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Weak solutions to degenerate complex Monge–Ampère flows II ☆

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A B S T R A C T

Studying the (long-term) behavior of the Kähler–Ricci flow on mildly singular varieties, one is naturally led to study weak solutions of degenerate parabolic complex Monge–Ampère equations.

The purpose of this article, the second of a series on this subject, is to develop a viscosity theory for degenerate complex Monge–Ampère flows on compact Kähler manifolds. Our general theory allows in particular to define and study the (normalized) Kähler–Ricci flow on varieties with canonical singularities, generalizing results of Song and Tian.

Contents

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0. Introduction

The study of the (long-term) behavior of the Kähler–Ricci flow on mildly singular varieties in relation to the Minimal Model Program was undertaken by J. Song and G. Tian [29,30] and it requires a theory of weak solutions for certain degenerate parabolic complex Monge–Ampère equations modeled on:

$$\frac{\partial \phi}{\partial t} + \phi = \log \frac{(dd^c \phi)^n}{V} \tag{0.1}$$

where V is volume form and ϕ a t-dependent Kähler potential on a compact Kähler manifold. The approach in [30] is to regularize the equation and take limits of the solutions of the regularized equation with uniform higher order estimates. But as far as the existence and uniqueness statements in [30] are concerned, we believe that a zeroth order approach would be both simpler and more efficient.

There is a well established pluripotential theory of weak solutions to elliptic complex Monge–Ampère equations, following the pioneering work of Bedford and Taylor [2,3] in the local case (domains in $\mathbb{C}^n$). A complementary viscosity approach has been developed only recently in [23,17,34,18] both in the local and the global case (compact Kähler manifolds).

Surprisingly no similar theory has ever been developed on the parabolic side. The most significant reference for a parabolic flow of plurisubharmonic functions on pseudoconvex domains is [20] but the flow studied there takes the form

$$\frac{\partial \phi}{\partial t} = ((dd^c \phi)^n)^{1/n} \tag{0.2}$$

which does not make sense in the global case. The purpose of this article, the second of a series on this subject, is to develop a viscosity theory for degenerate complex Monge–Ampère flows of the form (0.3).

This article focuses on solving this problem on compact Kähler manifolds, while its companion [19] concerns the local case (domains in $\mathbb{C}^n$). More precisely we study here the complex degenerate parabolic complex Monge–Ampère flows

$$e^{\dot{\varphi}_t + F(t,x,\varphi)} \mu(t,x) - (\omega_t + dd^c \varphi_t)^n = 0, \tag{0.3}$$

where

- $T \in]0, +\infty]$;
- $\omega = \omega(t,x)$ is a continuous family of semi-positive $(1,1)$-forms on X,

- $F(t, z, r)$ is continuous in $[0, T[\times X \times \mathbb{R}$ and non-decreasing in r,
- $\mu(t, z) \geq 0$ is a bounded continuous volume form on X,
- $\varphi : X_T := [0, T[\times X \to \mathbb{R}$ is the unknown function, with $\varphi_t := \varphi(t, \cdot)$.

Our plan is to adapt the viscosity approach developed by P.L. Lions et al. (see [25,14]) to the complex case, using the elliptic side of the theory which was developed in [17]. It should be noted that the method used in [30] is a version of the classical PDE method of vanishing viscosity which was superseded by the theory of viscosity solutions.

We develop the appropriate definitions of (viscosity) subsolution, supersolution and solution in the *first section*, and connect these to weak solutions of the Kähler–Ricci flow (normalized or not).

As is often the case in the viscosity theory, one of our main technical tools is the global comparison principle. We actually establish several comparison principles in the *second section*, in particular the following:

Theorem A. *Assume $t \mapsto \omega_t$ is non-decreasing or more generally regular in the sense of Definition 2.5. If φ (resp. ψ) is a bounded subsolution (resp. supersolution) of the above degenerate parabolic equation then*

$$\max_{X_T}(\varphi - \psi) \leq \max_{x \in X}(\varphi(0, x) - \psi(0, x))_+,$$

with the notation $a_+ = \max(a, 0)$, given a a real number.

We do not reproduce here the rather technical Definition 2.5 and refer the reader to section 2 instead. It is enough to record here that the condition is satisfied in all the situations arising from the Kähler–Ricci flow with singularities.

In the *third section* we specialize to the complex Monge–Ampère flows arising in the study of the (normalized) Kähler–Ricci flow on mildly singular varieties, assuming $F(t, x, \varphi) = \alpha\varphi$, and

$$\mu(x, t) = e^{u(x)} f(x, t) dV(x),$$

where $f > 0$ is a positive continuous density and u is quasi-plurisubharmonic function that is *exponentially continuous* (i.e. such that e^u is continuous).

We construct barriers at each point of the parabolic boundary and use the Perron method to eventually show the existence of a viscosity solution to the Cauchy problem:

Theorem B. *Let φ_0 be a continuous ω_0-plurisubharmonic function on X and assume F, μ are as above. The Cauchy problem for the parabolic complex Monge–Ampère equation with initial data φ_0 admits a unique viscosity solution $\varphi(t, x)$; it is the upper envelope of all subsolutions.*

We describe applications to the Kähler–Ricci flow on varieties with a definite first Chern class in the *fourth section*, showing in particular a generalization of Cao's theorem [8]:

Theorem C. *Let Y be a $\mathbb{Q}$-Calabi–Yau variety and S_0 a positive closed current with continuous potentials representing a Kähler class $\alpha \in H^{1,1}(Y, \mathbb{R})$. The Kähler–Ricci flow*

$$\frac{\partial \omega_t}{\partial t} = -\mathrm{Ric}(\omega_t)$$

can be uniquely run from S_0 and converges, as $t \to +\infty$, towards the unique Ricci flat Kähler–Einstein current S_{KE} in α.

We similarly handle the case of canonical models:

Theorem D. *Let Y be a canonical model, i.e. a general type projective algebraic variety with only canonical singularities such that K_Y is nef and big and S_0 a positive closed current with continuous potential representing a Kähler class $\alpha \in H^{1,1}(Y, \mathbb{R})$. The normalized Kähler–Ricci flow*

$$\frac{\partial \omega_t}{\partial t} = -\mathrm{Ric}(\omega_t) - \omega_t$$

can be uniquely run from S_0 and exists for all time. Moreover ω_t has continuous potentials on $\mathbb{R}^+ \times Y$ and converges, as $t \to +\infty$, towards the unique singular Kähler–Einstein metric S_{KE} on Y.

The convergence is here uniform at the level of potentials. The existence of S_{KE} is due to [16], while the continuity of its potentials follows from the elliptic viscosity approach of [17].

We also show that the weak Kähler–Ricci flows considered by Song and Tian [30] (when the measure μ is sufficiently regular) coincide with ours, this yields in particular the global continuity of the corresponding potentials which was not established in [30].

We conclude by proposing a (discontinuous) viscosity approach to understanding the behavior of the Kähler–Ricci flow over the flips. This requires to extend our results allowing for discontinuous densities, a promising line of research for the future. We plan to come back to that question in a forthcoming work.

We learnt of the possibility to use viscosity solutions in the present context via a hint in [9] where no attempt to fully justify this technique was made.

1. Complex Monge–Ampère flows on compact manifolds

1.1. Geometrical background for Complex Monge–Ampère flows

Let X be a n-dimensional compact complex manifold and $n = \dim_{\mathbb{C}}(X)$.

The sheaf $\mathcal{Z}^{1,1}_X$ of closed $(1,1)$-forms with continuous potential is, by definition, the quotient sheaf $\mathcal{Z}^{1,1}_X := \mathcal{C}^0_X/\mathcal{PH}_X$ of the sheaf $\mathcal{C}^0_X$ of real valued continuous functions on X by its subsheaf of pluriharmonic functions. Given a section of $\mathcal{Z}^{1,1}_X$ represented by a cocycle $(\phi_\beta)_{\beta\in B}$ where $\phi_\beta \in C^0(U_\beta,\mathbb{R})$ and $\mathfrak{U} = (U_\beta)_{\beta\in B}$ is covering of X, the currents $dd^c\phi_\beta$ defined on each U_β glue into a closed current of bidegree $(1,1)$ on X. Global sections of $\mathcal{Z}^{1,1}_X$ form the background geometry in the study of degenerate complex Monge–Ampère equations in the global case [16].

It is straightforward to formulate a parabolic analog. Let T be positive real number and consider the manifold with boundary $X_T := [0,T[\times X$ and denote by $\mathcal{C}^0_{X_T}$ the sheaf of continuous functions on X_T. Denote by $\mathcal{PH}_{X_T/[0,T[} \subset \mathcal{C}^0_{X_T}$ the sheaf of continuous real valued local functions whose restriction to each $X_t := \{t\}\times X \overset{i_t}{\hookrightarrow} X_T$ is a pluriharmonic real valued function.

Say a germ of real valued function on X_T is of class $C^{1,2}$ if it is of class C^1 admitting continuous second order partial derivatives in the X direction.

Definition 1.1. A family of closed real $(1,1)$-forms with continuous local potentials $\omega = (\omega_t)_{t\in[0,T[}$ is a global section of the sheaf $\mathcal{Z}^{1,1}_{X_T/[0,T[} := \mathcal{C}^0_{X_T}/\mathcal{PH}_{X_T/[0,T[}$.

A continuous family of closed real $(1,1)$-forms $\omega = (\omega_t)_{t\in[0,T[}$ is a global section of the sheaf $C^0\mathcal{Z}^{1,1}_{X_T/[0,T[} := \mathcal{C}^{1,2}_{X_T}/\mathcal{PH}_{X_T/[0,T[}$.

It is straightforward to see that there is a covering $\mathfrak{U} = (U_\beta)_{\beta\in B}$ of X such that, for every ω a global section of the sheaf $\mathcal{Z}^{1,1}_{X_T/\mathbb{R}} := \mathcal{C}^0_{X_T}/\mathcal{PH}_{X_T/\mathbb{R}}$, $\omega|_{[0,T[\times U_\beta}$ is represented by $\Phi_\beta \in C^0([0,T[\times U_\beta,\mathbb{R})$ such that $\Phi_{\beta\beta'} = \Phi_\beta - \Phi_{\beta'} \in C^0(U_{\beta\beta'})$ satisfies $\partial\bar\partial\Phi_{\beta\beta'} = 0$ and conversely such a cochain $(\Phi_\beta)_{\beta\in B}$ defines a global section of $\mathcal{Z}^{1,1}_{X_T/[0,T[}$. The covering $\mathfrak{U}$ will be fixed throughout the article for technical reasons but our results will not depend on this choice.

We have a natural map $\mathcal{Z}^{1,1}_{X_T/[0,T[} \to (i_t)_*\mathcal{Z}^{1,1}_X$ hence for every $t \in [0,T[$, ω defines a closed real $(1,1)$-form with continuous potentials ω_t by the prescription:

$$\Big(\omega \mapsto \omega_t := dd^c\Phi_\beta|_{\{t\}\times U_\beta}, \quad H^0(X_T,\mathcal{Z}^{1,1}_{X_T/[0,T[}) \to H^0(X,\mathcal{Z}^{1,1}_X)\Big)$$

and, taking the Bott Chern cohomology class $\{\omega_t\}$ of ω_t, we get a map

$$\{-_t\} : H^0(X_T,\mathcal{Z}^{1,1}_{X_T/[0,T[}) \to H^{1,1}_{BC}(X,\mathbb{R}) \simeq H^1(X,\mathcal{PH}_X)$$

such that $t \mapsto \{\omega_t\}$ is a continuous $H^{1,1}_{BC}(X,\mathbb{R})$-valued function. The resulting map $H^0(X_T,\mathcal{Z}^{1,1}_{X_T/[0,T[}) \to C^0([0,T[,H^{1,1}_{BC}(X,\mathbb{R}))$ is surjective. On the other hand, $H^0(X_T,C^0\mathcal{Z}^{1,1}_{X_T/[0,T[})$ maps onto $C^1([0,T[,H^{1,1}_{BC}(X,\mathbb{R}))$.

Let us remark that the previous definitions make sense for normal complex spaces. However, for the formulation of the flows to be given in the next paragraph, it is necessary to assume smoothness.

1.2. Complex Monge–Ampère flows

Definition 1.2. The complex Monge–Ampère flow associated to (ω, μ, F) where:

- $\omega \in H^0(X_T, C^0\mathcal{Z}^{1,1}_{X_T/[0,T[})$ is a continuous family of closed real $(1,1)$-forms on X in the sense of Definition 1.1,
- $0 \le \mu(t,x) \in C^0(X, \Omega^{n,n}_{X_T/[0,T[})$ is a continuous family of volume forms on X,
- $F : [0,T[\times X \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and non-decreasing in the last variable,

is the following parabolic equation:

$$(\omega + dd^c\phi)^n = e^{\frac{\partial\phi}{\partial t}+F(t,x,\phi)}\mu. \qquad (CMAF)_{X,\omega,\mu,F}$$

Here: $\phi : X_T \longrightarrow \mathbb{R}$ is the unknown function.

A classical solution of a complex Monge–Ampère flow is a function of class $C^{1,2}$ satisfying equation $(CMAF)_{X,\omega,\mu,F}$ pointwise in $]0,T[\times X$.

Define $F_\beta(t,x,r) := F(t,x,r-\Phi_\beta(x))$ and $\mu_\beta(t,x) := e^{-\frac{\partial\Phi_\beta(t,x)}{\partial t}}\mu(t,x)$ where $(t,x) \in]0,T[\times U_\beta$.

Definition 1.3. A function $\phi : X_T \to \mathbb{R}$ is a viscosity sub/super-solution of $(CMAF)_{\omega,\mu,F}$ iff, for each $\beta \in B$, $\psi = \phi + \Phi_\beta$ is a viscosity sub/super-solution of the following parabolic Monge–Ampère equation:

$$(PMA)_{\mu_\beta,F_\beta} \quad (dd^c\psi)^n = e^{\frac{\partial\psi}{\partial t}+F_\beta(t,x,\psi)}\mu_\beta \quad \text{on} \quad]0,T[\times U_\beta.$$

A bounded function $\phi : X_T \to \mathbb{R}$ is a viscosity solution of $(CMAF)_{\omega,\mu,F}$ iff it is both a sub- and a supersolution of $(CMAF)_{\omega,\mu,F}$. Such a function is continuous.

A bounded function $\phi : X_T \to \mathbb{R}$ is a discontinuous viscosity solution of $(CMAF)_{\omega,\mu,F}$ iff its upper semi-continuous regularization ϕ^* is a subsolution of $(CMAF)_{\omega,\mu,F}$ and its lower semicontinuous regularization ϕ_* is a supersolution of $(CMAF)_{\omega,\mu,F}$.

If a viscosity solution (resp. subsolution, resp. supersolution) is of class $C^{1,2}$, it is a classical solution (resp. subsolution, resp. supersolution). We refer the reader to [19] for a study of viscosity sub/super-solutions to local complex Monge–Ampère flows.

This definition is a special case of the general theory of [14] for viscosity solutions of general degenerate elliptic/parabolic equations. The reader is referred to this survey article for the first principles of the theory. The basic fact we certainly need to recall is that subsolutions are u.s.c whereas supersolutions are l.s.c.

Recall that if ω is a closed smooth $(1,1)$-form in X, then the complex Monge–Ampère measure $(\omega + dd^c\psi)^n$ is well-defined *in the pluripotential sense* for all bounded ω-psh functions ψ in X, as follows from the work of Bedford and Taylor (see [2,21]) and viscosity (sub)solutions of complex Monge–Ampère equations can be interpreted in pluripotential theory as explained in [17, Theorem 1.9].

On the other hand, it is not clear to us how to interpret viscosity solutions of Complex Monge–Ampère flows in terms of pluripotential theory. We will note however the following useful lemma which follows easily from [17, Theorem 1.9, Lemma 4.7].

Lemma 1.4. *Let $u \in C^0(X_T, \mathbb{R})$ such that:*

- *u admits a continuous partial derivative $\partial_t u$ with respect to t,*
- *for every $t \in]0, T[$, the restriction u_t of u to X_t satisfies*

$$(\omega_t + dd^c u_t)^n \geq e^{\partial_t u + F(t,x,u)} \mu(t, x)$$

  in the pluripotential sense on X_t.

