Study of periodic traveling waves solution to a nonlocal equation

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1 Introduction

My internship took place at the Institut de Mathématiques de Toulouse (IMT) from the 6^{th} of June 2016 to the 29^{th} of July 2016. I was directed by Grégory Faye, junior CNRS researcher, whom I thank for his advices. The purpose of this internship was to study periodic traveling waves solution to the equation:

$$\partial_t u(x,t) - \left[\int_{\mathbb{R}} J(y)u(x-y,t)dy - u(x,t)\right] = f(u(x,t),t) \text{ for all } x \in \mathbb{R}, \quad t \geq 0. \tag{1.1}$$

The hypothesis satisfied by the functions f and J and the definition of a periodic traveling wave will be detailed later. The periodic traveling waves are an interesting type of solutions, as proved by the work done in [1] on the equation:

$$\partial_t u(x,t) - \partial_{xx} u(x,t) = f(u(x,t),t) \text{ for all } x \in \mathbb{R}, \quad t \geq 0, \tag{1.2}$$

for which, under some conditions on the initial conditions, a solution u converges exponentially fast in L^∞ to a periodic traveling wave. This is called the exponential stability of periodic traveling waves, and that was our main motivation to work on periodic traveling waves solution to (1.1). The first specificity of this evolution equation, arising in the modelling of population dynamics, is that the diffusion term $\displaystyle\int_{\mathbb{R}} J(y)u(x-y,t)dy - u(x,t)$ is nonlocal, because of the convolution. The local counterpart (1.2), with a Laplace operator instead of the convolution diffusion, has already been studied in [1]. Moreover, in (1.1), f depends on the time. The autonomous case, that is to say the equation:

$$\partial_t u(x,t) - \left[\int_{\mathbb{R}} J(y)u(x-y,t)dy - u(x,t)\right] = f(u(x,t)) \text{ for all } x \in \mathbb{R}, \quad t \geq 0, \tag{1.3}$$

was the subject of the work done in [3]. In [3] and in [1], the authors proved the existence, the uniqueness and the exponential stability respectively of traveling waves and of periodic traveling waves solution to their equations. My purpose during this internship was to understand their methods, and to adapt them in order to work on (1.1) which shows the specificities of both (1.2) and (1.3). Thus, this report is organized as follows. In section 2, I will explain with more details the hypothesis satisfied by f and J, and the exact definition of a periodic traveling wave. Then, in section 3, I will prove a strong maximum principle and a comparison principle inspired by the well-known theorems for elliptic equations used in [1], which are essential for the sequel. I prove the uniqueness of periodic traveling wave solution to (1.1) in section 4 with a method inspired from [1]. In section 5, I prove the existence, firstly in every compact set, then in the wanted unbounded domain. In section 6, I will propose some extensions of this work, which I would have tackled if I had had more time.

2 Presentation of the problem

We are interested in the study of the following nonlocal evolution equation, arising in the modelling of population dynamics:

$$\partial_t u(x,t) - \left[\int_{\mathbb{R}} J(y)u(x-y,t)dy - u(x,t)\right] = f(u(x,t),t) \text{ for all } x \in \mathbb{R}, \quad t \geq 0,$$

where f and J verify the following assumptions.

Hypothesis (H1)

(i) *there exists* $T > 0$ *such that for all* $u \in \mathbb{R}$ *and* $t \in \mathbb{R}$, $f(u, t+T) = f(u, t)$,

(ii) $f \in C^{2,1}(\mathbb{R} \times [0;T])$, *and for all* $t \in [0;T]$, $f(\cdot, t)$ *is Lipschitz,*

(iii) *define, for all* $\alpha \in \mathbb{R}$, $w(\alpha, \cdot)$ *the unique solution of the Cauchy problem:*

$$\begin{cases} \partial_t w = f(w, t), \\ w(0) = \alpha. \end{cases} \tag{2.1}$$

We can define the Poincaré map P *such that for all* $\alpha \in \mathbb{R}$, $P(\alpha) = w(\alpha, T)$.

We assume that P *admits only three fixed points* $\alpha^- < \alpha^0 < \alpha^+$ *such that*

$$\frac{d}{d\alpha} P(\alpha^\pm) < 1 < \frac{d}{d\alpha} P(\alpha^0).$$

(iv) $\sup \{ \partial_u f(u, t) \mid u \in [W^-(t), W^+(t)], t \in [0;T] \} < 1$.

