fact in Section 4, when describing the range of the complex Monge–Ampère operator on classes $\mathcal{E}_\chi(X, \omega)$.

Proof. Observe that $\chi'(2t) \leqslant \chi'(t)$ for all $t < 0$, hence

$$\int\limits_X (-\chi) \circ \varphi \, \omega_\psi^j \wedge T = \int\limits_{-\infty}^0 \chi'(t) \omega_\psi^j \wedge T(\varphi < t) \, dt \leqslant 2 \int\limits_{-\infty}^0 \chi'(t) \omega_\psi^j \wedge T(\varphi < 2t) \, dt.$$

Now $(\varphi < 2t) \subset (\varphi < \psi + t) \cup (\psi < t)$, hence

$$\int\limits_X (-\chi) \circ \varphi \, \omega_\psi^j \wedge T \leqslant 2 \int\limits_{-\infty}^0 \chi'(t) \omega_\psi^j \wedge T(\varphi < \psi + t) \, dt + 2 \int\limits_X (-\chi) \circ \psi \, \omega_\psi^j \wedge T.$$

The comparison principle yields $\omega_\psi^j \wedge T(\varphi < \psi + t) \leqslant \omega_\varphi^j \wedge T(\varphi < \psi + t)$. The desired inequality follows by observing that $(\varphi < \psi + t) \subset (\varphi < t)$. □

2.2. Continuity of weighted Monge–Ampère operators

We are now in position to prove a strong version of Theorem 1.9.

Theorem 2.6. *Let* $\varphi_j \in PSH(X, \omega)$ *be a sequence decreasing towards* $\varphi \in \mathcal{E}_\chi(X, \omega)$. *Then* $\varphi_j \in \mathcal{E}_\chi(X, \omega)$ *and* $(\omega + dd^c \varphi_j)^n \to (\omega + dd^c \varphi)^n$. *Moreover for any* $\tilde{\chi} \in \mathcal{W}^-$ *such that* $\tilde{\chi} = o(\chi)$, *one has*

$$\tilde{\chi}(-|\varphi_j|)(\omega + dd^c \varphi_j)^n \to \tilde{\chi}(-|\varphi|)(\omega + dd^c \varphi)^n.$$

When $\varphi_j := \max(\varphi, -j)$ *is the canonical approximation, then*

$$\mathbf{1}_B \chi(-|\varphi_j|)(\omega + dd^c \varphi_j)^n \to \mathbf{1}_B \chi(-|\varphi|)(\omega + dd^c \varphi)^n$$

for all Borel subsets $B \subset X$.

Observe that Theorem 1.9 easily follows from this result together with Proposition 2.2.

Proof.
Step 1. Assume first that $\varphi_j := \max(\varphi, -j)$. For $B \subset X$,

$$\left| \int\limits_B |\chi| \circ \varphi_j \omega_{\varphi_j}^n - \int\limits_B |\chi| \circ \varphi \omega_\varphi^n \right| \leqslant \int\limits_{B \cap (\varphi \leqslant -j)} \left[|\chi| \circ \varphi_j \omega_{\varphi_j}^n + |\chi| \circ \varphi \omega_\varphi^n \right]$$

$$\leqslant 2 \int\limits_{(\varphi \leqslant -j)} |\chi| \circ \varphi \omega_\varphi^n \to 0,$$

since $\chi \circ \varphi \in L^1(\omega_\varphi^n)$. We have used here the upper bound:

$$
\int\limits_{(\varphi \leqslant -j)} (-\chi) \circ \varphi_j \omega_{\varphi_j}^n = (-\chi)(-j) \int\limits_{(\varphi \leqslant -j)} \omega_{\varphi_j}^n
$$

$$
= (-\chi)(-j) \int\limits_{(\varphi \leqslant -j)} \omega_\varphi^n \leqslant \int\limits_{(\varphi \leqslant -j)} (-\chi) \circ \varphi \omega_\varphi^n.
$$

This shows that $\mathbf{1}_B \chi \circ \varphi_j \omega_{\varphi_j}^n \to \mathbf{1}_B \chi \circ \varphi \omega_\varphi^n$. We infer that the fundamental inequality holds when $\varphi, \psi \in \mathcal{E}_\chi(X, \omega)$.

Step 2. We now consider the case of a general sequence (φ_j) that decreases towards φ. The continuity of the complex Monge–Ampère operator $(\omega + dd^c \cdot)^n$ along such sequences is due to E. Bedford and A. Taylor [5] when φ is bounded, and we shall reduce the problem to this case. We can assume without loss of generality that $\varphi, \varphi_j \leqslant 0$. Consider

$$
\varphi_j^K := \max(\varphi_j, -K) \quad \text{and} \quad \varphi^K := \max(\varphi, -K).
$$

The integer K being fixed, the sequence $(\varphi_j^K)_j$ is uniformly bounded and decreases towards φ^K, hence

$$
\left(\omega + dd^c \varphi_j^K\right)^n \xrightarrow{j \to +\infty} \left(\omega + dd^c \varphi^K\right)^n.
$$

Thus we will be done if we can show that $(\omega + dd^c \varphi_j^K)^n$ converges towards $(\omega + dd^c \varphi_j)^n$ as $K \to +\infty$, uniformly with respect to j. Let h be a continuous test function on X. Then

$$
\left| \left\langle h, \left(\omega + dd^c \varphi_j^K\right)^n - \left(\omega + dd^c \varphi_j\right)^n \right\rangle \right|
$$

$$
\leqslant \|h\|_{L^\infty(X)} \int\limits_{(\varphi_j \leqslant -K)} \left[\left(\omega + dd^c \varphi_j^K\right)^n + \left(\omega + dd^c \varphi_j\right)^n \right]
$$

$$
\leqslant \frac{\|h\|_{L^\infty(X)}}{(-\chi)(-K)} \left\{ \int\limits_X (-\chi) \circ \varphi_j^K \left(\omega + dd^c \varphi_j^K\right)^n + \int\limits_X (-\chi) \circ \varphi_j \left(\omega + dd^c \varphi_j\right)^n \right\}.
$$

Since $\varphi \leqslant \varphi_j \leqslant \varphi_j^K$, it follows now from Step 1 that the last two integrals are uniformly bounded from above by $2^n \int_X (-\chi) \circ \varphi \omega_\varphi^n$. This yields the desired uniformity. Note that the same proof shows that $\tilde{\chi} \circ \varphi_j \omega_{\varphi_j}^n \to \tilde{\chi} \circ \varphi \omega_\varphi^n$, whenever $\tilde{\chi} = o(\chi)$, so that a factor $\tilde{\chi}(-K)/\chi(-K) \to 0$ yields uniformity. □

Corollary 2.7. *The fundamental inequality holds when $\varphi, \psi \in \mathcal{E}_\chi(X, \omega)$. Moreover if $0 \geqslant \varphi_j \in \mathcal{E}_\chi(X, \omega)$ is a sequence of functions converging towards φ in $L^1(X)$ and such that $E_\chi(\varphi_j)$ is uniformly bounded, then $\varphi \in \mathcal{E}_\chi(X, \omega)$.*

Proof. Set $\Phi_j := (\sup_{l \geqslant j} \varphi_l)^*$. This is a sequence of ω-psh functions which decrease towards φ. Since $0 \geqslant \Phi_j \geqslant \varphi_j$, it follows from Lemma 2.3 and Theorem 2.6 that $E_\chi(\Phi_j)$ is uniformly bounded. Thus $\varphi \in \mathcal{E}_\chi(X, \omega)$ by Corollary 2.4. □

The class $\mathcal{E}^1(X, \omega)$ plays a special role in many respects, as will become clear in Section 3. We indicate here one specific property that will be useful when solving Monge–Ampère equation in Section 4.

Proposition 2.8. *Let* $\varphi_j, \varphi \in \mathcal{E}^1(X, \omega)$ *be such that* $\varphi_j \to \varphi$ *in* $L^1(X)$. *If* $\int_X |\varphi_j - \varphi| \omega_{\varphi_j}^n \to 0$, *then* $(\omega + dd^c \varphi_j)^n \to (\omega + dd^c \varphi)^n$.

Proof. We can assume without loss of generality that $\varphi_j, \varphi \leqslant 0$. Passing to a subsequence if necessary, we can assume $\int |\varphi_j - \varphi| \omega_{\varphi_j}^n \leqslant 1/j^2$. Consider

$$\Phi_j := \max(\varphi_j, \varphi - 1/j) \in \mathcal{E}^1(X, \omega).$$

It follows from Hartogs' lemma that $\Phi_j \to \varphi$ in capacity. This means that $\mathrm{Cap}_\omega(|\Phi_j - \varphi| > \varepsilon) \to 0$, for all $\varepsilon > 0$, where

$$\mathrm{Cap}_\omega(K) := \sup \left\{ \int\limits_K \left(\omega + dd^c u \right)^n \bigg| u \in PSH(X, \omega), \ -1 \leqslant u \leqslant 0 \right\}$$

is the Monge–Ampère capacity (see [5,19]). It is a well-known consequence of the quasicontinuity of ω-psh functions that $(\omega + dd^c \Phi_j)^n \to (\omega + dd^c \varphi)^n$, when the Φ_j's are uniformly bounded [27]. We can reduce to this case by showing that $(\omega + dd^c \Phi_j^K)^n$ converges towards $(\omega + dd^c \varphi^K)^n$ uniformly with respect to K, where $\Phi_j^K := \max(\Phi_j, -K)$ and $\varphi^K := \max(\varphi, -K)$. Indeed if θ is a test function, then

$$\left| \langle \omega_{\Phi_j^K}^n, \theta \rangle - \langle \omega_{\Phi_j}^n, \theta \rangle \right| \leqslant \sup_X |\theta| \left\{ \int\limits_{(\Phi_j \leqslant -K)} \omega_{\Phi_j^K}^n + \int\limits_{(\Phi_j \leqslant -K)} \omega_{\Phi_j}^n \right\}$$

$$\leqslant \frac{\sup_X |\theta|}{K} \left\{ E_1(\Phi_j^K) + E_1(\Phi_j) \right\}$$

$$\leqslant 2 \frac{\sup_X |\theta|}{K} E_1(\varphi - 1),$$

where $E_1 = E_\chi$ for $\chi(t) = t$. This follows from Lemma 2.3 and the lower bound $\Phi_j \geqslant \varphi - 1$. Thus we have shown that the measures $(\omega + dd^c \Phi_j)^n$ converge towards $(\omega + dd^c \varphi)^n$.

We now need to compare $(\omega + dd^c \Phi_j)^n$ and $(\omega + dd^c \varphi_j)^n$. It follows from Corollary 1.7, that

$$\omega_{\Phi_j}^n \geqslant \mathbf{1}_{\{\varphi_j \geqslant \varphi - 1/j\}} \cdot \omega_{\varphi_j}^n.$$

Let E_j denotes the set $X \setminus \{\varphi_j \geqslant \varphi - 1/j\}$, i.e. $E_j = \{\varphi - \varphi_j > 1/j\}$. Our assumption implies that $\mathbf{1}_{E_j} \omega_{\varphi_j}^n \to 0$. Indeed,

$$0 \leqslant \int\limits_{E_j} \omega_{\varphi_j}^n \leqslant j \int\limits_X |\varphi - \varphi_j| \omega_{\varphi_j}^n \leqslant \frac{1}{j}.$$

Therefore $0 \leqslant \omega_{\varphi_j}^n \leqslant \omega_{\Phi_j}^n + o(1)$, hence $\omega_\varphi^n = \lim \omega_{\varphi_j}^n$. $\quad \square$

Remark 2.9. Observe the following property which was used and proved above (when $\chi(t) = t$): if $\varphi \in PSH(X, \omega)$, $\varphi_j \in \mathcal{E}(X, \omega)$, $\varphi_j \to \varphi$ in capacity, and $\varphi_j \geqslant \psi$ for some fixed $\psi \in \mathcal{E}(X, \omega)$, then $\varphi \in \mathcal{E}(X, \omega)$ and $(\omega + dd^c \varphi_j)^n \to (\omega + dd^c \varphi)^n$.