Then u is a subsolution of $(CMAF)_{\omega,\mu,F}$.
Let $v \in C^0(X_T, \mathbb{R})$ such that:

- *The restriction v_t of v to X_t is ω_t-psh,*
- *v admits a continuous partial derivative $\partial_t v$ with respect to t,*
- *there exists a continuous function w such that, for every $t \in]0, T[$, the restriction v_t to X_t satisfies*

$$(\omega_t + dd^c v_t)^n \leq e^w \mu(t, x)$$

  in the pluripotential sense on X_t and $\partial_t v_t + F(t, x, v_t) \geq w$.

Then v is a supersolution of $(CMAF)_{\omega,\mu,F}$.

We also need to record a basic property from [19].

Proposition 1.5. *Let ϕ be a viscosity subsolution of $(CMAF)_{\omega,\mu,F}$. For each $t \in]0, T[$, we have $\phi_t \in \mathrm{PSH}(X, \omega_t)$.*

Let us remark that, for these applications of the results of [19] on weak solutions to local complex Monge–Ampère flows, it is actually enough to assume the technical condition that Φ_β is continuous and locally Lipschitz in the time variable (hence $\frac{\partial}{\partial t}\Phi_\beta$ exists a.e.) and μ is only measurable but:

$$\mu_\beta := e^{-\frac{\partial \Phi_\beta}{\partial t}} \mu|_{U_\beta} \text{ is continuous.} \tag{1.1}$$

We note that this condition allows the function $t \mapsto \{\omega_t\}$ be Lipschitz non-differentiable. However, we are not able to prove global results unless the following stronger regularity condition holds:

$$\Phi_\beta \in C^{1,2}, \;\; \mu \text{ is continuous,} \tag{1.2}$$

which is indeed what Definition 1.2 requires.

1.3. The Kähler–Ricci flow with canonical singularities

Normalized Kähler–Ricci flow. Let us now interpret in the present framework the Kähler–Ricci flow on varieties with canonical singularities that was defined in [30].

Let Y be an irreducible normal compact Kähler space with only canonical singularities and $n = \dim_{\mathbb{C}}(Y)$. Let $\pi : X \to Y$ be a log-resolution, i.e.: X is a compact Kähler manifold, π is a bimeromorphic projective morphism and $\mathrm{Exc}(\pi)$ is a divisor with simple normal crossings. Denote by $\{E\}_{E\in\mathcal{E}}$ the family of the irreducible components of $\mathrm{Exc}(\pi)$. With this notation, one has furthermore

$$K_X \equiv \pi^* K_Y + \sum_E a_E E$$

where $a_E \in \mathbb{Q}_{\geq 0}$, K_Y denote the first Chern class in Bott–Chern cohomology of the $\mathbb{Q}$-line bundle $O_Y(K_Y)$ on Y whose restriction to the smooth locus is the line bundle whose sections are holomorphic top dimensional forms (or according to the standard terminology canonical forms), K_X the canonical class of X and E also denotes with a slight abuse of language the cohomology class of E. This means that for every non-vanishing locally defined multivalued canonical form η defined over Y, the holomorphic multivalued canonical form $\pi^*\eta$ on X has a zero of order a_E along E.

Denote by $\mathcal{K}(Y) \subset H^1(Y, \mathcal{PH}_Y)$ the open convex cone of Kähler classes and let ω_0 be a semi-Kähler form on Y with C^2 potential (see [16] for the definitions of Kähler metrics and variants on normal complex spaces) such that $\{\omega_0\} + \epsilon K_Y \in \mathcal{K}(Y)$ for $1 \gg \epsilon > 0$. Assume h is a smooth hermitian metric on the holomorphic $\mathbb{Q}$-line bundle underlying $O_Y(K_Y)$. Then

$$\chi := -dd^c \log h$$

is a smooth representative of $K_Y \in H^1(Y, \mathcal{PH}_Y)$.

We are going to study the existence and the long term behavior of the normalized Kähler–Ricci flow (NKRF for short) on Y,

$$\frac{\partial \omega_t}{\partial t} = -\mathrm{Ric}(\omega_t) - \omega_t,$$

starting from the initial data ω_0. At the cohomological level, this yields a first order ODE showing that the cohomology class of ω_t evolves as

$$\{\omega_t\} = e^{-t}\{\omega_0\} + (1 - e^{-t})K_Y.$$

We thus let $T_{max} \in]0, +\infty]$ be defined by

$$T_{max} := \sup\{t > 0,\ e^{-t}\{\omega_0\} + (1 - e^{-t})K_Y \in \mathcal{K}(Y)\}$$

and denote by the following C^1 in $t \in [0, T[$ relative semi-Kähler form on Y_T,

$$\chi_t = e^{-t}\chi_0 + (1 - e^{-t})\chi,$$

where χ_0 is a smooth Kähler representative of the Kähler class ω_0 and χ is a smooth representant of the canonical class K_Y.

Then $\omega = (\omega_t)_{t \in [0,T[}$ the solution of the normalized Kähler–Ricci flow can be written as $\omega_t = \chi_t + dd^c\phi_t$, where $\phi : Y_T \longrightarrow \mathbb{R}$ is continuous in Y_T.

We now define

$$\omega_{NKRF} := \pi^*\omega \in H^0(X, \mathcal{Z}^{1,1}_{X_T/[0,T[})$$

and

$$\mu_{NKRF} = c_n \frac{\pi^*\eta \wedge \overline{\pi^*\eta}}{\pi^*\|\eta\|_h^2} \in C^0(X, \Omega_X^{n,n})$$

which we view as a continuous element of $C^0(X_T, \Omega^{n,n}_{X_T/[0,T[})$ and c_n is the unique complex number of modulus 1 such that the expression is positive. As the notation suggests, μ_{NKRF} is independent of the auxiliary multivalued holomorphic form η but depends on h an auxiliary smooth metric on Y. When it will be necessary to display this dependence we shall write $\mu_{NKRF}(h)$. Since the local potentials of χ are of class C^∞ the pair $(\omega_{NKRF}, \mu_{NKRF})$ satisfies the requirements of Definition 1.2.

In local coordinates μ_{NKRF} has a continuous density of the form

$$v_{NKRF} = \prod_E |f_E|^{2a_E} v$$

where $v > 0$ is smooth and f_E is an equation of E in these local coordinates.

Lemma 1.6. *Every viscosity solution ϕ_π of the Monge–Ampère flow*

$$(CMAF)_{X,\omega_{NKRF},\mu_{NKRF},r} \ (\omega_{NKRF} + dd^c\phi)^n = e^{\phi + \frac{\partial\phi}{\partial t}}\mu_{NKRF}$$

with Cauchy datum ϕ_0 descends to Y_T, i.e.: $\phi_\pi = \pi^*\phi$ and the element $\omega + dd^c\phi \in H^0(Y, \mathcal{C}^0_{Y_T}/\mathcal{PH}_{Y_T/[0,T[})$ obtained this way is independent of π and of h.

Proof. The fact that $\omega + dd^c\phi$ does not depend on the auxiliary hermitian metric h is obvious. The rest follows from the quasi-plurisubharmonicity of viscosity (sub)solutions established in [19], together with the argument in [16] for the static case which implies that ϕ_π is constant along the fibers of π. This also works for subsolutions. □

Definition 1.7. We say that $\omega + dd^c\phi \in H^0(Y, \mathcal{C}^0_{Y_T}/\mathcal{PH}_{Y_T/[0,T[})$ as in Lemma 1.6 is a solution of the normalized Kähler–Ricci flow on Y starting at ω_0.

Lemma 1.6 implies that the notion does not depend on the choice of the log resolution $\pi : Y \to X$.

A basic observation is that the normalized Kähler–Ricci flow can also be formulated as $(CMAF)_{\omega',\mu'}$ where

$$\omega' = \pi^*\omega + dd^c\Psi, \quad \mu' = e^{-\Psi - \frac{\partial\Psi}{\partial t}}\mu_{NKRF},$$

$\Psi \in C^\infty(X_T, \mathbb{R})$ being arbitrary.

Change of time variable. The important case of $(CMAF)_{\omega,\mu,F}$ when dealing with the Kähler–Ricci flow is thus when $F(t,x,r) = \alpha r$. Then the sign of α is crucial for the long term behavior of φ_t. We however observe that it plays no role for finite time:

Lemma 1.8. *The solutions of the flows:*

$$(\omega_1(t) + dd^c\phi)^n = e^{\alpha\phi + \frac{\partial\phi}{\partial t}}\mu_1(t), \;\; t \in [t_0, t_1[$$
$$(\omega_0(s) + dd^c\psi)^n = e^{\frac{\partial\psi}{\partial s}}\mu_0(s), \;\; s \in [s_0, s_1[$$

coincide if we do the following change of variables (when $\alpha > 0$):

$$s - s_0 = e^{\alpha(t-t_0)} - 1, \quad \psi(s) = (1 + s - s_0)\phi\left(t_0 + \frac{\log(1 + s - s_0)}{\alpha}\right)$$

where

$$\omega_0(s) = (1 + s - s_0)\,\omega_1\left(t_0 + \frac{\log(1 + s - s_0)}{\alpha}\right),$$
$$\mu_0(s) = (1 + s - s_0)^n\,\mu_1\left(t_0 + \frac{\log(1 + s - s_0)}{\alpha}\right).$$

The proof is a straightforward computation. In the sequel we will therefore often reduce to the case $\alpha = 0$.

Kähler–Ricci flow. The above formulation of the normalized Kähler–Ricci flow is adapted to the asymptotic behavior of solutions when $t \to +\infty$. Applying Lemma 1.8 below to this flow, one gets another equivalent flow which is nothing but the (classical, unnormalized) Kähler–Ricci flow. This equivalent formulation is given in the definition:

Definition 1.9. A flow on X_T of the form $(\omega + dd^c\phi)^n = e^{\frac{\partial\phi}{\partial t}}\mu$ is a Kähler–Ricci flow on Y iff

- $\omega_t \in \pi^*\mathcal{K}(Y)$ for $t > 0$;

- $\mu = \prod_E |s_E|_{h_E}^{2a_E} W$ where W is a volume form with continuous positive density on X, $s_E \in H^0(X, O_X(E))$ denotes the tautological section and h_E a smooth metric on $O_X(E)$;
- $\frac{\partial \omega}{\partial t} = dd^c \log \mu - a_E[E]$ in the sense of currents.

The corresponding cohomological flow takes the form $\frac{\partial\{\omega_t\}}{\partial t} = \pi^* K_Y$.

Klt pairs. Let us also mention without going into details that one may also replace Y with a pair (Y, Δ) having klt singularities. In that case,

$$K_X \equiv \pi^*(K_Y + \Delta) + \sum_E a_E E$$

with $a_E > -1$. In that case μ_{NKRF} has poles and the preceding discussion does not apply. However, using a construction of [16], allowing high ramification along the E's, we construct a compact complex orbifold $\mathcal{X}$ whose moduli space is $c : \mathcal{X} \to X$ and we may do the preceding construction replacing X by $\mathcal{X}$. Indeed $c^*\mu_{NKRF}$ is continuous in orbifold coordinates.

1.4. The Perron discontinuous viscosity solution

A very attractive feature of discontinuous viscosity solutions is that their existence is easily established.

Definition 1.10. A Cauchy datum for $(CMAF)_{\omega,\mu,F}$ is a continuous function $\phi_0 : X \to \mathbb{R}$ such that $\phi_0 \in \mathrm{PSH}(X, \omega_0)$.

We say $\phi \in USC(X_T, \mathbb{R} \cup -\infty)$ (resp. $LSC(X_T, \mathbb{R} \cup +\infty)$) is a subsolution (resp. supersolution) to the Cauchy problem:

$$(\omega + dd^c\phi)^n = e^{\frac{\partial\phi}{\partial t} + F(t,x,\phi)}\mu, \quad \phi|_{X\times\{0\}} = \phi_0 \qquad (CMAF)_{X,\omega,\mu,F}(\phi_0)$$

if ϕ is a subsolution (resp. supersolution) to $(CMAF)_{\omega,\mu,F}$ such that $\phi|_{X\times\{0\}} \le \phi_0$ (resp. $\ge$).

The Cauchy Problem $(CMAF)_{X,\omega,\mu,F}(\phi_0)$ is said to be admissible if it has a bounded subsolution and there exists a continuous function ψ such that $\phi_0 \le \psi|_{X\times\{0\}}$ and every subsolution is $\le \psi$.

For instance, if $(CMAF)_{X,\omega,\mu,F}(\phi_0)$ admits a classical *strict* supersolution ψ, this Cauchy problem is admissible.

Proposition 1.11. *If the Cauchy Problem $(CMAF)_{X,\omega,\mu,F}(\phi_0)$ is admissible, denoting by $\mathcal{S}$ the set of all its subsolutions, the usc regularization s^* of $s := \sup_{u\in\mathcal{S}} u$ is a discontinuous viscosity solution of $(CMAF)_{X,\omega,\mu,F}$.*

Proof. Omitted. See [14,24]. □

This construction raises two issues: whether s^* is continuous, hence a true viscosity solution and whether it is a solution to the Cauchy problem in the naïve sense namely whether $s^*|_{X\times\{0\}} = \phi_0$. The first issue is generally treated using a Comparison Principle and the second issue is taken care of by barrier constructions.

The Parabolic Comparison Principle (PCP) states that if ϕ (resp. ψ) is a subsolution (resp. a supersolution) to $(CMAF)_{X,\omega,\mu,F}(\phi_0)$ then $\phi \le \psi$. It implies that s^* as in Proposition 1.11 is the unique viscosity solution to $(CMAF)_{X,\omega,\mu,F}(\phi_0)$ and that it is continuous. If the (PCP) holds for sub/supersolutions with extra regularity (e.g.: classical, Lipschitz, ...) it implies that there is at most one viscosity solution with this extra regularity.

Comparison Principles are rather elaborate forms of the maximum principle. We believe that (PCP) should hold under condition (1.1) provided there exists a semipositive smooth closed $(1,1)$-form θ of positive volume such that $\omega_t \ge \theta$ for all $t \in [0,T[$. Unfortunately, proving (PCP) is rather technical and we will describe what we have been able to prove it in the next section. It is very encouraging that optimal results in that direction are available in the local case [19].

2. Parabolic comparison principles

Let X be a compact complex manifold of dimension n and ω_t a continuous family of closed real $(1,1)$-forms on X. We consider the complex Monge–Ampère flow on $X_T = [0,T[\times X$ associated to (ω, F, μ),

$$e^{\dot{\varphi}_t + F(t,x,\varphi)}\mu(t,x) - (\omega_t + dd^c\varphi_t)^n = 0, \tag{2.1}$$

according to Definition 1.2.

2.1. Statement of the global parabolic comparison principles

Let φ (resp. ψ) be a bounded subsolution (resp. supersolution) to the parabolic complex Monge–Ampère equation (2.1) in X_T associated to (ω, μ, F). Our goal in this section is to establish several versions of the global comparison principle, starting with the following:

Theorem 2.1. *Assume that $\mu(t,x) > 0$ is positive in X_T and that locally in $]0,T[\times X$ we have the inequality $\partial_t\psi \ge -C$ in the sense of viscosity, for some constant $C > 0$. Then for all $(t,x) \in [0,T[\times X$,*

$$\varphi(t,x) - \psi(t,x) \le \max_{x\in X}(\varphi(0,x) - \psi(0,x))_+.$$

In particular if $\varphi(0,x) \le \psi(0,x)$ in X then $\varphi(t,x) \le \psi(t,x)$ in $[0,T[\times X$.

Observe that as in the local case (see [19, Remark 2.4]), in order to apply the parabolic Jensen–Ishii's maximum principle, we need to assume that the supersolution satisfies a local lower bound on its time derivative even when $\mu > 0$, since our parabolic equation has a structural disymmetry.

