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Hölder continuous solutions to Monge–Ampère equations

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Abstract

We study the regularity of solutions to the Dirichlet problem for the complex Monge–Ampère equation $(dd^c u)^n = f dV$ on a bounded strongly pseudoconvex domain $\Omega \subset \mathbb{C}^n$. We show, under a mild technical assumption, that the unique solution u to this problem is Hölder continuous if the boundary data ϕ is Hölder continuous and the density f belongs to $L^p(\Omega)$ for some $p > 1$. This improves previous results by Bedford and Taylor and Kolodziej.

Introduction

Let Ω be a bounded strongly pseudoconvex open subset of $\mathbb{C}^n$. Given $\phi \in \mathcal{C}^0(\partial\Omega)$ and $f \in L^p(\Omega)$, we consider the Dirichlet problem

$$\mathrm{MA}(\Omega, \phi, f) : \begin{cases} (dd^c u)^n = f\beta_n & \text{in } \Omega, \\ u = \phi & \text{on } \partial\Omega, \end{cases}$$

where $u \in \mathrm{PSH}(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$. Here $\beta_n = dV$ denotes the euclidean volume form in $\mathbb{C}^n$, $d = \partial + \overline{\partial}$, $d^c = i(\overline{\partial} - \partial)$, $\mathrm{PSH}(\Omega)$ is the set of plurisubharmonic functions in Ω (the set of locally integrable functions u such that $dd^c u \geqslant 0$ in the sense of currents), and $(dd^c \cdot)^n$ denotes the complex Monge–Ampère operator: this operator is well defined on the subset of bounded (in particular continuous) plurisubharmonic functions, as follows from the work of Bedford and Taylor [**2**]. We refer the reader to [**10**] for a recent survey on its properties.

The equation $\mathrm{MA}(\Omega, \phi, f)$ has been studied intensively during the last decades. Bremermann [**3**], Walsh [**12**], and Bedford and Taylor [**1**] have shown that $\mathrm{MA}(\Omega, \phi, f)$ admits a unique continuous solution $u \in \mathrm{PSH}(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$ when $f \in \mathcal{C}^0(\overline{\Omega})$ is *continuous*.

It was further shown in [**1**] that $u \in \mathrm{Lip}_\alpha(\overline{\Omega})$ is α-Hölder continuous whenever $\phi \in \mathrm{Lip}_{2\alpha}(\partial\Omega)$ and $f^{1/n} \in \mathrm{Lip}_\alpha(\overline{\Omega})$. Higher regularity results have been established by Caffarelli, Kohn, Nirenberg and Spruck [**4**], assuming smoothness of the data ϕ, f and nondegeneracy of the density $f > 0$.

It has been proved by the second author [**7**, **8**] (see also [**5**]) that $\mathrm{MA}(\Omega, \phi, f)$ still admits a unique *continuous* solution $u \in \mathrm{PSH}(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$ under the much milder assumption $f \in L^p(\Omega)$, $p > 1$.

Our aim here is to show that this solution is actually Hölder continuous, when ϕ is so. A significant particular case of our results can be stated as follows.

Main Theorem. *Assume that ϕ is $C^{1,1}$ on $\partial\Omega$ and that $f \in L^p(\Omega)$ for some $p > 1$. Then the unique solution $u \in \mathrm{PSH}(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$ to $\mathrm{MA}(\Omega, \phi, f)$ is α-Hölder continuous on $\overline{\Omega}$, for any exponent*

$$\alpha < \alpha_p := \frac{2}{(qn+1)}, \quad \textit{where } \frac{1}{p} + \frac{1}{q} = 1.$$

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We can also prove that u is Hölder continuous on $\overline{\Omega}$ when $\phi \in \mathrm{Lip}_{2\alpha}(\overline{\Omega})$ is merely Hölder continuous, but we then need to add an extra technical assumption: Theorems 3.1 and 4.1, which we refer the reader to.

Let us stress that the exponent $\alpha_p = 2/(qn+1)$ (as well as further exponents α', α'' from Theorems 3.1 and 4.1) is not far from being optimal as we indicate in Examples 4.4 and 4.5.

1. *The stability estimate*

Our main tool is the following estimate which is proved in [6] in a compact setting (under growth, but no boundary, conditions, see [6, Proposition 3.3]). A similar, but weaker, estimate was established by S.Kolodziej in [9].

THEOREM 1.1. *Fix $0 \leqslant f \in L^p(\Omega)$, $p > 1$. Let φ, ψ be two bounded plurisubharmonic functions in Ω such that $(dd^c\varphi)^n = f\beta_n$ in Ω, and let $\varphi \geqslant \psi$ on $\partial\Omega$. Fix $r \geqslant 1$ and $0 \leqslant \gamma < r/[nq+r]$, $1/p + 1/q = 1$. Then there exists a uniform constant $C = C(\gamma, \|f\|_{L^p(\Omega)}) > 0$ such that*

$$\sup_\Omega(\psi - \varphi) \quad \leqslant C\ [\|(\psi - \varphi)_+\|_{L^r(\Omega)}]^\gamma,$$

where $(\psi - \varphi)_+ := \max(\psi - \varphi, 0)$.

The proof closely follows that given in [6], but for the reader's convenience, we will give it at the end of this section. The estimate of the theorem is a consequence of several results to follow.