2.3. Homogeneous weights

The weights $\chi_p(t) = -(-t)^p$, $0 < p \leqslant 1$, belong to $\mathcal{W}^-$. We shall use the notation

$$\mathcal{E}^p(X, \omega) := \mathcal{E}_\chi(X, \omega), \quad \text{when } \chi(t) = -(-t)^p.$$

These classes are easier to understand thanks to the homogeneity property of the weight function, $\chi(\varepsilon t) = \varepsilon^p \chi(t)$.

Also the class $\mathcal{E}^1(X, \omega)$ deserves special attention, as it is the turning point between low-energy and high-energy classes. All functions from $\mathcal{E}^1(X, \omega)$ have gradient in $L^2(X)$ (see Proposition 3.2), while most functions of lower energy do not (Example 2.14). We will show (Theorem 3.3) that solutions of complex Monge–Ampère equations have a unique solution in $\mathcal{E}^1(X, \omega)$, while it is a question that remains open in classes of lower energy.

Our aim here is to establish further properties of the classes $\mathcal{E}^p(X, \omega)$. It will allow us to give a complete characterization of the range of the complex Monge–Ampère operator on them (see Theorem 4.2).

Proposition 2.10. *There exists $C_p > 0$ such that for all $0 \geqslant \varphi_0, \ldots, \varphi_n \in PSH(X, \omega) \cap L^\infty(X)$,*

$$0 \leqslant \int\limits_X (-\varphi_0)^p \omega_{\varphi_1} \wedge \cdots \wedge \omega_{\varphi_n} \leqslant C_p \max_{0 \leqslant j \leqslant n} \left[\int\limits_X (-\varphi_j)^p \omega_{\varphi_j}^n \right].$$

In particular the class $\mathcal{E}^p(X, \omega)$ is starshaped and convex.

Proof. It follows from Proposition 2.5 applied with $\varphi = \varphi_0$, $\psi = \varphi_1$ and $T = \omega_{\varphi_2} \wedge \cdots \wedge \omega_{\varphi_n}$ that

$$\int\limits_X (-\chi) \circ \varphi_0 \omega_{\varphi_1} \wedge T \leqslant 2 \int\limits_X (-\chi) \circ \varphi_0 \omega_{\varphi_0} \wedge T + 2 \int\limits_X (-\chi) \circ \varphi_1 \omega_{\varphi_1} \wedge T, \qquad (7)$$

thus we can assume $\varphi_0 = \varphi_1$ in the sequel.

Set $u = \varepsilon \sum_{i=1}^n \varphi_i$, where $\varepsilon > 0$ is small enough ($\varepsilon < (2n)^{-1/p}$ will do). Observe that $\omega_u^n \geqslant \varepsilon^n \omega_{\varphi_1} \wedge \cdots \wedge \omega_{\varphi_n}$, hence it suffices to get control on $\int_X (-\chi) \circ \varphi_i \omega_u^n$ for all $1 \leqslant i \leqslant n$ to conclude. By using Proposition 2.5 again,

$$\int\limits_X (-\chi) \circ \varphi_i \omega_u^n \leqslant 2E_\chi(\varphi_i) + 2E_\chi(u),$$

where $E_\chi(u) := \int_X (-\chi) \circ u \omega_u^n$. By subadditivity and homogeneity of $-\chi$, we get $E_\chi(u) \leqslant \varepsilon^p \sum_{j=1}^n \int_X (-\chi) \circ \varphi_j \omega_u^n$, hence

$$\sum_{i=1}^n \int_X (-\chi) \circ \varphi_i \omega_u^n \leqslant \frac{2}{1 - 2n\varepsilon^p} \sum_{i=1}^n E_\chi(\varphi_i). \tag{8}$$

We deduce from (7), (8) and $\omega_u^n \geqslant \varepsilon^n \omega_{\varphi_1} \wedge \cdots \wedge \omega_{\varphi_n}$ that

$$0 \leqslant \int_X (-\varphi_0)^p \omega_{\varphi_1} \wedge \cdots \wedge \omega_{\varphi_n} \leqslant \frac{4n}{\varepsilon^n [1 - 2n\varepsilon^p]} \max_{1 \leqslant i \leqslant n} E_\chi(\varphi_i).$$

Each class $\mathcal{E}_\chi(X, \omega)$ is clearly starshaped. It follows from previous inequality that the classes $\mathcal{E}^p(X, \omega)$ are also convex. $\quad\square$

It follows from this result that if some functions $\psi_j \in \mathcal{E}^p(X, \omega)$, $\psi_j \leqslant 0$, have uniformly bounded energy $\sup_{j \geqslant 0} E_\chi(\psi_j) < +\infty$, then

$$\psi := \sum_{j \geqslant 1} 2^{-j} \psi_j \in \mathcal{E}^p(X, \omega).$$

This observation, together with the homogeneity property of $\chi(t) = -(-t)^p$ allows us to derive the following quantitative characterization of integrability properties with respect to a given measure μ.

Lemma 2.11. *Let μ be a probability measure on X. Then $\mathcal{E}^p(X, \omega) \subset L^p(\mu)$ if and only if there exists $C > 0$ such that for all $\psi \in PSH(X, \omega) \cap L^\infty(X)$ with $\sup_X \psi = -1$,*

$$0 \leqslant \int_X (-\psi)^p \, d\mu \leqslant C \left[\int_X (-\psi)^p \omega_\psi^n \right]^{\frac{p}{p+1}}. \tag{9}$$

Proof. Assume on the contrary that (9) is not satisfied, then there exists $\psi_j \in PSH(X, \omega) \cap L^\infty(X)$, $\sup_X \psi_j = -1$, such that

$$\int_X (-\chi) \circ \psi_j \, d\mu \geqslant 4^{jp} M_j^{\frac{p}{p+1}}, \quad \text{where } M_j := E_\chi(\psi_j).$$

If (M_j) is bounded we consider $\psi := \sum_{j \geqslant 1} 2^{-j} \psi_j$. This is a ω-psh function which belongs to $\mathcal{E}^p(X, \omega)$ by Proposition 2.10. Now

$$\int_X (-\chi) \circ \psi \, d\mu \geqslant \int_X (-\chi) (2^{-j} \psi_j) \, d\mu \geqslant 2^{jp} M_j^{\frac{p}{p+1}} \geqslant 2^{jp},$$

because $\psi_j \leqslant -1$, hence $M_j \geqslant 1$. Thus $\int_X (-\chi) \circ \psi \, d\mu = +\infty$, a contradiction.

We similarly obtain a contradiction if (M_j) admits a bounded subsequence, so we can assume $M_j \to +\infty$ and $M_j \geqslant 1$. Set $\varphi_j = \varepsilon_j \psi_j$, where $\varepsilon_j = M_j^{-1/(p+1)}$. We show here-below that $E_\chi(\varphi_j)$ is bounded, hence $\varphi := \sum_{j \geqslant 1} 2^{-j} \varphi_j \in \mathcal{E}^p(X, \omega)$ by Proposition 2.10. Now

$$\int\limits_X (-\chi) \circ \varphi \, d\mu \geqslant 2^{-jp} \int\limits_X (-\chi) \circ \varphi_j \, d\mu \geqslant 2^{-jp} \varepsilon_j^p \int\limits_X (-\chi) \circ \psi_j \, d\mu$$

$$\geqslant 2^{jp} \varepsilon_j^p M_j^{p/(p+1)} = 2^{jp},$$

hence $\int_X (-\chi) \circ \varphi \, d\mu = +\infty$, a contradiction.

It remains to check that $(E_\chi(\varphi_j))$ is indeed bounded. Observe that $\omega_{\varphi_j} \leqslant \varepsilon_j \omega_{\psi_j} + \omega$, thus

$$E_\chi(\varphi_j) = \varepsilon_j^p \int\limits_X (-\chi) \circ \psi_j \omega_{\varphi_j}^n$$

$$\leqslant \varepsilon_j^p \left[\int\limits_X (-\chi) \circ \psi_j \omega^n + 2^n \varepsilon_j E_\chi(\psi_j) \right]$$

$$\leqslant \int\limits_X (-\psi_j) \omega^n + 2^n.$$

We have used here the definition of ε_j, $\varepsilon_j^{p+1} E_\chi(\psi_j) = 1$, the bounds $\varepsilon_j \leqslant 1$, $|\chi|(t) \leqslant |t|$, and the inequalities

$$E_\chi(\psi_j) = \int\limits_X (-\chi) \circ \psi_j \omega \wedge \omega_{\psi_j}^{n-1} + \int\limits_X \chi' \circ \psi_j \, d\psi_j \wedge d^c \psi_j \wedge \omega_{\psi_j}^{n-1}$$

$$\geqslant \int\limits_X (-\chi) \circ \psi_j \omega \wedge \omega_{\psi_j}^{n-1} \geqslant \int\limits_X (-\chi) \circ \psi_j \omega^k \wedge \omega_{\psi_j}^{n-k},$$

for all $1 \leqslant k \leqslant n-1$. Now $\int_X (-\psi_j) \omega^n$ is bounded because $\sup_X \psi_j = -1$ (see Proposition 2.7 in [19]), hence $(E_\chi(\varphi_j))$ is bounded. $\quad\square$

Remark 2.12. These results actually apply for any weight $\chi \in \mathcal{W}^-$ close enough to a homogeneous weight, as we indicate in Section 3.3.

2.4. Some examples

The function $L(t) := -\log(1-t)$ belongs to $\mathcal{W}^-$, as well as $L_p(t) := L \circ \cdots \circ L(t)$ (p times). It is in fact necessary to consider functions χ with arbitrarily slow growth in order to understand the range of the complex Monge–Ampère operator on $\mathcal{E}(X, \omega)$, as the following example shows.

Example 2.13. Fix $h \in \mathcal{W}^- \cap \mathcal{C}^\infty(\mathbb{R}^-, \mathbb{R}^-)$ with $h'(-\infty) = 0$, and consider

$$\varphi \colon z \in \mathbb{C} \subset \mathbb{P}^1 = \mathbb{C} \cup \{\infty\} \mapsto h\left(\log|z| - \frac{1}{2}\log\left[1 + |z|^2\right] - 1\right) \in \mathbb{R}^-,$$

with the convention $\varphi(\infty) = h(-1)$. This is an ω-subharmonic function on the Riemann sphere $\mathbb{P}^1$, where ω denotes the Fubini–Study volume form. Note that φ is smooth but at the origin $0 \in \mathbb{C}$ and

$$\omega_\varphi = \left[f(z) + o\big(f(z)\big) \right] dV, \quad f(z) = \frac{c}{|z|^2} h''\big(\log|z|\big), \text{ near } 0,$$

where dV denotes the Euclidean volume form and $c > 0$. One thus checks that $\varphi \in \mathcal{E}_\chi(X, \omega)$ if χ does not grow too fast (e.g. $|\chi|(t) \lesssim [h' \circ h^{-1}(t)]^{-1/2}$), while $\chi \circ \varphi \notin L^1(\omega_\varphi)$ (hence $\varphi \notin \mathcal{E}_\chi(X, \omega)$) if e.g. $|\chi(t)| \geqslant [h' \circ h^{-1}(t)]^{-1}$.

As a concrete example, consider $h(t) = t \log(1 - t)$. The reader can check that h satisfies our requirements, and that in this case $\omega_\varphi = f \, dV$ is a measure with density such that

$$f(z) \sim \frac{c'}{|z|^2(-\log|z|^2)[\log(-\log|z|^2)]^2} \quad \text{near } 0.$$

In particular $|\varphi|^p \notin L^1(\omega_\varphi)$ for any $p > 0$, moreover $\log[1 - \varphi] \notin L^1(\omega_\varphi)$. One can obtain explicit examples with even slower growth by considering $h(t) = L^p(t) := L \circ \cdots \circ L(t)$ (p times), where $L(t) = -\log(1 - t)$.