The versions of the comparison principle that we will use in the sequel require that one weakens the hypothesis that μ be positive and lift the condition on ψ. It is not clear what is the optimal provable result in this direction. We will state and prove four variants which all require that we strengthen our working hypotheses as follows:

$$\begin{cases} a) \ X \text{ is Kähler}, \\ b) \text{ there exists a semipositive closed } (1,1) \text{ form } \theta \text{ on } X \text{ such that} \\ \quad \omega_t \geq \theta \text{ and } \{\theta\}^n > 0, \\ c) \ (t,x,r) \mapsto F(t,x,r) \text{ is uniformly Lipschitz in the } r \text{ variable}, \\ d) \ (t,x) \mapsto F(t,x,0) \text{ is uniformly bounded above.} \end{cases} \tag{2.1}$$

Since we also require F is non-decreasing in the r variable, the conditions for F are satisfied when $F(t,x,r) = \alpha r$ with $\alpha \geq 0$.

Corollary 2.2. *Assume (2.1) holds. Assume that $\mu(t,x) \geq 0$ in X_T and locally in $]0,T[\times X$, there exists a constant $C > 0$ such that $|\partial_t \varphi| \leq C$ and $\partial_t \psi \geq -C$. Then*

$$\varphi(t,x) - \psi(t,x) \leq \max_{x\in X} (\varphi(0,x) - \psi(0,x))_+,$$

for all $(t,x) \in [0,T[\times X$.

Corollary 2.3. *Assume (2.1) holds. Assume that $\mu(t,x) = \mu(x) \geq 0$ and $t \longmapsto \omega_t = \omega(t,\cdot)$ is constant. Then for all $(t,x) \in [0,T[\times X$,*

$$\varphi(t,x) - \psi(t,x) \leq \max_{x\in X} (\varphi(0,x) - \psi(0,x))_+.$$

The following generalization holds:

Corollary 2.4. *Assume (2.1) holds. Assume that $\mu(t,x) = \mu(x) \geq 0$ and $t \longmapsto \omega_t = \omega(t,\cdot)$ is monotone in t. Then for all $(t,x) \in [0,T[\times X$,*

$$\varphi(t,x) - \psi(t,x) \leq \max_{x\in X} (\varphi(0,x) - \psi(0,x))_+.$$

In order to lift this monotonicity condition, we introduce a slightly technical condition in addition to (2.1).

Definition 2.5. Say $t \mapsto \omega_t$ is regular, if the following holds:

For every positive real constant $\varepsilon > 0$ there exists $E(\varepsilon) > 0$ such that

$$\forall t \in [0, T - 2\varepsilon],\ \forall t' \in]t-\varepsilon, t+\varepsilon[,\ (1 + E(\varepsilon))\omega_t \geq \omega_{t'} \geq (1 - E(\varepsilon))\omega_t$$

and $E(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Corollary 2.6. *Assume that (2.1) holds, $t \mapsto \omega_t$ is regular in the sense of Definition 2.5 and $\mu(t,x) = \mu(x) \geq 0$. Then for all $(t,x) \in [0,T[\times X$,*

$$\varphi(t,x) - \psi(t,x) \leq \max_{x\in X}(\varphi(0,x) - \psi(0,x))_+.$$

Regularity in the sense of Definition 2.5 holds true if, in addition to (2.1), we require θ to be Kähler. It should be remarked that when ω_t is a smooth family of Kähler forms and μ is a smooth positive volume form, the optimal comparison principle holds true and follows from the existence of a classical solution to the Cauchy problem. However, one needs Corollary 2.6 to obtain it by the present methods, the other versions being too weak. On the other hand, the following generalization of this remark covers many cases of interest:

Lemma 2.7. *Let $\pi : X \to Y$ be a bimeromorphic morphism onto a normal Kähler variety. Assume ω_t^Y is a continuous family of* **smooth** *Kähler forms on Y. Then $\omega_t = \pi^*\omega_t^Y$ satisfies (2.1) and is regular in the sense of Definition 2.5.*

2.2. Proofs

We start by proving the Theorem and give the proof of the corollaries afterwards.

Proof. We first establish a slightly more general estimate (2.2) assuming $\mu > 0$ is positive.

Namely let $\mu(t,x) > 0$ and $\nu(t,x) \geq 0$ be two positive continuous volume forms on X_T and $F, G : \mathbb{R}^+ \times X \times \mathbb{R} \longrightarrow \mathbb{R}$ two continuous functions. Let φ be a bounded subsolution to the parabolic complex Monge–Ampère equation (2.1) associated to (ω, F, μ) in X_T and ψ be a bounded supersolution to the parabolic complex Monge–Ampère equation (2.1) associated to (ω, G, ν) in X_T. We assume furthermore that $\partial_t\psi \geq -C$ locally on X_T.

We are going to show that for any fixed $\delta > 0$ small enough, either there exists a point $(\hat{t}, \hat{x}) \in]0,T[\times X$ where the function defined by

$$\tilde{\varphi}(t,x) - \psi(t,x) := \varphi(t,x) - \frac{\delta}{T-t} - \psi(t,x)$$

achieves its maximum on X_T and the following inequality is satisfied

$$e^{\frac{\delta}{(T-\hat{t})^2} + F(\hat{t},\hat{x},\tilde{\varphi}(\hat{t},\hat{x}))}\mu(\hat{t},\hat{x}) \leq e^{G(\hat{t},\hat{x},\psi(\hat{t},\hat{x}))}\nu(\hat{t},\hat{x}), \tag{2.2}$$

or this maximum is achieved at some point $(0,\hat{x})$ on the parabolic boundary. This is a global version of [19, Lemma 3.1].

Choose a large constant $C > 0$ such that φ and ψ are both $\leq C/4$ in L^∞-norm and fix $\delta > 0$ arbitrarily small.

Since $\tilde{\varphi} - \psi$ is upper semicontinuous in $[0, T[\times X$ and tends to $-\infty$ when $t \to T^-$, the maximum of $\varphi - \psi$ is achieved at some point $(t_0, x_0) \in [0, T[\times \Omega$ i.e.

$$M := \sup_{(t,x) \in [0,T[\times X} (\tilde{\varphi}(t,x) - \psi(t,x)) = \tilde{\varphi}(t_0, x_0) - \psi(t_0, x_0)$$

and there exists $T' < T$ such that it cannot be achieved in $[T', T[\times X$ i.e. $t_0 \in [0, T'[\times X$.

If $t_0 = 0$ then we obtain for any $(t,x) \in X_T$,

$$\tilde{\varphi}(t,x) - \psi(t,x) \leq M = \tilde{\varphi}(0, x_0) - \psi(0, x_0) = \max_X (\tilde{\varphi}(0,x) - \psi(0,x)). \tag{2.3}$$

We now focus on the most delicate case when $t_0 \in]0, T'[$ and assume that the maximum M of $\tilde{\varphi}(t,x) - \psi(t,x)$ is not achieved in $\{0\} \times X$, nor in $[T', T[\times X$ i.e.

$$M > \max_{X'_T} \{\tilde{\varphi}(t,x) - \psi(t,x)\}, \text{ where } X'_T := \{0\} \times X \cup [T', T[\times X \tag{2.4}$$

The idea is to localize near the point x_0 and apply the parabolic Jensen–Ishii's maximum principle from [19]. Choose complex coordinates $z = (z^1, \ldots, z^n)$ near x_0 defining a biholomorphism identifying an open neighborhood of x_0 to the complex ball $B_4 := B(0,4) \subset \mathbb{C}^n$ of radius 4, sending x_0 to the origin in $\mathbb{C}^n$.

Observe that $\tilde{\varphi}$ is upper semi-continuous and satisfies, in $X_T =]0, T[\times X$, the viscosity differential inequality

$$e^{\partial_t \tilde{\varphi} + \frac{\delta}{(T-t)^2} + F(t,x,\tilde{\varphi} + \frac{\delta}{T-t})} \mu(t,x) \leq (\omega + dd^c \tilde{\varphi})^n.$$

We let $h(t,x)$ be a continuous local potential for ω such that $\partial_t h$ is continuous in $[0, T[\times B_4$ i.e. $dd^c h = \omega$ in $[0, T[\times B_4$. We may without loss of generality assume that C is chosen so large that $\|h\|_\infty < C/4$.

Consider the upper semi-continuous function

$$\tilde{u}(t, \zeta) := \tilde{\varphi}(t, z^{-1}(\zeta)) + h(t, z^{-1}(\zeta)).$$

Then $\tilde{u}$ satisfies the viscosity differential inequality

$$e^{\partial_t \tilde{u} + \frac{\delta}{(T-t)^2} + \tilde{F}(t,\zeta,\tilde{u})} \tilde{\mu}(t,\zeta) \leq (dd^c \tilde{u})^n, \quad \text{in }]0, T[\times B_4, \tag{2.5}$$

where $\tilde{\mu} := z_*(\mu) \geq 0$ is a continuous volume form on B_4 and

$$\tilde{F}(t, \zeta, r) = F\left(t, x, r - h(t,x) + \frac{\delta}{T-t}\right) - \partial_t h(t,x),$$

where $x := z^{-1}(\zeta)$.

In the same way, the lower semi-continuous function

$$v(t,\zeta) := \psi(t, z^{-1}(\zeta)) + h(t, z^{-1}(\zeta))$$

satisfies the viscosity differential inequality

$$e^{\partial_t v+\tilde{G}(t,z,v)}\tilde{\nu}(\zeta) \geq (dd^c v)^n, \quad \text{in } B_4, \tag{2.6}$$

where $\tilde{\nu} := z_*(\nu) \geq 0$ is a positive and continuous volume form on B_4 and $\tilde{G}(t,\zeta,r) = G(t,x,r-h(t,x)) - \partial_t h(t,x)$, with $x := z^{-1}(\zeta)$.

Observe that the functions $\tilde{F}$ and $\tilde{G}$ are continuous in $[0,T[\times B_4$ since $\partial_t h$ is continuous.

Then we have

$$M = \tilde{u}(t_0,0) - v(t_0,0) = \max_{[0,T']\times\bar{B}_3} (\tilde{u}(t,\zeta) - v(t,\zeta)). \tag{2.7}$$

We are going to estimate the number M by applying the parabolic version of Jensen–Ishii's maximum principle.

As in the local case we use a penalization method [14,19] but we need the localizing trick of [17] which consists in introducing a new localizing penalization function. For $\varepsilon > 0$, we consider the function defined in $[0,T[\times B_4 \times B_4$ by

$$(t,x,y) \longmapsto \tilde{u}(t,x) - v(t,y) - \sigma(x,y) - (1/2\varepsilon)|x-y|^2,$$

where σ is the localizing penalization function constructed in [17]. This is a non-negative smooth function $\sigma(x,y) \geq 0$ in X^2 which vanishes to high order only on the diagonal near the origin $(0,0)$ and is large enough on the boundary of the ball $B_3 \times B_3$ so that $\sigma \geq 3C$ on $\bar{B}_4^2 \setminus B_2^2$, to force the maximum to be attained at an interior point.

The role of the function σ is to force the maximum to be asymptotically attained along the diagonal (as in the degenerate elliptic case, see [17]). The fact that the second derivative of σ is a quadratic form on $\mathbb{R}^{2n}\times\mathbb{R}^{2n}$ which vanishes on the diagonal is going to be crucial in the sequel.

Observe that since $\sigma(0,0) = 0$, we also have

$$M = \max_{[0,T']\times\bar{B}_3} (\tilde{u}(t,\zeta) - v(t,\zeta) - \sigma(\zeta,\zeta). \tag{2.8}$$

Since we are maximizing an upper semi-continuous function on the compact set $[0,T[\times\bar{B}_3^2$, there exists $(t_\varepsilon,x_\varepsilon,y_\varepsilon) \in [0,T[\times\bar{B}_3\times\bar{B}_3$ such that

$$\begin{aligned} M_\varepsilon &:= \sup_{(t,x,y)\in[0,T']\times\bar{B}_3^2} \left\{ \tilde{u}(t,x) - v(t,y) - \sigma(x,y) - \frac{1}{2\varepsilon}|x-y|^2 \right\} \\ &= \tilde{u}(t_\varepsilon,x_\varepsilon) - v(t_\varepsilon,y_\varepsilon) - \sigma(x_\varepsilon,y_\varepsilon) - \frac{1}{2\varepsilon}|x_\varepsilon - y_\varepsilon|^2. \end{aligned}$$

Observe that φ,ψ,h are bounded by $C/4$ in the L^∞-norm in $[0,T[\times\bar{B}_4$, while $\sigma \geq 3C$ on $\bar{B}_4^2 \setminus B_2^2$. Therefore for any ε, we have

$$M_\varepsilon \geq M = \max_{\bar{B}_3}(\tilde{u}(t_0,x) - v(0,x)) \geq -3C/4 - \delta/T. \tag{2.9}$$

On the other hand, for $(t, x, y) \in [0, T[\times B_3^2 \setminus B_2^2$, we have

$$u(t,x) - v(t,y) - \sigma(x,y) - \frac{1}{2\varepsilon}|x-y|^2 \le +C - 3C = -2C. \tag{2.10}$$

Therefore if we assume $0 < \delta < CT/4$, then for any $\varepsilon > 0$ small enough, we have $(t_\varepsilon, x_\varepsilon, y_\varepsilon) \in [0, T'] \times B_2^2$.

The following result is classical (see [14, Proposition 3.7]):

Lemma 2.8. *We have* $|x_\varepsilon - y_\varepsilon|^2 = o(\varepsilon)$. *Every limit point* $(\hat{t}, \hat{x}, \hat{y})$ *of* $(t_\varepsilon, x_\varepsilon, y_\varepsilon)$ *satisfies* $\hat{x} = \hat{y}$, $(\hat{x}, \hat{x}) \in \Delta \cap \bar{B}_2^2$, $\hat{t} \in [0, T]$ *and*

$$\lim_{\varepsilon\to 0} M_\varepsilon = \lim_{\varepsilon\to 0}(\tilde{u}(t_\varepsilon, x_\varepsilon) - v(t_\varepsilon, y_\varepsilon) - \sigma(x_\varepsilon, y_\varepsilon) = \tilde{u}(\hat{t}, \hat{x}) - v(\hat{t}, \hat{x}) - \sigma(\hat{x}, \hat{x}).$$

Moreover $\sigma(\hat{x}, \hat{x}) = 0$ *and* $(\hat{t}, \hat{x}) \in]0, T[\times B_2$.

Proof. Observe that the first part of lemma is a consequence of [14, Proposition 3.7]. To prove the second part we use following easy observation. From the first part of the lemma, using (2.8) and (2.9), we deduce that

$$M \le \lim_{\varepsilon\to 0} M_\varepsilon = \tilde{u}(\hat{t}, \hat{x}) - v(\hat{t}, \hat{x}) - \sigma(\hat{x}, \hat{x}) \le M - \sigma(\hat{x}, \hat{x}),$$

hence $\sigma(\hat{x}, \hat{x}) = 0$. Since by construction $\Delta \cap \sigma^{-1}(0) \subset B_2^2$, it follows that $\hat{x} \in B_2$. □

It follows from (2.4) that $(\hat{t}, \hat{x}) \in]0, T'[\times B_2$, hence it is an interior point of $[0, T'] \times \bar{B}_3^2$. Thus there exists a sequence $(t_{\varepsilon_j}, x_{\varepsilon_j}, y_{\varepsilon_j}) \in]0, T'[\times B_2$ which converges to $(\hat{t}, \hat{x})$ such that the conditions of the Lemma are satisfied.

