Viscosity Solutions to Degenerate Complex Monge-Ampère Equations

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Abstract

Degenerate complex Monge-Ampère equations on compact Kähler manifolds have recently been studied intensively using tools from pluripotential theory. We develop an alternative approach based on the concept of viscosity solutions and systematically compare viscosity concepts with pluripotential theoretic ones.

This approach works only for a rather restricted type of degenerate complex Monge-Ampère equations. Nevertheless, we prove that the local potentials of the singular Kähler-Einstein metrics previously constructed by the authors are continuous plurisubharmonic functions. They were previously known to be locally bounded.

Another application is a lower-order construction with a C^0-estimate of the solution to the Calabi conjecture that does not use Yau's celebrated theorem.

Introduction

Pluripotential theory lies at the foundation of the approach to degenerate complex Monge-Ampère equations on compact Kähler manifolds, as developed in [7, 11, 19, 22, 23, 33, 36, 39] and many others. This method is global in nature, since it relies on some delicate integrations by parts.

On the other hand, a standard PDE approach to second-order degenerate elliptic equations is the method of viscosity solutions introduced in [34]; see [16] for a survey. This method is local in nature and solves existence and unicity problems for weak solutions very efficiently. Our main goal in this article is to develop the viscosity approach for complex Monge-Ampère equations on compact complex manifolds.

Whereas the viscosity theory for real Monge-Ampère equations has been developed by P.-L. Lions and others (see, e.g., [31]), the complex case hasn't been

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studied until very recently. There is a viscosity approach to the Dirichlet problem for the complex Monge-Ampère equation on a smooth hyperconvex domain in a Stein manifold in [28]. This recent article, however, does not prove any new results for complex Monge-Ampère equations since this case serves there as a motivation to develop a deep generalization of plurisubharmonic functions to Riemannian manifolds with some special geometric structure (e.g., the exceptional holonomy group). To the best of our knowledge, there is no reference on viscosity solutions to complex Monge-Ampère equations on compact Kähler manifolds.

There has been some recent interest in adapting viscosity methods to solve degenerate elliptic equations on compact or complete Riemannian manifolds [3]. This theory can be applied to complex Monge-Ampère equations only in very restricted cases since it requires the Riemann curvature tensor to be nonnegative. By using [35], a compact Kähler manifold with a nonnegative Riemannian curvature tensor has an étale cover that is a product of a symmetric space of compact type (e.g., $\mathbb{P}^n(\mathbb{C})$, Grassmannians) and a compact complex torus. In particular, [3] does not allow in general the construction of a viscosity solution to the elliptic equation

$$(\mathrm{DMA})_{\omega,v} \qquad (\omega + dd^c\varphi)^n = e^{\varphi} v$$

where ω is a smooth Kähler form and v a smooth volume on a general n-dimensional compact Kähler manifold X. However, a unique smooth solution has been known to exist for more than thirty years thanks to the celebrated work of Aubin [2] and Yau [38]. This is a strong indication that the viscosity method should work in this case to easily produce weak solutions.

In this article, we confirm this guess, defining and studying viscosity solutions to degenerate complex Monge-Ampère equations. Our main technical result is the following:

THEOREM A *Let X be a compact complex manifold, ω a continuous, closed, real $(1,1)$-form with $\mathcal{C}^2$ local potentials, and $v > 0$ a volume form with continuous density. Then the viscosity comparison principle holds for* $(\mathrm{DMA})_{\omega,v}$.

The viscosity comparison principle (see below for details) differs substantially from the pluripotential comparison principle of [5], which is the main tool in [23, 25, 33]. This technical statement is based on the Alexandroff-Bakelmann-Pucci maximum principle. We need, however, to modify the argument in [16] by a localization technique.

Although we need to assume v is positive in Theorem A, it is then easy to let it degenerate to a nonnegative density in the process of constructing weak solutions to degenerate complex Monge-Ampère equations. In this way we obtain the following:

COROLLARY B *Assume X is as above, and v is merely semipositive with $\int_X v > 0$. If $\omega \geq 0$ and $\int_X \omega^n > 0$, then there is a unique viscosity solution $\varphi \in C^0(X)$ to* $(\mathrm{DMA})_{\omega,v}$.

If X is a compact complex manifold in the Fujiki class, it coincides with the unique, locally bounded ω-plurisubharmonic (ω-psh for short) function φ on X such that $(\omega + dd^c\varphi)^n_{\mathrm{BT}} = e^{\varphi} v$ in the pluripotential sense [23].

Recall that φ is ω-psh if it is locally the sum of a smooth and a psh function, and such that $\omega + dd^c\varphi \geq 0$ in the weak sense of currents.

In this context Bedford and Taylor [5] showed that when φ is bounded, there exists a unique positive Radon measure $(\omega + dd^c\varphi)^n_{\mathrm{BT}}$ with the following property: if φ_j are smooth, locally ω-psh, and decreasing to φ, then the smooth measures $(\omega + dd^c\varphi_j)^n$ weakly converge towards the measure $(\omega + dd^c\varphi)^n_{\mathrm{BT}}$. If the measures $(\omega + dd^c\varphi_j)^n$ (locally) converge to $e^{\varphi} v$, we say that the equality $(\omega + dd^c\varphi)^n_{\mathrm{BT}} = e^{\varepsilon} v$ holds *in the pluripotential sense*.

Combining pluripotential and viscosity techniques, we can push our results further and obtain the following:

THEOREM C *Let X be a compact complex manifold in the Fujiki class. Let v be a semipositive probability measure with L^p-density, $p > 1$, and fix $\omega \geq 0$, a smooth, closed, real semipositive $(1,1)$-form such that $\int_X \omega^n = 1$. The unique locally bounded ω-psh function on X normalized by $\int_X \varphi = 0$ such that its Monge-Ampère measure satisfies $(\omega + dd^c\varphi)^n_{\mathrm{BT}} = v$ is continuous.*

This continuity statement was obtained in [23] under a regularization statement for ω-psh functions that we were not able to obtain in full generality. It could have been obtained using [3] in the cases covered by this reference. However, for rational homogeneous spaces, the regularization statement is easily proved by convolution [18], and [3] does not give anything new. A proof of the continuity when X is projective under mild technical assumptions has been obtained in [21].

We now describe the organization of the article. The first section is devoted to the local theory. It makes the connection between the complex Monge-Ampère operator and the viscosity subsolutions of inhomogenous complex Monge-Ampère equations. We have found no reference for these basic facts.

In the second section, we introduce the viscosity comparison principle and give a proof of the main theorem. The gain with respect to classical pluripotential theory is that one can consider supersolutions to prove continuity of pluripotential solutions for $(\omega + dd^c\varphi)^n = e^{\varphi} v$.

In the third section we apply these ideas to show that the singular Kähler-Einstein potentials constructed in [23] are globally continuous.

In the fourth and last section, we stress some advantages of our method:

- It gives an alternative proof of Kolodziej's C^0 Yau theorem that does not depend on [38].
- It allows us to easily produce the unique, negatively curved singular Kähler-Einstein metric in the canonical class of a projective manifold of general type, a result first obtained in [23] assuming [9, 27], and then in [11] by means of asymptotic Zariski decompositions.

Then we establish further comparison principles: these could be useful when studying similar problems where pluripotential tools do not apply. We end the article with some remarks on a possible interpretation of viscosity supersolutions in terms of pluripotential theory using plurisubharmonic projection.

The idea of applying viscosity methods to the Kähler-Ricci flow was proposed originally in a remark from the preprint [12]. We hope that the technique developed here will have further applications. In a forthcoming work it will be applied to the Kähler-Ricci flow.

1 Viscosity Subsolutions to $(dd^c\varphi)^n = e^{\varepsilon\varphi}v$

The purpose of this section is to make the connection, in a purely local situation, between the pluripotential theory of complex Monge-Ampère operators, as founded by Bedford and Taylor [5], and the concept of viscosity subsolutions developed by Lions et al. (see [16, 31]).

1.1 Viscosity Subsolutions of $(dd^c\varphi)^n = v$

Let $M = M^{(n)}$ be a (connected) complex manifold of dimension n and v a semipositive measure with continuous density. In this section B will denote the unit ball of $\mathbb{C}^n$ or its image under a coordinate chart in M.

DEFINITION 1.1 An upper semicontinuous function $\varphi : M \to \mathbb{R} \cup \{-\infty\}$ is said to be a *viscosity subsolution of the Monge-Ampère equation*

$$(dd^c\varphi)^n = v \tag{DMA$_v$}$$

if it satisfies the following conditions:

(i) $\varphi|_M \not\equiv -\infty$.

(ii) For every $x_0 \in M$ and any $\mathcal{C}^2$ function q defined on a neighborhood of x_0 such that $\varphi - q$ has a local maximum at x_0, then

$$(dd^c q)^n_{x_0} \geq v_{x_0}.$$

We will also say that φ satisfies the differential inequality $(dd^c\varphi)^n \geq v$ in the viscosity sense on M.

Note that if $v \geq v'$, then $(dd^c\varphi)^n \geq v$ in the viscosity sense implies $(dd^c\varphi)^n \geq v'$. This holds in particular if $v' = 0$.

Another basic observation is that the class of subsolutions is stable under taking the maximum:

LEMMA 1.2 *If φ_1 and φ_2 are subsolutions of $(dd^c\varphi)^n = v$, so is* $\sup(\varphi_1, \varphi_2)$.

The proof is straightforward and left to the reader. We now observe that a function φ satisfies $(dd^c\varphi)^n \geq 0$ in the viscosity sense if and only if it is plurisubharmonic:

PROPOSITION 1.3 *The viscosity subsolutions of $(dd^c\varphi)^n = 0$ are precisely the plurisubharmonic functions on M.*

PROOF. The statement is local and we can assume $M = B$. Let φ be a subsolution of $(dd^c\varphi)^n = 0$. Let $x_0 \in B$ be such that $\varphi(x_0) \neq -\infty$. Let $q \in \mathcal{C}^2(B)$ be such that $\varphi - q$ has a local maximum at x_0. Then the Hermitian matrix $Q = dd^c q_{x_0}$ satisfies $\det(Q) \geq 0$. For every Hermitian semipositive matrix H, we also have $\det(Q + H) \geq 0$ since, a fortiori for $q_H = q + H(x - x_0)$, $\varphi - q_H$ has a local maximum at x_0 too.

It follows from Lemma 1.4 below that $Q = dd^c q_{x_0}$ is actually semipositive. We infer that for every positive definite Hermitian matrix $(h^{i\bar{j}})$

$$\Delta_H q(x_0) := h^{i\bar{j}} \frac{\partial^2 q}{\partial z_i \partial \bar{z}_j}(x_0) \geq 0,$$

i.e., φ is a viscosity subsolution of $\Delta_H \varphi = 0$. In appropriate complex coordinates this constant-coefficient differential operator is nothing but the Laplace operator. Hence, [29, prop. 3.2.10′, p. 147] applies to the effect that φ is Δ_H-subharmonic and so is in $L^1_{\mathrm{loc}}(B)$ and satisfies $\Delta_H \varphi \geq 0$ in the sense of distributions. Let (w^i) be any vector in $\mathbb{C}^n$. Consider a positive Hermitian matrix $(h^{i\bar{j}})$ degenerating to the rank 1 matrix $(w^i \bar{w}^j)$. By continuity, we have $w^i \bar{w}^j \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} \geq 0$ in the sense of distributions. Thus φ is plurisubharmonic.

Conversely, assume φ is plurisubharmonic. Fix $x_0 \in B$ and $q \in \mathcal{C}^2(B))$ such that $\varphi - q$ has a local maximum at x_0. Then, for every small enough ball $B' \subset B$ centered at x_0, we have

$$\varphi(x_0) - q(x_0) \geq \frac{1}{V(B')} \int_{B'} (\varphi - q) dV;$$

hence

$$\frac{1}{V(B')} \int_{B'} q\, dV - q(x_0) \geq \frac{1}{V(B')} \int_{B'} \varphi\, dV - \varphi(x_0) \geq 0.$$

Letting the radius of B' tend to 0, it follows since q is $\mathcal{C}^2$ that $\Delta q_{x_0} \geq 0$. Using complex ellipsoids instead of balls (this amounts to a linear change of complex coordinates), we conclude that $\Delta_H q(x_0) \geq 0$ for every positive definite Hermitian matrix. Thus $dd^c q_{x_0} \geq 0$ and $(dd^c\varphi)^n \geq 0$ in the viscosity sense. □

The following lemma is easily proven by diagonalizing Q:

LEMMA 1.4 *Let Q be a Hermitian matrix such that, for every semipositive Hermitian matrix H,* $\det(Q + H) \geq 0$. *Then Q is semipositive.*

Recall that when φ is plurisubharmonic and locally bounded, its Monge-Ampère measure $(dd^c\varphi)^n_{\mathrm{BT}}$ is well-defined [5] (as the unique limit of the smooth measures $(dd^c\varphi_j)^n$, where φ_j is any sequence of smooth psh functions decreasing to φ). Our next result makes the basic connection between this pluripotential notion and its viscosity counterpart:

PROPOSITION 1.5 *Let φ be a locally bounded, upper semicontinuous function in M. It satisfies $(dd^c\varphi)^n \geq v$ in the viscosity sense if and only if it is plurisubharmonic and its Monge-Ampère measure satisfies $(dd^c\varphi)^n_{\mathrm{BT}} \geq v$ in the pluripotential sense.*

PROOF. We first recall the following classical formulation of the pluripotential comparison principle for the complex Monge-Ampère operator, acting on bounded plurisubharmonic functions [5]:

LEMMA 1.6 *Let $u, w \in \mathrm{PSH} \cap L^\infty(B)$. If $u \geq w$ near ∂B and $(dd^c u)^n_{\mathrm{BT}} \leq (dd^c w)^n_{\mathrm{BT}}$, then $u \geq w$.*

Assume $\varphi \in \mathrm{PSH} \cap L^\infty(B)$ satisfies $(dd^c\varphi)^n_{\mathrm{BT}} \geq v$. Consider q a $\mathcal{C}^2$ function such that $\varphi - q$ achieves a local maximum at x_0 and $\varphi(x_0) = q(x_0)$. Since φ satisfies $(dd^c\varphi)^n \geq 0$ in the viscosity sense, $(dd^c q)^n_{x_0} \geq 0$ and $dd^c q_{x_0} \geq 0$ by Lemma 1.4. Assume $(dd^c q)^n_{x_0} < v_{x_0}$. Let $q^\varepsilon := q + \varepsilon\|x - x_0\|^2$. Choosing $\varepsilon > 0$ small enough, we have $0 < (dd^c q^\varepsilon_{x_0})^n < v_{x_0}$. Since v has continuous density, we can choose a small ball B' containing x_0 of radius $r > 0$ such that $\overline{q}^\varepsilon = q^\varepsilon - \varepsilon(r^2/2) \geq \varphi$ near $\partial B'$ and $(dd^c\overline{q}^\varepsilon)^n_{\mathrm{BT}} \leq (dd^c\varphi)^n_{\mathrm{BT}}$. The comparison principle (Lemma 1.6) yields $\overline{q}^\varepsilon \geq \varphi$ on B'. But this fails at x_0. Hence $(dd^c q)^n_{x_0} \geq v_{x_0}$ and φ is a viscosity subsolution.