Definition 2.1. *A differentiable function* $g \in C^0(\mathbb{R})$ *is said bistable if it has only three zeros* $\alpha^- < \alpha^0 < \alpha^+$ *such that*

$$g'(\alpha^\pm) < 0 < g'(\alpha^0).$$

Definition 2.2. *A periodic traveling wave solution to*

$$\partial_t u(x,t) - \left[\int_{\mathbb{R}} J(y) u(x-y, t) dy - u(x,t) \right] = g(u(x,t)) \text{ for all } x \in \mathbb{R}, \quad t \geq 0,$$

where g *is bistable, is a solution* u *such that there exist* $c \in \mathbb{R}$ *and* U *such that for all* $x \in \mathbb{R}$, $t \in \mathbb{R}$, $\xi \in \mathbb{R}$, *we have* $u(x,t) = U(x - ct)$, $\lim\limits_{\xi \to +\infty} U(\xi) = \alpha^-$, $\lim\limits_{\xi \to -\infty} U(\xi) = \alpha^+$ *for all* $t \geq 0$, *and* $U(0,0) = \alpha^0$.

In [3], this problem was studied with f independent of time, that is to say $f \in C^2(\mathbb{R})$ and f bistable. Here, hypothesis (H1) is a generalization of the case f bistable, where f depends on time.

Hypothesis (H2)

(i) J *is a nonnegative even continuous and continuously differentiable function of* $\mathbb{R}$,

(ii) $\int_{\mathbb{R}} J(y) dy = 1$,

(iii) *there exists* $\mu_0 > 0$ *such that for all* $\mu \in (0; \mu_0)$, $\int_{\mathbb{R}} J(y) e^{\mu y} dy < +\infty$.

Here, J is an integral kernel taken smooth enough and even in order to simplify the following reasoning and calculation.

Remark 2.3. *By the assumptions on f, the only T-periodic solutions to (2.1) are $w(\alpha^{\pm},\cdot)$ and $w(\alpha^0,\cdot)$. We will note for all $t \in [0;T]$,* $\quad W^{\pm}(t) := w(\alpha^{\pm}, t)$.

Our model is the nonlocal counterpart of the following problem, studied in [1]:

$$\begin{cases} \partial_t u - \partial_{xx} u = f(u,t), \\ u(x,0) = g(x) \text{ for all } x \in \mathbb{R}, \end{cases} \tag{2.2}$$

with the same assumptions on f, where g is a given function in $L^\infty(\mathbb{R})$. Indeed, it was proved that if g satifies $\limsup_{z\to+\infty} g(z) < \alpha^0$ and $\liminf_{z\to+\infty} g(z) > \alpha^0$, then the solution of (2.2) converges in L^∞ as $t \to +\infty$ exponentially fast to a periodic traveling wave (as defined below) solution to (2.2). Here, we will rather study the existence and uniqueness of a periodic traveling wave solution to (1.1).

Definition 2.4. *A periodic traveling wave solution to (1.1) is a solution u to (1.1) such that there exist $c \in \mathbb{R}$ and U such that for all $x \in \mathbb{R}$, $t \in \mathbb{R}$, $\xi \in \mathbb{R}$, we have $u(x,t) = U(x-ct,t)$, $U(\xi, t+T) = U(\xi,t)$, $\lim_{\xi\to+\infty} U(\xi,t) = W^-(t)$, $\lim_{\xi\to-\infty} U(\xi,t) = W^+(t)$ for all $t \geq 0$, and $U(0,0) = \alpha^0$.*

Consequently, if we write $\xi = x - ct$, we are now interested in the study of the following system:

$$\begin{cases} \partial_t U - c\partial_\xi U - [J * U - U] = f(U,t), \\ U(\xi, t+T) = U(\xi,t) \text{ for all } \xi \in \mathbb{R}, t \geq 0, \\ \lim_{\xi\to+\infty} U(\xi,t) = W^-(t), \ \lim_{\xi\to-\infty} U(\xi,t) = W^+(t) \text{ for all } t \geq 0, \\ U(0,0) = \alpha_0, \end{cases} \tag{2.3}$$

where for all $(\xi,t) \in \mathbb{R}$, $J * U(\xi,t) := \displaystyle\int_{\mathbb{R}} J(y)U(\xi-y,t)dy$.

Remark 2.5. *Let (U,c) be a solution to (2.3), and $\tau \in \mathbb{R}$. We note U_τ the function such that for all $\xi \in \mathbb{R}$, $t \in \mathbb{R}$, $U_\tau(\xi,t) = U(\xi+\tau,t)$. Then (U_τ, c) is also a periodic traveling wave solution to (1.1).*

Remark 2.6. *u is a periodic traveling wave solution to (1.1) such that $u(0,0) = \alpha^0$ if and only if the couple (U,c) as defined in definition 2.4 is a solution to the system (2.3).*

Thus, the proof of the existence and uniqueness of periodic traveling waves solution to (1.1) is equivalent to the following theorem:

Theorem 1 (Existence and Uniqueness of a periodic traveling wave)**.** *There exists a unique function $U : \mathbb{R} \times [0;T] \longrightarrow \mathbb{R}$ and a unique constant $c \in \mathbb{R}$ such that $(U;c)$ satisfies the system (2.3).*

3 Maximum Principles and a comparison principle

To prove Theorem 1, we will need some preliminary results. The proof of similar results for (2.2) used a strong maximum principle and a comparison principle. In this section, we will prove the nonlocal counterparts of these principles.