We now apply the parabolic Jensen–Ishii's maximum principle (see [19]) to u and v with $\phi(t, x, y) = \frac{1}{2\varepsilon}|x-y|^2 + \sigma(x, y)$. For $j >> 1$, we get the following:

Lemma 2.9. *For any* $\gamma > 0$, *we can find* $(\tau_j^+, p_j^+, Q_j^+), (\tau_j^-, p_j^-, Q_j^-) \in \mathbb{R} \times \mathbb{C}^n \times Sym_{\mathbb{R}}^2(\mathbb{C}^n)$ *such that*

(1) $(\tau_j^+, p_j^+, Q_j^+) \in \overline{\mathcal{P}}^{2+}u(t_{\varepsilon_j}, x_{\varepsilon_j})$, $(\tau_j^-, p_j^-, Q_j^-) \in \overline{\mathcal{P}}^{2-}v(t_{\varepsilon_j}, y_{\varepsilon_j})$, *where*

$$p_j^+ = D_x\sigma(x_{\varepsilon_j}, y_{\varepsilon_j}) + \frac{(x_{\varepsilon_j} - y_{\varepsilon_j})}{2\varepsilon_j},$$

$$p_j^- = -D_y\sigma(x_{\varepsilon_j}, y_{\varepsilon_j}) - \frac{(x_{\varepsilon_j} - y_{\varepsilon_j})}{2\varepsilon_j},$$

$$\tau_j^+ = \tau_j^- + \frac{\delta}{(T - t_{\varepsilon_j})^2}.$$

(2) *The block diagonal matrix with entries* (Q_j^+, Q_j^-) *satisfies:*

$$-(\gamma^{-1} + \|A\|)I \leq \begin{pmatrix} Q_j^+ & 0 \\ 0 & -Q_j^- \end{pmatrix} \leq A + \gamma A^2,$$

where $A = D^2\phi(x_{\varepsilon_j}, y_{\varepsilon_j})$, *i.e.*

$$A = \varepsilon_j^{-1} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + D^2\sigma(x_{\varepsilon_j}, y_{\varepsilon_j})$$

and $\|A\|$ *is the spectral radius of* A *(maximum of the absolute values for the eigenvalues of this symmetric matrix).*

Proof. The proof, just like in [19, section 3], consists in applying [14, Theorem 8.3] with $u_1 = u$, $u_2 = -v$. Observe that since the situation is localized near $(\hat{t}, \hat{x})$ in $]0, T'[\times B_2$, the local lower bound on $\partial_t \psi$ and a local lower bound on $\partial_t h$ imply a local lower bound $\partial_t v \geq -C$ near the point $(\hat{t}, \hat{x})$ with a larger constant $C > 1$. This permits to fulfill condition (8.5) on $-v$ when applying [14, Theorem 8.3]. Also, we use $\mu > 0$ to see that condition (8.5) in [14] is satisfied by u. Observe also that $\tau_j^- \geq -C$ for $j >$ large enough (see the Remark following Proposition 1.6 in [19]). □

By construction, the Taylor series of σ at any point in $\Delta \cap \sigma^{-1}(0)$ vanishes up to order $2n + 2$. In particular,

$$D^2\sigma(x_{\varepsilon_j}, y_{\varepsilon_j}) = O(|x_{\varepsilon_j} - y_{\varepsilon_j}|^{2n}) = o(\varepsilon_j^n).$$

This implies $\|A\| \simeq 1/\varepsilon_j$. We choose $\gamma = \varepsilon_j$ and deduce

$$-(2\varepsilon_j^{-1})I \leq \begin{pmatrix} Q_j^+ & 0 \\ 0 & -Q_j^- \end{pmatrix} \leq \frac{3}{\varepsilon_j} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + o(\varepsilon_j^n)$$

Looking at the upper and lower diagonal terms we deduce that the eigenvalues of Q_j^+, Q_j^- are positive and $O(\varepsilon_j^{-1})$. Evaluating the inequality on vectors of the form (Z, Z) we deduce that the eigenvalues of $Q_j^+ - Q_j^-$ are $\leq o(\varepsilon_j^n)$.

For a fixed $Q \in Sym^2_{\mathbb{R}}(\mathbb{C}^n)$, denote by $H = Q^{1,1}$ its $(1,1)$-part. It is a hermitian matrix. Obviously the eigenvalues of $H_j^+ := (Q_j^+)^{1,1}, H_j^- := (Q^-)^{1,1}$ are $O(\varepsilon_j^{-1})$ but those of $H_j^+ - H_j^-$ are $\leq o(\varepsilon_j^n)$. Since $(\tau_j^+, p_j^+, Q_j^+) \in \overline{\mathcal{P}}^{2+}u(t_{\varepsilon_j}, x_{\varepsilon_j})$ we deduce from the viscosity differential inequality satisfied by u that H_j^+ is positive definite and that the product of its n eigenvalues is $\geq c > 0$ uniformly in j (see [19, Theorem 2.5]). In particular its smallest eigenvalue is $\geq c\varepsilon_j^{n-1}$. The relation $H_j^+ + o(\varepsilon_j^n) \leq H_j^-$ forces $H_j^- > 0$ for $j > 1$ large enough and $\det H_j^+ \leq \det H_j^- + o(\varepsilon_j)$.

From the viscosity differential inequalities satisfied by u and v, we deduce that

$$
\begin{aligned}
&e^{\tau_j^- + \frac{\delta}{(T-t_{\varepsilon_j})^2} + \tilde{F}(t_{\varepsilon_j}, x_{\varepsilon_j}, u(t_{\varepsilon_j}, x_{\varepsilon_j}))} \tilde{\mu}(t_{\varepsilon_j}, x_{\varepsilon_j}) \\
&\leq \det H_j^+ \leq \det H_j^- + o(\varepsilon_j) \\
&\leq e^{\tau_j^- + \tilde{G}(t_{\varepsilon_j}, y_{\varepsilon_j}, v(t_{\varepsilon_j}, y_{\varepsilon_j}))} \tilde{\nu}(t_{\varepsilon_j}, y_{\varepsilon_j}) + o(\varepsilon_j).
\end{aligned}
$$

Therefore for $j >$ large enough, we get

$$
e^{\frac{\delta}{(T-t_{\varepsilon_j})^2} + \tilde{F}(t_{\varepsilon_j}, x_{\varepsilon_j}, u(t_{\varepsilon_j}, x_{\varepsilon_j}))} \tilde{\mu}(t_{\varepsilon_j}, x_{\varepsilon_j}) \leq e^{\tilde{G}(t_{\varepsilon_j}, y_{\varepsilon_j}, v(t_{\varepsilon_j}, y_{\varepsilon_j}))} \tilde{\nu}(t_{\varepsilon_j}, y_{\varepsilon_j}) + e^{-C} o(\varepsilon_j).
$$

Then letting $j \to +\infty$, we infer the following (see [19, Lemma 3.1])

$$
e^{\frac{\delta}{(T-\hat{t})^2} + \tilde{F}(\hat{t}, \hat{x}, \tilde{u}(\hat{t}, \hat{x}))} \tilde{\mu}(\hat{t}, \hat{x}) \leq e^{\tilde{G}(\hat{t}, \hat{x}, v(\hat{t}, \hat{x}))} \tilde{\nu}(\hat{t}, \hat{x}). \tag{2.11}
$$

Back to φ and ψ we then get the required inequality (2.2).

If $\nu = \mu > 0$ and $G = F$, then we get

$$
\frac{\delta}{(T-\hat{t})^2} + F(\hat{t}, \hat{x}, \tilde{\varphi}(\hat{t}, \hat{x})) < F(\hat{t}, \hat{x}, \psi(\hat{t}, \hat{x}))
$$

Since F is non-decreasing in the last variable, it follows that

$$
\tilde{\varphi}(\hat{t}, \hat{x}) \leq \psi(\hat{t}, \hat{x}).
$$

Taking into account the inequality (2.3), we conclude that

$$
\varphi(t,x) - \psi(t,x) - \frac{\delta}{T-t} \leq \max_X (\varphi(0,x) - \psi(0,x)_+ - \frac{\delta}{T}.
$$

Letting $\delta \to 0$ we obtain the theorem. □

Proof of Corollary 2.2. We first establish a more general estimate.

Let $\mu(t,x) \geq 0$ and $\nu(t,x) \geq 0$ be two non-negative continuous volume forms on X_T and $F, G : \mathbb{R}^+ \times X \times \mathbb{R} \longrightarrow \mathbb{R}$ two continuous functions.

Assume that φ is a bounded subsolution to the parabolic complex Monge–Ampère equation (2.1) associated to (ω, F, μ) and ψ is a bounded supersolution to the parabolic complex Monge–Ampère equation (2.1) associated to (ω, G, μ) in X_T.

Let θ be as in (2.1.b) and let $\rho < 0$ be a bounded θ-psh function in X satisfying $(\theta + dd^c\rho)^n \geq \lambda_0 > 0$ for a fixed positive volume form λ_0 on X [16]. Fix $\varepsilon \in]0,1[$ and set

$$
\varphi^\varepsilon(t,x) := (1-\varepsilon)\varphi(t,x) + \varepsilon\rho - At,
$$

where $A = A(\varepsilon) > 0$ is a constant to be chosen later so that $A(\varepsilon) \to 0$ as $\varepsilon \to 0^+$. Then

$$
(\omega + dd^c\varphi^\varepsilon)^n \geq (1-\varepsilon)^n(\omega + dd^c\varphi)^n + \varepsilon^n\lambda_0.
$$

Since $\varphi^\varepsilon \leq \varphi + M\varepsilon$, where M is a bound for the L^∞-norm of φ and $\partial_t\varphi \geq -C$, it follows that $\partial_t\varphi^\varepsilon \leq \partial_t\varphi + C\varepsilon$, in the sense of viscosity and then

$$\begin{aligned} e^{\partial_t\varphi^\varepsilon + F(t,x,\varphi^\varepsilon)}\mu(t,x) &\leq e^{\partial_t\varphi - A + \varepsilon C + F(t,x,\varphi) + M\kappa\varepsilon}\mu(t,x), \\ &\leq (1-\varepsilon)^n(\omega + dd^c\varphi)^n, \end{aligned}$$

if we choose $A := \varepsilon C + M\kappa\varepsilon - n\log(1-\varepsilon)$. Here, we used (2.1.c) to introduce κ a uniform Lipschitz constant for F with respect to the variable r.

Therefore

$$(\omega + dd^c\varphi^\varepsilon)^n \geq e^{\partial_t\varphi^\varepsilon + F(t,x,\varphi^\varepsilon)}\mu(t,x) + \varepsilon^n\lambda_0.$$

Observe that since $\partial_t\varphi \leq C$,

$$\partial_t\varphi^\varepsilon + F(t,x,\varphi^\varepsilon) \leq C(1-\varepsilon) + B_0$$

where $B_0 > 0$ exists thanks to (2.1.d), and choosing $\eta := \varepsilon^n e^{-C(1-\varepsilon)-B_0}$ we obtain

$$(\omega + dd^c\varphi^\varepsilon)^n \geq e^{\partial_t\varphi^\varepsilon + F(t,x,\varphi^\varepsilon)}(\mu(t,x) + \eta\lambda_0).$$

Thus φ^ε is a subsolution to the parabolic equation associated to $(\omega, F, \mu(t,x) + \eta\lambda_0)$. Since the volume form $\mu(t,x) + \eta\lambda_0$ is positive, we can apply the inequality (2.2) to conclude that

$$e^{\frac{\delta}{(T-\hat{t}_\varepsilon)^2} + F(\hat{t}_\varepsilon,\hat{x}_\varepsilon,\tilde{\varphi}^\varepsilon(\hat{t}_\varepsilon,\hat{x}_\varepsilon))}(\mu(\hat{t}_\varepsilon,\hat{x}_\varepsilon) + \eta\lambda_0) \leq e^{G(\hat{t}_\varepsilon,\hat{x}_\varepsilon,\psi(\hat{t}_\varepsilon,\hat{x}_\varepsilon))}\nu(\hat{t}_\varepsilon,\hat{x}_\varepsilon), \tag{2.12}$$

when there exists a point $(\hat{t}_\varepsilon, \hat{x}_\varepsilon) \in]0,T[\times X$ where $\tilde{\varphi}^\varepsilon - \psi$ achieves its maximum on X_T. In particular, $\nu(\hat{t}_\varepsilon, \hat{x}_\varepsilon) > 0$.

If moreover $\mu = \nu$, it follows from (2.12) that $\mu(\hat{t}, \hat{x}) > 0$ and then

$$\frac{\delta}{(T-\hat{t})^2} + F(\hat{t},\hat{x},\tilde{\varphi}^\varepsilon(\hat{t},\hat{x})) \leq G(\hat{t},\hat{x},\psi(\hat{t},\hat{x})). \tag{2.13}$$

If moreover $F = G$, we conclude as before that

$$\tilde{\varphi}^\varepsilon - \psi \leq \max_X(\tilde{\varphi}_0^\varepsilon - \psi_0)_+ \leq \max_X(\varphi_0 - \psi_0)_+.$$

Letting $\delta \to 0$ and then $\varepsilon \to 0$, we obtain the required inequality $\varphi - \psi \leq \max_X(\varphi_0 - \psi_0)_+$. $\square$

Proof of Corollary 2.3. Here we assume that the forms ω do not depend on the time variable. We will try and do the proof when μ depends on t in order to stress the role of the hypothesis that μ is time independent.

We are going to regularize in the time variable to reduce to the previous case. Let (φ^k) be the upper Lipschitz regularization of φ and (ψ_k) the lower Lipschitz regularization of ψ in the variable t [19, Lemma 2.5]. Recall that

$$\varphi^k(t,x) := \sup\{\varphi(s,x) - k|s-t|, s \in [0,T[\},$$
$$\psi_k(t,x) := \inf\{\psi(s,x) + k|s-t|, s \in [0,T[\}.$$

Since ω does not depend on the time variable, if follows that for each $k > 1$, φ^k is a subsolution to the parabolic equation associated to (ω, F_k, μ_k) and ψ_k is a supersolution to the parabolic equation associated to (ω, F^k, μ^k) (see [19, Lemma 2.5]). Recall that

$$\begin{aligned}
F^k(t,x,r) &:= \sup\{F(s,x,r) - k|s-t|; s \in [0,T[,\ |s-t| \leq \alpha/k\},\\
F_k(t,x,r) &:= \inf\{F(s,x,r) + k|s-t|; s \in [0,T[,\ |s-t| \leq \alpha/k\},\\
\mu_k(t,x) &:= \inf\{\mu(s,x); |s-t| \leq \alpha/k\},\\
\mu^k(t,x) &:= \sup\{\mu(s,x); |s-t| \leq \alpha/k\},
\end{aligned}$$

for some $\alpha > 0$.

As in the proof of Corollary 2.2 define, for $0 < \varepsilon < 1$, $\varphi^{k,\varepsilon}(t,x) := (1-\varepsilon)\varphi^k(t,x) + \varepsilon\rho(x) - A_k(\epsilon)t$ and $\tilde{\varphi}^{k,\epsilon} := \varphi^{k,\varepsilon} - \frac{\delta}{T-t}$. Then we can apply the inequality (2.12) in the proof of Corollary 2.2 to deduce that:

$$e^{\frac{\delta}{T^2} + F_k(\hat{t},\hat{x},\tilde{\varphi}^{k,\varepsilon}(\hat{t},\hat{x}))}(\mu_k + \eta\lambda_0) \leq e^{F^k(\hat{t},\hat{x},\psi_k(\hat{t},\hat{x}))}\mu^k, \tag{2.14}$$

where $(\hat{t},\hat{x}) = (\hat{t}_{\delta,k,\varepsilon}, \hat{x}_{\delta,k,\varepsilon}) \in]0,T[\times X$ is a point where $\tilde{\varphi}^{k,\varepsilon} - \psi_k$ achieves its maximum on X_T.

By construction $\hat{t} \leq T_\delta < T$ where T_δ does not depend on k, ϵ. Since $F_k, F^k \to F$ locally uniformly and $\mu_k = \mu^k = \mu$,[1] for k large enough we get

$$\frac{\delta}{2T^2} + F(\hat{t},\hat{x},\tilde{\varphi}^{k,\varepsilon}(\hat{t},\hat{x})) \leq F(\hat{t},\hat{x},\psi_k(\hat{t},\hat{x})).$$

Since F is non-decreasing in the last variable, it follows that for $k > 1$ large enough and for all $0 < \epsilon < 1$,

$$\tilde{\varphi}^{k,\varepsilon}(\hat{t},\hat{x}) < \psi_k(\hat{t},\hat{x}).$$

Therefore we get

$$\max_{X_T}(\tilde{\varphi}^{k,\varepsilon} - \psi_k) \leq \max_X(\tilde{\varphi}^{k,\varepsilon}(0,\cdot) - \psi_k(0,\cdot))_+ \leq \max_X(\tilde{\varphi}^k(0,\cdot) - \psi_k(0,\cdot))_+$$

[1] Here we use the hypothesis that μ does not depend on t. Without this hypothesis an error term $\log(\frac{\mu^k}{\mu_k+\eta_k(\varepsilon)\lambda_0})$ appears that may diverge to $+\infty$ when $\epsilon \to 0$, k being kept fixed, say if $\mu(t,x) = 0$ for $t \leq \hat{t}$ but $\mu(t,x) > 0$ for $t > \hat{t}$.