Conversely, assume φ is a viscosity subsolution. Fix $x_0 \in B$ such that $\varphi(x_0) \neq -\infty$ and $q \in \mathcal{C}^2$ such that $\varphi - q$ has a local maximum at x_0. Then the Hermitian matrix $Q = dd^c q_{x_0}$ satisfies $\det(Q) \geq v_{x_0}$.

Recall that the classical trick (due to Krylov) of considering the complex Monge-Ampère equation as a Bellmann equation relies on the following:

LEMMA 1.7 [24] *Let Q be an $n \times n$ nonnegative Hermitian matrix; then*

$$\det(Q)^{1/n} = \inf\{\mathrm{tr}(HQ) \mid H \in H_n^+ \text{ and } \det(H) = n^{-n}\},$$

where H_n^+ denotes the set of positive Hermitian $n \times n$ matrices.

Applying this to our situation, it follows that for every positive definite Hermitian matrix $(h_{i\bar{j}})$ with $\det(h) = n^{-n}$,

$$\Delta_H q(x_0) := h_{i\bar{j}} \frac{\partial^2 q}{\partial z_i \partial \overline{z}_j}(x_0) \geq v^{1/n}(x_0);$$

i.e., φ is a viscosity subsolution of the linear equation $\Delta_H \varphi \geq v^{1/n}$.

This is a constant-coefficient linear partial differential equation. Assume $v^{1/n}$ is C^α with $\alpha > 0$ and choose a $\mathcal{C}^2$ solution of $\Delta_H \varphi = v^{1/n}$ in a neighborhood of x_0. Then $u = \varphi - f$ satisfies $\Delta_H u \geq 0$ in the viscosity sense. Once again, [29, prop. 3.2.10′, p. 147] applies to the effect that u is Δ_H-subharmonic; hence $\Delta_H \varphi \geq v^{1/n}$ in the sense of positive Radon measures.

Using convolution to regularize φ and setting $\varphi_\varepsilon = \varphi * \rho_\varepsilon$, we see that $\Delta_H \varphi_\varepsilon \geq (v^{1/n})_\varepsilon$. Another application of the above lemma yields

$$(dd^c\varphi_\varepsilon)^n \geq ((v^{1/n})_\varepsilon)^n.$$

Here $\tilde{\varphi}_k = \varphi_{1/k}$ is a decreasing sequence of smooth functions converging to φ. Continuity of $(dd^c\varphi)^n_{\mathrm{BT}}$ with respect to such a sequence [5] yields $(dd^c\varphi)^n_{\mathrm{BT}} \geq v$.

This settles the case when $v > 0$ and v is Hölder-continuous. In case $v > 0$ is merely continuous we observe that $v = \sup\{w \mid w \in \mathcal{C}^\infty,\ v \geq w > 0\}$. Taking into account the fact that any subsolution of $(dd^c\varphi)^n = v$ is a subsolution of $(dd^c\varphi)^n = w$ provided $v \geq w$, we conclude $(dd^c\varphi)^n_{\mathrm{BT}} \geq v$.

In the general case, we observe that $\psi_\varepsilon(z) = \varphi(z) + \varepsilon\|z\|^2$ satisfies $(dd^c\psi_\varepsilon)^n \geq v + \varepsilon^n\lambda$ in the viscosity sense with λ the euclidean volume form. Hence

$$(dd^c\psi_\varepsilon)^n_{\mathrm{BT}} \geq v,$$

from which we conclude that $(dd^c\varphi)^n_{\mathrm{BT}} \geq v$. □

Remark 1.8. The basic idea of the proof is closely related to the method in [4] and is the topic treated in [37]. The next section contains a more powerful version of this argument. However, it uses sup-convolution, which is not a conventional tool in pluripotential theory. We felt that keeping this version would improve the exposition.

We now relax the assumption that φ is bounded and connect viscosity subsolutions to pluripotential subsolutions through the following:

THEOREM 1.9 *Assume $v = (dd^c\rho)^n_{\mathrm{BT}}$ for some bounded plurisubharmonic function ρ. Let φ be an upper semicontinuous function such that $\varphi \not\equiv -\infty$ on any connected component. The following are equivalent*:

(i) *φ satisfies $(dd^c\varphi)^n \geq v$ in the viscosity sense*;

(ii) *φ is plurisubharmonic and for all $c > 0$, $(dd^c \sup[\varphi, \rho - c])^n_{\mathrm{BT}} \geq v$.*

Observe that these properties are local and that the semipositive measure v can always be written locally as $v = (dd^c\rho)^n_{\mathrm{BT}}$ for some bounded plurisubharmonic function ρ [33].

PROOF. Assume first that φ is a viscosity subsolution of $(dd^c\varphi)^n = v$. Since $\rho - c$ is also a subsolution, it follows from Lemma 1.2 that $\sup(\varphi, \rho - c)$ is a subsolution; hence Proposition 1.5 yields $(dd^c \sup(\varphi, \rho - c))^n_{\mathrm{BT}} \geq v$.

Conversely, fix $x_0 \in M$ and assume (i) holds. If φ is locally bounded near x_0, Proposition 1.5 implies that φ is a viscosity subsolution near x_0.

Assume $\varphi(x_0) \neq -\infty$ but that φ is not locally bounded near x_0. Fix $q \in \mathcal{C}^2$ such that $q \geq \varphi$ near x_0 and $q(x_0) = \varphi(x_0)$. Then for $c > 0$ big enough we have $q \geq \varphi_c = \sup(\varphi, \rho - c)$ and $q(x_0) = \varphi_c(x_0)$; hence $(dd^c q)^n_{x_0} \geq v_{x_0}$ by Proposition 1.5 again.

Finally, if $\varphi(x_0) = -\infty$, there is no q to be tested against the differential inequality; hence it holds for every test function q. □

Condition (ii) might seem a bit cumbersome. The point is that the Monge-Ampère operator cannot be defined on the whole space of plurisubharmonic functions. The above arguments actually work in any class of plurisubharmonic functions in which the Monge-Ampère operator is continuous by decreasing limits of locally bounded functions and the comparison principle holds. These are precisely the finite energy classes studied in [14, 26].

When φ belongs to its domain of definition, condition (ii) is equivalent in the pluripotential sense to $(dd^c\varphi)^n_{\mathrm{BT}} \geq v$. To be more precise, we have the following:

COROLLARY 1.10 *Let $\Omega \subset \mathbb{C}^n$ be a hyperconvex domain. Then $\varphi \in \mathcal{E}(\Omega)$ (see [15] for the notation) satisfies $(dd^c\varphi)^n \geq v$ in the viscosity sense if and only if its Monge-Ampère measure $(dd^c\varphi)^n_{\mathrm{BT}}$ satisfies $(dd^c\varphi)^n_{\mathrm{BT}} \geq v$.*

We do not want to recall the definition of the class $\mathcal{E}(\Omega)$. It suffices to say that when $n = 2$, a psh function φ belongs to this class if and only if $\nabla\varphi \in L^2_{\mathrm{loc}}$ [10].

1.2 Viscosity Subsolutions to $(dd^c\varphi)^n = e^{\varepsilon\varphi} v$

Let $\varepsilon > 0$ be a real number. Say that a u.s.c. function φ is a viscosity subsolution of $(dd^c\varphi)^n = e^{\varepsilon\varphi}v$ if φ is not identically $-\infty$, and for all $x_0 \in M$ and all $q \in \mathcal{C}^2(M)$ of x_0 such that $\varphi - q$ has a local maximum at x_0 and $\varphi(x_0) = q(x_0)$, one has $(dd^c q(x_0))^n \geq e^{\varepsilon q(x_0)} v(x_0)$.

PROPOSITION 1.11 *Let $\varphi : M \to \mathbb{R}$ be a bounded u.s.c. function. It satisfies $(dd^c\varphi)^n \geq e^{\varepsilon\varphi}v$ in the viscosity sense if and only if it is plurisubharmonic and it does in the pluripotential sense.*

PROOF. When φ is continuous, so is the density of $\tilde{v} = e^{\varepsilon\varphi}v$ and Proposition 1.5 above can be applied. When φ is not assumed to be continuous, the issue is more subtle.

We can assume without loss of generality that $\varepsilon = 1$ and $M = \Omega$ is a domain in $\mathbb{C}^n$. Assume φ is a viscosity subsolution. It follows from Proposition 1.3 that φ is psh. Set $v = f\beta_n$, where $f > 0$ is the continuous density of the volume form v with respect to the euclidean volume form on $\mathbb{C}^n$. We approximate φ by its sup-convolution:

$$\varphi^\delta(x) := \sup_y \left\{ \varphi(y) - \frac{1}{2\delta^2}|x-y|^2 \right\}, \quad x \in \Omega_\delta,$$

for $\delta > 0$ small enough, where $\Omega_\delta := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > A\delta\}$ and $A > 0$ is a large constant so that $A^2 > 2\,\mathrm{osc}_\Omega\,\varphi$.

This family of semiconvex functions decreases towards φ as δ decreases to 0. Furthermore, by [30], φ^δ satisfies the following inequality in the sense of viscosity on Ω_δ:

$$(dd^c\varphi^\delta)^n \geq e^{\varphi^\delta} f_\delta \beta_n \quad \text{with } f_\delta(x) = \inf\{f(y) \mid |y-x| \leq A\delta\}.$$

It follows from Proposition 1.3 that φ^δ is psh.[1] Since φ^δ is *continuous*, we can invoke Proposition 1.5 and get that

$$(dd^c\varphi^\delta)^n_{BT} \geq e^{\varphi^\delta} f_\delta\beta_n \geq e^{\varphi} f_\delta \beta_n$$

[1] This argument implies that a sup-convolution of a psh function is psh. This in turn is easily deduced from the change of variables $y = x - y'$ in the definition of $\varphi^\delta(x)$.

holds in the pluripotential sense. Since the complex Monge-Ampère operator is continuous along decreasing sequences of bounded psh functions, and since f_δ increases towards f, we finally obtain $(dd^c\varphi)^n_{BT} \geq e^{\varphi}v$ in the pluripotential sense.

We now treat the other implication. Let φ be a psh function satisfying the inequality

$$(dd^c\varphi)^n_{BT} \geq e^{\varphi}v$$

in the pluripotential sense on Ω. We want to prove that φ satisfies the above differential inequality in the sense of viscosity on Ω. If φ were continuous, then we could use Proposition 1.5. But since φ is not necessarily continuous, we first approximate φ using sup-convolution φ^δ as above. Lemma 1.12 below yields the following pointwise inequality in Ω_δ:

$$(dd^c\varphi^\delta)^n_{BT} \geq e^{\varphi^\delta} f_\delta \beta_n \tag{1.1}$$

in the sense of pluripotential theory.

Since φ^δ is continuous we can apply Proposition 1.5 to conclude that φ^δ is a viscosity subsolution of the equation $(dd^c u)^n = e^u f_\delta \beta_n$ on Ω_δ. From this we want to deduce that φ is a viscosity subsolution of the equation $(dd^c\varphi)^n = e^{\varphi} f\beta_n$ by passing to the limit as δ decreases to 0. This is certainly a well-known fact in viscosity theory, but let us give a proof here for convenience.

Let $x_0 \in \Omega$, q be a quadratic polynomial such that $\varphi(x_0) = q(x_0)$, and $\varphi \leq q$ on a neighborhood of x_0, say on a ball $2B$, where $B := B(x_0, r) \Subset \Omega$. Since φ is psh on Ω, it satisfies $(dd^c\varphi)^n \geq 0$ in the viscosity sense on Ω by Proposition 1.5 and then by Lemma 1.4, it follows that $dd^c q(x_0) \geq 0$. Replacing q by $q(x)+\varepsilon|x-x_0|^2$ and taking $r > 0$ small enough, we can assume that q is psh on the ball $2B$. We want to prove that $(dd^c q(x_0))^n \geq e^{\varphi(x_0)} f(x_0)\beta_n$.

Fix $\varepsilon > 0$ small enough. For $x \in B$, set

$$q_\varepsilon(x) := q(x) + 2\varepsilon(|x - x_0|^2 - r^2) + \varepsilon r^2.$$

Observe first that since $\varphi \leq q$ on $2B$, we have the following properties:

- If $x \in \partial B$, $\varphi^\delta(x) - q_\varepsilon(x) = \varphi^\delta(x) - q(x) - \varepsilon r^2 < 0$ for $0 < \delta \ll 1$.
- If $x = x_0$, then $\varphi_\delta(x_0) - q^\varepsilon(x_0) = \varphi_\delta(x_0) - q(x_0) + \varepsilon r^2$.