To state the results needed for the proof, it is useful to consider the Monge–Ampère capacity introduced and studied by Bedford and Taylor in [2]. Recall that for a Borel subset $K \Subset \Omega$,

$$\mathrm{Cap}(K) := \sup\left\{\int_K (dd^c v)^n \,/\, v \in \mathrm{PSH}(\Omega) \text{ with } -1 \leqslant v \leqslant 0\right\}.$$

PROPOSITION 1.2. *Fix $f \in L^p(\Omega)$, $p > 1$, and let φ, ψ be bounded plurisubharmonic functions in Ω such that $\varphi \geqslant \psi$ on $\partial\Omega$. If $(dd^c\varphi)^n = f\beta_n$, then for any $\alpha > 0$ there exists a uniform constant $A = A(\alpha, \|f\|_{L^p(\Omega)})$ such that for all* $\varepsilon > 0$,

$$\sup_\Omega(\psi - \varphi) \ \leqslant\ \varepsilon\ +\ A\ [\mathrm{Cap}(\{\varphi - \psi < -\varepsilon\})]^\alpha\,.$$

Before proving Proposition 1.2, we first establish three lemmas.

LEMMA 1.3. *Fix $\varphi, \psi \in \mathrm{PSH}(\Omega) \cap L^\infty(\Omega)$ such that $\underline{\lim}_{\zeta\to\partial\Omega}(\varphi - \psi) \geqslant 0$. Then for all* $t, s > 0$,

$$t^n \mathrm{Cap}(\{\varphi - \psi < -s - t\}) \ \leqslant\ \int_{\{\varphi-\psi<-s\}} (dd^c\varphi)^n.$$

Proof. Fix $v \in \mathrm{PSH}(\Omega)$ such that $-1 \leqslant v \leqslant 0$. Then for any $s > 0$ and $t > 0$, we have $\{\varphi - \psi < -s - t\} \subset \{\varphi < \psi - s + tv\} \subset \{\varphi < \psi - s\} \Subset \Omega$. By the comparison principle [2] we get

$$t^n \int_{\{\varphi-\psi<-s-t\}} (dd^c v)^n \leqslant \int_{\{\varphi<\psi-s+tv\}} (dd^c(-s+\psi+tv))^n \leqslant \int_{\{\varphi-\psi<-s\}} (dd^c\varphi)^n.$$

Taking the supremum over all the vs yields the desired result. ∎

LEMMA 1.4. *Assume $0 \leqslant f \in L^p(\Omega)$, $p > 1$. Then for all $\tau > 1$, there exists $D_\tau = D(\tau, \|f\|_{L^p(\Omega)}) > 0$ such that for any Borel subset $K \subset \Omega$,*

$$0 \leqslant \int_K f\, dV \;\leqslant\; D_\tau \;\, [\mathrm{Cap}(K)]^\tau .$$

Proof. By Hölder inequality we have

$$\int_K f\, dV \leqslant \|f\|_{L^p(\Omega)} [\mathrm{Vol}(K)]^{1/q}.$$

On the other hand, it is well known that

$$\mathrm{Vol}(K) \lesssim \exp[-\mathrm{Const} \cdot [\mathrm{Cap}(K)^{-1/n}]],$$

which is a much better control than what we actually need (see [**13**, Theorem 7.1]). The estimate of the lemma follows. □

LEMMA 1.5. *Let $g : \mathbb{R}^+ \to \mathbb{R}^+$ be a decreasing right-continuous function. Assume that there exist $\tau, B > 1$ such that g satisfies*

$$tg(s+t) \;\leqslant\; B\; [g(s)]^\tau \quad \forall s, t > 0.$$

Then $g(s) = 0$ for all $s \geqslant s_\infty$, where

$$s_\infty := \frac{2Bg(0)^{\tau-1}}{1 - 2^{1-\tau}}.$$

The proof, almost identical to that of [**6**, Lemma 2.3], is left to the reader.

Proof of Proposition 1.2. Combining Lemmas 1.3 and 1.4, we conclude that, given $\varepsilon > 0$, the function defined for $s > 0$ by $g(s) := \mathrm{Cap}(\{\varphi - \psi < -s - \varepsilon\})^{1/n}$ satisfies the conditions of Lemma 1.5 for any $\tau > 1$ with the constant $B := D_\tau^{1/n}$. Therefore applying this lemma we obtain that $\mathrm{Cap}(\{\varphi - \psi < -s_\infty - \varepsilon\}) = 0$, which means that $\psi - \varphi \leqslant \varepsilon + s_\infty$ almost everywhere on Ω. Then if we choose $\tau := 1 + \alpha n$, it follows that

$$\sup_\Omega (\psi - \varphi) \leqslant \varepsilon + A[\mathrm{Cap}(\{\varphi - \psi < -\varepsilon\})]^\alpha,$$

where $A := 2B/(1 - 2^{1-\tau})$. □

We finally give the proof of Theorem 1.1.

Proof of Theorem 1.1. Applying Lemma 1.3 with $s = t = \varepsilon > 0$ and using Hölder inequality, we get

$$\begin{aligned} \mathrm{Cap}(\{\varphi - \psi < -2\varepsilon\}) &\leqslant \varepsilon^{-n} \int_{\{\varphi - \psi < -\varepsilon\}} f\, dV \\ &\leqslant \varepsilon^{-n-r/q} \int_\Omega (\psi - \varphi)_+^{r/q} f\, dV \\ &\leqslant \varepsilon^{-n-r/q} \|(\psi - \varphi)_+\|_{L^r(\Omega)}^{r/q} \|f\|_{L^p(\Omega)}. \end{aligned}$$

Now fix α to be chosen later and apply Proposition 1.2 to get

$$\sup_\Omega (\psi - \varphi) \leqslant 2\varepsilon + A\varepsilon^{-\alpha(n + r/q)}\; \|f\|_{L^p(\Omega)}^\alpha \; \|(\psi - \varphi)_+\|_{L^r(\Omega)}^{\alpha r/q}.$$