First let $\epsilon \to 0$. It follows that:

$$\max_{X_T}(\tilde{\varphi}^k - \psi_k) \leq \max_X(\tilde{\varphi}^k(0,\cdot) - \psi_k(0,\cdot))_+$$

Now we let $k \to +\infty$ and use Dini–Cartan's lemma to conclude that $\max_{X_T}(\tilde{\varphi} - \psi) \leq \max_X(\tilde{\varphi}(0,\cdot) - \psi(0,\cdot))_+$, which implies the required estimate as $\delta \to 0$. □

Proof of Corollary 2.4. Now assume that ω depends on t. Then from the proof of Lemma 2.5 in [19], we see that for any $(t_0, x_0) \in]0, T[\times X$ there exists $t_0^* \in]t_0 - \alpha/k, t_0 + \alpha/k[$ such that φ^k satisfies the viscosity inequality

$$(\omega(t_0^*, x_0) + dd^c\varphi^k(t_0, x_0))^n \geq e^{\partial_t\varphi^k(t_0,x_0)+F(t_0^*,x_0,\varphi^k(t_0,x_0))}\mu(x_0),$$

where $\alpha > 0$ is a constant.

Now assume that $t \longmapsto \omega(t,\cdot)$ is non-decreasing. Then for $k > 1$ large enough, $(\omega(t_0^*, x_0) \leq (\omega(t_0 + \alpha/k, x_0)$ and then the function $u^k(t,x) := \varphi^k(t - \alpha/k, x)$ is a subsolution to the parabolic equation associated to $(\omega, \hat{F}_k, \mu)$ in $]\alpha/k, T[\times X$, where

$$\hat{F}_k(t,x,r) := F_k(t - \alpha/k, x, r).$$

In the same way, we see that the function $v_k := \psi_k(t + \alpha/k, x)$ is a supersolution to the parabolic equation associated $(\omega, \hat{F}^k, \mu)$ in $]0, T[\times X$, where

$$\hat{F}^k(t,x,r) := F^k(t + \alpha/k, x, r).$$

Then one easily modifies the proof of Corollary 2.3, with u^k replacing ϕ^k and u_k replacing ψ_k.

It is clear that the same argument works in the non-increasing case. □

Proof of Corollary 2.6. By Lemma 2.5 in [19], φ^k is a subsolution of the equation associated to $((1 + E(\alpha/k))\omega_t, F_k, \mu)$ whereas ψ_k is a supersolution of the equation associated to $((1 - E(\alpha/k))\omega_t, F_k, \mu)$ with $\alpha > 0$ as above. Hence $\varphi_\star^k = \frac{\varphi^k}{1+E(\alpha/k)}$ is a subsolution of the equation to $(\omega_t, F_k - \log(1 + E(\alpha/k)), \mu)$ and $\psi_{\star k}$ is a supersolution of the equation associated to $(\omega_t, F^k + \log(1 + E(\alpha/k)), \mu)$. We can now argue exactly as in the proof of Corollary 2.3, with $\varphi_\star^k$ replacing φ^k and $\psi_{\star k}$ replacing ψ_k. □

Remark 2.10. Renormalization in the time variable leads to twisted parabolic complex Monge–Ampère equation of the type

$$e^{h(t)\partial\varphi_t+F(t,\cdot,\varphi)}\mu - (\omega_t + dd^c\varphi_t)^n = 0 \tag{2.15}$$

in $[0, T[\times X$, where $h : [0, T[\longrightarrow]0, +\infty[$ is a continuous positive function.

The comparison principle Theorem 2.1 holds for the twisted parabolic complex Monge–Ampère equation (2.15) as in the local case (see [19]).

3. Barrier constructions

Let X be a compact Kähler manifold of dimension n and ω_0 is semipositive closed $(1,1)$ form with positive volume. We consider in this section the Cauchy problem on X_T

$$\begin{cases} e^{\partial_t \varphi + \alpha\ \varphi}\mu - (\omega_t + dd^c \varphi_t)^n = 0 \\ \varphi(0,x) = \varphi_0(x), \quad (0,x) \in \{0\} \times X, \end{cases} \tag{3.1}$$

where φ_0 is a given continuous ω_0-plurisubharmonic function on X and $\alpha \in \mathbb{R}^+$.

The Cauchy problem does not necessarily admit a solution when μ vanishes identically on an open set (see Proposition 3.7). We first treat the case when $\mu > 0$ is positive, and then allow μ to vanish along pluripolar sets. This latter setting contains as a particular case the Kähler–Ricci flow on varieties with canonical singularities.

We will mainly focus on the case $\alpha = 0$. The case $\alpha > 0$ is actually easier and can be reduced to the previous one by a change of time variable. We also need to impose some uniformity in the positivity properties of the forms we are dealing with:

We assume in the whole section that X is a compact Kähler manifold of dimension n and there exists a closed real $(1,1)$-form θ on X whose cohomology class is semi-positive and a Kähler form Θ such that for all $0 \le t \le T$, the background continuous family of closed $(1,1)$-forms satisfies:

$$\theta \le \omega_t \le \Theta. \tag{3.2}$$

3.1. Existence of sub/supersolutions

Lemma 3.1. *The Cauchy problem (3.1) admits a continuous subsolution $\underline{u}$, Lipschitz in the variable t.*

Assume $\mu(t,x) \ge f_0(x)dV$, where $f_0 \ge 0$ is a continuous density such that

$$\int_X f_0\, dV > 0. \tag{†}$$

Then, there exists a continuous supersolution $\bar{v}$, Lipschitz in the variable t.

Moreover we can choose these so that $\underline{u} \le \bar{v}$ in $[0,T[\times X$.

Proof. By [16], there exists a continuous θ-psh function ρ_1 in X such that $(\theta + dd^c\rho_1)^n = c_1 dV$ in the weak sense on X, where c_1 is a normalizing constant. We can normalize ρ_1 so that $\rho_1 \le \varphi_0$ in X. Define for $C_1 > 0$, the function

$$\underline{u} := -C_1 t + \rho_1(x).$$

Then, by Lemma 1.4, if $C_1 >> 1$ is chosen so large that $e^{-C_1} \sup_{X_T} \mu \le c_1 dV$, the function $\underline{u}$ is a subsolution to the Cauchy problem (3.1).

In the same way we construct a supersolution. Since $f_0 \geq 0$ is a bounded upper semi-continuous function on X and $\int_X f_0 dV > 0$, there exists a continuous Θ-psh ρ_2 satisfying

$$(\Theta + dd^c \rho_2)^n = c_2 f_0(x) \lambda_0$$

in the weak sense on X, where c_2 is a normalizing constant (by [27,17]). We normalize ρ_2 so that $\rho_2 \geq \varphi_0$ in X. Consider the function

$$\bar{v} := +C_2 t + \rho_2,$$

where $C_2 > -\log c_2$ is a positive constant.

Lemma 1.4 implies then that $\bar{v}$ is also a supersolution to the parabolic complex Monge–Ampère equation (3.1). Since $\bar{v} \geq \varphi_0$ in X we obtain a continuous supersolution to the Cauchy problem (3.1). □

Corollary 3.2. *Assume either $\mu > 0$ or the hypotheses of Corollaries 2.4 or 2.6 are satisfied in addition to those of Lemma 3.1(†). Then the Cauchy problem (3.1) is admissible.*

Fix $\underline{u}, \bar{v}$ a subsolution and a supersolution of the Cauchy problem (3.1). We are now in the position to apply Proposition 1.11. The natural candidate to be a solution is the upper envelope of subsolutions

$$\varphi := \sup\{u \mid u \in \mathcal{S}, \underline{u} \leq \psi \leq \bar{v}\}, \tag{3.3}$$

where $\mathcal{S}$ denotes the family of all subsolutions to the Cauchy problem (3.1). We let φ^* denote the upper semi-continuous regularization of φ and φ_* denote its lower semi-continuous regularization. It follows that:

Corollary 3.3. *Assume the hypotheses of Corollaries 2.4 or 2.6 are satisfied in addition to those of Lemma 3.1(†).*

The upper semi-continuous regularization φ^* is a discontinuous viscosity solution to the underlying parabolic Monge–Ampère equation in $]0, T[\times X$.

The lower semi-continuous regularization φ_* is thus a supersolution to the parabolic Monge–Ampère equation in $]0, T[\times X$ and they satisfy then for all $(t, x) \in]0, T[\times X$,

$$\varphi^*(t, x) - \varphi_*(t, x) \leq \max_{x \in X}(\varphi^*(0, x) - \varphi_*(0, x)). \tag{3.4}$$

If we could make sure that $\varphi^* \leq \varphi_0 \leq \varphi_*$ on the parabolic boundary $\{0\} \times X$, it would follow from the inequality (3.4) that $\varphi^* = \varphi_* = \varphi$ is a unique viscosity solution of the Cauchy problem. Establishing this classically requires the construction of barriers at each boundary point in $\{0\} \times X$.

3.2. Existence of barriers

Definition 3.4. Fix $(0, x_0) \in \{0\} \times X$ and $\varepsilon \geq 0$.

1. An upper semi-continuous function $u : X_T \longrightarrow \mathbb{R}$ is an ε-subbarrier to the Cauchy problem (3.1) at the boundary point $(0, x_0)$, if

- u is a subsolution to the Monge–Ampère flow (3.1) in $]0, T[\times X$,
- $u(0, \cdot) \leq \varphi_0$ in X,
- $u_*(0, x_0) \geq \varphi_0(x_0) - \varepsilon$.

When $\varepsilon = 0$, u is called a subbarrier.

2. A lower semi-continuous function $v : X_T \longrightarrow \mathbb{R}$ is an ε-superbarrier to the Cauchy problem (3.1) at the boundary point $(0, x_0)$, if

- v is a supersolution to the Monge–Ampère flow (3.1) in $]0, T[\times X$,
- $v(0, \cdot) \geq \varphi_0$ in X,
- $v^*(0, x_0) \leq \varphi_0(x_0) + \varepsilon$.

When $\varepsilon = 0$ v is called a superbarrier.

We now investigate the existence of sub/super-barriers.

Proposition 3.5.

1. Assume $\omega_0 \leq \omega_t$ and fix $\varepsilon > 0$. There exists a continuous function U_ε in $X_T := [0, T[\times X$, Lipschitz in t which is an ε-subbarrier to the Cauchy problem (3.1) at any point $(0, x_0) \in \{0\} \times X$.

2. Assume $\mu(t, x) > 0$ in X_T and fix $\varepsilon > 0$. There exists a continuous function V_ε in X_T, Lipschitz in t, which is a ε-superbarrier to the Cauchy problem (3.1) at any point $(0, x_0) \in \{0\} \times X$.

As the proof will show one can moreover impose that for all $(t, x) \in X_T$,

$$-C_1 t + \rho_1(x) \leq U_\varepsilon(t, x) \leq V_\varepsilon(t, x) \leq C_2 t + \rho_2(x),$$

where C_1, ρ_1, C_2, ρ_2 are independent of ε and given in Lemma 3.1.

Proof of Proposition 3.5. 1. By [17], since μ is continuous, there exists w_0 a continuous θ-psh function on X such that $(\theta + dd^c w_0)^n \geq e^{w_0}\mu$. Adding a negative constant we can always assume that $w_0 \leq \varphi_0$ in X.

Fix $\varepsilon > 0$, $\eta = \eta_\varepsilon > 0$, $C = C_\varepsilon > 0$ (to be chosen below) and set

$$u(t, x) := (1 - \eta)\varphi_0(x) + \eta w_0(x) - Ct, \; (t, x) \in X_T.$$

This is a continuous function in X_T such that for any $t \in [0, T[$. Since $\omega_0 \le \omega_t$, u_t is ω_t-psh in the space variable $x \in X$ and satisfies the differential inequalities

$$(\omega_t + dd^c u_t)^n \ge \eta^n (\theta + dd^c w_0)^n \ge \eta^n e^{w_0} \mu \text{ on } X,$$

while $\partial_t u = -C$ in X_T. We choose $C = C(\eta) > 1$ large enough so that $\eta^n e^{w_0} \ge e^{-C}$, hence for each $t \in]0, +T[$ we have

$$(\omega_t + dd^c u_t)^n \ge e^{\partial_t u(t,\cdot)} \mu$$

Note that $u_0 = \varphi_0 + \eta(w_0 - \varphi_0) \le \varphi_0$ in X. We can choose $\eta > 0$ so small that $\eta \sup_X (\varphi_0 - w_0) \le \varepsilon$ and Lemma 1.4 enables to conclude that u is an ε-subbarrier for the Cauchy problem (3.1) at any point $(0, x_0)$.

We can moreover use Lemma 3.1 to find a bounded subsolution $-C_1 t + \rho_1$ to the Cauchy problem (3.1) which is independent of ε. Set for $(t, x) \in X_T$,

$$U_\varepsilon(t, x) := \sup\{u(t, x), -C_1 t + \rho_1\}.$$

The function U is also an ε-subbarrier to the Cauchy problem (3.1) at any boundary point $(0, x_0) \in \{0\} \times X$.

2. Constructing superbarriers. Fix $\varepsilon > 0$. Since Θ is Kähler and φ_0 is in particular a Θ-psh function in X (recall that $\omega_0 \le \Theta$), there exists a C^∞-smooth Θ-psh function $\tilde{\varphi}_0$ in X such that $\varphi_0 \le \tilde{\varphi}_0 \le \varphi_0 + \varepsilon$ in X (see [15,6]). Thus there is a constant $C > 0$ such that

$$(\Theta + dd^c \tilde{\varphi}_0)^n \le e^C \mu$$

pointwise on X, as we are assuming $\mu > 0$.

Set $v(t, x) := \tilde{\varphi}_0(x) + Ct$ in X_T and observe that

$$(\Theta + dd^c v_t)^n = (\Theta + dd^c \tilde{\varphi}_0)^n \le e^C \mu \le e^{\partial_t v} \mu.$$

Since $\omega_t \le \Theta$ we infer that v_t also satisfies, in the viscosity sense:

$$(\omega_t + dd^c v_t)^n \le e^C \mu.$$

Therefore v is a continuous ε-superbarrier to the Cauchy problem (3.1) at any boundary point in $\{0\} \times X$.

Using Lemma 3.1 and the condition $\Theta \ge \omega_t$, we moreover obtain a supersolution $\rho_2 + C_2 t$ to the Cauchy problem (3.1) and set for $(t, x) \in X_T$,

$$V_\varepsilon(t, x) := \inf\{v(t, x), \rho_2(x) + C_2 t\}\cdot$$

This V is an ε-superbarrier to the Cauchy problem (3.1) at any boundary point $(0, x_0) \in \{0\} \times X$. $\square$

Remark 3.6.

1. If the Cauchy data φ_0 is a continuous θ-psh function on X satisfying $(\theta + dd^c\varphi_0)^n \geq e^{\varphi_0}\mu$, then we can take $w_0 = \varphi_0$ in the above construction of subbarriers. The corresponding function U is then a bounded continuous subsolution, which is uniformly Lipschitz in t and satisfies $U(0,\cdot) = \varphi_0$, i.e. U is a subbarrier to the Cauchy problem (3.1).

2. If the Cauchy data φ_0 is a continuous Θ-psh function on X such that $(\Theta + dd^c\varphi_0)^n$ has an L^∞-density, then we can take $\tilde{\varphi}_0 = \varphi_0$ and $\varepsilon = 0$ in the above construction of superbarriers. We thus obtain a bounded continuous supersolution V which is uniformly Lipschitz in t and such that $V(0,\cdot) = \varphi_0$ in X, i.e. V is a superbarrier to the Cauchy problem (3.1).

3.3. Non-negative densities

We explain in this section a non-existence result: when μ vanishes on an open set, there is no solution unless the initial data has special properties.

Proposition 3.7. *Assume that $\mu = f dV$, where $f \geq 0$ vanishes identically on $D \times [0,\delta]$, where $D \subset X$ is open.*

If the initial data φ_0 is not a maximal ω-psh function in D, then the Cauchy problem (3.1) has no viscosity solution.

