Weak complex Monge-Ampère flows

Joint work with P.Eyssidieux and A.Zeriahi

Vincent Guedj

Institut Universitaire de France & Institut de Mathématiques de Toulouse

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Studying the Kähler-Ricci flow

Let X be a compact Kähler manifold of complex dimension $n \geq 1$. Fix ω_0 a Kähler form and consider the Kähler-Ricci flow

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This flow admits a unique solution $\omega = \omega(t,x) = \omega_t(x)$ on a maximal domain $[0, T_{max}[\times X$, where

$$T_{max} = \sup\{t > 0\,;\, \{\omega_0\} - tc_1(X) \text{ is Kähler }\}.$$

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- try and restart the KRF on X_1 with initial data S_1;
- repeat finitely many times to reach a minimal model X_r;
- study the long term behavior of the NKRF (K_{X_r} is *nef*),

$$\begin{cases} \frac{\partial \omega}{\partial t} = -\mathrm{Ric}(\omega) - \omega_{\mathrm{t}} \\ \omega_{|t=0} = S_r \end{cases}$$

and show that (X_r, ω_t) converges to a canonical model (X_{can}, ω_{can}).

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  - Degenerate initial data (Kähler current rather than a Kähler form).
  - Define and study the KRF on mildly singular varieties.

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- This is not a quotient singularity if $n \geq 3$.

Complex Monge-Ampère flows

Solving the (normalized) Kähler-Ricci flow is equivalent to solving

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and $(t,x) \mapsto \varphi(t,x) = \varphi_t(x)$ is the unknown function.

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$$e^{\psi} = \prod_{j=1}^{N} |s_j|_h^2 \longleftrightarrow \text{canonical singularities.}$$

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If φ_0 is an arbitrary continuous ω_0-psh function, there exists a unique viscosity solution $(t, x) \mapsto \varphi_t(x)$ of (CMAF) with initial value φ_0. The function φ_t is the upper envelope of viscosity subsolutions. In particular $x \mapsto \varphi_t(x)$ is ω_t-plurisubharmonic for all $t \geq 0$.

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Theorem

On a $\mathbb{Q}$-Calabi-Yau variety (canonical singularities), the KRF continuously deforms any Kähler current S_0 to the unique KE current in $\{S_0\}$.

Classical sub/super/solutions

Definition

A function $\varphi \in \mathcal{C}^{1,2}$ *is a* **classical subsolution** *of (CMAF) if for all* $t \geq 0$ $x \mapsto \varphi_t(x)$ *is* ω_t*-psh and*

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PROBLEM: classical solutions usually do not exist !

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Given $u : X_T := (0, T) \times X \to \mathbb{R}$ *an u.s.c. bounded function and* $(t_0, x_0) \in X_T$, q *is a differential test form above for* u *at* (t_0, x_0) *if*

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An u.s.c. bounded function $u : X_T \to \mathbb{R}$ is a **viscosity subsolution** *of (CMAF) if for all $(t_0, x_0) \in X_T$ and all differential test q from above,*

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- One can also use sup-convolutions in space **locally**, considering

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The most difficult result here is the following maximum principle:

Theorem

Assume u is a subsolution to the Cauchy problem and
v is a supersolution to the Cauchy problem.
Then $u_0 \leq v_0 \Longrightarrow u_t \leq v_t$ for all $t > 0$.

Observe that this already implies uniqueness of solutions.

Step 4: Construction of barriers

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It follows then from Step 3 that

$$\varphi = \varphi^* = \varphi_* \text{ is the solution.}$$

Ricci-flat currents

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- The current $T_{KE} = \theta_0 + dd^c\varphi_{KE}$ is constructed by solving

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where the θ_0-psh function φ_{KE} is normalized by $\int_X \varphi_{KE}\, d\mu_{can} = 0$.

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  where the θ_0-psh function φ_{KE} is normalized by $\int_X \varphi_{KE} \, d\mu_{can} = 0$.
- Alternatively one can work on a desingularization $\pi : Y \to X$ with

$$(\pi^* \theta_0 + dd^c \varphi_{KE} \circ \pi)^n = \pi^* \mu_{can} = e^{\psi_{can}} dV.$$

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Theorem

The functions φ_t uniformly converge, as $t \to +\infty$, to $\varphi_{KE} \circ \pi$.

The proof

- We consider a perturbation of the flow: for $\varepsilon > 0$,

$$(\omega_0 + dd^c \varphi_t^\varepsilon)^n = e^{\dot{\varphi}_t^\varepsilon + \varepsilon \varphi_t^\varepsilon + \psi_{can}} dV,$$

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- Pluripotential stability: ψ^ε uniformly converges to $\varphi_{KE} \circ \pi$ as $\varepsilon \searrow 0$.

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The proof is complete.

The end

Thank you for your attention !