Next fix γ as in the theorem and set $\varepsilon := \|(\psi - \varphi)_+\|^{\gamma}_{L^r(\Omega)}$ in the last estimate. Then it is easy to check that the estimate of the theorem holds if we choose

$$\alpha := \frac{\gamma q}{r - \gamma(r + nq)}. \qquad \square$$

2. *Hölder continuous barriers*

For fixed $\delta > 0$ we consider $\Omega_\delta := \{z \in \Omega \,/\, \mathrm{dist}(z, \partial\Omega) > \delta\}$ and set

$$u_\delta(z) := \sup_{\|\zeta\| \leqslant \delta} u(z + \zeta), \quad z \in \Omega_\delta.$$

This is a plurisubharmonic function in Ω_δ, when u is plurisubharmonic in Ω, which measures the modulus of continuity of u. We would like to use Theorem 1.1 applied with $\psi = u_\delta$. However, u_δ is not globally defined in Ω, so we need to extend it with control on the boundary values. This is the content of our next result which makes heavy use of the pseudoconvexity assumption.

PROPOSITION 2.1. *Let $u \in \mathrm{PSH}(\Omega) \cap L^\infty(\Omega)$ be a plurisubharmonic function such that $u_{|\partial\Omega} = \phi \in \mathrm{Lip}_{2\alpha}(\partial\Omega)$. Then there exists a constant $C = C(u) > 0$ and $\delta_0 > 0$ small enough such that for any $0 < \delta < \delta_0$ the function defined on Ω by*

$$\tilde{u}_\delta = \begin{cases} \max\{u_\delta, u + C\delta^\alpha\} & \text{in } \Omega_\delta, \\ u + C\delta^\alpha & \text{in } \Omega \setminus \Omega_\delta, \end{cases}$$

is a bounded plurisubharmonic function on Ω and the family $(\tilde{u}_\delta)$ decreases to u as δ decreases to 0.

In particular, $\sup_{\Omega_\delta}(u_\delta - u) \leqslant \sup_\Omega(\tilde{u}_\delta - u)$ for $0 < \delta < \delta_0$.

The proof relies on the construction of Hölder continuous plurisubharmonic and plurisuperharmonic barriers for the Dirichlet problem $\mathrm{MA}(\Omega, \phi, f)$.

LEMMA 2.2. *Fix $\phi \in \mathrm{Lip}_{2\alpha}(\partial\Omega)$, $f \in L^p(\Omega)$, $p > 1$, and set $u := u(\Omega, \phi, f)$. Then there exist $v, w \in \mathrm{PSH}(\Omega) \cap \mathrm{Lip}_\alpha(\overline{\Omega})$ such that*
(1) $v(\zeta) = \phi(\zeta) = -w(\zeta), \forall \zeta \in \partial\Omega$,
(2) $v(z) \leqslant u(z) \leqslant -w(z), \forall z \in \Omega$.

Proof. Assume first that $\phi \equiv 0$. We are going to show that there exists a weak barrier $b_f \in \mathrm{PSH}(\Omega) \cap \mathrm{Lip}_1(\Omega)$ for the Dirichlet problem $\mathrm{MA}(\Omega, 0, f)$, that is, a plurisubharmonic function which satisfies
(i) $b_f(\zeta) = 0, \ \forall \zeta \in \partial\Omega$,
(ii) $b_f \leqslant u(\Omega, 0, f)$, in Ω,
(iii) $|b_f(z) - b_f(\zeta)| \leqslant C_1|z - \zeta|, \ \forall z \in \Omega, \ \forall \zeta \in \Omega$,
for some uniform constant $C_1 > 0$.

In order to construct b_f, we set $u_0 := u(\Omega, 0, f)$ and assume first that the density f is bounded near $\partial\Omega$: there exists a compact subset $K \subset \Omega$ such that $0 \leqslant f \leqslant M$ on $\Omega \setminus K$. Let ρ be a $\mathcal{C}^2$ strictly plurisubharmonic defining function for Ω. Then for $A > 0$ large enough the function $b_f := A\rho$ satisfies the condition $(dd^c b_f)^n \geqslant M\beta_n \geqslant f\beta_n$ on $\Omega \setminus K$. Moreover, taking A large enough we also have $A\rho \leqslant m \leqslant u_0$ on a neighbourhood of K, where $m := \min_\Omega u_0$. Therefore the function b_f is a $\mathcal{C}^2$ plurisubharmonic function on Ω satisfying the conditions $(dd^c b_f)^n \geqslant (dd^c u_0)^n$ on $\Omega \setminus K$ and $b_f \leqslant u_0$ on $\partial(\Omega \setminus K)$. This implies, by the comparison principle [2], that $b_f \leqslant u_0$ in $\Omega \setminus K$, and hence in Ω.

When f is not bounded near $\partial\Omega$, we can proceed as follows. Fix a large ball $\mathbb{B} \subset \mathbb{C}^n$ so that $\Omega \Subset \mathbb{B} \subset \mathbb{C}^n$. Define $\tilde{f} := f$ in Ω and $\tilde{f} = 0$ in $\mathbb{B} \setminus \Omega$. We can use our previous construction to find a barrier function $b_{\tilde{f}} \in \mathrm{PSH}(\mathbb{B}) \cap \mathcal{C}^2(\mathbb{B})$ for the Dirichlet problem $\mathrm{MA}(\mathbb{B}, 0, \tilde{f})$ for the ball $\mathbb{B}$. Let $h = u(\Omega, -b_{\tilde{f}}, 0)$ denote the Bremermann function in Ω with boundary values $-b_{\tilde{f}}$, for the zero density. Since $-b_{\tilde{f}} \in \mathcal{C}^2(\partial\Omega)$, the plurisubharmonic function h is Lipschitz on Ω (see [**1**]); therefore $b_f := h + b_{\tilde{f}} \in \mathrm{PSH}(\Omega) \cap \mathrm{Lip}_1(\Omega)$ is a barrier function for the Dirichlet problem $\mathrm{MA}(\Omega, 0, f)$.