Recall that a continuous ω-psh function u is *maximal* in D if it satisfies the homogeneous complex Monge–Ampère equation $(\omega + dd^c u)^n = 0$ there.

Proof of Proposition 3.7. Assume that the Cauchy problem (3.1) with initial data φ_0 has a solution φ in $[0,\delta] \times X$. Since $\mu = 0$ in $[0,\delta] \times D$, it follows that φ is a solution to the degenerate parabolic equation $(\omega_t + dd^c\varphi_t)^n = 0$ in $D \times [0,\delta]$.

We claim that for almost every $t > 0$, the function φ_t is a continuous ω_t-psh function on X, which is a viscosity solution of the elliptic equation

$$(\omega_t + dd^c\varphi_t)^n = 0.$$

This is clear if φ is a classical solution. To treat the general case we use inf convolution to approximate φ by an increasing sequence (φ_j) of semi-concave functions which satisfy the same equation on a slightly smaller domain that we still denote by $[0,\delta] \times D$ for simplicity. The functions φ_j admit a $(1,2)$-Taylor expansion almost everywhere, hence for a.e. (t,x),

$$(\omega + dd^c\varphi_j(t,x))^n = 0.$$

Fixing one such t, it follows that for almost every x,

$$(\omega + dd^c\varphi_j(t,\cdot))^n = 0.$$

It follows that the latter actually holds everywhere in D in the viscosity sense (see [19]).

Since φ_j increases to φ, it follows from the continuity of the complex Monge–Ampère operator along monotone sequences that for almost every t the function φ_t satisfies $(\omega + dd^c\varphi(t,\cdot))^n = 0$ in D.

Note finally that $\varphi_t \to \varphi_0$ uniformly, hence φ_0 is maximal in D. $\square$

3.4. Canonical vanishing: existence of solutions

We now restrict our attention to semi-positive measures

$$\mu(x,t) = e^{u(x)} f(x,t) dV(x),$$

where $f > 0$ is a positive continuous density and u is quasi-plurisubharmonic function that is *exponentially continuous* (i.e. such that e^u is continuous). The measure μ is thus allowed to vanish only along the closed pluripolar set $(u = -\infty)$, in a time independent fashion.

Lemma 3.8. *For any $\varepsilon > 0$ there exists a lower semi-continuous function $w : [0,T[\times X \longrightarrow \mathbb{R}$, which is an ε-superbarrier to the Cauchy problem (3.1) at any boundary point $(0,x_0)$ with $u(x_0) > -\infty$.*

Proof. We can assume without loss of generality that $u \leq 0$ is a Θ-psh function on X. Fix $\varepsilon > 0$. From the approximation theorem of Demailly (see [15,6]), it follows that there exists a smooth Θ-psh function $\tilde{\varphi}_0$ in X such that $\varphi_0 \leq \tilde{\varphi}_0 \leq \varphi_0 + \varepsilon$ in X. Set

$$v(t,x) := \tilde{\varphi}_0 - tu + Ct, \ (t,x) \in [0,T[\times\Omega,$$

where $\Omega := \{x \in X \,|\, u(x) > -\infty\}$ is open and $C > 0$ is a constant to be chosen later. Observe that v is continuous in $[0,T[\times\Omega$ and satisfies

$$\Theta + dd^c v_t = 2\Theta + dd^c\tilde{\varphi}_0 - t(dd^c u + \Theta) + (t-1)\Theta,$$

in the sense of currents in Ω. Since $dd^c u + \Theta \geq 0$, for $0 < t \leq T$, we have

$$\Theta + dd^c v_t \leq 2\Theta + dd^c\tilde{\varphi}_0$$

in the sense of currents in Ω.

We choose $C > 1$ so big that $(2\Theta + dd^c\tilde{\varphi}_0)^n \leq e^C dV$. This we can do since $2\Theta + dd^c\tilde{\varphi}_0$ is a smooth positive form on X.

Note that $e^{\partial_t v} = e^{C-u}$ thus it follows from Lemma 3.9 that v satisfies the viscosity parabolic differential inequality $(\Theta + dd^c v_t)^n \leq e^{\partial_t v}\mu$ in $[0,T[\times\Omega$. As $\omega_t \leq \Theta$, Lemma 3.9 also implies that v is a supersolution to the parabolic Monge–Ampère equation (3.1) in $[0,T[\times\Omega$.

On the other hand we know that there exists a (continuous) supersolution $\bar{v}$ to the parabolic Monge–Ampère equation (3.1) in $\mathbb{R}^+ \times X$ such that $\bar{v}_0 \geq \varphi_0$ in X. The function $w := \inf\{v, \bar{v}\}$ is continuous on $[0, T[\times X \setminus \{u = -\infty\}$ and uniformly Lipschitz in t. It can thus be extended as a lower semi-continuous function on $[0, T[\times X$ by setting

$$w(0, x_0) := \inf\{\tilde{\varphi}_0(x_0), \bar{v}_0(x_0)\}$$

for any point $(0, x_0)$ with $u(x_0) = -\infty$. We let the reader check that this extension, which we still denote by w, is a supersolution to the parabolic Monge–Ampère equation (3.1) in $]0, T[\times X$ such that $\varphi_0 \leq w_0 \leq \varphi_0 + \varepsilon$ in X.

Fix a point $x_0 \in X$ such that $u(x_0) > -\infty$. Then $w^*(0, x_0) = w(0, x_0) \leq \varphi_0(x_0) + \varepsilon$, hence w is an ε-superbarrier at such a point. □

In the proof above, we have used the following technical result:

Lemma 3.9. *Let $\mu \geq 0$ be a continuous volume form on some domain D. Let ψ be a bounded lower semi-continuous function in $D \subset X$ and ρ a C^2-smooth function in D such that $dd^c\psi \leq dd^c\rho$ in the sense of currents. Then $(dd^c\psi)^n \leq (dd^c\rho)_+^n$ in the viscosity sense in D.*

If Θ_1 and Θ_2 are smooth closed real $(1, 1)$-forms in X such that $\Theta_1 \leq \Theta_2$ and $(\Theta_2 + dd^c\psi)^n \leq \mu$ in the viscosity sense, then $(\Theta_1 + dd^c\psi)^n \leq \mu$ in the viscosity sense.

Recall that $(dd^c\rho)_+$ is the $(1, 1)$-form defined pointwise by $(dd^c\rho)_+(x_0) := dd^c\rho(x_0)$ if $dd^c\rho(x_0) \geq 0$ and 0 otherwise.

Proof of Lemma 3.9. If q a C^2 lower test function for ψ at a point $x_0 \in D$, i.e. $q \leq_{x_0} \psi$, then $\rho - \psi \leq_{x_0} \rho - q$. Since $dd^c\psi \leq dd^c\rho$, it follows that $\rho - \psi$ is plurisubharmonic in D. Hence $dd^c(\rho - q)(x_0) \geq 0$, i.e. $dd^c\rho(x_0) \geq dd^c q(x_0)$. If $dd^c q(x_0) \geq 0$ it follows that $dd^c\rho(x_0) \geq 0$ and $(dd^c q(x_0))^n \leq (dd^c\rho(x_0))^n$. This proves the first statement.

The proof of the second statement goes along the same lines. □

Definition 3.10. Say $t \mapsto \omega_t$ is very regular if it is regular in the sense of Definition 2.5 and there exists $\eta > 0$, a function of class C^1 $\epsilon : [0, T[\to [0, 1 - \eta]$ such that $\epsilon(0) = 0$ and $\omega_t \geq (1 - \epsilon(t))\omega_0$.

As we will see in the next section, this condition is satisfied in many geometric situations and the following result will be important for our applications.

Theorem 3.11. *Assume that $\mu = e^u f dV$ is as above and $t \mapsto \omega_t$ is non-decreasing or is very regular in the sense of Definition 3.10. Then the maximal subsolution φ constructed in Proposition 1.11 is a unique viscosity solution to the Cauchy problem (3.1).*

Proof. We first assume $t \mapsto \omega_t$ is non-decreasing. By Proposition 3.5, given $\varepsilon > 0$ there exists a continuous ε-subbarrier U at any point $(0, x_0) \in \{0\} \times X$ i.e. $U \leq \varphi$ and

$U(0,x_0) \geq \varphi_0(x_0) - \varepsilon$. Since U is continuous, it follows that $U \leq \varphi_*$ in $\mathbb{R}^+ \times X$, hence $\varphi_*(0,x_0) \geq \varphi_0(x_0)$ for any $x_0 \in X$. This shows that φ_* is a supersolution to the Cauchy problem (3.1).

We claim that $\varphi^*(0,\cdot) \leq \varphi_0$ in X. Indeed if we fix $\varepsilon > 0$, by Lemma 3.8 there exists an ε-superbarrier w to the Cauchy problem (3.1) in $[0,T[\times X$ at any point $(0,x_0)$ with $u(x_0) > -\infty$ and which is uniformly Lipschitz in t.

Since w is a supersolution to the Cauchy problem (3.1) in $[0,T[\times X$, it follows from the comparison principle (Corollary 2.6) and the continuity of w that $\varphi \leq w$ in $]0,T[\times X$. Since w is continuous up to the boundary,

$$\varphi^*(0,x_0) \leq w(0,x_0) \leq \varphi_0(x_0) + \varepsilon$$

for any $x_0 \in X$ with $u(x_0) > -\infty$.

Therefore $\varphi^*(0,\cdot) \leq \varphi_0$ almost everywhere in X, since the set $\{u = -\infty\}$ has Lebesgue measure 0. Since the slice function $\varphi^*(t,\cdot)$ is ω_t-plurisubharmonic for all $t > 0$ [19, Theorem 2.5], and φ^* is upper semicontinuous on $[0,T[\times X$ it follows that $\varphi^*(0,\cdot)$ is ω_0-plurisubharmonic. Hence $\varphi^*(0,x) \leq \varphi_0(x)$ for all $x \in X$.

We have shown that $\varphi^*(0,\cdot) \leq \varphi_0 \leq \varphi^*$. It follows therefore from Corollary 3.3 that $\varphi^* = \varphi_* = \varphi = \psi$ in $[0,T[\times X$ is the unique solution to the Cauchy problem (3.1).

Definition 3.10 is an ad hoc definition whose only virtue is to allow the construction of a subbarrier in Proposition 3.5 be carried out by:

$$u(t,x) := (1 - \eta - \epsilon(t))\phi_0(x) + \eta w(x) - Ct.$$

The superbarrier construction is completely insensitive to this difficulty and the theorem follows. □

3.5. Comparison with the vanishing viscosity method

In this section we consider the following ε-perturbation of Cauchy problem (3.1) on X_T with canonical vanishing given by a quasi-plurisubharmonic function w:

$$\begin{cases} e^{\partial_t \varphi + \alpha\varphi} e^{w} f dV - (\varepsilon\Theta + \omega_t + dd^c\varphi_t)^n = 0 \\ \varphi(0,x) = \varphi_0(x), \quad (0,x) \in \{0\} \times X, \end{cases}$$

where φ_0 is a given continuous ω_0-plurisubharmonic function on X.

Here $\varepsilon \geq 0$ is a non-negative constant and Θ is a Kähler form. Then, if $t \mapsto \omega_t$ is very regular, $t \mapsto \varepsilon\Theta + \omega_t$ is very regular too. In particular, Theorem 3.11 applies and for every $\varepsilon \geq 0$ we have a viscosity solution $\phi(\varepsilon)$ of the above ε-perturbed complex Monge–Ampère flow.

Proposition 3.12. *$\phi(\varepsilon)$ converges locally uniformly to $\phi(0)$ in $\mathbb{R}^+ \times X$ as $\varepsilon \to 0$.*

Proof. Since $\phi(\varepsilon')$ is a supersolution of ε-perturbed complex Monge–Ampère flow whenever $\varepsilon' \geq \varepsilon$, the comparison principle implies that

$$\phi(0) \leq \phi(\varepsilon) \leq \phi(\varepsilon') \text{ if } 0 \leq \varepsilon \leq \varepsilon'.$$

Using [14, section 6] (see also [19, Lemma 1.7]) we conclude with the comparison principle for ε-perturbed complex Monge–Ampère flows. □

Remark 3.13. One could also perturb μ to a smooth positive volume form.

4. Applications

In this section we show that our hypotheses are satisfied when studying the (normalized) Kähler–Ricci flow on a variety with canonical singularities. We prove the existence and study the behavior of the normalized Kähler–Ricci flow (NKRF for short) on such varieties starting from an arbitrary closed positive current with continuous potential.

4.1. The normalized Kähler–Ricci flow on varieties with canonical singularities

Let Y be an irreducible compact Kähler normal complex analytic space with only canonical singularities. Let χ_0 be a Kähler form on Y. We study the existence of the normalized Kähler–Ricci flow on Y,

$$\frac{\partial \omega_t}{\partial t} = -\mathrm{Ric}(\omega_t) - \omega_t,$$

starting from an initial data $\omega_0 = \chi_0 + dd^c\phi_0$ with ϕ_0 being a continuous potential which is plurisubharmonic with respect to the given Kähler form χ_0 on Y. At the cohomological level, this yields a first order ODE showing that the cohomology class of ω_t evolves as

$$\{\omega_t\} = e^{-t}\{\omega_0\} + (1 - e^{-t})K_Y.$$

We thus defined by

$$T_{max} := \sup\{t > 0,\ e^{-t}\{\omega_0\} + (1 - e^{-t})K_Y \in \mathcal{K}(Y)\}$$

the maximal time of existence of the flow.

Recall that given a Kähler class on Y with a smooth positive representative χ_0 and $\phi_0 \in PSH(Y, \chi_0)$ a continuous function, the Cauchy problem with initial data $S_0 := \chi_0 + dd^c\phi_0$ for the normalized Kähler–Ricci flow is defined after a desingularization $\pi : X \to Y$ as the Cauchy problem with initial data $\varphi_0 := \pi^*\phi_0$ for the flow $(CMAF)_{X,\omega_{NKRF},\mu_{NKRF},r}$ (see Definition 1.7). We prove the following general version of Tian–Zhang's existence theorem for the Kähler–Ricci flow:

Theorem 4.1. *The Cauchy problem with initial data $S_0 := \chi_0 + dd^c\phi_0$ for the normalized Kähler–Ricci flow on Y has a unique viscosity solution defined on $[0, T_{max}[\times Y$.*

Proof. Fix $T < T_{max}$. Since for any $t \in [0,T]$, $e^{-t}\{\omega_0\} + (1-e^{-t})K_Y \in \mathcal{K}(Y)$, one can show that there exists a smooth family of Kähler forms $(\chi_t)_{0\le t\le T} \in \mathcal{K}(Y)$ such that for any $t \in [0,T]$, $\{\chi_t\} = \{\omega_t\}$. Observe that if $\mathcal{K}_Y$ is semi-ample then $T_{max} = +\infty$ and we can take $\chi_t := e^{-t}\chi_0 + (1-e^{-t})\chi$, where χ is a smooth semi-positive representative of the canonical class $\mathcal{K}_Y$.

In any case we can write $\omega_t = \chi_t + dd^c\phi_t$, where ϕ is a solution to the corresponding Monge–Ampère flow at the level of potentials,

$$(\chi_t + dd^c\phi_t)^n = e^{\partial_t\phi + \phi_t} dV_Y, \tag{4.1}$$

on Y_T for some admissible volume form dV_Y on Y, or equivalently

$$(\theta_t + dd^c\varphi_t)^n = e^{\partial_t\varphi + \varphi_t}\mu_{NKRF},$$

on a log resolution $\pi : X \to Y$, where μ_{NKRF} is a volume form on X with canonical vanishing i.e. locally $\mu_{NKRF} = \Pi_E|f_E|^{2a_E}dV_X$. Here we write $\varphi := \pi^*\phi$ and $\theta_t := \pi^*\chi_t$.

Since $(\chi_t)_{0\le t\le T}$ is a smooth family of Kähler forms on Y, it follows that the family of forms $[0,T[\ni t \longmapsto \theta_t$ is very regular on X in the sense of Definition 3.10. Therefore we can apply Theorem 3.11 to get a unique solution to the Monge–Ampère flow on X_T for any fixed $T < T_{max}$ starting at φ_0. By uniqueness all these solutions glue into a unique solution of the Monge–Ampère flow on $[0, T_{max}[\times X$ starting at φ_0. Pushing this solution down to Y we obtain a solution to the NKRF starting at S_0. □

We have recovered by a zeroth order method one of the main results in [30]. Our viscosity solution can be identified with their weak solution thanks to Proposition 3.12.