3.1 The weak maximum principle

Theorem 2 (Weak maximum principle)**.** *Let R be defined by $R := \partial_t - L$ where $Lu = J * u - u + c\partial_\xi u$. Assume J satisfies hypothesis (H2). Suppose that $Ru \leq 0$ in $Q := \mathbb{R} \times [0;T]$.*

Then, $\max\limits_{\bar{Q}} u = \max\limits_{\partial Q} u$.

Proof. First, suppose that $Ru < 0$ in Q. Assume, by way of contradiction, that u reaches its maximum at a point (x_0, t_0) in the interior of Q. Then, $Ru(x_0, t_0) = -[J * u - u](x_0, t_0) \geq 0$. This is a contradiction.

In the general case, define an auxiliary function v with $v(x,t) = e^{\mu x}$ in Q, where μ is a constant which will be determined later. Thus,

$$Rv(x,t) = -\int_{\mathbb{R}} J(x-y)(e^{\mu y} - e^{\mu x})dy - c\mu e^{\mu x} = -e^{\mu x}\left(\int_{\mathbb{R}} J(y)e^{-\mu y}dy - 1 + c\mu\right)$$

Define $\bar{J}(\mu) := \int_{\mathbb{R}} J(y)e^{-\mu y}dy + c\mu$. Then, we have:

- $\bar{J}(0) = 1$,
- $\bar{J}'(0) = -\int_{\mathbb{R}} yJ(y)dy + c = c$,
- $\bar{J}''(0) = \int_{\mathbb{R}} y^2 J(y)dy > 0$.

Thus, in accordance with the sign of $\bar{J}'(0)$, that is to say the sign of c, we can choose μ such that $\bar{J}(\mu) > 1$, and therefore $Rv < 0$ in Q. Define $\varepsilon > 0$ and $w := u + \varepsilon v$. Thus, $Rw < 0$ in Q. The conclusion is found when $\varepsilon \to 0$. □

3.2 The nonlinear comparison principle

Theorem 3 (Nonlinear nonlocal comparison principle)**.** *Assume $f : \mathbb{R} \times [0;T]$ satisfies hypothesis (H1) ans J satisfies hypothesis (H2).*

Assume there exist u and v verifying:

$$\begin{cases} \partial_t u - [J * u - u] - c\partial_x u \geq f(u,t), \\ \partial_t v - [J * v - v] - c\partial_x v \leq f(v,t), \\ u(x,0) \geq v(x,0) \text{ for all } x \in \mathbb{R}, \\ \lim\limits_{x\to+\infty} u(x,t) \geq \lim\limits_{x\to+\infty} v(x,t) \text{ for all } t \in [0,T], \\ \lim\limits_{x\to-\infty} u(x,t) \geq \lim\limits_{x\to-\infty} v(x,t) \text{ for all } t \in [0,T]. \end{cases}$$

Then, $u(x,t) \geq v(x,t)$ *for all* $t \in [0,T], \quad x \in \mathbb{R}$.

Proof. Define $w := u - v$, and γ such that for all $x \in \mathbb{R}$, and all $t \in [0,T]$,

$$\gamma(x,t) = \begin{cases} \frac{f(u(x,t),t) - f(v(x,t),t)}{u(x,t) - v(x,t)} \text{ if } u(x,t) \neq v(x,t), \\ \partial_s f(u(x,t),t) \text{ if } u(x,t) = v(x,t). \end{cases}$$

Thus, for all $x \in \mathbb{R}, t \in [0,T]$, $f(u,t) - f(v,t) = \gamma(x,t)w$. Notice that with these assumptions, $\gamma \in L^\infty(\mathbb{R} \times [0,T])$, and we get:

$$\begin{cases} \partial_t w - [J * w - w] - c\partial_x w - \gamma(x,t)w \geq 0 \text{ in } Q, \\ w(0,x) \geq 0 \text{ in } \mathbb{R}, \\ \lim\limits_{x \to \pm\infty} w(x,t) \geq 0 \text{ in } [0,T]. \end{cases}$$

Using the weak maximum principle with the operator R defined with $Ru := \partial_t u - [J*u-u] - c\partial_x u - \gamma(x,t)u$, we get $w \geq 0$ in Q. We do not have any problem with the sign of γ to use the maximum principle since $\gamma \in L^\infty(\mathbb{R} \times [0,T])$, and therefore it suffices to change the variable w into $e^{\lambda t} w$ where $\lambda \geq ||\gamma||_{L^\infty(\mathbb{R}\times[0,T])}$. Hence, we conclude the proof of the nonlinear comparison principle. □