Our next example shows that it is possible to define and control the complex Monge–Ampère measure of functions with slightly attenuated singularites, although these functions need not have gradient in $L^2(X)$.

Example 2.14. For $\varphi \in PSH(X, \omega)$, $\varphi \leqslant -1$, and $0 < p < 1$, we set $\varphi_p := -(-\varphi)^p$. Then

$$\omega_{\varphi_p} = p(-\varphi)^{p-1}\omega_\varphi + \left[1 - p(-\varphi)^{p-1} \right]\omega + p(1-p)(-\varphi)^{p-2}\, d\varphi \wedge d^c\varphi \geqslant 0,$$

hence $\varphi_p \in PSH(X, \omega)$. One can compute similarly $(\omega + dd^c\varphi_p)^n$ and check that there exists $\alpha(p, n) > 0$ small enough and $C(p, n) > 0$ independent of φ such that

$$\int_X (-\varphi_p)^{\alpha(p,n)}\big(\omega + dd^c\varphi_p\big)^n \leqslant C(p, n).$$

This shows that $\varphi_p \in \mathcal{E}_\chi(X, \omega)$ with $\chi(t) = -(-t)^{\alpha(p,n)}$, as soon as $p < 1$. In particular the complex Monge–Ampère operator of $\varphi_p = -(-\varphi)^p$ is well defined for any ω-psh function $\varphi \leqslant -1$ and for any $p < 1$, although $\nabla\varphi_p$ does not belong to $L^2(X)$ when ω_φ is the current of integration along a complex hypersurface and $1/2 \leqslant p < 1$.

3. High energy classes

We now consider further classes of ω-plurisubharmonic with milder singularities. Not only are they interesting in themselves, but we also actually need first to understand the range of the complex Monge–Ampère operator on them, before being able to describe the corresponding range on previous classes (see the proofs of Theorems 4.1, 4.2).

We let $\mathcal{W}^+$ denote the set of *concave* increasing functions $\chi : \mathbb{R}^- \to \mathbb{R}^-$ such that $\chi(0) = 0$, $\chi(-\infty) = -\infty$ and we set

$$\mathcal{W}_M^+ := \left\{ \chi \in \mathcal{W}^+ \mid 0 \leqslant |t\chi'(t)| \leqslant M|\chi(t)|, \text{ for all } t \in \mathbb{R}^- \right\},$$

where $M > 0$ is independent of t. For $\chi \in \mathcal{W}^+$ we consider

$$\mathcal{E}_\chi(X, \omega) := \left\{ \varphi \in \mathcal{E}(X, \omega) \mid \chi(-|\varphi|) \in L^1\big((\omega + dd^c\varphi)^n\big) \right\}.$$

Observe that for all weights $\chi_1 \in \mathcal{W}^-$, $\chi_2 \in \mathcal{W}^+$, we have

$$PSH(X, \omega) \cap L^\infty(X) \subset \mathcal{E}_{\chi_2}(X, \omega) \subset \mathcal{E}^1(X, \omega) \subset \mathcal{E}_{\chi_1}(X, \omega) \subset \mathcal{E}(X, \omega).$$

Thus the class $\mathcal{E}^1(X, \omega)$ is the turning point between classes of low energy and those of high energy.

3.1. Gradient of quasiplurisubharmonic functions

The classes $\mathcal{E}_\chi(X, \omega)$, $\chi \in \mathcal{W} := \mathcal{W}^- \cup \mathcal{W}^+$, form a whole scale among unbounded ω-plurisubharmonic functions, joining the maximal class $\mathcal{E}(X, \omega)$ of ω-psh functions φ whose Monge–Ampère measure $(\omega + dd^c\varphi)^n$ does not charge pluripolar sets (Proposition 2.2) to the class $PSH(X, \omega) \cap L^\infty(X)$ of bounded ω-psh functions.

Proposition 3.1.

$$PSH(X, \omega) \cap L^\infty(X) = \bigcap_{\chi \in \mathcal{W}^+} \mathcal{E}_\chi(X, \omega).$$

Proof. Clearly a bounded ω-psh function belongs to $\mathcal{E}_\chi(X, \omega)$ for all $\chi \in \mathcal{W}$.

Conversely assume $\varphi \in \mathcal{E}_\chi(X, \omega)$ for any weight $\chi \in \mathcal{W}$. We claim there exists $t_\infty > 0$ such that $(\omega + dd^c\varphi)^n(\varphi < -t_\infty) = 0$. Otherwise we could set

$$\chi(t) := -h(-t), \quad \text{where } h'(t) = \frac{1}{\omega_\varphi^n(\varphi < -t)}$$

is well defined for all $t > 0$. Now $\chi \in \mathcal{W}^+$ and, assuming $\varphi \leqslant 0$,

$$\int_X (-\chi) \circ \varphi \, \omega_\varphi^n = \int_0^{+\infty} h'(t) \omega_\varphi^n(\varphi < -t) \, dt = +\infty,$$

contradicting $\varphi \in \mathcal{E}_\chi(X, \omega)$.

We infer $(\omega + dd^c\varphi)^n(\varphi < -t) = 0$ for all $t \geqslant t_\infty$. It follows from the comparison principle that $\text{Vol}_\omega(\varphi < -t_\infty) \leqslant \int_{(\varphi < -t_\infty)} \omega_\varphi^n$. Thus $\varphi \geqslant -t_\infty$ almost everywhere, hence everywhere. $\quad\square$

One of the main differences between functions with finite low-energy and those with finite high-energy resides in the following observation.

Proposition 3.2. *Fix $\chi \in \mathcal{W}^+$. Then any function $\varphi \in \mathcal{E}_\chi(X, \omega)$ is such that $\nabla\varphi \in L^2(\omega^n)$.*

Proof. It suffices to establish this result when $\chi(t) = t$, i.e. when $\varphi \in \mathcal{E}^1(X, \omega)$. Let T be a positive current of bidimension $(1, 1)$ and fix $\varphi \in PSH(X, \omega) \cap L^\infty(X)$ such that $\varphi \leqslant 0$. Then

$$\int_X (-\varphi)\omega_\varphi \wedge T = \int_X (-\varphi)\omega \wedge T + \int_X d\varphi \wedge d^c\varphi \wedge T \geqslant \int_X (-\varphi)\omega \wedge T.$$

A repeated application of this inequality, applied with $T = \omega_\varphi^j \wedge \omega^{n-1-j}$, therefore yields

$$0 \leqslant \int_X d\varphi \wedge d^c\varphi \wedge \omega^{n-1} \leqslant \int_X (-\varphi)\omega_\varphi \wedge \omega^{n-1} \leqslant \int_X (-\varphi)\omega_\varphi^n. \tag{10}$$

Fix now $\varphi \in \mathcal{E}^1(X, \omega)$ and set $\varphi_j := \max(\varphi, -j)$. We can assume without loss of generality $\varphi, \varphi_j \leqslant 0$. It follows from (10) that the sequence of gradients $(\nabla\varphi_j)$ has uniformly bounded L^2-norm. Since $\nabla\varphi_j$ converges towards $\nabla\varphi$ in the weak sense of distributions, it follows from the weak L^2-compactness of $\nabla\varphi_j$ that $\nabla\varphi \in L^2(X)$. $\quad\square$

We can prove similarly that $\varphi \in \mathcal{E}_\chi(X, \omega)$ satisfies all intermediate local boundedness conditions required in order for the local complex Monge–Ampère operator $(\omega + dd^c\varphi)^n$ to be well defined (see [9]). It follows however from Example 2.14 that there are functions $\varphi \in \mathcal{E}_\chi(X, \omega)$ such that $\nabla\varphi \notin L^2(X)$ when $\chi \in \mathcal{W}^-$.

3.2. Extension of Calabi's uniqueness result

When φ, ψ are smooth functions such that $\omega_\varphi, \omega_\psi$ are Kähler forms, it has been shown by E. Calabi [10] that

$$\left(\omega + dd^c\varphi\right)^n = \left(\omega + dd^c\psi\right)^n \quad\Longrightarrow\quad \varphi - \psi \equiv \text{constant}.$$

Calabi's proof has been generalized by Z. Blocki [7] to the case where $\varphi, \psi \in PSH(X, \omega) \cap L^\infty(X)$. In both cases, the proof consists in showing that $\nabla(\varphi - \psi) = 0$, by using ingenious integration by parts.

We push this argument further by showing that there is still uniqueness in the class $\mathcal{E}^1(X, \omega)$ (Theorem B in the introduction). We rely heavily on the fact that $\nabla\varphi \in L^2(\omega^n)$ when $\varphi \in \mathcal{E}^1(X, \omega)$. It is an interesting open question to establish the uniqueness of solutions in classes of lower energy.

Theorem 3.3. *Assume $(\omega + dd^c\varphi)^n \equiv (\omega + dd^c\psi)^n$, where $\varphi \in \mathcal{E}(X, \omega)$ and $\psi \in \mathcal{E}^1(X, \omega)$. Then $\varphi - \psi$ is constant.*

Proof. We assume first that both φ and ψ are in $\mathcal{E}^1(X, \omega)$. Set $f = (\varphi - \psi)/2$ and $h = (\varphi + \psi)/2$. It follows from Proposition 2.10 that $h \in \mathcal{E}^1(X, \omega)$. We can assume without loss of generality $\varphi, \psi \leqslant -C_\omega$, where $C_\omega > 0$ is chosen so that $\int_X (-h)\omega_h^n \geqslant 1$. We are going to

prove that $\nabla f = 0$ by showing that $\int_X df \wedge d^c f \wedge \omega^{n-1} = 0$. This will be done by establishing an upper bound involving the E_1-energy of h and by using the following formal computation,

$$0 \leqslant \int_X df \wedge d^c f \wedge \omega_h^{n-1} \leqslant \int_X df \wedge d^c f \wedge \sum_{l=0}^{n-1} \omega_\varphi^l \wedge \omega_\psi^{n-1-l} = \int_X \frac{-f}{2} (\omega_\varphi^n - \omega_\psi^n) = 0.$$

Of course we should (and could) justify these integration by parts and make sense of all terms involved in this computation. We are rather going to establish the following a priori bound, when φ, ψ are bounded,

$$\int_X df \wedge d^c f \wedge \omega^{n-1} \leqslant 3^{n-1} \left(\int_X df \wedge d^c f \wedge \omega_h^{n-1} \right)^{1/2^{n-1}} \int_X (-h)\omega_h^n. \tag{†}$$

Approximating φ, ψ by $\varphi_j = \max(\varphi, -j)$, $\psi_j = \max(\psi, -j)$, it will then follow from Theorem 2.6 and (†) that $2f = \varphi - \psi$ is constant, if $\omega_\varphi^n \equiv \omega_\psi^n$.

The a priori bound (†) follows by applying inductively the following inequality, where $T = \omega_h^l \wedge \omega^{n-2-l}$ is a closed positive current of bidimension $(2, 2)$ and $l = n - 2, \ldots, 0$,

$$\int_X df \wedge d^c f \wedge \omega \wedge T \leqslant 3 \left(\int_X df \wedge d^c f \wedge \omega_h \wedge T \right)^{1/2} \left(\int_X (-h)\omega_h^2 \wedge T \right)^{1/2}. \tag{$†_T$}$$

Indeed observe that for $T = \omega_h^l \wedge \omega^{n-2-l}$,

$$\int_X (-h)\omega_h^{2+l} \wedge \omega^{n-2-l} \leqslant \int_X (-h)\omega_h^n, \quad \text{for all } l = n - 2, \ldots, 0$$

and use that $\int_X (-h)\omega_h^n \geqslant 1$ since $h \leqslant -C_\omega$, to derive (†) from ($†_T$).