If Y is minimal, i.e.: K_Y is nef, the flow is defined up to existence time $T = +\infty$, and it is natural to enquire about its long-term behavior. The sequel of this and the following section will be mainly devoted to the study of this problem.

Turning briefly our attention to the case when $-K_Y$ is ample, it follows from Lemma 1.8 that a similar result holds when Y is a $\mathbb{Q}$-Fano variety. We refer the reader to [4] for background on $\mathbb{Q}$-Fano varieties. The Normalized Kähler–Ricci flow is here

$$\frac{\partial\omega_t}{\partial t} = -\mathrm{Ric}(\omega_t) + \omega_t$$

and the cohomology class is again constant (equal to $c_1(Y)$) if we start from an initial data $S_0 = \chi + dd^c\phi_0$, where χ is a Kähler form representing $c_1(Y)$. The flow can be written, at the level of potentials,

$$(\chi + dd^c\phi_t)^n = e^{\partial_t\phi - \phi_t} dV_Y$$

for some admissible volume form dV_Y, or equivalently

$$(\theta_0 + dd^c\varphi_t)^n = e^{\partial_t \varphi - \varphi_t}\mu_{NKRF},$$

on a log resolution $\pi : X \to Y$, where $\theta_0 := \pi^*(\chi)$ and μ_{NKRF} is a volume form with canonical vanishing i.e. locally $\mu_{NKRF} = \Pi_E |f_E|^{2a_E} dV_X$.

Theorem 4.1 then guarantees that this complex Monge–Ampère flow can be started from an arbitrary continuous θ_0-psh potential φ_0 and exists for all times $t > 0$. The long term behavior is however much more difficult to understand on $\mathbb{Q}$-Fano varieties and is related to the (mildly) singular version of the Yau–Tian–Donaldson conjecture (see [4,13,33]).

4.2. Canonically polarized varieties

We work in this section on a minimal model of general type, i.e. Y has canonical singularities and K_Y is big and nef (hence semi-ample by a classical result of Kawamata). This contains in particular the case when Y is a canonical model, i.e. a general type projective algebraic variety with only canonical singularities such that K_Y is ample (see [5] for the existence of a unique canonical model in every birational class of complex projective manifolds of the general type).

4.2.1. Starting from the canonical class

In this paragraph, we assume K_Y is ample. If we start the normalized Kähler–Ricci flow from an initial data $S_0 = \chi_0 + dd^c\phi_0$ whose cohomology class $\{\chi_0\} = c_1(K_Y)$ is the canonical class, then $\{\omega_t\} \equiv c_1(K_Y)$ is constantly equal to the canonical class of Y. Thus $\omega_t = \chi_0 + dd^c\phi_t$ and the flow can be written, at the level of potentials,

$$(\chi_0 + dd^c\phi_t)^n = e^{\partial_t \phi + \phi_t} dV_Y$$

on $\mathbb{R}^+ \times Y$ for some admissible volume form dV_Y.

Theorem 4.1 gives a unique viscosity solution to this complex Monge–Ampère flow with initial data $\phi_0 \in PSH(X, \chi_0) \cap \mathcal{C}^0(X)$. This shows in particular that the Kähler–Ricci flow can be run on Y from an initial data S_0 which is an arbitrary positive current in $c_1(K_Y)$ with continuous potentials.

It follows from [16, Theorem 7.8] that Y admits a unique singular Kähler–Einstein current $S_{KE} \in c_1(K_Y)$, which is a smooth bona fide Kähler–Einstein metric on the regular part Y_{reg} of Y, and admits globally continuous potentials at singular points Y_{sing} [17].

Theorem 4.2. *Given any initial data S_0 which is an arbitrary positive current with continuous potentials in $c_1(K_Y)$, the normalized Kähler–Ricci flow*

$$\frac{\partial \omega_t}{\partial t} = -\mathrm{Ric}(\omega_t) - \omega_t$$

can be run from S_0 and converges, as $t \to +\infty$, towards S_{KE}.

The convergence is uniform at the level of (properly normalized) potentials. One can further show that the convergence holds in the $\mathcal{C}^\infty$-sense in Y_{reg} (see [29]), if S_0 is a smooth Kähler form on Y.

Proof of Theorem 4.2. We work on a log resolution $\pi : X \to Y$. Let $\theta_0 := \pi^*(\chi_0)$. Recall from [16,17] that

$$\pi^* S_{KE} = \theta_0 + dd^c \varphi_{KE},$$

where $\varphi_{KE} \in PSH(X, \theta_0) \cap \mathcal{C}^0(X)$ is a viscosity/pluripotential solution of the elliptic degenerate complex Monge–Ampère equation

$$(\theta_0 + dd^c \varphi_{KE})^n = e^{\varphi_{KE}} \mu_{NKRF}.$$

Thus φ_{KE} is a fixed point (= static solution) of the NKRF and the comparison principle yields

$$\|\varphi_t - \varphi_{KE}\|_{L^\infty(\mathbb{R}^+ \times X)} \leq \|\varphi_0 - \varphi_{KE}\|_{L^\infty(X)}.$$

We can actually reinforce this uniform control by applying the comparison principle to the functions $u(t,x) = e^t \varphi(t,x)$ and $u_{KE}(t,x) = e^t \varphi_{KE}(x)$ which are $e^t\theta_0$-psh in X: observe indeed that $t \mapsto e^t \theta_0$ is non-decreasing and the u_t's satisfy the twisted parabolic Monge–Ampère equation

$$(e^t \theta_0 + dd^c u_t)^n = e^{e^{-t}\partial_t u_t + nt} \mu_{NKRF}.$$

It follows therefore from Remark 2.10 that for all $t > 0$,

$$\|\varphi_t - \varphi_{KE}\|_{L^\infty(X)} \leq e^{-t} \|\varphi_0 - \varphi_{KE}\|_{L^\infty(X)},$$

from which the conclusion follows. □

4.2.2. Starting from an arbitrary class

Here we come back to the general case when K_Y is nef and big.

Theorem 4.3. *Given any initial data S_0 which is an arbitrary positive current with continuous potentials in the Kähler class $\{\chi_0\}$, the Kähler–Ricci flow*

$$\frac{\partial \omega_t}{\partial t} = -\mathrm{Ric}(\omega_t) - \omega_t \tag{4.2}$$

can be run from S_0 and converges, as $t \to +\infty$, towards S_{KE}.

Again the convergence is uniform at the level of (properly normalized) potentials. One can further show that the convergence holds in the $\mathcal{C}^\infty$-sense in Y_{reg} (see [29]), if S_0 is a smooth Kähler form on Y.

Proof of Theorem 4.3. Theorem 4.1 implies that the equation (4.2) has a unique solution starting from S_0. It is clear that at the level of cohomology classes $\{\omega_t\} \to c_1(K_Y)$ as $t \to +\infty$. We want to show that this is the case for the flow itself. This can be done using the comparison principle at the level of potentials.

We work on a log resolution $\pi : X \to Y$ so that (4.2) is equivalent to the following Monge–Ampère flow:

$$(\theta_t + dd^c\varphi_t)^n = e^{\partial_t\varphi+\varphi_t}\mu_{NKRF},$$

where $\theta_t := \pi^*(\chi_t)$ and μ_{NKRF} is a volume form with canonical vanishing i.e. locally $\mu_{NKRF} = \Pi_E|f_E|^{2a_E}dV_X$.

Let ϕ_{KE} be the potential of the singular Kähler–Einstein metric S_{KE} on Y given by [16] i.e. $S_{KE} = \chi + dd^c\phi_{KE}$ and $(\chi + dd^c\phi_{KE})^n = e^{\phi_{KE}}dV_Y$. Define $\theta_\infty := \pi^*(\chi)$ and $\varphi_{KE} := \pi^*(\phi_{KE})$. Then the Kähler–Einstein equation can be written as

$$(\theta_\infty + dd^c\varphi_{KE})^n = e^{\varphi_{KE}}\mu_{NKRF}.$$

The proof will be completed in three steps.

Step 1: We first establish a lower bound for the solution φ by finding an appropriate subsolution to the Cauchy problem for the flow (4.1). Consider

$$u(t,x) := e^{-t}\varphi_0 + (1-e^{-t})\varphi_{KE} + h(t)e^{-t},$$

on $\mathbb{R}^+ \times Y$, where h is a C^1 function in $\mathbb{R}^+$ to be chosen so that u is a subsolution to the Cauchy problem for the flow (4.1).

Observe that $u(0,x) = \varphi_0$ if $h(0) = 0$ and for all $t > 0$,

$$\theta_t + dd^c u_t = e^{-t}(\theta_0 + dd^c\varphi_0) + (1-e^{-t})(\theta_\infty + dd^c\varphi_{KE}) \geq 0$$

in the weak sense of currents, hence u_t is θ_t-psh and satisfies the inequality

$$(\theta_t + dd^c u_t)^n \geq (1-e^{-t})^n(\theta_\infty + dd^c\varphi_{KE})^n = (1-e^{-t})^n e^{\varphi_{KE}}dV_Y.$$

in the pluripotential sense on X.

On the other hand $\partial_t u + u = \varphi_{KE} + h'(t)e^{-t}$ thus u is a subsolution if $(1-e^{-t})^n \leq e^{h'(t)e^{-t}}$. We therefore choose h to be the unique solution of the ODE $h'(t) = ne^t\log(1-e^{-t})$ with $h(0) = 0$. We let the reader check that

$$h(t) = n\left\{(e^t-1)\log(e^t-1) - e^t\log(e^t)\right\} = O(t) \text{ as } t \to +\infty.$$

It follows therefore from Lemma 1.4 that u is a subsolution to the Cauchy problem for the normalized Monge–Ampère flow (4.1). By the comparison principle we have $u \leq \phi$ in $\mathbb{R}^+ \times X$ i.e.

$$\varphi_{KE}(x) - \varphi(t,x) \leq h(t)e^{-t} = O(te^{-t}), \tag{4.3}$$

for all $(t,x) \in \mathbb{R}^+ \times X$.

The proof of the upper bound is done by constructing an appropriate supersolution to the Cauchy problem. The construction is more involved and uses our earlier results in the degenerate elliptic case. We proceed in two steps.

Step 2: We first assume that K_Y is ample. Fix β an arbitrary Kähler form on X and set $\theta_t := e^{-t}\theta_0 + (1-e^{-t})\beta$. Let φ be the solution to the Monge–Ampère flow

$$(\theta_t + dd^c\varphi_t)^n = e^{\partial_t\varphi+\varphi}\mu_{NKRF}, \tag{4.4}$$

and let ψ be the solution to the degenerate elliptic equation

$$(\beta + dd^c\psi)^n = e^{\psi}\mu_{NKRF}. \tag{4.5}$$

Assume moreover that $\theta_0 \leq \beta$ and consider the function

$$v(t,x) := \psi + Ce^{-t},$$

defined on $\mathbb{R}^+ \times X$, where $C := \max_X(\phi_0 - \psi) > 0$ is chosen so that $v_0 = C + \psi \geq \phi_0$ in X. This implies that $dd^c v_t + \theta_t \leq dd^c\psi + \beta$ hence for all $t > 0$,

$$(dd^c v_t + \theta_t)^n \leq (dd^c\psi + \beta)^n = e^{\psi} = e^{\partial_t v + v}\mu_{NKRF},$$

in the sense of viscosity on X.

Therefore v is a supersolution to the flow (4.4) and the comparison principle yields the upper bound

$$\varphi(t,x) \leq \psi(x) + e^{-t}\max_X(\varphi_0 - \psi)$$

When β is an arbitrary Kähler form on X, it follows from the definition of θ_t that there exists $T >> 1$ such that $\theta_t \leq 2\beta$ for $t \geq T$. The Kähler–Ricci flow starting from the current $\theta_T + dd^c\varphi_T$ has a unique solution given by $\phi(t,x) := \varphi(t+T,x)$ for $(t,x) \in \mathbb{R}^+ \times X$. Translating in time we can thus assume that $\theta_0 \leq 2\beta$. Set

$$v(t,x) := (1+e^{-t})\psi(x) + h(t)e^{-t} + Be^{-t},$$

where h is a smooth function, $h(0) = 0$ and $B := \max_X(\varphi_0 - 2\psi)$ so that $v_0 \geq \varphi_0$. We want v to be a supersolution of the flow (4.4). Since $dd^c v_t + \theta_t \leq (1 + e^{-t})(dd^c\psi + \beta)$ we get

$$(dd^c v_t + \theta_t)^n \leq (1 + e^{-t})^n (dd^c\psi + \beta)^n = (1 + e^{-t})^n e^{\psi}.$$

Since $\partial_t v + v = \psi - h'(t)e^{-t}$ we impose $-h'(t)e^{-t} = n \log(1 + e^{-t})$. Observe again that $h(t) = O(t)$. By the comparison principle we conclude that $\varphi(t, x) \leq v(t, x)$ hence

$$\varphi(t, x) \leq \psi + (\max_X(\varphi_0 - 2\psi) + \max_X \psi)e^{-t} + h(t)e^{-t}. \tag{4.6}$$

From (4.3) and (4.6) we conclude, when K_Y is ample that $|\varphi_t - \varphi_{KE}| = O(te^{-t})$ as $t \to +\infty$.

Step 3: We now establish the upper bound when K_Y is merely nef and big. We set $\beta = \theta_\infty := \pi^*(\chi)$, where χ is semi-positive and big and represents the canonical class K_Y. The solution to the corresponding (4.5) is the function $\psi = \varphi_{KE}$.

We approximate β by Kähler forms $\beta_\varepsilon := \beta + \varepsilon\eta$ for $\varepsilon >$ small enough, where $\eta > 0$ is a fixed Kähler form on X. Set $\theta_t^\varepsilon := e^{-t}\theta_0 + (1 - e^{-t})\beta_\varepsilon$ and solve as in Step 2 the corresponding complex Monge–Ampère flow

$$(\theta_t^\varepsilon + dd^c\varphi_t^\varepsilon)^n = e^{\partial_t\varphi^\varepsilon + \varphi_t^\varepsilon}\mu_{NKRF}, \tag{4.7}$$

with Cauchy data $\varphi_0^\varepsilon = \varphi_0$ which is θ_0^ε-psh in X since $\theta_0^\varepsilon = \theta_0$. Let ψ^ε be the continuous β_ε-psh solution of the degenerate elliptic equation

$$(\beta_\varepsilon + dd^c\psi^\varepsilon)^n = e^{\psi^\varepsilon}\mu_{NKRF},$$

which exists by [16]. It follows from Step 2 that there exists $t_\varepsilon > 1$ such that for $t \geq t_\varepsilon$ and $x \in X$,

$$\varphi^\varepsilon(t, x) \leq \psi^\varepsilon(x) + e^{-t}\max_X(\varphi(t_\varepsilon, x) - 2\psi^\varepsilon(x)) + h(t)e^{-t},$$

where h is a smooth function satisfying the $h'(t)e^{-t} = n\log(1 + 2e^{-t})$ with $h(0) = 0$.

Since $\theta \leq \theta^\varepsilon$, the function φ is a supersolution to the parabolic equation (4.7) with the same Cauchy condition. Moreover the family $t \longmapsto \theta_t^\varepsilon$ is very regular in the sense of Definition 3.10. The comparison principle yields $\varphi \leq \varphi^\varepsilon$ on $\mathbb{R}^+ \times X$. Therefore

$$\begin{aligned}\varphi(t, x) - \varphi_{KE}(x) \leq\ & \psi^\varepsilon(x) - \varphi_{KE}(x)\\ & + \max_X(\varphi(t_\varepsilon, x) - 2\psi^\varepsilon(x))\\ & + \max_X \psi^\varepsilon + h(t))e^{-t},\end{aligned} \tag{4.8}$$

for $t \geq t_\varepsilon$ and $x \in X$. The comparison principle shows that the family $(\psi_\varepsilon)_{\varepsilon>0}$ is non-increasing and $\psi^\varepsilon \to \varphi_{KE}$ pointwise in X as $\varepsilon \to 0$ (see [17]). The convergence $\psi^\varepsilon \to \varphi_{KE}$ is uniform on X, as follows from Dini's lemma.