It remains to construct the functions v, w satisfying Conditions (1) and (2) above. It follows from [**1**] that the plurisubharmonic functions $u(\Omega, \pm\phi, 0)$ are Hölder continuous of order α. We let the reader check that the functions $v := u(\Omega, \phi, 0) + b_f$ and $w := u(\Omega, -\phi, 0) + b_f$ do the job. □

We are now ready for the proof of the proposition.

Proof of Proposition 2.1. It follows from Lemma 2.2 that

$$|u(z) - u(\zeta)| \leqslant C|z - \zeta|^\alpha \quad \forall \zeta \in \partial\Omega,\ \forall z \in \Omega.$$

For $\delta > 0$ small enough, the function $u_\delta(z) := \sup_{\|\zeta\| \leqslant \delta} u(z + \zeta)$ is plurisubharmonic in Ω_δ. Observe that if $z \in \partial\Omega_\delta$ and $\zeta \in \mathbb{C}^n$ with $\|\zeta\| \leqslant \delta$ then $z + \zeta \in \partial\Omega$, and hence $u_\delta \leqslant u(z) + C\delta^\alpha$. Thus the functions

$$\tilde{u}_\delta(z) := \begin{cases} \sup\{u_\delta(z), u(z) + C\delta^\alpha\} & \text{in } \Omega_\delta, \\ u + C\delta^\alpha & \text{in } \Omega \setminus \Omega_\delta \end{cases}$$

are plurisubharmonic and bounded in Ω and decrease to u as δ decreases to 0. □

Our construction of barriers allows us to control the total mass of the Laplacian of solutions to $\mathrm{MA}(\Omega, \phi, f)$. This will be important in Section 4.

PROPOSITION 2.3. *Fix $0 \leqslant f \in L^p(\Omega)$ $(p > 1)$ and $\phi \in C^0(\partial\Omega)$. Then*
(1) *if $\phi \in C^{1,1}(\partial\Omega)$, then $\Delta u(\Omega, \phi, 0)$ has finite mass in Ω;*
(2) *$\Delta u(\Omega, 0, f)$ has finite mass in Ω. Moreover, if $\Delta u(\Omega, \phi, 0)$ has finite mass in Ω, then $\Delta u(\Omega, \phi, f)$ also has finite mass in Ω.*

Proof. Fix a strictly plurisubharmonic exhaustion ρ for Ω.

(1) Assume first that $\phi \in C^2(\partial\Omega)$. Consider any smooth extension of ϕ in a neighbourhood of $\overline{\Omega}$ and correct it by adding $A\rho$, $A \gg 1$, in order to obtain a smooth plurisubharmonic extension $\hat{\phi}$ that is plurisubharmonic in a neighbourhood of $\overline{\Omega}$. Since $\hat{\phi}$ is a subsolution to $\mathrm{MA}(\Omega, \phi, 0)$ whose Laplacian has finite mass in Ω, it follows from the comparison principle that $\Delta u(\Omega, \phi, 0)$ also has finite mass in Ω.

Now if $\phi \in C^{1,1}(\partial\Omega)$ then it has a $C^{1,1}$ extension to a neighbourhood of $\overline{\Omega}$ which we still denote by ϕ. Then $dd^c\phi$ is a positive current with bounded coefficients on a neighbourhood of $\overline{\Omega}$, and then for $A > 1$ big enough, the function $\hat{\phi} := \phi + A\rho$ is plurisubharmonic on a neighbourhood of $\overline{\Omega}$. We conclude as before, since by construction $\hat{\phi}$ is a subsolution to $\mathrm{MA}(\Omega, \phi, 0)$, whose Laplacian has finite mass in Ω.

(2) Let $\tilde{f}$ be the trivial extension of f to a large ball $\mathbb{B}$ containing Ω. Let $b_{\tilde{f}} \in \mathcal{C}^2(\mathbb{B})$ be a plurisubharmonic barrier for $\mathrm{MA}(\mathbb{B}, 0, \tilde{f})$ (see the proof of Lemma 2.2). Then $b_f := u(\Omega, -b_{\tilde{f}}, 0) + b_{\tilde{f}}$ is a plurisubharmonic barrier for $\mathrm{MA}(\Omega, 0, f)$. Its Laplacian has finite mass in Ω since $b_{\tilde{f}}$ is smooth, so it follows from the comparison principle that $\Delta u(\Omega, 0, f)$ has finite mass in Ω.

Now set $v := u(\Omega, 0, f) + u(\Omega, \phi, 0)$. This is a plurisubharmonic function in Ω such that $v = \phi$ on $\partial\Omega$ and $(dd^c v)^n \geqslant f\, dV$ in Ω. If $\Delta u(\Omega, \phi, 0)$ has finite mass in Ω, then Δv has finite mass in Ω, and hence $\Delta u(\Omega, \phi, f)$ also has finite mass in Ω. □

3. *Gradient estimates*

This section is devoted to the proof of the following result.