Since $\varphi_\delta(x_0) - q(x_0) \to \varphi(x_0) - q(x_0) + \varepsilon r^2 = \varepsilon r^2$ as $\delta \to 0$, it follows that for δ small enough, the function $\varphi^\delta(x) - q_\varepsilon(x)$ takes its maximum on $\bar{B}$ at some interior point $x_\delta \in B$ and this maximum satisfies the inequality

$$\lim_{\delta\to 0} \max_{\bar{B}}(\varphi_\delta - q^\varepsilon) = \lim_{\delta\to 0}(\varphi_\delta(x_\delta) - q^\varepsilon(x_\delta)) \geq \varepsilon r^2. \tag{1.2}$$

Moreover, we claim that $x_\delta \to x_0$ as $\delta \to 0$. Indeed, we have

$$\begin{aligned}\varphi^\delta(x_\delta) - q_\varepsilon(x_\delta) &= \varphi^\delta(x_\delta) - q(x_\delta) - 2\varepsilon(|x_\delta - x_0|^2 - r^2) - \varepsilon r^2\\ &\leq q^\delta(x_\delta) - q(x_\delta) - 2\varepsilon|x_\delta - x_0|^2 + \varepsilon r^2.\end{aligned}$$

Since $q^\delta(x_\delta) - q(x_\delta)$ converges to 0, it follows that if x_0' is a limit point of the family (x_δ) in $\bar{B}$, then $\max_{\bar{B}}(\varphi_\delta - q^\varepsilon)$ will converge to a limit that is less or equal to $-2\varepsilon|x_0' - x_0|^2 + \varepsilon r^2$. By the inequality (1.2), this limit is $\geq \varepsilon r^2$. Therefore we obtain the inequality $-2\varepsilon|x_0' - x_0|^2 \geq 0$, which implies that $x_0' = x_0$ and our claim is proved.

Since $\varphi^\delta - q_\varepsilon$ takes its maximum on $\bar{B}$ at the point $x_\delta \in B$ and φ^δ is a viscosity subsolution of the equation $(dd^c u) \geq e^u f_\delta \beta_n$, it follows that

$$(dd^c q_\varepsilon(x_\delta))^n \geq e^{\varphi^\delta(x_\delta)} f_\delta(x_\delta)\beta_n = e^{\varphi^\delta(x_\delta) - q_\varepsilon(x_\delta)} e^{q_\varepsilon(x_\delta)} f_\delta(x_\delta)\beta_n.$$

Now observe that $\varphi^\delta - q_\varepsilon = (\varphi^\delta - q) + (q - q_\varepsilon)$ and by Dini's lemma

$$\limsup_{\delta\to 0} \max_{\bar{B}}(\varphi^\delta - q) = \max_{\bar{B}}(\varphi - q) = 0.$$

Therefore

$$\limsup_{\delta\to 0}(\varphi^\delta(x_\delta) - q_\varepsilon(x_\delta)) \geq \min_{\bar{B}}(q - q_\varepsilon) = \min_{\bar{B}}(-2\varepsilon|x - x_0|^2 + \varepsilon r^2) = -\varepsilon r^2.$$

It follows immediately that

$$(dd^c q_\varepsilon(x_0))^n \geq e^{q(x_0) - 2\varepsilon r^2} f(x_0)\beta_n.$$

Letting $\varepsilon \to 0$, we obtain the required inequality

$$(dd^c q(x_0))^n \geq e^{\varphi(x_0)} f(x_0)\beta_n,$$

since $q(x_0) = \varphi(x_0)$. □

LEMMA 1.12 *Let φ be a bounded plurisubharmonic function in a domain $\Omega \Subset \mathbb{C}^n$ such that*

$$(dd^c\varphi)^n_{BT} \geq e^\varphi f\beta_n$$

in the pluripotential sense in Ω, where $f \geq 0$ is a continuous density. Then the sup-convolutions (φ^δ) satisfy

$$(dd^c\varphi^\delta)^n_{BT} \geq e^{\varphi^\delta} f_\delta\beta_n,$$

in the pluripotential sense in Ω_δ, where $f_\delta(x) := \inf\{f(y) : |y - x| \leq A\delta\}$.

PROOF. Fix $\delta > 0$ small enough. For $y \in B(0, A\delta)$, denote by $\psi_y(x) := \varphi(x - y) - (1/2\delta^2)|y|^2$, $x \in \Omega_\delta$, and observe that ψ_y is a bounded psh function on Ω_δ that satisfies the following inequality in the pluripotential sense on Ω_δ,

$$(dd^c\psi_y)^n_{BT} \geq e^{\psi_y} f_\delta\beta_n,$$

thanks to the invariance of the complex Monge-Ampère operator by translation.

Since φ is the upper envelope of the family $\{\psi_y : y \in B(0, A\delta)\}$, it follows from a well-known topological lemma of Choquet that there is a sequence of points $(y_j)_{j\in\mathbb{N}}$ in the ball $B(0, A\delta)$ such that $\varphi^\delta = (\sup_j \psi_{y_j})^*$ on Ω_δ. For $j \in \mathbb{N}$, denote by $\theta_j := \sup_{0\leq k\leq j} \psi_{y_k}$. Then (θ_j) is an increasing sequence of bounded psh functions on Ω_δ that converges a.e. to φ^δ on Ω_δ. We claim that θ_j is also a pluripotential subsolution of the same equation, i.e.,

$$(dd^c\theta_j)^n_{BT} \geq e^{\theta_j} f_\delta \beta_n \tag{1.3}$$

in the pluripotential sense in Ω_δ. This follows from Demailly's inequality (see [17]), which we recall for convenience: if w_1 and w_2 are two bounded psh functions on Ω, then

$$(dd^c \sup\{w_1, w_2\})^n_{BT} \geq \mathbb{1}_{\{w_1>w_2\}}(dd^c w_1)^n_{BT} + \mathbb{1}_{\{w_1\leq w_2\}}(dd^c w_2)^n_{BT}.$$

If, moreover, $(dd^c w_i)^n_{BT} \geq e^{w_i} v$ for $i = 1, 2$, then $w := \sup\{w_1, w_2\}$ satisfies the same differential inequality, i.e., $(dd^c w)^n_{BT} \geq e^w v$. This proves our claim that θ_j satisfies inequality (1.3) in the pluripotential sense. Now by continuity of the complex Monge-Ampère operator along increasing sequences of bounded psh functions and the fact that $\sup_j \theta_j = \varphi^\delta$ quasi everywhere (see [5]), it follows from (1.3) that $(dd^c\varphi^\delta)^n_{BT} \geq e^{\varphi^\delta} f_\delta \beta_n$ in the weak sense on Ω_δ. □

2 Viscosity Comparison Principle for $(\omega + dd^c\varphi)^n = e^{\varepsilon\varphi} v$

We now set the basic frame for the viscosity approach to the equation

$$(\omega + dd^c\varphi)^n = e^{\varepsilon\varphi} v, \tag{DMA_v^ε}$$

where ω is a closed, smooth, real $(1, 1)$-form on an n-dimensional, connected complex manifold X, v is a volume form with nonnegative continuous density, and $\varepsilon \in \mathbb{R}_+$. Here the emphasis is on global properties.

The global comparison principle lies at the heart of the viscosity approach. Once it is established, Perron's method can be applied to produce viscosity solutions. Our main goal in this section is to establish the global comparison principle for $(\mathrm{DMA}_v^\varepsilon)$. We generally assume X is compact (and $\varepsilon > 0$): the structure of $(\mathrm{DMA}_v^\varepsilon)$ allows us to avoid any restrictive curvature assumption on X (unlike, e.g., in [3]).

2.1 Definitions for the Compact Case

To fit in with the viscosity point of view, we rewrite the Monge-Ampère equation as

$$e^{\varepsilon\varphi} v - (\omega + dd^c\varphi)^n = 0. \tag{$\mathrm{DMA}_v^\varepsilon$}$$

Let $x \in X$. If $\kappa \in \Lambda^{1,1}T_x X$, we define κ^n_+ to be κ^n if $\kappa \geq 0$ and 0 otherwise. For a technical reason, we will also consider a slight variant of $(\mathrm{DMA}_v^\varepsilon)$,

$$e^{\varepsilon\varphi} v - (\omega + dd^c\varphi)^n_+ = 0. \tag{$(\mathrm{DMA}_v^\varepsilon)_+$}$$

We let $\mathrm{PSH}(X, \omega)$ denote the set of all ω-psh functions on X: these are integrable upper semicontinuous functions $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ such that $dd^c\varphi \geq -\omega$ in the sense of currents.

LEMMA 2.1 *Let $\Omega \subset X$ be an open subset and $z : \Omega \to \mathbb{C}^n$ be a holomorphic coordinate chart. Let h be a smooth local potential for ω defined on Ω. Then* $(\mathrm{DMA}_v^{\varepsilon})$ *reduces in these z-coordinates to the scalar equation*

$$(\mathrm{DMA}_{v|z}^{\varepsilon}) \qquad e^{\varepsilon u} W - \det(u_{z\bar{z}}) = 0$$

where $u = (\varphi + h)|_{\Omega} \circ z^{-1}$, $z_* v = e^{\varepsilon h_{|\Omega} \circ z^{-1}} W \, d\lambda$, and λ is the Lebesgue measure on $z(\Omega)$.

On the other hand, $(\mathrm{DMA}_v^{\varepsilon})_+$ *reduces to the scalar equation*

$$(\mathrm{DMA}_{v|z}^{\varepsilon})_+ \qquad e^{\varepsilon u} W - \det(u_{z\bar{z}})_+ = 0.$$

The proof is straightforward. Note in particular that [31, condition (1.2), p. 27] (i.e., degenerate ellipticity) and [31, (2.11), p. 32; (2.18), p. 34] (properness) are satisfied by $(\mathrm{DMA}_{v|z}^{\varepsilon})_+$. If $v > 0$ and $\varepsilon > 0$, [31, (2.17), p. 33] is also satisfied, so that we can apply the tools described in [16, 31].

Subsolutions

If $\varphi_x^{(2)}$ is the 2-jet at $x \in X$ of a $\mathcal{C}^2$ real-valued function φ, we set

$$F_+(\varphi_x^{(2)}) = F_{+,v}^{\varepsilon}(\varphi_x) = e^{\varepsilon\varphi(x)} v_x - (\omega_x + dd^c \varphi_x)_+^n.$$

Recall the following definition from [31]:

DEFINITION 2.2 A *subsolution of* $(\mathrm{DMA}_v^{\varepsilon})_+$ is an upper semicontinuous function $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ such that $\varphi \not\equiv -\infty$ and the following property is satisfied: if $x_0 \in X$ and $q \in \mathcal{C}^2$, defined in a neighborhood of x_0, is such that $\varphi(x_0) = q(x_0)$ and

$$\varphi - q \ \text{ has a local maximum at } x_0,$$

then $F_+(q_{x_0}^{(2)}) \leq 0$.

Actually, this concept of a subsolution seems to be a bit too weak. It is not the same concept for $\epsilon = 0$ as in Section 1 and does not behave well if $v = 0$ since any u.s.c. function is then a viscosity subsolution of $(dd^c\varphi)_+^n = 0$. It behaves well, however, if $v > 0$. We introduce it nevertheless in order to be able to use the reference [31].

We now introduce what we believe to be the right definition, which leads to a slightly stronger statement. If $\varphi_x^{(2)}$ is the 2-jet at $x \in X$ of a $\mathcal{C}^2$ real-valued function φ, we set

$$F(\varphi_x^{(2)}) = F_v^{\varepsilon}(\varphi_x) = \begin{cases} e^{\varepsilon\varphi(x)} v_x - (\omega_x + dd^c\varphi_x)^n & \text{if } \omega + dd^c\varphi_x \geq 0, \\ +\infty & \text{otherwise.} \end{cases}$$

Recall the following definition from [16]:

DEFINITION 2.3 A *subsolution of* $(\mathrm{DMA}_v^{\varepsilon})$ is an upper semicontinuous function $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ such that $\varphi \not\equiv -\infty$ and the following property is satisfied: if

$x_0 \in X$ and $q \in \mathcal{C}^2$, defined in a neighborhood of x_0, is such that $\varphi(x_0) = q(x_0)$ and

$$\varphi - q \ \text{ has a local maximum at } x_0,$$

then $F(q^{(2)}_{x_0}) \leq 0$.

Remark 2.4. The function F^{ε}_{v} is lower semicontinuous and satisfies conditions (0.1) and (0.2) in [16].

Note that it is easy to compare subsolutions of $(\mathrm{DMA}^{\varepsilon}_{v})$ and $(\mathrm{DMA}^{\varepsilon}_{v})_{+}$:

LEMMA 2.5 *Every subsolution φ of $(\mathrm{DMA}^{\varepsilon}_{v})$ is a subsolution of $(\mathrm{DMA}^{\varepsilon}_{v})_{+}$; it is ω-plurisubharmonic.*

A locally bounded u.s.c. function is ω-psh and satisfies $(\omega + dd^{c}\varphi)^{n}_{\mathrm{BT}} \geq e^{\epsilon\varphi} v$ if and only if it is a (viscosity) subsolution of $(\mathrm{DMA}^{\varepsilon}_{v})$.

If $v > 0$, subsolutions of $(\mathrm{DMA}^{\varepsilon}_{v})_{+}$ are subsolutions of $(\mathrm{DMA}^{\varepsilon}_{v})$.

PROOF. The proof is an immediate consequence of the definitions, Theorem 1.9, and Proposition 1.11. One just has to choose a local potential ρ such that $dd^{c}\rho = \omega$ and set $\varphi' = \varphi + \rho$ and $v' = e^{-\varepsilon\rho} v$ to apply the local results of Section 1. □

Actually, the discussion after Theorem 1.9 fits well in the theory developed in [11] and we get the following:

COROLLARY 2.6 *Let X be a compact Kähler manifold and ω a smooth, closed, real $(1,1)$-form whose cohomology class $[\omega]$ is big. Let φ be any ω-plurisubharmonic function. Then φ satisfies $(\omega + dd^{c}\varphi)^{n} \geq e^{\varepsilon\varphi} v$ in the viscosity sense if and only if $\langle(\omega + dd^{c}\varphi)^{n}\rangle \geq e^{\varepsilon\varphi} v$, where $\langle(\omega + dd^{c}\varphi)^{n}\rangle$ is the nonpluripolar (pluripotential) Monge-Ampère measure of φ.*

(Super)solutions

DEFINITION 2.7 A *supersolution of* $(\mathrm{DMA}^{\varepsilon}_{v})$ is a supersolution of $(\mathrm{DMA}^{\varepsilon}_{v})_{+}$, that is, a lower semicontinuous function $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ such that $\varphi \not\equiv +\infty$ and the following property is satisfied: if $x_0 \in X$ and $q \in \mathcal{C}^2$, defined in a neighborhood of x_0, is such that $\varphi(x_0) = q(x_0)$ and $\varphi - q$ has a local minimum at x_0, then $F_{+}(q^{(2)}_{x_0}) \geq 0$.