We now establish ($†_T$). Note that

$$df \wedge d^c f \wedge \omega = df \wedge d^c f \wedge \omega_h - df \wedge d^c f \wedge dd^c h,$$

hence integrating by parts yields

$$\int_X df \wedge d^c f \wedge \omega \wedge T = \int_X df \wedge d^c f \wedge \omega_h \wedge T + \int_X df \wedge d^c h \wedge \frac{(\omega_\varphi - \omega_\psi)}{2} \wedge T.$$

It follows from the Cauchy–Schwarz inequality that

$$\left| \int_X df \wedge d^c h \wedge \omega_\varphi \wedge T \right| \leqslant 2 \left(\int_X df \wedge d^c f \wedge \omega_h \wedge T \right)^{1/2} \cdot \left(\int_X dh \wedge d^c h \wedge \omega_h \wedge T \right)^{1/2}.$$

Note that we can get a similar control on $\int_X df \wedge d^c f \wedge \omega_\psi \wedge T$. Thus $(\dagger_T)$ follows from the following last observation

$$\int_X df \wedge d^c f \wedge S = \frac{1}{4} \int_X (\psi - \varphi)(\omega_\varphi - \omega_\psi) \wedge S \leqslant \int_X (-h)\omega_h \wedge S,$$

where S is any positive closed current of bidimension $(1, 1)$.

It remains to treat the case where $\psi \in \mathcal{E}^1(X, \omega)$ but φ a priori merely belongs to $\mathcal{E}(X, \omega)$. We normalize φ, ψ by requiring $\sup_X \varphi = \sup_X \psi$, hence we need to show that $\varphi \equiv \psi$. Set $\varphi_j := \max(\varphi, \psi - j)$. It follows from Lemma 2.3 that $\varphi_j \in \mathcal{E}^1(X, \omega)$. By Proposition 3.4 below, $(\omega + dd^c \varphi_j)^n = (\omega + dd^c \psi)^n$. Now $\sup_X \varphi_j = \sup_X \varphi = \sup_X \psi$ if $j \geqslant 0$, hence $\varphi_j \equiv \psi$ by previous analysis. Letting $j \to +\infty$, we infer $\varphi \equiv \psi$. □

Regarding uniqueness in the class $\mathcal{E}(X, \omega)$, we have the following observation, of independent interest.

Proposition 3.4. *Assume* $\varphi, \psi \in \mathcal{E}(X, \omega)$ *are such that* $(\omega + dd^c \varphi)^n = (\omega + dd^c \psi)^n$. *Then*

$$\left(\omega + dd^c \max[\varphi, \psi]\right)^n = \left(\omega + dd^c \varphi\right)^n = \left(\omega + dd^c \psi\right)^n.$$

Proof. Applying Corollary 1.10 with $\mu = \omega_\varphi^n = \omega_\psi^n$ yields

$$\left(\omega + dd^c \max[\varphi, \psi]\right)^n \geqslant \mu = \left(\omega + dd^c \varphi\right)^n,$$

whence equality, since these measures have the same mass. □

3.3. Quasihomogeneous weights

In this section we fix a weight $\chi \in \mathcal{W}_M^+$, $M \geqslant 1$. Many results in this section are inspired by their local analogues, obtained by U. Cegrell [11] when $\chi(t) = -(-t)^p$, $p \geqslant 1$.

The following result is the $\mathcal{W}_M^+$-version of Lemma 2.3.

Lemma 3.5. *Let* $\varphi, \psi \in PSH(X, \omega) \cap L^\infty(X)$ *with* $\varphi \leqslant \psi \leqslant 0$. *Then*

$$0 \leqslant \int_X (-\chi) \circ \psi \left(\omega + dd^c \psi\right)^n \leqslant (M + 1)^n \int_X (-\chi) \circ \varphi \left(\omega + dd^c \varphi\right)^n.$$

Proof. The proof follows from a repeated application of the following inequality: if T is a positive closed current of bidimension $(1, 1)$, then

$$0 \leqslant \int_X (-\chi) \circ \varphi \omega_\psi \wedge T \leqslant (M + 1) \int_X (-\chi) \circ \varphi \omega_\varphi \wedge T. \qquad (11)$$

Integrating by parts yields

$$\int_X (-\chi) \circ \varphi \omega_\psi \wedge T = \int_X (-\chi) \circ \varphi \omega \wedge T + \int_X (-\psi) dd^c (\chi \circ \varphi) \wedge T.$$

The first integral in the right-hand side is bounded from above by $\int_X (-\chi) \circ \varphi \omega_\varphi \wedge T$ because $\int_X \chi' \circ \varphi \, d\varphi \wedge d^c\varphi \wedge T \geqslant 0$. Observe now that

$$dd^c \chi \circ \varphi = \chi'' \circ \varphi \, d\varphi \wedge d^c\varphi + \chi' \circ \varphi \, dd^c\varphi \leqslant \chi' \circ \varphi \omega_\varphi,$$

thus

$$\int_X (-\psi) dd^c (\chi \circ \varphi) \wedge T \leqslant \int_X (-\psi) \chi' \circ \varphi \omega_\varphi \wedge T \leqslant M \int_X (-\chi) \circ \varphi \omega_\varphi \wedge T,$$

since $(-\psi)\chi' \circ \varphi \leqslant (-\varphi)\chi' \circ \varphi \leqslant M(-\chi) \circ \varphi$. This yields (11). $\quad\square$

We let the reader check that Corollary 2.4, Theorem 2.6 and Corollary 2.7 hold when $\chi \in \mathcal{W}_M^+$, with exactly the same proof. We now establish the high-energy version of Proposition 2.5.

Proposition 3.6. *Let T be a positive closed current of bidimension (j, j) on X, $0 \leqslant j \leqslant n$, and let $\varphi, \psi \leqslant 0$ be bounded ω-psh functions. Then*

$$0 \leqslant \int_X (-\chi) \circ \varphi \omega_\psi^j \wedge T \leqslant 2M \int_X (-\chi) \circ \varphi \omega_\varphi^j \wedge T + 2M \int_X (-\chi) \circ \psi \omega_\psi^j \wedge T.$$

Proof. The proof is the same as that of Proposition 2.5 except that we have to replace inequality $0 \leqslant \chi'(2t) \leqslant \chi'(t)$ by $0 \leqslant \chi'(2t) \leqslant M\chi'(t)$. The latter follows from the concavity property of χ which (together with $\chi(0) = 0$) yields $|\chi(t)| \leqslant |t|\chi'(t)$ and $|\chi(2t)| \leqslant 2|\chi(t)|$ for all $t \in \mathbb{R}^-$. Therefore

$$0 \leqslant \frac{\chi'(2t)}{\chi'(t)} = \left| \frac{2t\chi'(2t)}{\chi(2t)} \right| \cdot \left| \frac{\chi(2t)}{2\chi(t)} \right| \cdot \left| \frac{\chi(t)}{t\chi'(t)} \right| \leqslant M. \quad\square$$

Our next observation is useful to establish convexity properties of the class $\mathcal{E}_\chi(X, \omega)$.

Lemma 3.7. *For all $0 \leqslant \varepsilon \leqslant 1$ and for all $t < -1$,*

$$0 \leqslant \varepsilon^M |\chi(t)| \leqslant |\chi(\varepsilon t)| \leqslant \varepsilon |\chi(t)|.$$

The proof follows easily from the concavity of χ, the normalization $\chi(0) = 0$ and the definition of $\mathcal{W}_M^+$. These inequalities can be interpreted as a weak-homogeneity property satisfied by the weights $\chi \in \mathcal{W}_M^+$. This allows us to show that $\mathcal{E}_\chi(X, \omega)$ is always convex in this case.

Proposition 3.8. *Fix $M > 0$ and $\chi \in \mathcal{W}_M^+$. There exists $C_\chi > 0$ such that for all $0 \geqslant \varphi_0, \ldots, \varphi_n \in PSH(X, \omega) \cap L^\infty(X)$,*

$$0 \leqslant \int_X (-\chi) \circ \varphi_0 \omega_{\varphi_1} \wedge \cdots \wedge \omega_{\varphi_n} \leqslant C_\chi \max_{0 \leqslant j \leqslant n} \left[\int_X (-\chi) \circ \varphi_j \omega_{\varphi_j}^n \right].$$

In particular the class $\mathcal{E}_\chi(X, \omega)$ is starshaped and convex.

Proof. The proposition follows from the inequality

$$0 \leqslant \int_X (-\chi) \circ \varphi_0 \omega_{\varphi_1} \wedge \cdots \wedge \omega_{\varphi_n} \leqslant \frac{4nM}{\varepsilon^n [1 - 2n\varepsilon M]} \max_{0 \leqslant j \leqslant n} E_\chi(\varphi_j). \tag{12}$$

The proof is identical to that of Proposition 2.10 except that Proposition 2.5 has to be replaced by Proposition 3.6, and the subadditivity and homogeneity of $x \in \mathbb{R}^+ \mapsto x^p \in \mathbb{R}^+$, $0 \leqslant p \leqslant 1$, has to be replaced by the convexity property of $x \in \mathbb{R}^+ \mapsto (-\chi)(-x) \in \mathbb{R}^+$, which yields

$$E_\chi \left(\varepsilon \sum_{i=1}^n \varphi_i \right) \leqslant \varepsilon \sum_{i=1}^n \int_X (-\chi) \circ \varphi_i \omega_u^n,$$

for $u := \varepsilon \sum_{i=1}^n$ and $0 < \varepsilon < [2nM_\chi]^{-1}$.   □

We now give a high-energy version of Lemma 2.11. Note that our proof of Lemma 2.11 uses the full homogeneity of $x \mapsto x^p$. Let us say that a weight $\chi \in \mathcal{W}^+$ is *quasihomogeneous* if there exist $C, M \geqslant 1$ and $0 \leqslant q < 1$ such that for all $0 \leqslant \varepsilon \leqslant 1$ and for all $t \leqslant -1$,

$$0 \leqslant C^{-1}\varepsilon^M |\chi(t)| \leqslant |\chi(\varepsilon t)| \leqslant C\varepsilon^{M-q} |\chi(t)|.$$

The functions $\chi_p(t) = -(-t)^p$, $p \geqslant 1$, belong to $\mathcal{W}^+$ and obviously satisfy the previous condition. Here again we shall use the notation

$$\mathcal{E}^p(X, \omega) := \mathcal{E}_\chi(X, \omega), \quad \text{when } \chi(t) = -(-t)^p.$$

These classes have been studied in a local context by U. Cegrell [11]. They are easier to understand thanks to the homogeneity property of the weight.

Lemma 3.9. *Let $\chi \in \mathcal{W}_M^+$ be a quasihomogeneous weight and let μ be a probability measure on X.*
Then $\chi \circ \mathcal{E}_\chi(X, \omega) \subset L^1(\mu)$ if and only if there exists $C > 0$ such that for all functions $\varphi \in PSH(X, \omega) \cap L^\infty(X)$ normalized by $\sup_X \varphi = -1$, one has

$$0 \leqslant \int_X (-\chi) \circ \varphi \, d\mu \leqslant C \left(\int_X (-\chi) \circ \varphi \omega_\varphi^n \right)^\gamma,$$

where $0 < \gamma := M/(M - q + 1) < 1$.

The proof, in the same vein as that of Lemma 2.11, is left to the reader.

Example 3.10. The weight $\chi(t) = -(-t)^p [\log(e - t)]^a \in \mathcal{W}^+$, $a > 0$, is an example of quasihomogeneous weight which is not homogeneous.

4. Range of the complex Monge–Ampère operator

We now turn to the central question of describing the range of the complex Monge–Ampère operator on classes $\mathcal{E}_\chi(X, \omega)$, and on class $\mathcal{E}(X, \omega)$.