THEOREM 3.1. *Assume that $f \in L^p(\Omega)$, for some $p > 1$, and $\phi \in \mathrm{Lip}_{2\alpha}(\partial\Omega)$, with $\nabla u(\Omega, \phi, 0) \in L^2(\Omega)$. Then*

$$u(\Omega, \phi, f) \in \mathrm{Lip}_{\alpha'}(\overline{\Omega}), \quad \textit{for all } \alpha' < \min(\alpha, 2/[qn + 2]),$$

where $1/p + 1/q = 1$.

The condition $\nabla u(\Omega, \phi, 0) \in L^2(\Omega)$ is automatically satisfied if $\phi \in \mathcal{C}^{1,1}(\partial\Omega)$: in this case $u(\Omega, \phi, 0) \in \mathrm{Lip}_1(\overline{\Omega})$, and hence $\nabla u(\Omega, \phi, 0)$ is actually bounded in Ω (see [**1**]). What really matters here is that there should exist a subsolution $v \in \mathcal{B}(\Omega, \phi, 0)$ such that $\nabla v \in L^2(\Omega)$. This implies (see Lemma 3.1) that $u(\Omega, \phi, 0)$ and $u(\Omega, \phi, f)$ both have gradient in $L^2(\Omega)$.

We could not avoid the use of this additional technical hypothesis on the homogenous solution $u(\Omega, \phi, 0)$. Also the exponent α' is probably not optimal. We can get a better exponent by assuming that $\Delta u(\Omega, \phi, 0)$ has finite mass in Ω (this is automatically satisfied when $\phi \in \mathcal{C}^2(\partial\Omega)$).

Proof. Since $f \in L^p(\Omega)$, $p > 1$, it follows from [**8**] that the solution $u = u(\Omega, \phi, f) \in \mathrm{PSH}(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$ is a continuous plurisubharmonic function. Our aim is to show that u is Hölder continuous on $\overline{\Omega}$.

Let $\tilde{u}_\delta$ be the functions given by Proposition 2.1. The stability estimate (Theorem 1.1) applied with $r = 2$ yields

$$\sup_{\Omega_\delta}(u_\delta - u) \leqslant \sup_{\Omega}(\tilde{u}_\delta - u) \leqslant C_1\delta^\alpha + C_2\|u_\delta - u\|^\gamma_{L^2(\Omega_\delta)},$$

for $\gamma < 2/(nq + 2)$. To conclude the proof of the theorem, it remains to show that $\|u_\delta - u\|_{L^2(\Omega_\delta)} = \mathrm{O}(\delta)$ as $\delta \downarrow 0$.

It will be a consequence of Lemma 3.2 below that $\nabla u \in L^2(\Omega)$. Assuming this for the moment, we derive the following precise uniform upper-bound:

$$\|u_\delta - u\|_{L^2(\Omega_\delta)} \;\leqslant\; 2^{n+1}\; \delta\; \|\nabla u\|_{L^2(\Omega)}.$$

Indeed, fix $\delta > 0$ small enough, $z \in \Omega_\delta$, and $|\zeta| \leqslant \delta$. Using the mean value inequality for u on the euclidean ball of centre $z + \zeta$ and radius $\delta > 0$ and averaging the gradient of u on the corresponding lines, we obtain the following estimate:

$$|u(z + \zeta) - u(z)| \;\leqslant\; 2\delta \int_0^1 dt \int_{|\eta| \leqslant \delta} \|\nabla u(z + t(\zeta + \eta))\|\; \frac{dV(\eta)}{\tau_{2n}\delta^{2n}}.$$

Observe that the reasoning above works only if u is smooth, for example, C^1 in a neighbourhood of $\overline{\Omega}_{3\delta}$ with $\delta > 0$ small enough. In our case by regularization we can approximate u on a neighbourhood of $\overline{\Omega}_{3\delta}$ by a decreasing sequence (u_j) of smooth plurisubharmonic functions. Then it is well known that the sequence (∇u_j) of gradients converges in $L^1_{\mathrm{loc}}(\Omega)$ and then it has a subsequence which converges almost everywhere on Ω. Therefore the inequality will follow from the smooth case by the Lebesgue convergence theorem.

Now a simple computation using Jensen's convexity inequality and a change of variables yields

$$|u_\delta(z) - u(z)|^2 \leqslant 4\delta^2 \int_0^1 dt \int_{|\xi| \leqslant 2t\delta} \|\nabla u(z+\xi)\|^2 \frac{dV(\xi)}{\tau_{2n} t^{2n} \delta^{2n}}.$$

Then integrating over Ω_δ, we get

$$\int_{\Omega_\delta} |u_\delta(z) - u(z)|^2 \, dV(z) \leqslant 4^{n+1} \delta^2 \, \|\nabla u\|^2_{L^2(\Omega_{3\delta})},$$

which proves the required estimate.

This ends the proof of the theorem up to the fact, to be established now, that u has gradient in $L^2(\Omega)$. □

Since u is plurisubharmonic and bounded, $\nabla u \in L^2_{\text{loc}}(\Omega)$. It follows from Lemma 3.2 below that $\nabla u \in L^2(\Omega)$ as soon as u is bounded from below by a bounded plurisubharmonic function v such that $v \leqslant u$ in Ω, $v = u = \phi$ on $\partial\Omega$, and $\nabla v \in L^2(\Omega)$. Our extra assumption in Theorem 4.1 precisely yields such a function v. Indeed set $v := u(\Omega, \phi, 0) + b_f$, where b_f is the plurisubharmonic barrier constructed in the proof of Lemma 2.2: this is a plurisubharmonic function such that

(1) $v = \phi + 0 = u$ on $\partial\Omega$;
(2) $(dd^c v)^n \geqslant (dd^c b_f)^n \geqslant f\beta_n$ in Ω, and thus $v \leqslant u$ in Ω;
(3) $\nabla u(\Omega, \phi, 0) \in L^2(\Omega)$ and $\nabla b_f \in L^\infty(\Omega)$, and hence $\nabla v \in L^2(\Omega)$.