DEFINITION 2.8 A *viscosity solution of* $(\mathrm{DMA}^{\varepsilon}_{v})$ is a function that is both a sub- and a supersolution. In particular, viscosity solutions are automatically continuous.

A *pluripotential solution of* $(\mathrm{DMA}^{\varepsilon}_{v})$ is a u.s.c. function $\varphi \in L^{\infty} \cap \mathrm{PSH}(X,\omega)$ such that $(\omega + dd^{c}\varphi)^{n}_{\mathrm{BT}} = e^{\varepsilon\varphi} v$.

Classical sub-/supersolutions are $\mathcal{C}^2$ viscosity sub-/supersolutions.

2.2 Local Comparison Principle

DEFINITION 2.9

(1) The *local* (*viscosity*) *comparison principle for* $(\mathrm{DMA}_v^\varepsilon)$ is said to hold if the following holds true: let $\Omega \subset X$ be an open subset such that $\bar{\Omega}$ is biholomorphic to a bounded, smooth, strongly pseudoconvex domain in $\mathbb{C}^n$; let $\underline{u}$ (respectively, $\overline{u}$) be a bounded subsolution (respectively, supersolution) of $(\mathrm{DMA}_v^\varepsilon)$ in Ω satisfying

$$\limsup_{z \to \partial\Omega} [\underline{u}(z) - \overline{u}(z)] \leq 0.$$

Then $\underline{u} \leq \overline{u}$.

(2) The *global* (*viscosity*) *comparison principle for* $(\mathrm{DMA}_v^\varepsilon)$ is said to hold if X is compact and the following holds true: let $\underline{u}$ (respectively, $\overline{u}$) be a bounded subsolution (respectively, supersolution) of $(\mathrm{DMA}_v^\varepsilon)$ in X. Then $\underline{u} \leq \overline{u}$.

We set the same definition with $(\mathrm{DMA}_v^\varepsilon)_+$ in place of $(\mathrm{DMA}_v^\varepsilon)$. Observe that $(\mathrm{DMA}_v^\varepsilon)_+$ may have extra subsolutions; the comparison principle for $(\mathrm{DMA}_v^\varepsilon)_+$ thus implies the comparison principle for $(\mathrm{DMA}_v^\varepsilon)$.

The local viscosity comparison principle does not hold for $(\mathrm{DMA}_0^0)_+$. Indeed, every u.s.c. function is a subsolution; the condition to be tested is actually empty. It is not clear whether it holds for (DMA_0^0) since it is actually a statement that differs substantially from the (pluripotential) comparison principle for the complex Monge-Ampère equation of [5].

PROPOSITION 2.10 *The local viscosity comparison principle for* $(\mathrm{DMA}_v^\varepsilon)$ *holds if* $\varepsilon > 0$ *and* $v > 0$.

PROOF. The proposition actually follows from corollary 4.8 in [3]. We include for the reader's convenience a proof that is an adaptation of arguments in [16].

We may assume without loss of generality that $\varepsilon = 1$. Let $\underline{u}$ be a bounded subsolution and $\overline{u}$ be a bounded supersolution of $(\mathrm{DMA}_v^\varepsilon)$ in some smoothly bounded, strongly pseudoconvex open set Ω such that $\underline{u} \leq \overline{u}$ on $\partial\Omega$. Replacing first $\underline{u}, \overline{u}$ by $\underline{u} - \delta, \overline{u} + \delta$, we can assume that the inequality is strict and holds in a small neighborhood of $\partial\Omega$.

As in the proof of Proposition 1.11, we regularize $\underline{u}$ and $\overline{u}$ using their sup/inf convolutions. Since $\underline{u}, \overline{u}$ are bounded, multiplying by a small constant, we can assume that for $\alpha > 0$ small enough and $x \in \Omega_\alpha$, we have

$$\underline{u}^\alpha(x) := \sup_{y \in \Omega} \left\{ \underline{u}(y) - \frac{1}{2\alpha^2}|y - x|^2 \right\} = \sup_{|y-x| \leq \alpha} \left\{ \underline{u}(y) - \frac{1}{2\alpha^2}|y - x|^2 \right\}$$

and

$$\overline{u}_\alpha(x) := \inf_{y \in \Omega_\alpha} \left\{ \overline{u}(y) + \frac{1}{2\alpha^2}|y - x|^2 \right\} = \inf_{|y-x| \leq \alpha} \left\{ \overline{u}(y) + \frac{1}{2\alpha^2}|y - x|^2 \right\}.$$

Then for $\alpha > 0$ small enough, $\underline{u}^\alpha(x) \leq \overline{u}_\alpha(x)$ near the boundary of Ω_α. Observe that, if we set $M_\alpha := \sup_{\overline{\Omega}_\alpha}[\underline{u}^\alpha - \overline{u}_\alpha]$, then

$$\liminf_{\alpha \to 0^+} M_\alpha \geq \sup_\Omega[\underline{u} - \overline{u}].$$

Arguing by contradiction, assume that $\sup_\Omega[\underline{u} - \overline{u}] > 0$. Then for $\alpha > 0$ small enough, the supremum M_α is > 0 and then it is attained at some point $x_\alpha \in \Omega_\alpha$.

The function $\underline{u}^\alpha$ is semiconvex and $\overline{u}_\alpha$ is semiconcave. In particular, they are twice differentiable almost everywhere on Ω_α by a theorem of Alexandrov [1] (see also [16]) in the following sense:

Definition 2.11 A real-valued function u defined on an open set $\Omega \subset \mathbb{C}^n$ is *twice differentiable* at almost every point $z_0 \in \Omega$ if and only if for every point $z_0 \in \Omega$ outside a Borel set of Lebesgue measure 0 in Ω, there exists a quadratic form $Q_{z_0}u$ on $\mathbb{R}^{2n}$, whose polar symmetric bilinear form will be denoted by $D^2u(z_0)$, such that for any $\xi \in \mathbb{R}^{2n}$ with $|\xi| \ll 1$, we have

$$u(z_0 + \xi) = u(z_0) + Du(z_0) \cdot \xi + \frac{1}{2} D^2u(z_0) \cdot (\xi, \xi) + o(|\xi|^2). \tag{2.1}$$

We first deduce a contradiction under the unrealistic assumption that $\underline{u}_\alpha$ and $\overline{u}_\alpha$ are twice differentiable at the point x_α. Then by the classical maximum principle we have

$$D^2\underline{u}^\alpha(x_\alpha) \leq D^2\overline{u}_\alpha(x_\alpha)$$

in the sense of quadratic forms on $\mathbb{R}^{2n}$. Applying this inequality for vectors of the form (Z, Z) and (iZ, iZ) and adding, we get the same inequality for Levi forms on $\mathbb{C}^n$, i.e.,

$$0 \leq dd^c\underline{u}^\alpha(x_\alpha) \leq dd^c\overline{u}_\alpha(x_\alpha),$$

where the first inequality follows from the fact that $\underline{u}^\alpha$ is plurisubharmonic on Ω_α since $\underline{u}$ is. From this inequality between nonnegative Hermitian forms on $\mathbb{C}^n$, it follows that the same inequality holds between their determinants, i.e.,

$$(dd^c\underline{u}^\alpha)^n(x_\alpha) \leq (dd^c\overline{u}_\alpha)^n(x_\alpha).$$

We know that

$$(dd^c\overline{u}_\alpha)^n(x_\alpha) \leq e^{\overline{u}_\alpha(x_\alpha)} f^\alpha(x_\alpha)\beta_n,$$

where f_α increases pointwise towards f, with $v = f\beta_n$, and

$$(dd^c\underline{u}^\alpha)^n(x_\alpha) \geq e^{\underline{u}^\alpha(x_\alpha)} f_\alpha(x_\alpha)\beta_n,$$

where f^α decreases towards f pointwise. Therefore we have for α small enough,

$$e^{\underline{u}^\alpha(x_\alpha)} f_\alpha(x_\alpha) \leq e^{\overline{u}_\alpha(x_\alpha)} f^\alpha(x_\alpha). \tag{2.2}$$

From this inequality we deduce immediately that

$$\sup_\Omega[\underline{u} - \overline{u}] \leq \overline{\lim}_{\alpha \to 0} M_\alpha \leq 0 = \lim_{\alpha \to 0} \log \frac{f^\alpha(x_\alpha)}{f_\alpha(x_\alpha)},$$

which contradicts our assumption that $\sup_\Omega[\underline{u} - \overline{u}] > 0$.

When $\underline{u}^\alpha, \overline{u}_\alpha$ are not twice differentiable at point x_α for a fixed $\alpha > 0$ small enough, we prove that inequality (2.2) is still valid by approximating x_α by a sequence of points where the functions are twice differentiable and not far from attaining their maximum at those points. For each $k \in \mathbb{N}^*$, the semiconvex function $\underline{u}^\alpha - \overline{u}_\alpha - \frac{1}{2k}|x - x_\alpha|^2$ attains a strict maximum at x_α. By Jensen's lemma ([32]; see also [16, lemma A.3, p. 60]), there exists a sequence $(p_k)_{k\geq 1}$ of vectors converging to 0 in $\mathbb{R}^n$ and a sequence of points (y_k) converging to x_α in Ω_α such that the functions $\underline{u}^\alpha$ and $\overline{u}_\alpha$ are twice differentiable at y_k and, if we set $q_k(x) = \frac{1}{2k}|x - x_\alpha|^2 + \langle p_k, x\rangle$, the function $\underline{u}^\alpha - \overline{u}_\alpha - q_k$ attains its maximum on Ω_α at the point y_k.

Applying the classical maximum principle for fixed α at each point y_k, we get

$$D^2\underline{u}^\alpha(y_k) \leq D^2\overline{u}_\alpha(y_k) + \frac{1}{k}I_n$$

in the sense of quadratic forms on $\mathbb{R}^{2n}$. As before we obtain the following inequalities between Levi forms:

$$0 \leq dd^c\underline{u}^\alpha(y_k) \leq dd^c\overline{u}_\alpha(y_k) + \frac{1}{k}dd^c|x|^2 \tag{2.3}$$

in the sense of positive Hermitian forms on $\mathbb{C}^n$, where the first inequality follows from the fact that $\underline{u}^\alpha$ is plurisubharmonic on Ω_α. The inequality (2.3) between positive Hermitian forms implies the same inequality between their determinants, so that

$$(dd^c\underline{u}^\alpha(y_k))^n \leq \left(dd^c\overline{u}_\alpha(y_k) + \frac{1}{k}dd^c|x|^2\right)^n. \tag{2.4}$$

Recall that $\overline{u}_\alpha - [1/(2\alpha^2)]|x|^2$ is concave on Ω_α; hence

$$D^2\overline{u}_\alpha(y_k) \leq \frac{1}{\alpha^2}I_n,$$

in the sense of quadratic forms on $\mathbb{R}^{2n}$. Therefore,

$$dd^c\overline{u}_\alpha(y_k) \leq \frac{1}{2\alpha^2}dd^c|x|^2. \tag{2.5}$$

From (2.3) and (2.5), it follows that, with α being fixed,

$$\left(dd^c\overline{u}_\alpha(y_k) + \frac{1}{k}dd^c|x|^2\right)^n = (dd^c\overline{u}_\alpha(y_k))^n + O(1/k). \tag{2.6}$$

We know by definition of subsolutions and supersolutions that

$$\begin{aligned} (dd^c\underline{u}^\alpha(y_k))^n &\geq e^{\underline{u}^\alpha(y_k)}f_\alpha(y_k)\beta_n, \\ (dd^c\overline{u}_\alpha(y_k))^n &\leq e^{\overline{u}_\alpha(y_k)}f^\alpha(y_k)\beta_n. \end{aligned} \tag{2.7}$$

Therefore from inequalities (2.4), (2.6), and (2.7), it follows that for any $k \geq 1$ we have

$$e^{\underline{u}^\alpha(y_k)}f_\alpha(y_k) \leq e^{\overline{u}_\alpha(y_k)}f^\alpha(y_k) + O(1/k),$$

which implies inequality (2.2) as $k \to +\infty$. The same argument as above then gives a contradiction. $\square$

2.3 Global Viscosity Comparison Principle

The global comparison principle can be deduced from [3] when X carries a Kähler metric with positive sectional curvature. This global curvature assumption is very strong: as explained in the introduction, [35] reduces us to a situation where one can regularize ω-psh functions with no loss of positivity [18], so that the viscosity approach is not needed to achieve continuity (see [23]). On the other hand, [3] considers very general degenerate elliptic equations, whereas we are considering a rather restricted class of complex Monge-Ampère equations.

In the general case, neither [3] nor [28] allows us to establish a global comparison principle even in the simplest case that we now consider:

PROPOSITION 2.12 *The global comparison principle holds when the cohomology class of ω is Kähler and v is continuous and positive.*

PROOF. We can assume without loss of generality that $\varepsilon = 1$.

Assume first that $v > 0$ and smooth. By [2, 38], there is $\varphi_Y \in C^2(M)$, a classical solution of (DMA_v^1). If $\underline{u}$ is a subsolution, then $\underline{u} \leq \varphi_Y$, as follows from Lemma 2.13 below. Similarly, if $\overline{u}$ is a supersolution, $\overline{u} \geq \varphi$. If $v > 0$ is merely continuous, fix $\delta > 0$. Then, if $\underline{u}$ is a subsolution of (DMA_v^1), $\underline{u} - \delta$ is a subsolution of $(\mathrm{DMA}_{e^\delta v}^1)$. Similarly, if $\overline{u}$ is a supersolution of (DMA_v^1), $\overline{u} + \delta$ is a supersolution of $(\mathrm{DMA}_{e^{-\delta} v}^1)$. Choose v^* a smooth volume form such that $e^{-\delta} v < v^* < e^\delta v$. Then $\underline{u} - \delta$ is a subsolution of $(\mathrm{DMA}_{v^*}^1)$ and $\overline{u} + \delta$ a supersolution. Hence, $\underline{u} - \delta \leq \overline{u} + \delta$. Letting $\delta \to 0$, we conclude the proof. $\square$

LEMMA 2.13 *Assume $v > 0$. Let $\underline{u}$ be a bounded subsolution of* $(\mathrm{DMA}_v^\varepsilon)$ *on X. If $\overline{u}$ is a $\mathcal{C}^2$ supersolution on X, then $\underline{u} \leq \overline{u}$.*

Note in particular that if $(\mathrm{DMA}_v^\varepsilon)$ has a classical solution, then it dominates (respectively, minorates) every subsolution (respectively, supersolution); hence the global viscosity principle holds.