An obvious necessary condition for solving the Monge–Ampère equation

$$\left(\omega + dd^c \varphi\right)^n = \mu, \qquad \qquad \text{(MA}_\mu\text{)}$$

is that μ should be a positive Radon measure of total mass $\int_X \omega^n$ on X. For simplicity we assume throughout the rest of this section that ω is a *Kähler form*, normalized by $\int_X \omega^n = 1$, and that μ is a probability measure.

4.1. The classes $\mathcal{E}_\chi(X, \omega)$

We fix here an increasing function $\chi : \mathbb{R}^- \to \mathbb{R}^-$ such that $\chi \in \mathcal{W}^- \cup \mathcal{W}_M^+$, $M \geqslant 1$. It follows from Propositions 2.5 and 3.6 that if $\mu = (\omega + dd^c \varphi)^n$ for some function $\varphi \in \mathcal{E}_\chi(X, \omega)$, then

(I_χ) $\chi \circ \mathcal{E}_\chi(X, \omega) \subset L^1(\mu)$.

We now show that the converse is true under a quantitative version of (I_χ).

Theorem 4.1. *Suppose there exists $F : \mathbb{R}^+ \to \mathbb{R}^+$ with $\lim_{+\infty} F(t)/t = 0$, such that for all $\psi \in PSH(X, \omega) \cap L^\infty(X)$, $\sup_X \psi = -1$,*

(II_χ) $0 \leqslant \int_X (-\chi) \circ \psi \, d\mu \leqslant F(E_\chi(\psi))$.

Then there exists $\varphi \in \mathcal{E}_\chi(X, \omega)$ such that

$$\mu = \left(\omega + dd^c \varphi\right)^n \quad and \quad \sup_X \varphi = 0.$$

It is an interesting problem to determine whether conditions (I_χ) and (II_χ) are equivalent (obviously (II_χ) implies (I_χ)). This is the case when $\chi(t) = -(-t)^p$, $p > 0$, thanks to Lemmas 2.11, 3.9, with $F(t) = t^{p/(p+1)}$. Thus we obtain a complete characterization of the range of the Monge–Ampère operator in this case (Theorem C in our introduction).

Theorem 4.2. *Let μ be a probability measure on X and $p > 0$. There exists $\varphi \in \mathcal{E}^p(X, \omega)$ such that $\mu = (\omega + dd^c \varphi)^n$ if and only if $\mathcal{E}^p(X, \omega) \subset L^p(\mu)$.*

The proof of Theorem 4.1 will occupy the rest of Section 4.1. The proof follows the lines of U. Cegrell's one, in the local case [11]:

- We approximate μ by smooth probability volume forms μ_j using local convolutions and a partition of unity.
- We invoke Yau's solution of the Calabi conjecture to find uniquely determined ω-psh functions φ_j such that $\mu_j = \omega_{\varphi_j}^n$, $\sup_X \varphi_j = -1$.
- Since ω-psh functions φ normalized by $\sup_X \varphi = -1$ form a compact subset of $L^1(X)$, we can assume that $\varphi_j \to \varphi$ in $L^1(X)$.

- The quantitative integrability condition (II_χ) guarantees, at least when $\chi(t) = t$,

$$\sup_j \int (-\chi) \circ \varphi_j \omega_{\varphi_j}^n < +\infty,$$

  hence yields $\varphi \in \mathcal{E}_\chi(X, \omega)$.
- The delicate point is then to show that $\omega_{\varphi_j}^n \to \omega_\varphi^n$. This is done by showing that $\int |\varphi_j - \varphi| \, d\mu_j \to 0$ and invoking Proposition 2.8. We need here to assume first that μ is suitably dominated by the Monge–Ampère capacity (in the spirit of [21]).
- We then treat the general case by using a Radon–Nikodym decomposition of the measure μ.

Here follow the technical details. Let μ be an arbitrary probability measure on X. Let $\{U_i\}$ be a finite covering of X by open sets U_i which are biholomorphic to the unit ball of $\mathbb{C}^n$. In each U_i we let $\mu_\varepsilon^{U_i} := \mu_{|U_i} * \varrho_\varepsilon$ denote local regularization of $\mu_{|U_i}$ by means of convolution with radial non-negative smooth approximations ϱ_ε of the Dirac mass. Let $\{\theta_i\}$ be a partition of unity subordinate to $\{U_i\}$ and set

$$\mu_j := c_j \left[\sum_i \theta_i \mu_{\varepsilon_j}^{U_i} + \varepsilon_j \omega^n \right],$$

where $\varepsilon_j \searrow 0$ and $c_j \nearrow 1$ is chosen so that $\mu_j(X) = 1$. Thus the μ_j's are smooth probability volume forms which converge weakly towards μ. It follows from the solution of the Calabi conjecture [28], that there exists a unique function $\varphi_j \in PSH(X, \omega) \cap \mathcal{C}^\infty(X)$ such that

$$\mu_j = \omega_{\varphi_j}^n \quad \text{and} \quad \sup_X \varphi_j = -1.$$

Recall from Proposition 1.7 in [19] that $\mathcal{F} := \{\varphi \in PSH(X, \omega) \mid \sup_X \varphi = -1\}$ is a compact subset of $L^1(X)$. Passing to a subsequence if necessary, we can therefore assume $\varphi_j \to \varphi$ in $L^1(X)$, where $\varphi \in PSH(X, \omega)$ with $\sup_X \varphi = -1$.

Lemma 4.3. *There exists $C > 1$ such that for all $j \in \mathbb{N}$,*

$$\int_X (-\varphi_j) \omega_{\varphi_j}^n \leqslant C \int_X (-\varphi_j) \, d\mu.$$

If $\mathcal{E}^1(X, \omega) \subset L^1(\mu)$, then $\sup_j \int_X (-\varphi_j) \omega_{\varphi_j}^n < +\infty$, hence $\varphi \in \mathcal{E}^1(X, \omega)$.

Proof. Since $c_j \to 1$ and $\varepsilon_j \to 0$, we can write

$$\int_X (-\varphi_j) \omega_{\varphi_j}^n = \sum_i \int_X \theta_i (-\varphi_j) \, d\mu_{\varepsilon_j}^{U_i} + o(1),$$

where

$$\int_X \theta_i (-\varphi_j) \, d\mu_{\varepsilon_j}^{U_i} \leqslant \int_{U_i} (-\varphi_j * \varrho_{\varepsilon_j}) \, d\mu.$$

Now $\varphi_j = u^i_j - \gamma_i$ in U_i, where γ_i is a smooth local potential of ω in U_i and u^i_j is psh in U_i. Therefore $-u^i_j * \varrho_{\varepsilon_j} \leqslant -u^i_j$, while $\gamma_i * \varrho_{\varepsilon_j}$ converges uniformly towards γ_i. We infer

$$\int_{U_i} (-\varphi_j * \varrho_{\varepsilon_j}) \, d\mu \leqslant \int_{U_i} (-\varphi_j) \, d\mu + o(1)$$

hence

$$\int (-\varphi_j) \, d\mu_j = \int (-\varphi_j) \omega^n_{\varphi_j} \leqslant C \int (-\varphi_j) \, d\mu.$$

When $\mathcal{E}^1(X, \omega) \subset L^1(\mu)$, it follows from Lemma 2.11 (case $p = 1$) that

$$\int_X (-\varphi_j) \omega^n_{\varphi_j} \leqslant C' \left(\int_X (-\varphi_j) \omega^n_{\varphi_j} \right)^{1/2},$$

hence $\sup_j \int_X (-\varphi_j) \omega^n_{\varphi_j} \leqslant (C')^2 < +\infty$. Thus $\varphi \in \mathcal{E}^1(X, \omega)$. $\quad\square$

We now would like to apply Proposition 2.8 to insure that $\mu = \omega^n_\varphi$. For this we first assume that μ belongs to the compact convex set $\mathcal{M}_A$ of probability measures ν on X which satisfy

$$\nu(K) \leqslant A \, \mathrm{Cap}_\omega(K), \quad \text{for all Borel set } K \subset X.$$

Here $A > 0$ is a fixed constant.

Lemma 4.4. *Assume $\mu \in \mathcal{M}_A$, then*

$$\int_X \varphi_j \, d\mu \to \int_X \varphi \, d\mu \quad \text{and} \quad \int_X |\varphi_j - \varphi| \, d\mu_j \to 0.$$

Proof. When the φ_j's are uniformly bounded, the first convergence follows from standard arguments (see [11]). Set

$$\varphi^{(k)}_j := \max(\varphi_j, -k) \quad \text{and} \quad \varphi^{(k)} := \max(\varphi, -k).$$

We will be done with the first convergence if we can show that $\int |\varphi^{(k)}_j - \varphi_j| \, d\mu \to 0$ uniformly in j as $k \to +\infty$. This is where we use our assumption on μ, since

$$\int_X |\varphi^{(k)}_j - \varphi_j| \, d\mu \leqslant 2 \int_{(\varphi_j < -k)} (-\varphi_j) \, d\mu \leqslant \frac{2}{\sqrt{k}} \int_X (-\varphi_j)^{3/2} \, d\mu \leqslant \frac{C}{\sqrt{k}}.$$

Indeed Proposition 5.3 shows that $\mathcal{E}^1(X, \omega) \subset L^1(\mu)$, hence $\sup_j \int_X (-\varphi_j)\omega_{\varphi_j}^n < +\infty$ by previous lemma. Moreover

$$\int_X (-\varphi_j)^{3/2}\,d\mu = 1 + \frac{3}{2}\int_1^{+\infty} \sqrt{t}\,\mu(\varphi_j < -t)\,dt \leqslant 1 + \frac{3A}{2}\int_1^{+\infty}\sqrt{t}\,\mathrm{Cap}_\omega(\varphi_j < -t)\,dt.$$

It follows now from Lemma 5.1 that

$$\mathrm{Cap}_\omega(\varphi_j < -t) \leqslant C't^{-2}\int_X (-\varphi_j)\omega_{\varphi_j}^n \leqslant C''t^{-2},$$

which proves that $\int_X (-\varphi_j)^{3/2}\,d\mu$ is uniformly bounded from above.

It remains to prove a similar convergence when μ is replaced by μ_j. It actually suffices to consider the case of measures $\mu_j^U := \mu_{|U} * \varrho_{\varepsilon_j}$. Now

$$\int_U |\varphi_j - \varphi|\,d\mu_j^U = \int_U \left(\int_U |u_j(\zeta) - u(\zeta)|\varrho_{\varepsilon_j}(z - \zeta)\,d\lambda(\zeta) \right) d\mu(z),$$

where as above, u_j, u are psh functions in U such that $\varphi_j = u_j - \gamma$ and $\varphi = u - \gamma$ in U, γ is a local potential of ω in U and $d\lambda$ denotes the Lebesgue measure in U. The lemma will be proved if we can show that $\int w_j\,d\mu \to 0$, where

$$w_j(z) := \int_U |u_j(\zeta) - u(\zeta)|\varrho_{\varepsilon_j}(z - \zeta)\,d\lambda(\zeta).$$

Define $\tilde{u}_j := (\sup_{k \geqslant j} u_k)^*$. This is a sequence of psh functions in U which decrease towards u. Observe that $\tilde{u}_j \geqslant \max(u, u_j)$ so that

$$w_j \leqslant 2\tilde{u}_j * \varrho_{\varepsilon_j} - u * \varrho_{\varepsilon_j} - u_j * \varrho_{\varepsilon_j} \leqslant 2(\tilde{u}_j * \varrho_{\varepsilon_j} - u) + (\varphi - \varphi_j).$$

It follows from the monotone convergence theorem that $\int (\tilde{u}_j * \varrho_{\varepsilon_j} - u)\,d\mu \to 0$, while $\int (\varphi_j - \varphi)\,d\mu \to 0$ by the first part of lemma. Therefore $\int w_j\,d\mu \to 0$ and we are done. $\square$

What we have shown so far is that for any measure $\nu \in \mathcal{M}_A$, $A > 0$, there exists a unique $\varphi \in \mathcal{E}^1(X, \omega)$ such that $\nu = (\omega + dd^c\varphi)^n$ and $\sup_X \varphi = -1$.