By using (4.3) and (4.8), we conclude that $\varphi_t \to \varphi_{KE}$ uniformly on X as $t \to +\infty$. Thus $\theta_t + dd^c\varphi_t \to \theta_\infty + dd^c\varphi_{KE}$. Pushing down to Y we conclude that $\omega_t \to S_{KE}$ weakly on Y. $\square$

4.3. Calabi–Yau varieties

Let Y be a $\mathbb{Q}$-Calabi–Yau variety, i.e. a Gorenstein Kähler space of finite index with trivial first Chern class (see [16, Definition 7.4]).

Fix χ_0 a Kähler form on Y and $S_0 = \chi_0 + dd^c\phi_0$ a positive closed current with a continuous potential $\phi_0 \in PSH(Y, \chi_0) \cap \mathcal{C}^0(Y)$. The Kähler–Ricci flow

$$\frac{\partial \omega_t}{\partial t} = -\mathrm{Ric}(\omega_t)$$

preserves the cohomology class $\{\chi_0\}$ since $c_1(Y) = 0$. Thus $\omega_t = \chi_0 + dd^c\phi_t$ and the KRF can be written at the level of potentials as the complex Monge–Ampère flow

$$(\chi_0 + dd^c\phi_t)^n = e^{\partial_t\phi} dV_Y$$

for some admissible volume form dV_Y

It follows from Theorem 3.11 that the corresponding complex Monge–Ampère flow on a log resolution $\pi : X \longrightarrow Y$ with initial data $\varphi_0 := \phi_0 \circ \pi$ has a unique viscosity solution φ. This shows in particular that the Kähler–Ricci flow in the sense of Definition 1.9 can be run on Y from an initial data S_0 which is an arbitrary positive current with continuous potentials. The solution exists for all times $t > 0$. Again, we recover one of the main results of [30].

It follows from [16, Theorem 7.5] that Y admits a unique singular Ricci flat Kähler–Einstein current S_{KE} in the Kähler class $\{\theta_0\}$, which is a smooth bona fide Kähler–Einstein metric on the regular part Y_{reg} of Y, and admits globally continuous potentials at singular points Y_{sing}, thanks to [17].

Theorem 4.4. *Let Y be a $\mathbb{Q}$-Calabi–Yau variety and fix $\alpha_0 \in \mathcal{K}(Y)$ a Kähler class. Given any initial data $S_0 \in \alpha_0$ which is an arbitrary positive current with continuous potentials on Y, the Kähler–Ricci flow*

$$\frac{\partial \omega_t}{\partial t} = -\mathrm{Ric}(\omega_t) \tag{4.9}$$

can be run from S_0 and converges, as $t \to +\infty$, towards the singular Ricci flat Kähler–Einstein current $S_{KE} \in \alpha_0$.

The convergence here is uniform on Y at the level of (properly normalized) potentials. A parabolic version of Yau's $\mathcal{C}^2$-estimate, together with Tsuji's trick and parabolic Evan–Krylov's + Schauder theory allow to show that the convergence holds in the $\mathcal{C}^\infty$-sense in Y_{reg} (see [29]) when S_0 is a smooth Kähler form on Y.

Proof of Theorem 4.4. The Kähler–Ricci flow (4.9) is equivalent to the following complex Monge–Ampère flow on X, a log resolution $\pi : X \longrightarrow Y$

$$(\theta_0 + dd^c\varphi_t)^n = e^{\partial_t\varphi}\mu_{NKRF}, \tag{4.10}$$

starting at φ_0 with the usual notations.

By Theorem 3.11, this flow has a unique solution φ defined in $\mathbb{R}^+ \times X$. Observe that the solution φ is uniformly bounded in $\mathbb{R}^+ \times X$. Indeed let ρ be a solution to the degenerate elliptic equation $(\theta_0 + dd^c\rho)^n = dV_Y$ on Y normalized by $\max_X(\varphi_0 - \rho) = 0$, which exists by [16]. The function $\psi(t, x) := \rho(x)$ is a solution to the Monge–Ampère flow (4.10) with Cauchy condition $\psi_0 = \rho$. By the comparison principle we conclude that for any $(t, x) \in \mathbb{R}^+ \times X$, we have

$$\rho(x) - \max_X(\rho - \varphi_0) \leq \varphi(t, x) \leq \rho(x).$$

This shows that there exist uniform constants m_0, M_0 such that $m_0 \leq \varphi(t, x) \leq M_0$ for all $(t, x) \in \mathbb{R}^+ \times X$.

The proof of the convergence theorem goes by approximating by perturbed complex Monge–Ampère flows and by using the comparison principle as in the proof of [19, Theorem 5.2].

We first prove an upper bound. Consider the flows

$$(\theta_0 + dd^c\phi_t)^n = e^{\partial_t\phi+\varepsilon(\phi-M_0)}dV_Y, \tag{4.11}$$

starting at φ_0, where $\varepsilon > 0$ is a parameter that we shall eventually let converge to zero.

By Theorem 3.11, the flow (4.13) has a unique viscosity solution φ^ε on $\mathbb{R}^+\times X$. Observe that φ is a subsolution to this flow by the choice of M_0. The comparison principle thus insures

$$\varphi(t, x) \leq \varphi^\varepsilon(t, x), \text{ in } \mathbb{R}^+ \times X.$$

It remains to estimate φ^ε from above. For $\varepsilon > 0$ fixed, the solution of the perturbed flow uniformly converges, as $t \to +\infty$, to the solution of the static equation

$$(\theta_0 + dd^c u^\varepsilon)^n = e^{\varepsilon(u^\varepsilon-M_0)}dV_Y,$$

using a similar reasoning as in the previous section.

By the strong version of the comparison principle for the equation (4.13) as in the proof of Theorem 4.2, we have

$$\max_{\mathbb{R}^+\times X} |\phi^\varepsilon(t,x) - u^\varepsilon(x)| \le e^{-\varepsilon t} \max_X |\varphi_0(x) - u^\varepsilon(x)|.$$

Moreover by stability of solutions to degenerate complex Monge–Ampère equations established in [17] we know that $u^\varepsilon \to u$ uniformly on X to the solution u of the equation $(\theta_0 + dd^c u)^n = dV_Y$, normalized by the condition $\int_Y u dV_Y = 0$. We infer

$$\varphi(t,x) - u(x) \le e^{-\varepsilon t} \max_X |\varphi_0(x) - u^\varepsilon(x)| + \max_X |u^\varepsilon(x) - u(x)|. \tag{4.12}$$

We now take care of the lower bound. Consider for $\varepsilon > 0$

$$(\theta_0 + dd^c \psi_t)^n = e^{\partial_t \psi + \varepsilon(\psi - m_0)} dV_Y, \tag{4.13}$$

starting at φ_0. Observe that φ is a supersolution to this flow by the choice of m_0. Theorem 3.11 guarantees that this flow has a unique viscosity solution ψ^ε. The comparison principle thus yields

$$\psi^\varepsilon(t,x) \le \varphi(t,x), \text{ in } \mathbb{R}^+ \times X.$$

We now estimate ψ^ε from below. For $\varepsilon > 0$ fixed, the solution of the perturbed flow uniformly converges, as $t \to +\infty$, to the solution of the static equation

$$(\theta_0 + dd^c v^\varepsilon)^n = e^{\varepsilon(v^\varepsilon} - m_0) dV_Y.$$

Again by stability of solutions to degenerate complex Monge–Ampère equations established in [17] we know that $v^\varepsilon \to u$ uniformly on X, where u is the unique solution of the equation $(\theta_0 + dd^c u)^n = dV_Y$, normalized by the condition $\int_X u\mu_{NKRF} = 0$. As above we obtain the lower bound

$$u(x) - \varphi(t,x) \le \varepsilon^{-\varepsilon t} \max_X |v^\varepsilon(x) - \varphi_0(x)| + \max_X |u(x) - v^\varepsilon(x)|. \tag{4.14}$$

It is now clear from (4.12) and (4.14) that $\varphi_t \to u$ uniformly in X as $t \to +\infty$.

Pushing down everything to Y we see that $\omega_t = \theta_0 + dd^c \phi_t \to \theta_0 + dd^c u = S_{KE}$, as $t \to +\infty$, as claimed. $\square$

4.4. Smoothing properties of the Kähler–Ricci flow

Smoothing properties of the Kähler–Ricci flow have been observed and used by many authors in the last thirty years (see e.g. [1,32,28]).

Attempts to run the Kähler–Ricci flow from a degenerate initial data have motivated several recent works [10–12,30,31]. The best result (before [22]) is that of Song and

Tian [30] who showed that on a projective variety Y with canonical singularities, the Kähler–Ricci flow

$$\frac{\partial \omega_t}{\partial t} = -\mathrm{Ric}(\omega_t)$$

can be run from an initial data $T_0 = \chi_0 + dd^c\phi_0$ which is a positive current with continuous potentials.[2] It is then classical that the flow exists on a maximal interval of time $[0, T_{max}[$, where

$$T_{max} = \sup\{t > 0 \,|\, \{\omega_0\} - tc_1(Y) \text{ is Kähler }\}.$$

The parabolic viscosity approach we have developed in this article allows us to show that the potentials constructed in all these works are globally continuous on $[0, T_{max}[\times Y$.

Theorem 4.5. *Let Y be a projective variety with at worst canonical singularities. Fix χ a smooth closed form representing $c_1(K_Y)$, χ_0 a Kähler form on Y and let $S_0 = \chi_0 + dd^c\phi_0$ be a positive current with a continuous potential on Y. The Kähler–Ricci flow with initial data S_0*

$$\frac{\partial \omega_t}{\partial t} = -\mathrm{Ric}(\omega_t)$$

admits a unique solution $\omega_t = \chi_0 + t\chi + dd^c\phi_t$, with

- *for all $0 < t < T_{max}$, the function $x \mapsto \varphi_t(x)$ is a χ_t-psh function on Y which is smooth in Y_{reg};*
- *$(t, x) \mapsto \varphi(t, x)$ continuous on $[0, T_{max}[\times Y$.*

Proof. When K_Y is semi-ample, we can assume $\chi \geq 0$ hence $t \mapsto \theta_t = \theta_0 + t\chi$ is non-decreasing. In the general case since $\theta_0 > 0$ is Kähler, there exists a constant $A > 0$ such that $-\chi \leq A\theta_0$ in Y. Therefore the family $t \longmapsto \theta_t = \theta_0 + t\chi$ is very regular in the sense of Definition 3.10. The result is thus an immediate consequence of Theorem 3.11. $\square$

The continuity of φ at singular points of Y_{sing} is the novelty here: this is the parabolic analogue of the main application of [17].

For complex Monge–Ampère flows starting from even more degenerate initial data, we refer the reader to [22], where the work of Song and Tian is extended so as to allow the Kähler–Ricci flow to be run from a positive current with zero Lelong numbers. Our viscosity approach can also be used in this latter context to show that the maximal solution of the Kähler–Ricci flow becomes immediately smooth on Y_{reg}, for $t > 0$, with globally continuous potentials on Y.

[2] The precise assumption in [30] is a bit more restrictive but can easily be extended to this statement as observed in [7].

5. Concluding remarks: the Kähler–Ricci flow over flips

The extinction time of the KRF on Y can be expressed as

$$T_0 = \sup\{t > 0, \{\omega_0\} + tK_Y\} \in \mathcal{K}(Y).$$

Let us assume that $(Y, \{\omega_0\})$ satisfies the following assumptions:

- Y has terminal singularities.
- $T_0 < \infty$.
- $\{\omega_{T_0}\} = \{\omega_0\} + T_0 K_Y$ is *a non-trivial pull back from a Kähler class*, i.e.: that there exists a non-biholomorphic proper bimeromorphic holomorphic map $\psi^- : Y \to Z$ such that Z is a normal Kähler complex space and $\{\omega_0\} + T_0 K_Y \in (\psi^-)^*\mathcal{K}(Z)$.
- For $N \in \mathbb{N}^*$ divisible enough the sheaf of graded algebras

$$\mathcal{P}(Y/Z) := \bigoplus_{n\in\mathbb{N}} \psi^-_* O_Y(nNK_Y)$$

  is locally finitely generated over O_Z.

The last condition is fulfilled thanks to [5, Thm. 1.2 (3)] if Y and Z are projective varieties. We then denote by $\psi^+ : Y^+ \to Z$ the relative canonical model of $\psi^- : Y \to Z$, namely $Y^+ := \mathrm{Proj}(\mathcal{P}(Y/Z))$. It is known thanks to the classical work of M. Reid that Y^+ is normal (and has canonical singularities) and it is trivial to see that ψ^+ is a proper bimeromorphic mapping.

It follows from [26, Lemma 3.38][3] that Y^+ has terminal singularities. Also, if Y, Z are projective and Y is $\mathbb{Q}$-factorial, then Y^+ is $\mathbb{Q}$-factorial. One can construct a diagram:

$$\begin{tikzcd}
& X \arrow[dl, "\pi^-"'] \arrow[dr, "\pi^+"] & \\
Y \arrow[dr, "\psi^-"'] & & Y^+ \arrow[dl, "\psi^+"] \\
& Z &
\end{tikzcd}$$

where X is smooth, π^-, π^+ are log-resolutions such that $\mathrm{Exc}(\pi^+) \cup \mathrm{Exc}(\pi^+)$ is a divisor with simple normal crossings, ψ^+, ψ^- are proper bimeromorphic holomorphic maps. By construction, $-K_Y$ is ψ^--ample, K_{Y^+} is ψ^+-ample and one has the following properties:

[3] Stated for algebraic varieties. The proof however goes through in the complex analytic category since ψ^- is a projective morphism due to the fact that $-K_Y$ is ψ^--ample.

Lemma 5.1. *There exists a real number $\epsilon > 0$ such that for $t \in]T_0, T_0 + \epsilon[$,*

$$\{\omega_{T_0}\} + (t - T_0)K_{Y^+} \in \mathcal{K}(Y^+).$$

Proof. Immediate consequence of the fact that K_{Y^+} is ψ^+-ample. □

Lemma 5.2. *The exceptional divisors of π^- are exceptional for π^+.*

Proof. The bimeromorphic map $Y^+ \overset{(\psi^-)^{-1}\circ\psi^+}{\dashrightarrow} Y$ contracts no divisor, since a log-canonical model is a contraction and $Y^+ \to Z$ is the log-canonical model of $Y \to Z$ see [5, section 3]. □

Furthermore, for an exceptional divisor E of π^+, we have $a_E \leq a_E(Y^+) := a_E^+$ where $a_E = 0$ if E is not π^--exceptional by [26, Lemma 3.38] and we define $\delta_E = a_E^+ - a_E \geq 0$.

We define a measurable volume form with semipositive continuous density on X by

$$\mu = \left(\mathbb{I}_{t\leq T_0} + \mathbb{I}_{\{t>T_0\}}(\prod_E |s_E|^{2\delta_E}_{h_E})\right)\mu_{NKRF}(h^-)$$

and $\bar{\omega} \in H^0(X, \mathcal{Z}^{1,1}_{X_{T_0+\epsilon}/[0,T_0+\epsilon[})$ by

$$\bar{\omega}_t = \omega_0 + \int_0^t du\ (dd^c \log(\mu) - a_E[E] - \mathbb{I}_{\{u>T_0\}}\delta_E[E]).$$

The fact that $\bar{\omega}$ has continuous local potentials is straightforward. The pair $(\bar{\omega}, \mu)$ defines a Kähler–Ricci flow on $Y = Y^-$ for $t < T_0$ and a Kähler–Ricci flow on Y^+ for $t > T_0$. On the other hand the flow $(CMAF)_{\bar{\omega},V}$ does not satisfy condition (1.1) at T_0. Indeed in every coordinate system one can find a potential in such a way that this flow has the following expression:

$$(dd^c\phi)^n = e^{\frac{\partial\phi}{\partial t}}|z_E|^{2a_E+2\mathbb{I}_{\{t>T_0\}}\delta_E}.$$

We believe a large part of the theory developed here should hold in spite of the breakdown of condition (1.1) but we shall not treat any further this topic in the present article and hope to return to that problem in a later work.

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