PROOF. If $\overline{u}$ is classical, the fact that $\underline{u} \leq \overline{u}$ is a trivial consequence of the definition of subsolution at a maximum of $\underline{u} - \overline{u}$. Indeed, let $x_0 \in X$ such that $\underline{u}(x_0) - \overline{u}(x_0) = \max_X(\underline{u} - \overline{u}) = m$. Use $q = \overline{u} + m$ as a test function in the definition of $\underline{u}$ being a viscosity subsolution to deduce

$$(\omega + dd^c \overline{u})^n_{x_0} \geq e^{\overline{u}(x_0) + m} v.$$

On the other hand, since $\overline{u}$ is a classical supersolution, we have

$$(\omega + dd^c \overline{u})^n_{x_0} \leq e^{\overline{u}(x_0)} v.$$

Hence $m \leq 0$. $\square$

The above remarks only have academic interest since we use existence of a classical solution to deduce a comparison principle whose main consequence is existence of a viscosity solution (cf. infra). We need to establish an honest global comparison principle that will allow us to produce solutions without invoking [2, 38]. We now come to this result:

THEOREM 2.14 *The global viscosity comparison principle for* $(\mathrm{DMA}_v^\varepsilon)$ *holds, provided ω is a closed, real $(1,1)$-form, $v > 0$, $\varepsilon > 0$, and X is compact.*[2]

Since $v > 0$, the subsolutions of $(\mathrm{DMA}_v^\varepsilon)_+$ are those of $(\mathrm{DMA}_v^\varepsilon)$; hence this could be stated as the global viscosity comparison principle for $(\mathrm{DMA}_v^\varepsilon)_+$.

PROOF. As above, we assume $\varepsilon = 1$. Let u^* be a bounded supersolution and u_* be a bounded subsolution. We choose $C > 0$ such that both are $\leq C/1000$ in the L^∞ norm. Since $u_* - u^*$ is upper semicontinuous on the compact manifold M, it follows that its maximum is achieved at some point $\hat{x}_1 \in M$. Choose complex coordinates $(z^1, \dots, z^n)$ near $\hat{x}_1$ that define a biholomorphism identifying an open neighborhood of $\hat{x}_1$ to the complex ball $B(0,4)$ of radius 4 sending $\hat{x}_1$ to 0.

Using a partition of unity, construct a Riemannian metric on M that coincides with the flat Kähler metric $\sum_k (\sqrt{-1}/2) dz^k \wedge d\bar{z}^k$ on the ball of center 0 and radius 3. For $(x,y) \in M \times M$ define $d(x,y)$ to be the Riemannian distance function. The continuous function d^2 is of class $\mathcal{C}^2$ near the diagonal and > 0 outside the diagonal $\Delta \subset M^2$.

Next, construct a smooth nonnegative function φ_1 on $M \times M$ by the following formula:

$$\varphi_1(x,y) = \chi(x,y) \sum_{i=1}^n |z^i(x) - z^i(y)|^{2n},$$

where χ is a smooth, nonnegative cutoff function with $1 \geq \chi \geq 0$, $\chi \equiv 1$ on $B(0,2)^2$ and $\chi = 0$ near $\partial B(0,3)^2$.

Finally, consider a second smooth function on $M \times M$ with $\varphi_2|_{B(0,1)^2} < -1$, $\varphi_2|_{M^2 - B(0,2)^2} > 3C$. Choose $1 \gg \eta > 0$ such that $-\eta$ is a regular value of both φ_2 and $\varphi_2|_\Delta$. We perform a convolution of $(\xi, \xi') \mapsto \max(\xi, \xi')$ by a smooth semipositive function ρ such that $B_{\mathbb{R}^2}(0,\eta) = \{\rho > 0\}$ and get a smooth function $\max_\eta$ on $\mathbb{R}^2$ such that

- $\max_\eta(\xi, \xi') = \max(\xi, \xi')$ if $|\xi - \xi'| \geq \eta$,
- $\max_\eta(\xi, \xi') > \max(\xi, \xi')$ if $|\xi - \xi'| < \eta$.

We define $\varphi_3 \in \mathcal{C}^\infty(M^2, \mathbb{R})$ to be $\varphi_3 = \max_\eta(\varphi_1, \varphi_2)$. Observe that

- $\varphi_3 \geq 0$,
- $\varphi_3^{-1}(0) = \Delta \cap \{\varphi_2 \leq -\eta\}$,
- $\varphi_3|_{M^2 - B(0,2)^2} > 3C$.

[2] We do *not* assume that X is Kähler. However, this statement seems to be useless outside the Fujiki class.

We define $h_\omega \in \mathcal{C}^2(\overline{B(0,4)}, \mathbb{R})$ to be a local potential, smooth up to the boundary for ω, and extend it smoothly to M. We may without losing generality assume that $\|h_\omega\|_\infty < C/10$. In particular, $dd^c h_\omega = \omega$ and $w_* = u_* + h_\omega$ is a viscosity subsolution of

$$(dd^c\varphi)^n = e^\varphi W \quad \text{in } B(0,4)$$

with W positive and continuous. On the other hand, $w^* = u^* + h_\omega$ is a viscosity supersolution of the same equation.

Now fix $\alpha > 0$. Consider $(x_\alpha, y_\alpha) \in M^2$ such that

$$\begin{aligned} M_\alpha &= \sup_{(x,y)\in\overline{B(0,4)}^2} w_*(x) - w^*(y) - \varphi_3(x,y) - \frac{1}{2}\alpha d^2(x,y) \\ &= w_*(x_\alpha) - w^*(y_\alpha) - \varphi_3(x_\alpha, y_\alpha) - \frac{1}{2}\alpha d^2(x_\alpha, y_\alpha). \end{aligned}$$

The sup is achieved since we are maximizing a u.s.c. function. We also have, taking into account that $\phi_3(\hat{x}_1, \hat{x}_1) = 0$,

$$2C + \frac{C}{5} \geq M_\alpha \geq w_*(\hat{x}_1) - w^*(\hat{x}_1) \geq 0.$$

By construction, we see that $(x_\alpha, y_\alpha) \in B(0,2)^2$.

Using [16, prop. 3.7], we deduce the following:

LEMMA 2.15 *We have* $\lim_{\alpha\to\infty} \alpha d^2(x_\alpha, y_\alpha) = 0$. *Every limit point* $(\hat{x}, \hat{y})$ *of* (x_α, y_α) *satisfies* $\hat{x} = \hat{y}$, $\hat{x} \in \Delta \cap \{\varphi_2 \leq -\eta\}$, *and*

$$\begin{aligned} w_*(\hat{x}) - w^*(\hat{x}) &= u_*(\hat{x}) - u^*(\hat{x}) \\ &= \max_{x\in\overline{B(0,4)}} w_*(x) - w^*(x) - \varphi_3(x,x) \\ &= \max_{x\in M^2} u_*(x) - u^*(x) - \varphi_3(x,x) \\ &= u_*(\hat{x}_1) - u^*(\hat{x}_1) = w_*(\hat{x}_1) - w^*(\hat{x}_1), \end{aligned}$$

$$\liminf_{\alpha\to+\infty} w_*(x_\alpha) - w^*(y_\alpha) \geq w_*(\hat{x}_1) - w^*(\hat{x}_1).$$

Next, we use [16, theorem 3.2] with $u_1 = w_*$, $u_2 = -w^*$, and $\varphi = \frac{1}{2}\alpha d^2 + \varphi_3$. For $\alpha \gg 1$, everything is localized to $B(0,2)$; hence d reduces to the euclidean distance function. Using the usual formula for the first and second derivatives of its square, we get the following:

LEMMA 2.16 $\forall\varepsilon > 0$, *we can find* $(p_*, X_*), (p^*, X^*) \in \mathbb{C}^n \times \mathrm{Sym}^2_{\mathbb{R}}(\mathbb{C}^n)$ *such that*

(i) $(p_*, X_*) \in \overline{J^{2+}} w_*(x_\alpha)$,

(ii) $(-p^*, -X^*) \in \overline{J^{2-}} w^*(y_\alpha)$, *and*

(iii) *the block diagonal matrix with entries* $(X_*, -X^*)$ *satisfies*

$$-(\varepsilon^{-1} + \|A\|)I \leq \begin{pmatrix} X_* & 0 \\ 0 & -X^* \end{pmatrix} \leq A + \varepsilon A^2,$$

where $A = D^2\varphi(x_\alpha, y_\alpha)$, *i.e.,*

$$A = \alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + D^2\varphi_3(x_\alpha, y_\alpha)$$

and $\|A\|$ *is the spectral radius of* A (*maximum of the absolute values for the eigenvalues of this symmetric matrix*).

By construction, the Taylor series of φ_3 at any point in $\Delta \cap \{\varphi_2 < -\eta\}$ vanishes up to order $2n$. By transversality, $\Delta \cap \{\varphi_2 < -\eta\}$ is dense in $\Delta \cap \{\varphi_2 \leq -\eta\}$, and this Taylor series vanishes up to order $2n$ on $\Delta \cap \{\varphi_2 \leq -\eta\}$. In particular,

$$D^2\varphi_3(x_\alpha, y_\alpha) = O(d(x_\alpha, y_\alpha)^{2n}) = o(\alpha^{-n}).$$

This implies $\|A\| \simeq \alpha$. We choose $\alpha^{-1} = \varepsilon$ and deduce

$$-(2\alpha)I \leq \begin{pmatrix} X_* & 0 \\ 0 & -X^* \end{pmatrix} \leq 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + o(\alpha^{-n}).$$

Looking at the upper and lower diagonal terms we deduce that the eigenvalues of X_*, X^* are $O(\alpha)$. Evaluating the inequality on vectors of the form (Z, Z), we deduce from the $\leq$ that the eigenvalues of $X_* - X^*$ are $o(\alpha^{-n})$.

Fix $X \in \mathrm{Sym}^2_{\mathbb{R}}(\mathbb{C}^n)$ and denote by $X^{1,1}$ its $(1,1)$ part. It is a Hermitian matrix. Obviously the eigenvalues of $X_*^{1,1}$ and $X^{*1,1}$ are $O(\alpha)$ and those of $X_*^{1,1} - X^{*1,1}$ are $o(\alpha^{-n})$. Since $(p_*, X_*) \in \overline{J^{2+}} w_*(x_\alpha)$, we deduce from the definition of viscosity solutions that $X_*^{1,1}$ is positive definite and that the product of its n eigenvalues is $\geq c > 0$ uniformly in α. In particular, its smallest eigenvalue is $\geq c\alpha^{-n+1}$. The relation $X_*^{1,1} + o(\alpha^{-n}) \leq X^{*1,1}$ forces $X^{*1,1} > 0$ and $\det(X^{*1,1})/\det(X_*^{1,1}) \geq 1 + o(\alpha^{-1})$.

Now, since $(p_*, X_*) \in \overline{J^{2+}} w_*(x_\alpha)$ and $(-p^*, -X^*) \in \overline{J^{2-}} w^*(y_\alpha)$, we get by the definition of viscosity solutions

$$\frac{\det(X^{*1,1})}{\det(X_*^{1,1})} \leq \frac{e^{w^*(y_\alpha)} W(y_\alpha)}{e^{w_*(x_\alpha)} W(x_\alpha)}.$$

Upon passing to the superior limit as $\alpha \to +\infty$, we get $1 \leq e^{\limsup w^*(y_\alpha) - w_*(x_\alpha)}$. Taking Lemma 2.15 into account, we have $w^*(\hat{x}) \geq w_*(\hat{x})$ and thus $u^*(\hat{x}) \geq u_*(\hat{x})$. □

Remark 2.17. The miracle with the complex Monge-Ampère equation studied here is that the equation does not depend on the gradient in complex coordinates. In fact, it takes the form $F(X) - f(x) = 0$. The localization technique would fail without this structural feature.

2.4 Perron's Method

Once the global comparison principle holds, one easily constructs continuous (viscosity = pluripotential) solutions by Perron's method as we now explain.

THEOREM 2.18 *Assume the global comparison principle holds for* (DMA_v^1) *and that* (DMA_v^1) *has a bounded subsolution* $\underline{u}$ *and a bounded supersolution* $\overline{u}$. *Then*

$$\varphi = \sup\{w \mid \underline{u} \le w \le \overline{u} \text{ and } w \text{ is a viscosity subsolution of } (\mathrm{DMA}_v^1)\}$$

is the unique viscosity solution of (DMA_v^1).

In particular, it is a continuous ω*-plurisubharmonic function. Moreover,* φ *is also a solution of* (DMA_v^1) *in the pluripotential sense.*

PROOF. See [16, pp. 22–24], with a grain of salt. Indeed, lemma 4.2 there implies that the upper envelope φ of the subsolutions of (DMA_v^1) is a subsolution of (DMA_v^1) since F is l.s.c. Hence φ is a subsolution of $(\mathrm{DMA}_v^1)_+$.

The trick is now to consider its l.s.c. envelope φ_*. We are going to show that it is a supersolution of (DMA_v^1); otherwise, we find $x_0 \in X$ and q a $\mathcal{C}^2$ function such that $\varphi_* - q$ has 0 as a local minimum at x_0 and $F_+(q_x^{(2)}) < 0$. This forces $v_{x_0} > 0$. Then, proceeding as in [16, p. 24] we can construct a subsolution U such that $U(x_1) > \varphi(x_1)$ for some $x_1 \in X$.

This contradiction leads to the conclusion that φ_* is a supersolution and, by the viscosity comparison principle, that $\varphi_* \ge \varphi$. Since $\varphi = \varphi^* \ge \varphi_*$, it follows that $\varphi = \varphi_* = \varphi^*$ is a continuous viscosity solution.