We now proceed with the proof of Theorem 4.1. For this we need the following observation.

Lemma 4.5. *Let μ be a probability measure which does not charge pluripolar sets. Then there exists $u \in PSH(X, \omega) \cap L^\infty(X)$ and $0 \leqslant f \in L^1(\omega_u^n)$ such that $\mu = f(\omega + dd^c u)^n$.*

Proof. Recall that $\mathcal{M}_1$ is a compact convex subset of the set of all probability measures on X. Let μ be a probability measure which does not charge pluripolar sets. It follows from a generalization of Radon–Nikodym theorem [23] that

$$\mu = f_1\nu + \sigma, \quad \text{where } \nu \in \mathcal{M}_1, \ 0 \leqslant f_1 \in L^1(\nu), \text{ and } \sigma \perp \mathcal{M}_1.$$

Now σ is carried by a pluripolar set since $\mathcal{M}_1$ contains all measures ω_u^n, $0 \leqslant u \leqslant 1$, hence $\sigma = 0$. By our previous analysis, $\nu = (\omega + dd^c v)^n$, where $v \in \mathcal{E}^1(X, \omega)$, $\sup_X v = 0$. Set $u := \exp v$. This is again a ω-psh function since

$$\omega_u = e^v \omega_v + [1 - e^v]\omega + e^v \, dv \wedge d^c v \geqslant e^v \omega_v \geqslant 0.$$

Observe that $0 \leqslant u \leqslant 1$ and $\omega_u^n \geqslant e^{nv}\omega_v^n$, thus μ is absolutely continuous with respect to ω_u^n.   □

We now go on with the proof of Theorem 4.1. Let μ be a probability measure which satisfies $(\text{II})_\chi$. Then μ does not charge pluripolar sets, hence it writes $\mu = f\omega_u^n$, where $u \in PSH(X, \omega) \cap L^\infty(X)$ is so that $0 \leqslant u \leqslant 1$. Consider

$$\mu_j := \delta_j \min(f, j)(\omega + dd^c u)^n,$$

where $\delta_j \searrow 1$ so that $\mu_j(X) = 1$. Note that

$$\mu_j \leqslant j\delta_j \, \text{Cap}_\omega, \quad \text{since } 0 \leqslant u \leqslant 1,$$

thus $\mu_j \in \mathcal{M}_{j\delta_j}$. It follows therefore from previous analysis that there exists a unique $\varphi_j \in \mathcal{E}^1(X, \omega)$ with $\sup_X \varphi_j = -1$ and $\mu_j = \omega_{\varphi_j}^n$. We can assume $\varphi_j \to \varphi$ in $L^1(X)$ and $\delta_j \leqslant 2$.

Suppose first that $\chi \in \mathcal{W}^-$, so that $\varphi_j \in \mathcal{E}_\chi(X, \omega) \supset \mathcal{E}^1(X, \omega)$. It follows from Theorem 2.6 that (II_χ) can be applied to the functions φ_j, hence

$$\int (-\chi) \circ \varphi_j \omega_{\varphi_j}^n \leqslant \delta_j \int (-\chi \circ \varphi_j) \, d\mu \leqslant 2CF\left(\int_X (-\chi) \circ \varphi_j \omega_{\varphi_j}^n\right).$$

Thus $\sup_j E_\chi(\varphi_j) < +\infty$, hence $\varphi \in \mathcal{E}_\chi(X, \omega)$ by Corollary 2.7. We set

$$\Phi_j := \left(\sup_{k \geqslant j} \varphi_k\right)^* \quad \text{and} \quad F_j := \inf_{k \geqslant j} \delta_k \min(f, k).$$

Clearly $\Phi_j \in \mathcal{E}_\chi(X, \omega)$ with $\Phi_j \searrow \varphi$ and $F_j \nearrow f$. It follows therefore from Corollary 1.10 that

$$\omega_{\Phi_j}^n \geqslant F_j \nu.$$

We infer $\omega_\varphi^n \geqslant \mu$, whence equality since these are both probability measures. Thus the proof of Theorem 4.1 is complete when $\chi \in \mathcal{W}^-$.

Assume now $\chi \in \mathcal{W}_M^+$. We can apply previous reasoning if we can show that $\varphi_j \in \mathcal{E}_\chi(X, \omega)$ for all j. This is the case, as we claim that the φ_j's actually belong to $\mathcal{E}^p(X, \omega)$ for all $p \geqslant 1$. Indeed recall that

$$\mu_j = (\omega + dd^c \varphi_j)^n \leqslant j\delta_j(\omega + dd^c u)^n \leqslant A_j \, \text{Cap}_\omega,$$

hence $\mu_j \in \mathcal{M}_{A_j}$. Observe that for all $p \geqslant 1$,

$$\mathcal{E}^p(X, \omega) \subset L^{p+1/2}(\nu), \quad \text{for all } \nu \in \mathcal{M}_A. \tag{$*$}$$

This has been established when $p = 1$ in the proof of Lemma 4.4. Since $\varphi_j \in \mathcal{E}^1(X, \omega)$ and $\mu_j = (\omega + dd^c\varphi_j)^n \in \mathcal{M}_{A_j}$, we infer $\varphi_j \in L^{3/2}([\omega + dd^c\varphi_j]^n)$, i.e. $\varphi_j \in \mathcal{E}^{3/2}(X, \omega)$. We can thus use $(*)$ inductively to obtain $\varphi_j \in \cap_p \mathcal{E}^p(X, \omega)$. The proof of Theorems 4.1 and 4.2 is now complete.

4.2. Non-pluripolar measures

We now describe the range of the complex Monge–Ampère operator on the class $\mathcal{E}(X, \omega)$ (Theorem A).

Theorem 4.6. *There exists $\varphi \in \mathcal{E}(X, \omega)$ such that $\mu = (\omega + dd^c\varphi)^n$ if and only if μ does not charge pluripolar sets.*

Proof. One implication is obvious. Namely if $\mu = (\omega + dd^c\varphi)^n$ for some function $\varphi \in \mathcal{E}(X, \omega)$, then μ does not charge pluripolar sets, as follows from Theorem 1.3.

Assume now that μ does not charge pluripolar sets. It follows from Lemma 4.5 that we can find $u \in PSH(X, \omega) \cap L^\infty(X)$ and $0 \leqslant f \in L^1(\omega_u^n)$ such that $\mu = f\omega_u^n$. Set

$$\mu_j := c_j \min(f, j)\omega_u^n,$$

where $c_j \searrow 1$ is such that $\mu_j(X) = \mu(X) = 1$. We can assume without loss of generality that $1 \leqslant c_j \leqslant 2$.

It follows from Chern–Levine–Nirenberg inequalities (see [19, Proposition 3.1]) that

$$\mathcal{E}^1(X, \omega) \subset PSH(X, \omega) \subset L^1(\mu_j),$$

because $\mu_j \leqslant 2j\omega_u^n$ and u is bounded. Thus by Theorem 4.2, case $p = 1$, there exists a unique function $\varphi_j \in \mathcal{E}^1(X, \omega)$ such that

$$\mu_j = \left(\omega + dd^c\varphi_j\right)^n \quad \text{and} \quad \sup_X \varphi_j = 0.$$

Passing to a subsequence if necessary, we can assume $\varphi_j \to \varphi$ in $L^1(X)$, for some function $\varphi \in PSH(X, \omega)$ such that $\sup_X \varphi = 0$.

We claim that $\varphi \in \mathcal{E}_\chi(X, \omega)$ for some $\chi \in \mathcal{W}^-$, and that $\mu = (\omega + dd^c\varphi)^n$. The function χ is defined as follows. Let $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ be a convex increasing function with $\lim_{z \to +\infty} \gamma(z)/z = +\infty$ such that $\gamma \circ f$ still belongs to $L^1(\omega_u^n)$ (see [24] for the construction of γ). Let γ^* be the Young-conjugate function of γ,

$$\gamma^*\colon z \in \mathbb{R}^+ \mapsto \sup\{zy - \gamma(y) \mid y \in \mathbb{R}^+\} \in \mathbb{R}^+,$$

and set $\chi(t) := -(\gamma^*)^{-1}(-t)$: this is a convex increasing function such that $\chi(-\infty) = -\infty$, which satisfies

$$(-\chi)(-t) \cdot f(x) \leqslant -t + \gamma \circ f(x), \quad \text{for all } (t, x) \in \mathbb{R}^- \times X, \tag{13}$$

since $zy \leqslant \gamma^*(z) + \gamma(y)$ for all $z, y \in \mathbb{R}^+$.

Note that we can assume $\chi(0) = 0$ without loss of generality, by imposing $\gamma(0) = 0$. Applying (13) with $t = \varphi_j(x)$ and averaging against ω_u^n, we infer

$$0 \leqslant \int\limits_X (-\chi) \circ \varphi_j \omega_{\varphi_j}^n \leqslant 2 \int\limits_X (-\chi) \circ \varphi_j f \omega_u^n \leqslant 2 \int\limits_X (-\varphi_j) \omega_u^n + 2 \int\limits_X \gamma \circ f \omega_u^n.$$

Again it follows from Chern–Levine–Nirenberg inequalities that $\int_X (-\varphi_j) \omega_u^n$ is uniformly bounded from above, because u and $\int_X (-\varphi_j) \omega^n$ are bounded. Thus the sequence of integrals $(\int_X (-\chi) \circ \varphi_j \omega_{\varphi_j}^n)$ is bounded.

Set $\phi_j := (\sup_{l \geqslant j} \varphi_l)^* \in \mathcal{E}^1(X, \omega)$. It follows from Corollary 1.10 that

$$\omega_{\phi_j}^n \geqslant \inf_{l \geqslant j} \omega_{\varphi_l}^n \geqslant \min(f, j) \omega_u^n.$$

The sequence (ϕ_j) decreases towards φ and satisfies $\varphi_j \leqslant \phi_j \leqslant 0$, hence by the fundamental inequality,

$$0 \leqslant \int\limits_X (-\chi) \circ \phi_j \omega_{\phi_j}^n \leqslant 2^n \int\limits_X (-\chi) \circ \varphi_j \omega_{\varphi_j}^n \leqslant M_0 < +\infty.$$

Therefore $\varphi \in \mathcal{E}_\chi(X, \omega)$ and

$$\omega_\varphi^n = \lim_{j \to +\infty} \omega_{\phi_j}^n \geqslant \lim_{j \to +\infty} \min(f, j) \omega_u^n = \mu.$$

Since μ and ω_φ^n are both probability measures, this actually yields equality, hence the proof is complete. $\quad \square$

Remark 4.7. Note that when $\mu = f \omega^n$ is a measure with density $0 \leqslant f$ such that $f \log f \in L^1(\omega^n)$, our proof shows that $\mu = (\omega + dd^c \varphi)^n$ with $\varphi \in \mathcal{E}_\chi(X, \omega)$, for $\chi(t) = -\log(1 - t)$ (this is a critical case in the Orlicz classes considered by S. Kolodziej [21]).

5. Examples and applications

5.1. Capacity of sublevel sets

Recall that the Monge–Ampère capacity associated to ω is defined by

$$\mathrm{Cap}_\omega(K) := \sup \left\{ \int\limits_K \omega_u^n \,\bigg|\, u \in PSH(X, \omega), \ 0 \leqslant u \leqslant 1 \right\},$$

where K is any Borel subset of X. This capacity vanishes on pluripolar sets, more precisely $\mathrm{Cap}_\omega(\varphi < -t) \leqslant C_\varphi / t$ for every fixed ω-psh function φ. This estimate is sharp in the sense that $\mathrm{Cap}_\omega(\varphi < -t) \geqslant C'_\varphi / t$ when ω_φ is the current of integration along a hypersurface. However when φ belongs to $\mathcal{E}_\chi(X, \omega)$, one can establish finer estimates as our next result shows.