It is easy to check that $\nabla u(\Omega, \phi, 0) \in L^\infty(\Omega) \subset L^2(\Omega)$ when $\phi \in \mathcal{C}^2(\partial\Omega)$. We refer the reader to [**1**] for a proof of the more delicate result that this still holds when $\phi \in \mathcal{C}^{1,1}(\partial\Omega)$.

Lemma 3.2. *Let $u, v \in \mathrm{PSH}(\Omega) \cap C^0(\overline{\Omega})$ such that $v \leqslant u$ on Ω and $v = u$ on $\partial\Omega$. Then $\int_\Omega du \wedge d^c u \wedge \beta^{n-1} \leqslant \int_\Omega dv \wedge d^c v \wedge \beta^{n-1}$, where $\beta := dd^c|z|^2$.*

We thank the referee for simplifying our original argument.

Proof. First assume that $u = v$ near the boundary $\partial\Omega$. Then integration by parts yields

$$\begin{aligned}\int_\Omega dv \wedge d^c v \wedge \beta^{n-1} - \int_\Omega du \wedge d^c u \wedge \beta^{n-1} &= \int_\Omega d(v-u) \wedge d^c(v+u) \wedge \beta^{n-1} \\ &= \int_\Omega (v-u) \wedge dd^c(v+u) \wedge \beta^{n-1} \geqslant 0.\end{aligned}$$

Now if we only know that $u = v$ on $\partial\Omega$, then we can define for $\varepsilon > 0$ small enough, $u_\varepsilon := \sup\{u - \varepsilon, v\}$. Then $v \leqslant u_\varepsilon$ on Ω and $u_\varepsilon = v$ near the boundary of Ω. Therefore we have

$$\int_\Omega dv \wedge d^c v \wedge \beta^{n-1} \geqslant \int_\Omega du_\varepsilon \wedge d^c v_\varepsilon \wedge \beta^{n-1}.$$

Now by Bedford and Taylor's convergence theorem [**1**], we know that $du_\varepsilon \wedge d^c u_\varepsilon \wedge \beta^{n-1} \to du \wedge d^c u \wedge \beta^{n-1}$ as $\varepsilon \downarrow 0$. Thus we have

$$\int_\Omega dv \wedge d^c v \wedge \beta^{n-1} \geqslant \int_\Omega du \wedge d^c u \wedge \beta^{n-1},$$

which proves the required inequality. □

4. *Laplacian estimates*

This section is devoted to the proof of the following result.

THEOREM 4.1. *Assume $f \in L^p(\Omega)$, for some $p > 1$, and $\phi \in \mathrm{Lip}_{2\alpha}(\partial\Omega)$ is such that the positive measure $\Delta u(\Omega, \phi, 0)$ has finite mass in Ω. Then*

$$u(\Omega, \phi, f) \in \mathrm{Lip}_{\alpha''}(\overline{\Omega}) \quad \textit{for all } \alpha'' < \min\left(\alpha, \frac{2}{[qn+1]}\right),$$

where $1/p + 1/q = 1$.

Observe that the hypothesis of the theorem is satisfied with $\alpha = 1$ when $\phi \in C^{1,1}(\partial\Omega)$ thanks to Proposition 2.3. In this case the theorem implies that $u(\Omega, \phi, f) \in \mathrm{Lip}_{\alpha''}(\overline{\Omega})$, for all $\alpha'' < 2/[qn+1]$, which implies immediately our Main Theorem stated in the introduction.

To prove the above theorem, we use the same method as above. The finiteness of the total mass of $\Delta u(\Omega, \phi, 0)$ allows a good control (see Lemma 4.2) on the terms $\hat{u}_\delta - u$, where

$$\hat{u}_\delta(z) := \frac{1}{\tau_{2n}\delta^{2n}} \int_{|\zeta - z| \leqslant \delta} u(\zeta)\, dV_{2n}(\zeta), \quad z \in \Omega_\delta,$$

where τ_{2n} denotes the volume of the unit ball in $\mathbb{C}^n$. We shall compare $\hat{u}_\delta$ with u_δ in Lemma 4.2 below.

It follows from the construction of plurisubharmonic Hölder continuous barriers that the solution $u = u(\Omega, \phi, f)$ is Hölder continuous near the boundary, that is, for $\delta > 0$ small enough, we have

$$u(z) - u(\zeta) \leqslant c_0 \delta^\alpha, \tag{1}$$

for $z, \zeta \in \overline{\Omega}$ with $\mathrm{dist}(z, \partial\Omega) \leqslant \delta, \mathrm{dist}(\zeta, \partial\Omega) \leqslant \delta$, and $|z - \zeta| \leqslant \delta$.

The link between u_δ and $\hat{u}_\delta$, is made by the following lemma.