For the reader's convenience, we briefly summarize the construction of U. Let $(z^1, .., z^n)$ be a coordinate system centered at x_0 giving a local isomorphism with the complex unit ball, and assume $v > 0$ on this complex ball neighborhood. Then, for $\gamma, \delta, r > 0$ small enough, $q_{\gamma,\delta} = q + \delta - \gamma\|z\|^2$ satisfies $F_+(q_{\gamma,\delta}^{(2)}) < 0$ for $\|z(x)\| \le r$.

Choose $\delta = (\gamma r^2)/8$, $r > 0$ small enough. Since $\varphi_*(x) - q(x) \ge 0$ for $\|z(x)\| \le r$, we have $\varphi(x) \ge \varphi_*(x) > q_{\gamma,\delta}(x)$ if $r/2 \le \|z(x)\| \le r$. It follows that U defined by

$$U(x) = \max(\varphi(x), q_{\delta,\gamma}(x))$$

if $\|z(x)\| \le r$ and $U(x) = \varphi(x)$ otherwise is a subsolution of $(\mathrm{DMA}_v^1)_+$ and in fact of (DMA_v^1) since we may assume that $v > 0$ on the relevant part of X. Choose a sequence (x_n) converging to x_0 so that $\varphi(x_n) \to \varphi_*(x_0)$. Then $q_{\gamma,\delta}(x_n) \to \varphi_*(x_0) + \delta$. Hence, for $n \gg 0$, $U(x_n) = q_{\gamma,\delta}(x_n) > \varphi(x_n)$.

It remains to be seen that φ is also a solution of (DMA_v^1) in the pluripotential sense. It follows from the previous argument in pluripotential theory. In fact, since φ is a viscosity subsolution, we know that $(\omega + dd^c\varphi)_{\mathrm{BT}}^n \ge e^{\epsilon\varphi}v$ by Proposition 1.11. Now argue by contradiction. Namely, choose $B \subset X$, a ball on which $(\omega + dd^c\varphi)_{\mathrm{BT}}^n \ne e^{\epsilon\varphi}v$. Solve a Dirichlet problem to get a continuous psh-function ψ on $\overline{B}$ with $(\omega + dd^c\psi)_{\mathrm{BT}}^n = e^{\epsilon\psi}v$ and $\psi = \varphi$ on ∂B. The Bedford-Taylor comparison principle gives $\psi \geq \varphi$ and $\psi \neq \varphi$ by hypothesis. Also, ψ is a viscosity subsolution. This contradicts the definition of φ as an envelope. $\square$

In situations where the global comparison principle is not available, one can always use the following substitute (this natural idea is used in the recent work [28]).

PROPOSITION 2.19 *Assume that* (DMA_v^1) *has a bounded subsolution* $\underline{u}$ *and a classical supersolution* $\overline{u}$. *Then,*

$$\varphi = \sup\{w \mid \underline{u} \leq w \leq \overline{u} \text{ and } w \text{ is a subsolution of } (\mathrm{DMA}_v^1)\}$$

is the unique maximal viscosity subsolution of (DMA_v^1). *In particular, it is a bounded* ω*-plurisubharmonic function.*

Remark 2.20. Assume X is a complex projective manifold such that K_X is ample. Let $\omega > 0$ be a Kähler representative of $[K_X]$ and v a volume form with $\mathrm{Ric}(v) = -\omega$. Then the Monge-Ampère equation $(\omega + dd^c\varphi)^n = e^\varphi v$ satisfies all the hypotheses of Theorem 2.18 and has a unique viscosity solution φ. On the other hand, the Aubin-Yau theorem [2, 38] implies that it has a unique smooth solution φ_{KE} (and $\omega + dd^c\varphi_{\mathrm{KE}}$ is the canonical Kähler-Einstein metric on X). Uniqueness of the viscosity solution implies $\varphi = \varphi_{\mathrm{KE}}$; hence the potential of the canonical KE metric on X is the envelope of the (viscosity = pluripotential) subsolutions to $(\omega + dd^c\varphi)^n = e^\varphi v$.

In the global case when $\varepsilon = 0$, i.e., for $(\omega + dd^c\varphi)^n = v$ on a compact Kähler manifold, classical strict sub-/supersolutions do not exist and using the Perron method directly seems doomed to failure.

3 Regularity of Potentials of Singular KE Metrics

In this section we apply the viscosity approach to show that the canonical singular Kähler-Einstein metrics constructed in [23] have continuous potentials.

3.1 Manifolds of General Type

Assume X is compact Kähler and v is a continuous volume form with semipositive density. Fix β a Kähler form on X. We consider the following condition on (the cohomology class of) ω:

$$(\dagger) \qquad \exists \eta > 0,\ \exists \psi \in L^\infty \cap \mathrm{PSH}(X, \omega), \quad (\omega + dd^c\psi)^n \geq \eta\beta^n.$$

When X is a compact Kähler manifold and ω is a semipositive $(1,1)$-form with $\int_X \omega^n > 0$, then (X, ω) satisfies (†), as follows from [11, 23]. However, the latter articles rely on [2, 38], and we will show in the proof of Theorem 3.3 how to check (†) directly, thus providing a new approach to the "continuous Aubin-Yau theorem."

Note that the inequality can be interpreted in the pluripotential or viscosity sense since these agree by Theorem 1.9.

LEMMA 3.1 *Assume* (†) *is satisfied and* v *has positive density. If* $c \gg 0$ *is large enough, then* $\psi - c$ *is a subsolution of* (DMA_v^1) *and the constant function* $\varphi = c$ *is a supersolution of* (DMA_v^1).

PROOF. Existence of ψ is needed for the subsolution, whereas the supersolution exists under the condition that $\exists C > 0$ such that $\omega^n \leq Cv$, which follows here from our assumption that v has positive density. ☐

COROLLARY 3.2 *When* (†) *is satisfied and* v *is positive,* (DMA_v^1) *has a unique viscosity solution* φ*, which is also the unique solution in the pluripotential sense.*

PROOF. Indeed, the global comparison principle holds and we finish the proof by applying Theorem 2.18. ☐

We are now ready to establish that the (pluripotential) solutions of some Monge-Ampère equations constructed in [23] are continuous:

THEOREM 3.3 *Assume* X *is a compact Kähler manifold,* ω *is a semipositive* $(1,1)$*-form with* $\int_X \omega^n > 0$*, and* v *is a semipositive continuous probability measure on* X*. Then* (†) *is satisfied and there exists a unique continuous* ω*-plurisubharmonic function* φ *that is the viscosity* (*equivalently, pluripotential*) *solution to the degenerate complex Monge-Ampère equation*

$$(\omega + dd^c\varphi)^n = e^{\varphi} v.$$

COROLLARY 3.4 *The function* $\varphi_P \in L^\infty \cap \mathrm{PSH}(X, \omega)$ *such that*

$$(\omega + dd^c\varphi_P)^n = e^{\varphi_P} v$$

in the pluripotential sense constructed in [23, theorem 4.1] *is a viscosity solution*; *hence it is continuous.*

PROOF. Observe that (†) is obviously satisfied when the cohomology class of ω is Kähler. If, moreover, v has positive density, the result is an immediate consequence of Corollary 3.2 together with the unicity statement [23, prop. 4.3].

We treat the general case by approximation. We first still assume that v is positive, but the cohomology class $\{\omega\}$ is now merely semipositive and big (i.e., $\int_X \omega^n > 0$). This is a situation considered in [23], where it is shown that (†) holds; however, we would like to make clear that the proof is independent of [38], so we (re)produce the argument.

By the above there exists, for each $0 < \varepsilon \leq 1$, a unique continuous $(\omega + \varepsilon\beta)$-psh function u_ε such that

$$(\omega + \varepsilon\beta + dd^c u_\varepsilon)^n = e^{u_\varepsilon} v.$$

We first observe that (u_ε) is relatively compact in $L^1(X)$. By [25], this is equivalent to checking that $\sup_X u_\varepsilon$ is bounded, as $\varepsilon \searrow 0^+$. Note that

$$e^{\sup_X u_\varepsilon} \geq \frac{\int_X \omega^n}{v(X)} = \int_X \omega^n;$$

hence $\sup_X u_\varepsilon$ is uniformly bounded from below. Set $w_\varepsilon := u_\varepsilon - \sup_X u_\varepsilon$. This is a relatively compact family of $(\omega+\beta)$-psh functions; hence there exists $C > 0$ such that for all $0 < \varepsilon \leq 1$, $\int_X w_\varepsilon \, dv \geq -C$ [25]. It follows from the concavity of the logarithm that

$$\log \int_X (\omega+\beta)^n \geq \sup_X u_\varepsilon + \log \int_X (e^{w_\varepsilon} \, dv) \geq \sup_X u_\varepsilon - C.$$

Thus $(\sup_X u_\varepsilon)$ is bounded as claimed.

We now assert that (u_ε) is decreasing as ε decreases to 0^+. Indeed, assume that $0 < \varepsilon' \leq \varepsilon$ and fix $\delta > 0$. Note that $u_{\varepsilon'}, u_\varepsilon$ are both $(\omega+\varepsilon\beta)$-plurisubharmonic. It follows from the (pluripotential) comparison principle that

$$\int_{(u_{\varepsilon'} \geq u_\varepsilon + \delta)} (\omega + \varepsilon\beta + dd^c u_{\varepsilon'})^n \leq \int_{(u_{\varepsilon'} \geq u_\varepsilon + \delta)} (\omega + \varepsilon\beta + dd^c u_\varepsilon)^n.$$

Since

$$(\omega + \varepsilon\beta + dd^c u_{\varepsilon'})^n \geq (\omega + \varepsilon'\beta + dd^c u_{\varepsilon'})^n \geq e^\delta (\omega + \varepsilon\beta + dd^c u_\varepsilon)^n$$

on the set $(u_{\varepsilon'} \geq u_\varepsilon + \delta)$, this shows that the latter set has zero Lebesgue measure. As $\delta > 0$ was arbitrary, we infer $u_{\varepsilon'} \leq u_\varepsilon$.

We let $u = \lim_{\varepsilon \to 0} u_\varepsilon$ denote the decreasing limit of the functions u_ε. By construction this is an ω-psh function. It follows from proposition 1.2, theorem 2.1, and proposition 3.1 in [23] that u is bounded and a (pluripotential) solution of the Monge-Ampère equation

$$(\omega + dd^c u)^n = e^u \, v.$$

This shows that (†) is satisfied; hence we can use Corollary 3.2 to conclude that u is actually continuous and is a viscosity solution.

It remains to relax the positivity assumption made on v. From now on $\{\omega\}$ is semipositive and big, and v is a probability measure with semipositive continuous density. We can solve

$$(\omega + dd^c \varphi_\varepsilon)^n = e^{\varphi_\varepsilon}[v + \varepsilon\beta^n],$$

where φ_ε are continuous ω-psh functions and $0 < \varepsilon \leq 1$. Observe that

$$e^{\sup_X \varphi_\varepsilon} \geq \frac{\int_X \omega^n}{1 + \int_X \beta^n};$$

hence $\sup_X \varphi_\varepsilon$ is bounded below.

It follows again from the concavity of the logarithm that $M_\varepsilon := \sup_X \varphi_\varepsilon$ is also bounded from above. Indeed, set $\psi_\varepsilon := \varphi_\varepsilon - M_\varepsilon$. This is a relatively compact family of nonpositive ω-psh functions [25]; thus there exists $C > 0$ such that $\int_X \psi_\varepsilon (v + \beta^n) \geq -C$. Now

$$\log\left(\int e^{\psi_\varepsilon} \frac{v + \varepsilon\beta^n}{\int_X v + \varepsilon\beta^n}\right) \geq \int \psi_\varepsilon \frac{v + \varepsilon\beta^n}{\int_X v + \varepsilon\beta^n} \geq \int \psi_\varepsilon (v + \beta^n) \geq -C$$

yields

$$\log \int_X \omega^n \geq M_\varepsilon + \log\left[1 + \varepsilon \int_X \beta^n\right] - C$$

so that (M_ε) is uniformly bounded.

We infer that (φ_ε) is relatively compact in $L^1(X)$. It follows from proposition 2.6 and proposition 3.1 in [23] that (φ_ε) is actually uniformly bounded as ε decreases to 0.

Lemma 2.3 in [23], together with the uniform bound on (φ_ε), yields, for any $0 < \delta \ll 1$,

$$\begin{aligned}\mathrm{Cap}_\omega(\varphi_\varepsilon - \varphi_{\varepsilon'} < -2\delta) &\leq \frac{C}{\delta^n} \int_{(\varphi_\varepsilon - \varphi_{\varepsilon'} < -\delta)} (\omega + dd^{\mathrm{c}}\varphi_\varepsilon)^n \\ &\leq \frac{C}{\delta^{n+1}} \int_X |\varphi_\varepsilon - \varphi_{\varepsilon'}|(\omega + dd^{\mathrm{c}}\varphi_\varepsilon)^n \\ &\leq \frac{C'}{\delta^{n+1}} \int_X |\varphi_\varepsilon - \varphi_{\varepsilon'}|(v + \beta^n).\end{aligned}$$

Using proposition 2.6 in [23] again and optimizing the value of δ yields the following variant of proposition 3.3 in [23]:

$$\|\varphi_\varepsilon - \varphi_{\varepsilon'}\|_{L^\infty} \leq C(\|\varphi_\varepsilon - \varphi_{\varepsilon'}\|_{L^1})^{\frac{1}{n+2}}.$$

Thus, if (ϵ_n) is a sequence decreasing to 0 as n goes to $+\infty$ such that $(\varphi_{\varepsilon_n})_n$ converges in L^1, $(\varphi_{\varepsilon_n})$ is actually a Cauchy sequence of continuous functions; hence it uniformly converges to the unique continuous pluripotential solution φ of (DMA_v^1). From this it follows that (φ_ε) has a unique cluster value in L^1 when ϵ decreases to 0 and hence converges in L^1. The preceding argument yields uniform convergence.