Lemma 5.1. *Fix* $\chi \in \mathcal{W}^- \cup \mathcal{W}_M^+$, $M \geqslant 1$. *If* $\varphi \in \mathcal{E}_\chi(X, \omega)$, *then*

$$\exists C_\varphi > 0, \ \forall t > 1, \quad \mathrm{Cap}_\omega(\varphi < -t) \leqslant C_\varphi |t\chi(-t)|^{-1}.$$

Conversely if there exists $C_\varphi, \varepsilon > 0$ *such that for all* $t > 1$,

$$\mathrm{Cap}_\omega(\varphi < -t) \leqslant C_\varphi |t^{n+\varepsilon}\chi(-t)|^{-1},$$

then $\varphi \in \mathcal{E}_\chi(X, \omega)$.

The proof relies on the comparison principle, as it was used in different contexts by U. Cegrell and S. Kolodziej (see [11,12,21]).

Proof. Fix $\varphi \in \mathcal{E}_\chi(X, \omega)$, $\varphi \leqslant -1$ and $u \in PSH(X, \omega)$ with $-1 \leqslant u \leqslant 0$. For $t \geqslant 1$, observe that $\varphi/t \in \mathcal{E}(X, \omega)$ and $(\varphi < -2t) \subset (\varphi/t < u - 1) \subset (\varphi < -t)$. It therefore follows from the comparison principle that

$$\int_{(\varphi < -2t)} (\omega + dd^c u)^n \leqslant \int_{(\varphi < -t)} (\omega + t^{-1} dd^c \varphi)^n$$

$$\leqslant \int_{(\varphi < -t)} \omega^n + t^{-1} \sum_{j=1}^n \binom{n}{j} \int_{(\varphi < -t)} \omega_\varphi^j \wedge \omega^{n-j}.$$

Recall now that $\mathrm{Vol}_\omega(\varphi < -t)$ decreases exponentially fast [19], and observe that for all $1 \leqslant j \leqslant n$,

$$\int_{(\varphi < -t)} \omega_\varphi^j \wedge \omega^{n-j} \leqslant \frac{1}{|\chi(-t)|} \int_X (-\chi) \circ \varphi \omega_\varphi^j \wedge \omega^{n-j} \leqslant \frac{1}{|\chi(-t)|} E_\chi(\varphi).$$

This yields our first assertion.

The second assertion follows from similar considerations. Namely for $\varphi \in PSH(X, \omega)$, $\varphi \leqslant -1$, set $\varphi_t := \max(\varphi, -t)$. Then $u = \varphi_t/t \in PSH(X, \omega)$ with $-1 \leqslant u \leqslant 0$ for all $t \geqslant 1$. Since $\omega_u^n \geqslant t^{-n} \omega_{\varphi_t}^n$, we infer

$$t^{-n} \omega_{\varphi_t}^n(\varphi < -t) \leqslant \omega_u^n(\varphi < -t) \leqslant \mathrm{Cap}_\omega(\varphi < -t).$$

If $\mathrm{Cap}_\omega(\varphi < -t) \leqslant C t^{-n-\varepsilon}|\chi(-t)|^{-1}$, this shows that $\omega_{\varphi_t}(\varphi < -t) \to 0$, hence $\varphi \in \mathcal{E}(X, \omega)$. Moreover

$$\omega_\varphi^n(\varphi \leqslant -t) = \int_X \omega^n - \omega_\varphi^n(\varphi > -t) = \int_X \omega^n - \omega_{\varphi_t}^n(\varphi > -t) = \omega_{\varphi_t}^n(\varphi \leqslant -t)$$

thus $\omega_\varphi^n(\varphi \leqslant -t) \leqslant C t^{-\varepsilon} |\chi(-t)|^{-1}$. This yields

$$\int\limits_X (-\chi) \circ \varphi \omega_\varphi^n = \int\limits_1^{+\infty} \chi'(-t) \omega_\varphi^n(\varphi < -t)\, dt \leqslant C_\varphi \int\limits_1^{+\infty} \frac{t\chi'(-t)}{|\chi(-t)|} \frac{1}{t^{1+\varepsilon}}\, dt < +\infty,$$

so that $\varphi \in \mathcal{E}_\chi(X, \omega)$. $\square$

These estimates allow us to give several examples of functions which belong to the classes $\mathcal{E}_\chi(X, \omega)$. We simply mention the following ones.

Example 5.2. Let φ be any ω-plurisubharmonic function such that $\varphi \leqslant 0$. Then $\psi := -\log(1 - \varphi) = \chi \circ \varphi \in PSH(X, \omega)$ since

$$dd^c \chi \circ \varphi = \chi'' \circ \varphi\, d\varphi \wedge d^c \varphi + \chi' \circ \varphi\, dd^c \varphi$$

and $\chi \in \mathcal{W}^-$ with $\chi'(t) \leqslant 1$ when $t \leqslant 0$.

Observe that the capacity of the sets $(\psi < -t)$ decrease exponentially fast, thus $\psi \in \bigcap_{p \geqslant 1} \mathcal{E}^p(X, \omega)$ (Lemma 5.1). Together with [19, Theorem 7.2], this shows that $\mathcal{E}^p(X, \omega)$ characterizes pluripolar sets (for any $p \geqslant 1$).

We now want to give some example of probability measures that satisfy the assumptions of Theorems 4.1, 4.2, 4.6.

When $\mu = f\omega^n$ has density $f \in L^r(X)$, $r > 1$, S. Kolodziej has proved [21] that $\mu = \omega_\psi^n$ for some *bounded* ω-psh function ψ. This is because μ is strongly dominated by Cap_ω in this case (see Proposition 5.3 below). When the density is only in L^1, this does not hold. Consider for instance $\mu = f\omega^n$, where $f \in \mathcal{C}^\infty(X \setminus \{a\})$ is such that

$$f(z) \simeq \frac{1}{\|z\|^4 (-\log \|z\|)^2} - 1$$

near the point $a = 0$, in a local chart. Observe that

$$\varphi(z) := \varepsilon \chi(z) \log \|z\| \in PSH(X, \omega)$$

if χ is a cut-off function so that $\chi \equiv 1$ near $a = 0$, and $\varepsilon > 0$ is small enough. Now $\varphi \notin L^1(\mu)$ but still $\mathcal{E}^1(X, \omega) \subset L^1(\mu)$, as follows from Proposition 5.3, thus there exists $\psi \in \mathcal{E}^1(X, \omega)$ such that $\mu = \omega_\psi^n$ (Theorem 4.2).

Observe also that there are measures $\mu = f\omega^n$ with L^1-density such that $\mathcal{E}^1(X, \omega) \not\subset L^1(\mu)$: one can consider for instance f_ε that looks locally near $a = 0$ like $[\|z\|^4(-\log \|z\|)^{1+\varepsilon}]^{-1}$, for $\varepsilon > 0$ small enough. However this measure μ does not charge pluripolar sets, hence is a Monge–Ampère measure of some function $\varphi \in \mathcal{E}_\chi(X, \omega)$, $\chi \in \mathcal{W}^-$ with slower growth.

Proposition 5.3. *Let μ be a probability measure on X.*
Assume there exist $\alpha > p/(p+1)$ and $A > 0$ such that

$$\mu(E) \leqslant A \operatorname{Cap}_\omega(E)^\alpha, \tag{14}$$

for all Borel sets $E \subset X$. Then $\mathcal{E}^p(X, \omega) \subset L^p(\mu)$.

Conversely assume $\mathcal{E}^p(X, \omega) \subset L^p(\mu)$, $p > 1$. Then there exist $0 < \alpha < 1$ and $A > 0$ such that (14) *is satisfied.*

Proof. We can assume without loss of generality that $\alpha < 1$. Let $\varphi \in \mathcal{E}^p(X, \omega)$ with $\sup_X \varphi = -1$. It follows from Hölder inequality that

$$0 \leqslant \int_X (-\varphi)^p \, d\mu = 1 + p \int_1^{+\infty} t^{p-1} \mu(\varphi < -t) \, dt$$

$$\leqslant 1 + pA \int_1^{+\infty} t^{p-1} \big[\operatorname{Cap}_\omega(\varphi < -t)\big]^\alpha \, dt$$

$$\leqslant 1 + pA \left[\int_1^{+\infty} t^{\frac{p-\alpha(p+1)}{1-\alpha}-1} \, dt \right]^{1-\alpha} \cdot \left[\int_1^{+\infty} t^p \operatorname{Cap}_\omega(\varphi < -t) \, dt \right]^\alpha.$$

The first integral in the last line converges since $p - \alpha(p + 1) < 0$ and it follows from the proof of Lemma 5.1 that the last one is bounded from above by $C_{p,\alpha}(\int(-\varphi)^p \omega_\varphi^n)^\alpha$. Therefore $\mathcal{E}^p(X, \omega) \subset L^p(\mu)$.

Assume conversely that $\mathcal{E}^p(X, \omega) \subset L^p(\mu)$. It follows from Theorem 4.2 that $\mu = \omega_\psi^n$ for some function $\psi \in \mathcal{E}^p(X, \omega)$ such that $\sup_X \psi = -1$. We claim then that there exist $\gamma_p \in \,]0, 1[$ and $A > 0$ such that for all functions $\varphi \in PSH(X, \omega)$ with $-1 \leqslant \varphi \leqslant 0$, one has

$$0 \leqslant \int_X (-\varphi)^p \omega_\psi^n \leqslant A \left[\int_X (-\varphi)^p \omega_\varphi^n \right]^{\gamma_p}. \tag{15}$$

We leave the proof of this claim to the reader (the exponent $\gamma_p = (1 - 1/p)^n$ would do). We apply now (15) to the extremal function $\varphi = h_{E,\omega}^*$ introduced in [19]. It follows from Theorem 3.2 in [19] that

$$0 \leqslant \mu(E) \leqslant \int_X \big(-h_{E,\omega}^*\big)^p \, d\mu \leqslant A \operatorname{Cap}_\omega(E)^{\gamma_p}. \qquad \square$$

This proposition allows to produce several examples of measures satisfying $\mathcal{E}^p(X, \omega) \subset L^p(\mu)$ as in the local theory (see [21,29]). It can also be used, together with Theorem 4.2, to prove that functions from the local classes of Cegrell $\mathcal{E}^r(\Omega)$, Ω a bounded hyperconvex domain of $\mathbb{C}^n$, can be sub-extended as global functions $\varphi \in \mathcal{E}^p(\mathbb{P}^n, A\omega_{FS})$, for all $p < r/n$ and some $A > 0$ (see [12] for similar results).

5.2. Complex dynamics

5.2.1. Large topological degree

Let $f : X \to X$ be a meromorphic endomorphism whose topological degree $d_t(f)$ is large in the sense that $d_t(f) > r_{k-1}(f)$, where $r_{k-1}(f)$ denotes the spectral radius of the linear action induced by f^* on $H^{k-1,k-1}(X, \mathbb{R})$. In this case there exists a unique invariant measure μ_f of maximal entropy which can be decomposed as

$$\mu_f := \Theta + dd^c(\mathcal{T}),$$

as was proved by the first author in [18]. Here Θ is a smooth probability measure and $\mathcal{T} \geqslant 0$ is a positive current of bidegree $(k-1, k-1)$. In particular

$$\mathcal{E}^1(X, \omega) \subset PSH(X, \omega) \subset L^1(\mu),$$

as follows from Stokes theorem (see [18, Theorem 2.1] and [19, Example 2.8]). Here ω denotes any fixed Kähler form on X, normalized by $\int_X \omega^n = \mu(X) = 1$. It follows therefore from Theorems 4.2 and 3.3 that there exists a unique function $g_f \in \mathcal{E}^1(X, \omega)$ such that $\sup_X g_f = -1$ and

$$\mu_f = \left(\omega + dd^c g_f \right)^k.$$

It is an interesting problem to establish further regularity properties of g_f in order for example to estimate the pointwise dimension of the measure μ_f. When $f : \mathbb{P}^n \to \mathbb{P}^n$ is an *holomorphic* endomorphism of the complex projective space $X = \mathbb{P}^n$, it is known [25] that g_f is an Hölder-continuous function.