LEMMA 4.2. *Given $\alpha \in]0,1[$, the following two conditions are equivalent.*

(i) *There exist $\delta_0, A > 0$ such that for any $0 < \delta \leqslant \delta_0$,*

$$u_\delta - u \leqslant A\delta^\alpha \quad \textit{on } \Omega_\delta.$$

(ii) *There exist $\delta_1, B > 0$ such that for any $0 < \delta < \delta_1$,*

$$\hat{u}_\delta - u \leqslant B\delta^\alpha \quad \textit{on } \Omega_\delta.$$

Proof. Observe that $\hat{u}_\delta \leqslant u_\delta$ in Ω_δ, and hence (i) $\Rightarrow$ (ii) follows immediately.

We now prove that (ii) $\Rightarrow$ (i). We need to show that there exist $A, \delta_0 > 0$ such that for $0 < \delta \leqslant \delta_0$,

$$\omega(\delta) := \sup_{z \in \Omega_\delta} [u_\delta(z) - u(z)] \leqslant A\delta^\alpha.$$

Fix $\delta_\Omega > 0$ small enough so that $\Omega_\delta \neq \emptyset$ for $\delta \leqslant 3\delta_\Omega$. Since u is uniformly continuous, for any fixed $0 < \delta < \delta_\Omega$,

$$\nu(\delta) := \sup_{\delta < t \leqslant \delta_\Omega} \omega(t) t^{-\alpha} < +\infty.$$

We claim that there exists a $\delta_0 > 0$ small enough so that for any $0 < \delta \leqslant \delta_0$,

$$\omega(\delta) \leqslant A\delta^\alpha \quad \text{with } A = (1 + 4^\alpha)c_0 + 2^\alpha 4^n B + \nu(\delta_\Omega),$$

where c_0 is the constant arising in inequality (1), while B is the constant from condition (ii). Assume that this is not the case. Then there exists a $0 < \delta < \delta_\Omega$ such that

$$\omega(\delta) > A\delta^\alpha. \tag{2}$$

Set $\delta := \sup\{t < \delta_\Omega \,/\, ø(t) > At^\alpha\}$. Then

$$\frac{ø(\delta)}{\delta^\alpha} \geqslant A \geqslant \frac{ø(t)}{t^\alpha} \quad \text{for all } t \in [\delta, \delta_\Omega]. \tag{3}$$

Since u is continuous, we can find $z_0 \in \overline{\Omega_\delta}$, $\zeta_0 \in \overline{\Omega}$ with $|z_0 - \zeta_0| \leqslant \delta$ such that

$$\omega(\delta) = \sup_{z \in \Omega_\delta} \left[\sup_{w \in B(z,\delta)} u(w) - u(z) \right] = u(\zeta_0) - u(z_0).$$

We first derive a contradiction if z_0 is close enough to the boundary of Ω. Assume that $\operatorname{dist}(z_0, \partial\Omega) \leqslant 3\delta$. Take $z_1 \in \partial\Omega$ such that $\operatorname{dist}(z_0, \partial\Omega) = \operatorname{dist}(z_0, z_1) \leqslant 4\delta$. It follows from (1) that

$$\omega(\delta) = u(\zeta_0) - u(z_0) = [u(\zeta_0) - u(z_1)] + [u(z_1) - u(z_0)] \leqslant [1 + 4^\alpha] c_0 \delta^\alpha.$$

This contradicts (3).

Thus we can assume that $\operatorname{dist}(z_0, \partial\Omega) > 3\delta$. Fix $b > 1$ so that $\operatorname{dist}(z_0, \partial\Omega) > (2b+1)\delta$. Thus any $z \in \mathbb{B}(\zeta_0, b\delta)$ satisfies $z \in \mathbb{B}(z_0, [b+1]\delta)$, and hence $z \in \Omega_{b\delta}$. By using inequality (3) with $t = b\delta$, we get $u(\zeta_0) - u(z) \leqslant b^\alpha ø(\delta)$; hence

$$u(z) \geqslant u(\zeta_0) - b^\alpha ø(\delta) \quad \text{for all } z \in \mathbb{B}(\zeta_0, b\delta). \tag{4}$$

Observe now that $\mathbb{B}(\zeta_0, \delta) \subset \mathbb{B}(z_0, [b+1]\delta)$, and hence

$$\begin{aligned} \hat{u}_{(b+1)\delta}(z_0) &= \left(\frac{b}{b+1}\right)^{2n} \hat{u}_{b\delta}(\zeta_0) + \frac{1}{\tau_n (b+1)^{2n} \delta^{2n}} \int_{\mathbb{B}(z_0,(b+1)\delta) \setminus \mathbb{B}(\zeta_0, b\delta)} u \, dV \\ &\geqslant \left(\frac{b}{b+1}\right)^{2n} u(\zeta_0) + \left[(1 - \frac{b^{2n}}{(b+1)^{2n}} \right] [u(\zeta_0) - b^\alpha \omega(\delta)] \\ &= u(\zeta_0) - b^\alpha \left[1 - \frac{b^{2n}}{(b+1)^{2n}} \right] ø(\delta), \end{aligned}$$

where we have used the subharmonicity of u together with inequality (4). Since $u(\zeta_0) = u(z_0) + ø(\delta)$, we infer, letting $b \to 1$,

$$\hat{u}_{2\delta}(z_0) \geqslant u(z_0) + 4^{-n} ø(\delta).$$

We now use assumption (ii), only considering small enough values of $\delta > 0$: since $\hat{u}_{2\delta}(z_0) \leqslant u(z_0) + B 2^\alpha \delta^\alpha$, we get

$$ø(\delta) \leqslant 4^n 2^\alpha B \delta^\alpha < A\delta^\alpha.$$

This contradicts the definition of δ, and hence we have proved that (ii) $\Rightarrow$ (i). □

It is straightforward to check that if assumption (i) is satisfied, then u belongs to $\operatorname{Lip}_\alpha(\overline{\Omega})$. Thus Theorem 4.1 will be proved if we can establish assumption (ii). It follows from Theorem 1.1 that it suffices to get control on the L^1-average of $\hat{u}_\delta - u$. This is the content of our next result.