Theorem 2.18 insures that the φ is also a viscosity subsolution. Remark 6.3 [16, p. 35] actually enables one to conclude that φ is indeed a viscosity solution. □

COROLLARY 3.5 *If X^{can} is a canonical model of a general-type projective manifold, then the canonical singular KE metric on X^{can} of* [23] *has continuous potentials.*

PROOF. This is a straightforward consequence of the above theorem, working in a log resolution of X^{can}, where $\omega = c_1(K_X, h)$ is the pullback of the Fubini-Study form from X^{can} and $v = v(h)$ has continuous semipositive density, since X^{can} has canonical singularities. □

3.2 Continuous Ricci Flat Metrics

We now turn to the study of the degenerate equations (DMA_v^0)

$$(\omega + dd^{\mathrm{c}}\varphi)^n = v$$

on a given compact Kähler manifold X. Here v is a continuous volume form with semipositive density and ω is a smooth semipositive, closed, real $(1,1)$-form on X. We assume that v is normalized so that

$$v(X) = \int_X \omega^n.$$

This is an obvious necessary condition in order to solve the equation

$$(\omega + dd^c\varphi)^n = v$$

on X. Bounded solutions to such equations have been provided in [23] when v has L^p-density, $p > 1$, by adapting the arguments of [33]. Our aim here is to show that these are actually *continuous*. We treat here the case of continuous densities, as this is required in the viscosity context, and refer the reader to Section 4.2 for more general cases.

THEOREM 3.6 *The pluripotential solutions to* (DMA_v^0) *are viscosity solutions*; *hence they are continuous.*

The plan is to combine the viscosity approach for the family of equations $(\omega + dd^c\varphi)^n = e^{\varepsilon\varphi}v$, together with the pluripotential tools developed in [14, 22, 23, 25, 33].

PROOF. For $\varepsilon > 0$ we let φ_ε denote the unique viscosity (or, equivalently, pluripotential) ω-psh continuous solution of the equation

$$(\omega + dd^c\varphi_\varepsilon)^n = e^{\varepsilon\varphi_\varepsilon}v.$$

Set $M_\varepsilon := \sup_X \varphi_\varepsilon$ and $\psi_\varepsilon := \varphi_\varepsilon - M_\varepsilon$. The latter form a relatively compact family of ω-psh functions [25]; hence there exists $C > 0$ such that

$$\int_X \psi_\varepsilon dv = \int_X (\varphi_\varepsilon - M_\varepsilon) dv \geq -C \quad \text{for all } \varepsilon > 0.$$

Observe that $M_\varepsilon \geq 0$ since $v(X) = \int_X \omega^n =: V$. The concavity of the logarithm yields

$$0 = \log\left(\int_X e^{\varepsilon\varphi_\varepsilon} \frac{dv}{V}\right) \geq \frac{1}{V}\int \varepsilon\varphi_\varepsilon\, dv;$$

therefore

$$0 \geq \int \varphi_\varepsilon\, dv \geq -C + VM_\varepsilon,$$

i.e., (M_ε) is uniformly bounded. We infer that (φ_ε) is relatively compact in L^1 and the Monge-Ampère measures $(\omega + dd^c\varphi_\varepsilon)^n$ have uniformly bounded densities in L^∞. Once again proposition 2.6 and (a variant of) proposition 3.3 in [23] show that this family of continuous ω-psh functions is uniformly Cauchy and hence converges to a *continuous* pluripotential solution of (DMA_v^0). This pluripotential solution is also a viscosity solution by remark 6.3 in [16].

It is well-known that the solutions of (DMA_v^0) are unique up to an additive constant. It is natural to wonder which solution is reached by the family φ_ε. Observe that $\int_X e^{\varepsilon\varphi_\varepsilon}\,dv = \int_X\, dv = \int_X \omega^n$; thus

$$0 = \int_X \frac{e^{\varepsilon\varphi_\varepsilon} - 1}{\varepsilon}\,dv = \int_X \varphi_\varepsilon\,dv + o(1);$$

hence the limit φ of φ_ε as ε decreases to 0 is the unique solution of (DMA_v^0) that is normalized by $\int_X \varphi\,dv = 0$. □

Note that the way we have produced solutions (by approximation through the nonflat case) is independent of [2, 38].

COROLLARY 3.7 *Let X be a compact $\mathbb{Q}$-Calabi-Yau Kähler space. Then the Ricci-flat singular metrics constructed in* [23, theorem 7.5] *have continuous potentials.*

4 Concluding Remarks

4.1 Continuous Calabi Conjecture

The combination of viscosity methods and pluripotential techniques yields a soft approach to solving degenerate complex Monge-Ampère equations of the form

$$(\omega + dd^c\varphi)^n = e^{\varepsilon\varphi}v$$

when $\varepsilon \geq 0$. Recall that here X is a compact Kähler n-dimensional manifold, v is a semipositive volume form with continuous density, and ω is a smooth, closed real $(1,1)$-form whose cohomology class is semipositive and big (i.e., $\{\omega\}^n > 0$).

Altogether this combination provides an alternative and independent approach to Yau's solution of the Calabi conjecture [38]: we have only used upper envelope constructions (both in the viscosity and pluripotential sense), a global (viscosity) comparison principle, and Kolodziej's pluripotential techniques [23, 33]. Our approach applies to degenerate equations but yields solutions that are merely continuous (Yau's work yields smooth solutions, assuming the cohomology class $\{\omega\}$ is Kähler and the measure v is both positive and smooth).

Note that a third (variational) approach has been studied recently in [7]. It applies to even more degenerate situations, providing solutions with less regularity (that belong to the so-called class of finite energy).

4.2 More Continuous Solutions

Let X be a compact Kähler manifold, $v = f\,dV_0$ a nonnegative measure that is absolutely continuous with respect to some volume form dV_0 on X, and ω a smooth semipositive closed real $(1,1)$-form on X with positive volume. We assume v is normalized so that

$$v(X) = \int_X \omega^n$$

where $n = \dim_{\mathbb{C}} X$.

When ω is Kähler, Kolodziej [33] has shown that there exists a unique continuous ω-plurisubharmonic function φ such that

$$(\omega + dd^c\varphi)^n = v \quad \text{and} \quad \int_X \varphi \, dV_0 = 0,$$

as soon as the density f is "good enough" (i.e., belongs to some Orlicz class; e.g., L^p, $p > 1$, is good enough).

This result has been extended to the case where ω is merely semipositive in [23], but for the continuity statement that now follows from the viscosity point of view developed in the present article, it suffices to approximate the density f by smooth positive densities f_ε (using normalized convolutions) and to show, as in the proof of Theorems 3.3 and 3.6, that the corresponding continuous solutions form a Cauchy family of continuous functions. We leave the details to the reader.

4.3 The Case of a Big Class

Our approach applies equally well to a slightly more degenerate situation. We still assume here that (X, ω_X) is a compact Kähler manifold of dimension n, but $v = f \, dV_0$ is merely assumed to have density $f \geq 0$ in L^∞, and moreover the smooth, real, closed $(1,1)$-form ω is no longer assumed to be semipositive: we simply assume that its cohomology class $\alpha := [\omega] \in H^{1,1}(X, \mathbb{R})$ is *big*, i.e., contains a Kähler current.

It follows from the work of Demailly [18] that one can find a Kähler current in α with analytic singularities: there exists an ω-psh function ψ_0 that is smooth in a Zariski open set Ω_α and has logarithmic singularities of analytic type along $X \setminus \Omega_\alpha = \{\psi_0 = -\infty\}$ such that $T_0 = \omega + dd^c\psi_0 \geq \varepsilon_0\omega_X$ dominates the Kähler form $\varepsilon_0\omega_X$, $\varepsilon_0 > 0$.

We refer the reader to [11] for more preliminary material on this situation. Our aim here is to show that one can solve (DMA_v^1) in a rather elementary way by observing that the (unique) solution is the upper envelope of subsolutions. We let

$$\mathcal{F} := \{\varphi \in \mathrm{PSH}(X, \omega) \cap L^\infty_{\mathrm{loc}}(\Omega_\alpha) \mid (\omega + dd^c\varphi)^n \geq e^\varphi v \text{ in } \Omega_\alpha\}$$

denote the set of all (pluripotential) subsolutions to (DMA_v^1) (which makes sense only in Ω_α).

Observe that $\mathcal{F}$ is not empty: since T_0^n dominates a volume form and v has density in L^∞, the function $\psi_0 - C$ belongs to $\mathcal{F}$ for C large enough. We assume for simplicity $C = 0$ (so that $\psi_0 \in \mathcal{F}$) and set

$$\mathcal{F}_0 := \{\varphi \in \mathcal{F} \mid \varphi \geq \psi_0\}.$$

PROPOSITION 4.1 *The class $\mathcal{F}_0$ is uniformly bounded on X. It is compact* (*for the L^1-topology*) *and convex.*

PROOF. We first show that $\mathcal{F}_0$ is uniformly bounded from above (by definition it is bounded from below by ψ_0). We can assume without loss of generality that v

is normalized so that $v(X) = 1$. Fix $\psi \in \mathcal{F}_0$. It follows from the convexity of the exponential that

$$\exp\left(\int \psi \, dv\right) \leq \int e^{\psi} \, dv \leq \int (\omega + dd^{c}\psi)^n \leq \mathrm{Vol}(\alpha).$$

All integrals here are computed on the Zariski open set Ω_α. We refer the reader to [11] for the definition of the volume of a big class.

We infer

$$\sup_X \psi \leq \int \psi \, dv + C_v \leq \log \mathrm{Vol}(\alpha) + C_v,$$

where C_v is a uniform constant that depends only on the fact that all ω-psh functions are integrable with respect to v (see [25]). This shows that $\mathcal{F}_0$ is uniformly bounded from above by a constant that depends only on v and $\mathrm{Vol}(\alpha)$.

We now check that $\mathcal{F}_0$ is compact for the L^1-topology. Fix $\psi_j \in \mathcal{F}_0^{\mathbb{N}}$. We can extract a subsequence that converges in L^1 and almost everywhere to a function $\psi \in \mathrm{PSH}(X, \omega)$. Since $\psi \geq \psi_0$, it has a well-defined Monge-Ampère measure in Ω_α, and we need to check that $(\omega + dd^{c}\psi)^n \geq e^{\psi} v$.

Set $\psi'_j := (\sup_{l \geq j} \psi_l)^*$. These are functions in $\mathcal{F}_0$ that decrease to ψ. It follows from a classical inequality due to Demailly that

$$(\omega + dd^{c}\psi'_j)^n \geq e^{\inf_{l \geq j} \psi_l} v.$$

Letting $j \to +\infty$ shows that $\psi \in \mathcal{F}_0$, as claimed.

The convexity of $\mathcal{F}_0$ can be shown along the same lines. We won't need it here so we let the reader check that this easily follows from the inequalities obtained in [20]. $\square$

It follows that

$$\psi := \sup\{\varphi \mid \varphi \in \mathcal{F}_0\},$$

the upper envelope of pluripotential subsolutions to (DMA_v^1), is a well-defined ω-psh function that is locally bounded in Ω_α.

THEOREM 4.2 *The function ψ is a pluripotential solution to* (DMA_v^1).

PROOF. In what follows we shall say (for short) that an ω-psh function φ is bounded if and only if it is locally bounded in the Zariski open set Ω_α.

By Choquet's lemma, we can find a sequence $\psi_j \in \mathcal{F}_0$ of bounded ω-psh (pluripotential) subsolutions such that

$$\psi = (\lim_{j \to +\infty} \psi_j)^*.$$

Observe that the family of bounded pluripotential subsolutions is stable under taking the maximum: assume w_1, w_2 are two such subsolutions and set $W_c := \max(w_1 + c, w_2)$; then the (pluripotential and local) comparison principle yields

$$\begin{aligned}(\omega + dd^{c}W_c)^n &\geq \mathbb{1}_{\{w_1 + c > w_2\}}(\omega + dd^{c}w_1)^n + \mathbb{1}_{\{w_1 + c < w_2\}}(\omega + dd^{c}w_2)^n \\ &\geq \mathbb{1}_{\{w_1 + c \neq w_2\}} e^{W_c} v.\end{aligned}$$

Now for all but countably many c's, the sets $(w_1 + c = w_2)$ have zero v-measure; thus by continuity of the Monge-Ampère operator under decreasing sequences, we infer

$$(\omega + dd^c \max[w_1, w_2])^n \geq e^{\max[w_1,w_2]} v.$$

This shows that we can assume that the ψ_j's form an increasing sequence of subsolutions. Finally, we use a local balayage procedure to show that ψ is indeed a pluripotential solution to (DMA_v^1). Fix B, an arbitrary small "ball" in X (image of a euclidean ball under a local biholomorphism), and let ψ_j' denote the solution of the local Dirichlet problem

$$(\omega + dd^c \psi_j')^n = e^{\psi_j'} v \text{ in } B \quad \text{and} \quad \psi_j' \equiv \psi_j \text{ on } \partial B.$$

We extend ψ_j' to X by setting $\psi_j' \equiv \psi_j$ in $X \setminus B$. That such a problem indeed has a solution follows from an adaptation of the corresponding "flat" Dirichlet problem of Bedford and Taylor [4, 5], as was considered by Cegrell [13].

Note that the ψ_j' are still subsolutions. It follows from the (pluripotential) comparison principle that $\psi_j' \geq \psi_j$ and $\psi_{j+1}' \geq \psi_j'$. Thus the increasing limit of the ψ_j' equals again ψ. Since the Monge-Ampère operator is continuous under increasing sequences [5], this shows that ψ is a pluripotential solution of (DMA_v^1) in B, and hence in all of X, as B was arbitrary. □

Remark 4.3. The situation considered above covers in particular the construction of a Kähler-Einstein current on a variety V with ample canonical bundle K_V and canonical singularities, since the canonical volume form becomes, after passing to a desingularization X, a volume form $v = f\, dV_0$ with density $f \in L^\infty$.

The more general case of log-terminal singularities yields density $f \in L^p$, $p > 1$. One can treat this case by an easy approximation argument: setting $f_j = \min(f, j) \in L^\infty$, one first solves $(\omega + dd^c \varphi_j)^n = e^{\varphi_j} f_j\, dV_0$ and observes (by using the comparison principle) that the φ_j' form a decreasing sequence that converges to the unique solution of $(\omega + dd^c \varphi)^n = e^{\varphi} f\, dV_0$.

Once again the problem (DMA_v^0) can be reached by first solving $(\mathrm{DMA}_v^\varepsilon)$, $\varepsilon > 0$, and then letting ε decrease to 0.