5.2.2. Small topological degree

Let T be a positive closed current of bidegree (p, p) on X. Let ω denote a positive closed $(1, 1)$ current with bounded potentials, and such that $\int_X T \wedge \omega^{n-p} > 0$. Note that the wedge product $T \wedge \omega^{n-p}$ is a well defined positive Radon measure because ω has bounded potentials [5]. Fix $\chi \in \mathcal{W}^-$.

Definition 5.4. We let $\mathcal{E}_\chi(T, \omega)$ denote the class of ω-psh functions with finite χ-energy with respect to T. This is the set of function $\varphi \in PSH(X, \omega)$ for which there exists a sequence $\varphi_j \in PSH(X, \omega) \cap L^\infty(X)$ such that

$$\varphi_j \searrow \varphi \quad \text{and} \quad \sup_j \int_X \left| \chi \circ \left(\varphi_j - \sup_X \varphi_j \right) \right| T \wedge \left(\omega + dd^c \varphi_j \right)^{n-p} < +\infty.$$

When $T = [X]$ is the current of integration on X ($p = 0$), this is the same notion as that of Definition 1.1. We let the reader check that the following properties hold, with the same proof as in the case $T = [X]$:

(1) $\forall \varphi, \psi \in PSH(X, \omega) \cap L^\infty(X)$, $\varphi \leqslant \psi \leqslant 0$ implies

$$0 \leqslant \int_X (-\chi) \circ \psi \, \omega_\psi^{n-p} \wedge T \leqslant 2^{n-p} \int_X (-\chi) \circ \varphi \, \omega_\varphi^{n-p} \wedge T;$$

(2) the operator $\omega_\varphi^{n-p} \wedge T$ is well defined on $\mathcal{E}_\chi(T, \omega)$ and continuous under decreasing sequences;

(3) the function $\chi \circ \varphi$ is integrable with respect to the positive Radon measure $\omega_\varphi^{n-p} \wedge T$.

This is of great interest in the dynamical study of meromorphic endomorphisms. Assume for instance that $f : X \to X$ is a meromorphic endomorphism of some compact Kähler *surface* X ($n = 2$), whose topological degree is smaller than the first dynamical degree $\lambda = \lambda_1(f) := \lim_{j \to +\infty}[r_1(f^j)]^{1/j}$, where $r_1(f)$ denotes the spectral radius of the linear action induced by f^* on $H^{1,1}(X, \mathbb{R})$ (see [15]). When f is 1-stable (i.e. when the iterates f^j of f do not contract any curve into the indeterminacy locus I_f of f), it is known that

$$\frac{1}{\lambda^j}(f^j)^* \omega^+ \to T^+ = \omega^+ + dd^c g^+,$$

where ω^+ is a positive closed $(1,1)$-current with bounded potentials, whose cohomology class is f^*-invariant, and T^+ is a canonical f^*-invariant positive current, $f^* T^+ = \lambda T^+$ (see [16]). Similarly

$$\frac{1}{\lambda^j}(f^j)_* \omega^- \to T^- = \omega^- + dd^c g^-,$$

where ω^- is a positive closed $(1,1)$-current with bounded potentials, whose cohomology class is f_*-invariant, and T^- is the canonical f_*-invariant positive current, $f_* T^- = \lambda T^-$ (see [16]).

It is a difficult and important question to define the dynamical wedge product $T^+ \wedge T^-$: this product is expected to yield the unique measure of maximal entropy. When $g^+ \in \mathcal{E}_\chi(T^-, \omega^+)$, one can use our ideas above and show that not only is the measure $\mu_f = T^+ \wedge T^-$ well defined, but also $\chi \circ g^+ \in L^1(\mu_f)$. In particular the measure μ_f does not charge the set $(g^+ = -\infty)$. This has several interesting dynamical consequences, as shown in the paper [16] to which we refer the reader. Note that when $\chi(t) = t$, this condition was introduced and extensively studied by E. Bedford and J. Diller in [2]: in this case the functions g^+, g^- have necessarily gradients in $L^2(X)$. Our weaker condition allows us to handle functions whose gradient does not necessarily belong to $L^2(X)$, as already noted above (Example 2.14).

To illustrate these ideas we now consider the special case where $X = \mathbb{P}^2$ is the complex projective plane equipped with the Fubini–Study Kähler form ω, and $f : \mathbb{P}^2 \to \mathbb{P}^2$ is a 1-stable *birational* map ($d_t(f) = 1$), with $\lambda_1(f) > 1$. The invariant currents write $T^\pm = \omega + dd^c g^\pm$, where $g^\pm \in PSH(X, \omega)$. The functions g^+, g^- do not belong to the class $\mathcal{E}(\mathbb{P}^2, \omega)$ because they have positive Lelong numbers at points of indeterminacy of the mappings f^n, $n \in \mathbb{Z}$, however we have the following result.

Proposition 5.5. *If* $g^+ \in \mathcal{E}^1(T^-, \omega)$ *then* $g := \max(g^+, g^-) \in \mathcal{E}^1(\mathbb{P}^2, \omega)$.

Proof. We can assume without loss of generality that $g^+, g^- \leqslant 0$, hence $g \leqslant 0$. The Bedford–Diller condition $g^+ \in \mathcal{E}^1(T^-, \omega)$ implies that $\nabla g^+, \nabla g^- \in L^2(\mathbb{P}^2)$, hence $\nabla g \in L^2(\mathbb{P}^2)$ and the complex Monge–Ampère measure $(\omega + dd^c g)^2$ is well defined (see [4] and [2]). Now

$$0 \leqslant \int_X (-g)\omega_g^2 \leqslant \int_X (-g^+)\omega_g^2 = \int_X (-g^+)\omega \wedge \omega_g + \int_X (-g^+) dd^c g \wedge \omega_g.$$

It follows from Cauchy–Schwarz inequality that the first integral in the right-hand side is finite. We can bound the last one from above by using Stokes theorem,

$$\int_X \left(-g^+\right) dd^c g \wedge \omega_g = \int_X (-g)\, dd^c g^+ \wedge \omega_g \leqslant \int_X (-g)\omega_{g^+} \wedge \omega_g.$$

Now

$$\int_X (-g)\omega_{g^+} \wedge \omega_g \leqslant \int_X (-g^-)\omega_{g^+} \wedge \omega_g$$

$$= \int_X (-g^-)\omega_{g^+} \wedge \omega + \int_X (-g^-)\omega_{g^+} \wedge dd^c g$$

$$\leqslant O(1) + \int_X (-g)\omega_{g^+} \wedge \omega_{g^-}$$

$$\leqslant O(1) + \int_X \left(-g^+\right)\omega_{g^+} \wedge T^- < +\infty,$$

the last integral being finite because $g^+ \in \mathcal{E}^1(T^-, \omega)$. $\quad\square$

It is an interesting problem to determine whether $\mu_f = (\omega + dd^c g)^2$. This is the case when e.g. f is a complex Hénon mapping, and it would imply—by Theorem 4.2—that $\mathcal{E}^1(\mathbb{P}^2, \omega) \subset L^1(\mu_f)$, hence in particular μ_f does not charge pluripolar sets.

5.3. Singular Kähler–Einstein metrics

It is well known that solving Monge–Ampère equations

$$\left(\omega + dd^c \varphi\right)^n = \mu \tag{MA$_{(X,\omega,\mu)}$}$$

is a way to produce Kähler–Einstein metrics (see [1,10,21,26,28]). In the classical case, the measure $\mu = f\omega^n$ admits a smooth density $f > 0$. When the ambient manifold has some singularities (which is often the case in dimension $\geqslant 3$), one has to allow the equation (MA$_{(X,\omega,\mu)}$) to degenerate in two different ways: resolving the singularities $\pi : \tilde{X} \to X$ of X yields a new equation (MA$_{(\tilde{X},\tilde{\omega},\tilde{\mu})}$), where

(1) $\{\tilde{\omega}\} = \{\pi^*\omega\}$ is a semi-positive and big class (one looses strict positivity along the exceptional locus $\mathcal{E}_\pi$);
(2) $\tilde{\mu} = \tilde{f}\tilde{\omega}^n$ is a measure with density $0 \leqslant \tilde{f} \in L^p$, $p > 1$, which may have zeroes and poles along some components of $\mathcal{E}_\pi$.

In this paper we have focused on the second type of degeneracy. We would like to mention that our techniques are supple enough so that we can produce solutions $\varphi \in \mathcal{E}_\chi(\tilde{X}, \tilde{\omega})$ to the Monge–Ampère equations (MA$_{(\tilde{X},\tilde{\omega},\tilde{\mu})}$), even when $\{\tilde{\omega}\}$ is merely big and semi-positive rather

than Kähler, as was assumed for simplicity throughout Section 4. This is done by perturbing the form $\tilde{\omega}$, adding small multiples of a Kähler form Ω, $\omega_j := \tilde{\omega} + \varepsilon_j \Omega$, and trying to understand what happens at the limit.

An elementary but crucial observation here is that classes $\mathcal{E}_\chi(X, \omega)$ have good stability properties in the following sense.

Proposition 5.6. *Fix* $\chi \in \mathcal{W} := \mathcal{W}^- \cup \mathcal{W}_M^+$, $M \geqslant 1$. *Let* ω_j *be a sequence of positive closed* $(1, 1)$-*currents with bounded potentials, that decrease towards* ω, *another current with bounded potentials. Assume* $\varphi_j \in \mathcal{E}_\chi(X, \omega_j)$ *decreases towards* φ *pointwise and* $\sup_j \int_X |\chi(|\varphi_j|)|(\omega_j + dd^c\varphi_j)^n < +\infty$.

Then $\varphi \in \mathcal{E}_\chi(X, \omega)$ *and* $(\omega_j + dd^c\varphi_j)^n \to (\omega + dd^c\varphi)^n$.

Proof. We can assume without loss of generality that $\varphi \leqslant \varphi_j \leqslant 0$. Set

$$\varphi_j^K := \max(\varphi_j, -K) \quad \text{and} \quad \varphi^K := \max(\varphi, -K).$$

It follows from the fundamental inequalities that the (χ, ω_j)-energy of the functions φ_j^K is uniformly bounded independently of both j and K.

Let ν be any cluster point of the bounded sequence of positive measures $\nu_j := (-\chi) \circ \varphi_j^K(\omega_j + dd^c\varphi_j^K)^n$. Observe that $(\omega_j + dd^c\varphi_j^K)^n \to (\omega + dd^c\varphi^K)^n$ as $j \to +\infty$, since φ_j^K decreases towards φ^K and these functions are uniformly bounded [5]. It follows therefore from the upper-semicontinuity of φ^K that

$$0 \leqslant (-\chi) \circ \varphi^K(\omega + dd^c\varphi^K)^n \leqslant \nu.$$

In particular $E_\chi(\varphi^K)$ is bounded from above by $\nu(X)$, hence $\varphi \in \mathcal{E}_\chi(X, \omega)$ by Corollary 2.4.

The convergence of $(\omega_j + dd^c\varphi_j)^n$ towards $(\omega + dd^c\varphi)^n$ follows again from the fact that $(\omega_j + dd^c\varphi_j^K)^n - (\omega_j + dd^c\varphi_j)^n$ converges towards zero as $K \to +\infty$, uniformly with respect to j. $\quad\square$

We refer the reader to [17] for an application to the construction of Kähler–Einstein metrics on canonical/minimal models in the sense of Mori.

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