Lemma 4.3. *Assume that Δu has finite mass in Ω. Then for $\delta > 0$ small enough, we have*

$$\int_{\Omega_\delta} [\hat{u}_\delta(z) - u(z)] \, dV_{2n}(z) \leqslant c_n \|\Delta u\| \delta^2,$$

where $c_n > 0$ is a uniform constant.

Proof. It follows from Jensen's formula that for $z \in \Omega_\delta$ and $0 < r < \delta$,

$$\frac{1}{\sigma_{2n-1}} \int_{|\xi|=1} u(z + r\xi)\, dS_{2n-1} = u(z) + \int_0^r t^{1-2n} \left(\int_{|\zeta| \leqslant t} dd^c u \wedge \beta_{n-1} \right) dt.$$

Using polar coordinates we get, for $z \in \Omega_\delta$,

$$\hat{u}_\delta(z) - u(z) = \frac{1}{\sigma_{2n-1}\delta^{2n}} \int_0^\delta r^{2n-1}\, dr \int_0^r t^{1-2n} \left(\int_{|\zeta - z| \leqslant t} dd^c u \wedge \beta_{n-1} \right) dt.$$

Finally, Fubini's theorem yields

$$\begin{aligned}\int_{\Omega_\delta} (\hat{u}_\delta - u)\, dV_{2n} &\leqslant a_n \delta^{-2n} \int_0^\delta r^{2n-1}\, dr \int_0^r t^{1-2n} \left(\int_{|\zeta| \leqslant t} \left(\int_\Omega \Delta u \right) \right) dt \\ &\leqslant c_n \delta^2 \|\Delta u\|. \end{aligned}$$

□

To complete the proof of Theorem 4.1, we use the same gluing construction as in Proposition 2.1 to construct global plurisubharmonic approximants (v_δ) decreasing to u in Ω as $\delta \downarrow 0$ such that $v_\delta = u + C\delta^\alpha$ on $\Omega \setminus \Omega_\delta$ and $\hat{u}_\delta - u \leqslant v_\delta - u \leqslant \hat{u}_\delta - u + C\delta^\alpha$ on Ω_δ. Now we can use Lemma 4.3 since by Proposition 2.3, $\Delta u = \Delta u(\Omega, \phi, f)$ has finite mass in Ω. Then using Theorem 1.1 (with $\psi = v_\delta, \varphi = u$, $r = 1$) we get

$$\sup_{\Omega_\delta} (\hat{u}_\delta - u) \leqslant \sup_\Omega (v_\delta - u) + C\delta^\alpha \leqslant C(\delta^{2\gamma} + \delta^\alpha),$$

where $C > 0$ is a constant, which proves our theorem due to Lemma 4.2.

We now give examples which show that the Hölder exponent in our theorems cannot be better that $2/nq$, where $q = p/(p-1)$. The first (simple) example explains why the exponent is optimal.

Example 4.4. Consider the function defined on $\mathbb{C}^n$ by $u(z_1, \ldots, z_n) := |z_1|^\alpha \cdot |z'|^2$, where $z' := (z_2, \ldots, z_n)$. This is a plurisubharmonic function in $\mathbb{C}^n$ which is Hölder-continuous of exponent $\alpha \in]0, 1[$. We let the reader check that

$$(dd^c u)^n = f\, dV \quad \text{with } f(z) = \frac{1}{|z_1|^{2-n\alpha}} g(z_2, \ldots, z_n),$$

where $g > 0$ is a smooth density.

Given $p > 1$, f belongs to $L^p_{\text{loc}}(\mathbb{C}^n)$ whenever $\alpha = \varepsilon + 2/nq$, for some $\varepsilon > 0$.

The next example was communicated to us by Plis [**11**]. It shows that one cannot expect a better exponent than $2/nq$ in the unit ball with zero boundary data.

Example 4.5. Consider the function

$$\eta(t) = \begin{cases} 0 & \text{if } |t| \geqslant 1, \\ \exp(-1/(1-t^2)) & \text{if } |t| < 1, \end{cases} \tag{5}$$

and let

$$f(z) := \eta\left(\frac{|z_n|}{|z'|^\alpha}\right) |z'|^\beta,$$

where $z = (z', z_n) \in \mathbb{B}_n$, $n \geqslant 2$, $\alpha > 0$, and $\beta \in \mathbb{R}$. Then by [11], if u is a continuous plurisubharmonic function on $\mathbb{B}_n$ such that

$$\begin{aligned} &(dd^c u)^n = f\beta_n \quad \text{in } \mathbb{B}_n, \\ &u = 0 \quad \text{on } \partial\mathbb{B}_n \end{aligned} \tag{6}$$

then there exist a sequence $\varepsilon_k \searrow 0$ and a constant $C > 0$ such that

$$u(0, \varepsilon_k) - u(0) \geqslant \varepsilon_k^{(2\alpha+2(n-1)+\beta)/n\alpha}.$$

Let $p > 1$ and $\varepsilon > 0$. Then if we set $\beta := -(2(\alpha + (n-1) + \varepsilon))/p$, we obtain a density $f \in L^p(\mathbb{B}_n)$ and for any $\delta > 0$, the solution u is not $(\delta + 2/nq)$-Hölder continuous on $\mathbb{B}_n$ if $\alpha > 0$ is big enough, where $q = p/(p-1)$.

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