4.4 More Comparison Principles

Let again $B \subset \mathbb{C}^n$ denote the open unit ball and let $B' = (1 + \eta)B$ with $\eta > 0$ be a slightly larger open ball. Let u, $u' \in \mathrm{PSH}(B')$, be plurisubharmonic functions. By convolution with an adequate nonnegative kernel of the form $\rho_\epsilon(z) = \epsilon^{-2n}\rho_1(z/\epsilon)$, we construct $(u_\epsilon)_{\eta > \epsilon > 0}$, a family of smooth plurisubharmonic functions decreasing to u as ϵ decreases to 0.

LEMMA 4.4 *For all points $z \in B$*

$$u(z) + u'(z) = \lim_{n \to \infty} \sup \left\{ u'(x) + u_{1/j}(x) \;\middle|\; j \geq n,\ |x - z| \leq \frac{1}{n} \right\}.$$

PROOF. Indeed, we have, if $2/n < \eta$:

$$\begin{aligned} u(z) + u'(z) &\le \sup\{u'(z) + u_{1/j}(z) \mid j \ge n\} \\ &\le \sup\left\{u'(x) + u_{1/j}(x) \;\Big|\; j \ge n,\ |x - z| \le \frac{1}{n}\right\} \\ &\le \sup\left\{u'(x) + u(x) \;\Big|\; |x - z| \le \frac{2}{n}\right\}. \end{aligned}$$

Since $u + u'$ is upper semicontinuous, we have

$$u(z) + u'(z) = (u + u')^*(z) = \lim_{n\to\infty} \sup\left\{u + u'(x) \;\Big|\; |x - z| \le \frac{2}{n}\right\}.$$

□

LEMMA 4.5 *Let φ be a bounded psh function on B, and v a continuous nonnegative volume form such that $e^{-\varphi}(dd^c\varphi)^n \ge v$ in the viscosity sense. Let ψ be a bounded psh function and w a continuous positive volume form, both defined on B' such that $(dd^c\psi)^n \ge w$.*

Then there exist constants $C, c > 0$ depending only on $\|\psi\|_{L^\infty}$ and $\|\varphi\|_{L^\infty}$ such that, for every $\epsilon \in [0, 1]$, $\Phi = \varphi + \epsilon\psi$ satisfies

$$e^{-\Phi}(dd^c\Phi)^n \ge (1 - \epsilon)^n e^{-C\epsilon} v + c\epsilon^n w$$

in the viscosity sense in B.

PROOF. We may assume $\epsilon > 0$ and w to be smooth. Let us begin with the case when ψ is of class C^2. Let $x_0 \in B$ and $q \in C^2$ such that $q(x_0) = \Phi(x_0)$ and $\Phi - q$ has a local maximum at x_0. Then $\varphi - (q - \epsilon\psi)$ has a local maximum at x_0.

We deduce

$$dd^c(q - \epsilon\psi)_{x_0} \ge 0,$$
$$e^{-q(x_0)+\epsilon\psi(x_0)}(dd^c(q - \epsilon\psi))_{x_0})^n \ge v_{x_0}.$$

Using the inequality $(dd^c q)^n_{x_0} \ge (dd^c(q - \epsilon\psi))_{x_0})^n + \epsilon^n(dd^c\psi)^n$, we prove the lemma for this case.

We now treat the general case. Since ψ is defined on B', we can construct by the above classical mollification a sequence of C^2 psh-functions $(\psi_{1/k})$ converging to ψ as k goes to $+\infty$. We know from the proof of Proposition 1.5 that $(dd^c\psi_k)^n \ge ((w^{1/n})_{1/k})^n = w_k$ in both the pluripotential and viscosity sense. We conclude from the previous case that $\Phi_k = \varphi + \epsilon\psi_k$ satisfies

$$c\epsilon^n w_k + (1 - \epsilon)^n e^{-C\epsilon} v \le e^{-\Phi_k}(dd^c\Phi_k)^n$$

in the viscosity sense. Since $w_k > 0$, there is no difference between the subsolution of DMA and of DMA_+; hence we have

$$c\epsilon^n w_k + (1 - \epsilon)^n e^{-C\epsilon} v - e^{-\Phi_k}(dd^c\Phi_k)^n_+ \le 0$$

in the viscosity sense.

By lemma 6.1 (p. 34) and remark 6.3 (p. 35) in [16], we conclude that

$$\overline{\Phi} = \limsup_{n\to\infty} \sup\left\{\Phi_j(x) \;\Big|\; j \geq n,\ |x-z| \leq \frac{1}{n}\right\}$$

satisfies the limit inequality

$$e^{-\overline{\Phi}}(dd^{\mathrm{c}}\overline{\Phi})^n_+ \geq (1-\epsilon)^n e^{-C\epsilon} v + c\epsilon^n w$$

in the viscosity sense. (Here we use the fact that DMA_+ is a continuous equation.) Now Lemma 4.4 implies that $\overline{\Phi} = \Phi$. Since $w > 0$, the proof is complete. □

THEOREM 4.6 *Let X be a compact Kähler manifold and $\omega \geq 0$ be a semi-Kähler smooth form. Then the global viscosity comparison principle holds for* (DMA^1_v) *for any nonnegative continuous probability measure v.*

PROOF. This is a variant of the argument sketched in [31, sec. V.3, p. 56].

Let $\overline{u}$ be a supersolution and $\underline{u}$ be a subsolution. Perturb the supersolution $\overline{u}$, setting $\overline{u}_\delta = \overline{u} + \delta$. This $\overline{u}_\delta$ is a supersolution to $(\mathrm{DMA}^1_{\tilde{w}})$ for every continuous volume form $\tilde{w}$ such that $\tilde{w} \geq e^{-\delta} v$. Choose $w > 0$ a continuous positive probability measure. Assuming without loss of generality that $\int_X \omega^n = 1$, we can construct ψ, a continuous quasi-plurisubharmonic function such that, in the vicosity sense,

$$(\omega + dd^{\mathrm{c}}\psi)^n = w.$$

Perturb the subsolution $\underline{u}$ setting

$$\underline{u}_\epsilon = (1-\epsilon)\underline{u} + \epsilon\psi.$$

By Lemma 4.5, $\underline{u}_\epsilon$ satisfies, in the viscosity sense,

$$e^{-(1+\epsilon)u}(\omega + dd^{\mathrm{c}}u)^n \geq \left(\frac{1-\epsilon}{1+\epsilon}\right)^n e^{-C\epsilon} v + c\left(\frac{\epsilon}{1+\epsilon}\right)^n w.$$

This in turn implies that $\underline{u}_\epsilon$ satisfies, in the viscosity sense,

$$e^{-u}(\omega + dd^{\mathrm{c}}u)^n \geq e^{-\epsilon\|u\|_\infty}\left[\left(\frac{1-\epsilon}{1+\epsilon}\right)^n e^{-C\epsilon} v + c\left(\frac{\epsilon}{1+\epsilon}\right)^n w\right].$$

Hence $\underline{u}_\epsilon$ satisfies, in the viscosity sense,

$$e^{-u}(\omega + dd^{\mathrm{c}}u)^n \geq \tilde{w}$$

whenever

$$\tilde{w} \leq e^{-\epsilon\|u\|_\infty}\left(\left(\frac{1-\epsilon}{1+\epsilon}\right)^n e^{-C\epsilon} v + c\left(\frac{\epsilon}{1+\epsilon}\right)^n w\right).$$

Choosing $1 \gg \delta \gg \epsilon > 0$, we find a continuous volume form $\tilde{w} > 0$ such that $\overline{u}_\delta$ is a supersolution and $\underline{u}_\epsilon$ is a viscosity subsolution of $e^{-u}(\omega + dd^{\mathrm{c}}u)^n = \tilde{w}$. Using the viscosity comparison principle for $\tilde{w}$, we conclude that $\overline{u}_\delta \geq \underline{u}_\epsilon$. Letting $\delta \to 0$, we infer $\overline{u} \geq \underline{u}$. □

This comparison principle has been inserted here for completeness. It could have been used instead of the pluripotential-theoretic arguments to establish existence of a viscosity solution in the case $v \geq 0$ of Theorem 3.3. This could be useful in dealing with similar problems where pluripotential tools are less efficient.

4.5 Viscosity Supersolutions of $(dd^c\varphi)^n = v$

Assume Ω is an euclidean ball. Given φ a bounded function, its plurisubharmonic projection

$$P(\varphi)(x) = P_\Omega(\varphi)(x) := (\sup\{\psi(x) \mid \psi \text{ psh on } \Omega \text{ and } \psi \leq \varphi\})^*,$$

is the greatest psh function that lies below φ on Ω. If φ is upper semicontinuous on Ω, there is no need of upper regularization and the upper envelope is $\leq \varphi$ on Ω.

LEMMA 4.7

(1) *Let ψ be a bounded plurisubharmonic function satisfying $(dd^c\psi)^n_{\mathrm{BT}} \leq v$ on Ω. Then its lower semicontinuous regularization ψ_* is a viscosity supersolution of the equation $(dd^c\varphi)^n = v$ on Ω.*

(2) *Let φ be a continuous viscosity supersolution of the equation $(dd^c\varphi)^n = v$ on Ω. Then $\psi := P(\varphi)$ is a continuous plurisubharmonic viscosity supersolution of the equation $(dd^c\psi)^n = v$ on Ω.*

(3) *Let φ be a C^2-smooth viscosity supersolution of $(dd^c\varphi)^n = v$ in Ω. Its plurisubharmonic projection $P(\varphi)$ satisfies $(dd^cP(\varphi))^n_{\mathrm{BT}} \leq v$.*

PROOF.

(1) We use the same idea as in the proof of Proposition 1.5. Assume $\psi \in \mathrm{PSH} \cap L^\infty(\Omega)$ satisfies $(dd^c\psi)^n_{\mathrm{BT}} \leq v$ in the pluripotential sense on Ω. Consider q a $\mathcal{C}^2$ function such that $\psi_*(x_0) = q(x_0)$ and $\psi_* - q$ achieves a local minimum at x_0. We want to prove that $(dd^cq(x_0))^n_+ \leq v(x_0)$. Assume that $(dd^cq(x_0))^n_+ > v_{x_0}$. Then $dd^cq(x_0) \geq 0$ and $(dd^cq(x_0))^n > v_{x_0} > 0$, which implies that $dd^cq(x_0) > 0$. Let $q^\varepsilon := q - 2\varepsilon(\|x - x_0\|^2 - r^2) - \varepsilon r^2$. Since v has continuous density, we can choose $\varepsilon > 0$ small enough and a small ball $B(x_0, r)$ containing x_0 of radius $r > 0$ such that $dd^cq^\varepsilon > 0$ in $B(x_0, r)$ and $(dd^cq^\varepsilon)^n > v$ on the ball $B(x_0, r)$. Thus we have $q^\varepsilon = q - \varepsilon r^2 < \psi_* \leq \psi$ near $\partial B(x_0, r)$ while $(dd^cq^\varepsilon)^n_{\mathrm{BT}} \geq v \geq (dd^c\psi)^n_{\mathrm{BT}}$ on $B(x_0, r)$. The comparison principle (Lemma 1.6) yields $q^\varepsilon \leq \psi$ on $B(x_0, r)$; hence

$$q^\varepsilon(x_0) = \liminf_{x \to x_0} q^\varepsilon(x) \leq \liminf_{x \to x_0} \psi(x) = \psi_*(x_0),$$

i.e., $q(x_0) + \varepsilon r^2 \leq \psi_*(x_0) = q(x_0)$, which is a contradiction.

(2) Set $\psi := P(\varphi)$, fix a point $x_0 \in \Omega$, and consider a super test function q for ψ at x_0; i.e., q is a C^2 function on a small ball $B(x_0, r) \subset \Omega$ such that $\psi(x_0) = q(x_0)$ and $\psi - q$ attains its minimum at x_0. We want to prove that $(dd^cq(x_0))^n_+ \leq v(x_0)$. Since $\psi \leq \varphi$, there are two cases:

- if $\psi(x_0) = \varphi(x_0)$, then q is also a super test function for φ at x_0 and then $(dd^c q(x_0))_+^n \le v(x_0)$ since φ is a supersolution of the same equation;
- if $\psi(x_0) < \varphi(x_0)$, by continuity of φ there exists a ball $B(x_0, s)$, $0 < s < r$, such that $\psi = P(\varphi) < \varphi$ on the ball $B(x_0, s)$ and then $(dd^c\psi)^n = 0$ on $B(x_0, s)$ since $(dd^c P(\varphi))^n$ is supported on the set $\{P(\varphi) = \varphi\}$. Therefore ψ is a continuous psh function satisfying the inequality $(dd^c\psi)^n = 0 \le v$ in the sense of pluripotential theory on the ball $B(x_0, s)$. Assume that $(dd^c q(x_0))_+^n > v(x_0)$. Then by definition, we have $dd^c q(x_0) > 0$ and $(dd^c q(x_0))^n > v(x_0)$. Taking $s > 0$ small enough and $\varepsilon > 0$ small enough, we can assume that $q^\varepsilon := q - \varepsilon(|x - x_0|^2 - s^2)$ is psh on $B(x_0, s)$ and $(dd^c q^\varepsilon)^n > v \ge (dd^c\psi)^n$ on the ball $B(x_0, s)$, while $q^\varepsilon = q \le \psi$ on $\partial B(x_0, s)$. By the pluripotential comparison principle for the complex Monge-Ampère operator, it follows that $q^\varepsilon \le \psi$ on $B(x_0, s)$; thus $q(x_0) + \varepsilon s^2 \le \psi(x_0)$, which is a contradiction.

(3) It is classical (see [4, 6, 17]) that under these hypotheses, $P(\varphi)$ is a $C^{1,1}$-smooth function, and its Monge-Ampère measure $(dd^c P(\varphi))_{\mathrm{BT}}^n$ is concentrated on the set where $P(\varphi) = \varphi$ and satisfies

$$(dd^c P(\varphi))_{\mathrm{BT}}^n = \mathbb{1}_{\{P(\varphi)=\varphi\}}(dd^c\varphi)^n.$$

The conclusion follows from the definition of viscosity supersolutions. □

We do not know whether part 3 of the lemma is valid for less regular supersolutions.

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