3.3 The strong maximum principle

Lemma 3.1. *Let R and J be as above. Assume $Ru \leq 0$ in Q. Assume there exist $x_0 \in \mathbb{R}$ and $0 < t_0 \leq T$ such that $u(x_0,t_0) = \sup\limits_Q u$. Then, $u(x,t_0) = u(x_0,t_0)$ for all $x \in \mathbb{R}$.*

Proof. We know that $Ru(x_0,t_0) \leq 0$ by assumption. Moreover, $\partial_\xi u(x_0,t_0) = 0$, $\partial_t u(x_0,t_0) = 0$, and

$$\int_{\mathbb{R}} J(x_0 - y)[u(y,t_0) - u(x_0,t_0)]dy \leq 0.$$

Thus, $\int_{\mathbb{R}} J(x_0 - y)[u(y,t_0) - u(x_0,t_0)]dy = 0$.

By assumption, J is non negative, even and continuous. Therefore, there exists $0 < a < b$ such that $[-b;-a] \cup [a;b] \subset \mathrm{Supp}(J)$. Hence, if $\Gamma := \{y \in \mathbb{R} \mid u(y,t_0) = u(x_0,t_0)\}$,

$$(x_0 - [-b,-a] \cup [a,b]) \subset \Gamma.$$

Let A and $C \in [a,b]$ be such that $\frac{A}{C} \notin \mathbb{Q}$. So $x_0 - C \in \Gamma$ and $x_0 + A \in \Gamma$. Now, repeating this argument with $x_0 - A$ and for $x_0 + C$, we have

$$(x_0 - A - [-b,-a] \cup [a,b]) \subset \Gamma \text{ and } (x_0 + C - [-b,-a] \cup [a,b]) \subset \Gamma.$$

Thus, $\{x_0 + pC - qA | (p,q) \in \{0;1;2\}^2\} \subset \Gamma$. By induction, $\{x_0 + pC - qA | (p,q) \in \mathbb{N}^2\} \subset \Gamma$. Therefore, Γ is closed, non empty, and contains a dense partition of $\mathbb{R}$. Hence, $\Gamma = \mathbb{R}$.

□

Lemma 3.2. *Let D^- be a half sphere centered at $(x_0,t_0) \in Q$:*

$$D^- = \left\{(x,t); (t-t_0)^2 + (x-x_0)^2 < \rho^2, t \leq t_0\right\} \subset Q,$$

and assume that $Ru \leq 0$ and $u < \sup\limits_Q u$ in $\{(x,t) \in D^-$ such that $t < t_0\}$. Then $u(x_0,t_0) < \sup\limits_Q u$.

Proof. Assume by contradiction that $u(x_0,t_0) = \sup\limits_Q u$. Let α be a positive constant. Let v be defined on Q by $v(x,t) = e^{-[(x-x_0)^2 + \alpha(t-t_0)]} - 1$. Hence

$$Rv(x,t) = e^{-[(x-x_0)^2+\alpha(t-t_0)]}[-\alpha + 2c(x-x_0) + 1 - \int_{\mathbb{R}} J(y) e^{(x-x_0)^2 - (x-y-x_0)^2} dy],$$

therefore $Rv < 0$ in D^- with α sufficiently large. Notice that $v(x_0, t_0) = 0$. Let E be the subset of Q defined by :

$$E := \{(x, t) \in D^- | (x - x_0)^2 + \alpha(t - t_0) < 0\}.$$

Also define $S_1 := \partial E \cap D^-$ and $S_2 := \bar{E} \cap \partial D_\rho(x_0, t_0)$ where $D_\rho(x_0, t_0)$ is the sphere centered at (x_0, t_0) of radius ρ. $u < \sup\limits_Q u$ in S_2, so there exists a positive constant δ such that $u \leq \sup\limits_Q u - \delta$ in S_2. Let ε be a positive constant. Define $w := u + \varepsilon v$. Therefore, for small enough ε, $w < \sup\limits_Q u$ in S_2. It is also worth noticing that $Rw < 0$ in E, $w < \sup\limits_Q u$ on $\partial E \backslash \{(x_0, t_0)\}$ and $w(x_0, t_0) = \sup\limits_Q u$. Since $Rw < 0$ in E, w cannot reach an interior maximum in E according to the weak maximum principle. Hence,

$$\max_{\bar{E}} w = \max_{\partial E} w = w(x_0, t_0).$$

This implies $\frac{\partial w}{\partial t}(x_0, t_0) \geq 0$. So,

$$\frac{\partial u}{\partial t}(x_0, t_0) \geq -\varepsilon \frac{\partial u}{\partial t}(x_0, t_0) \geq \varepsilon\alpha > 0.$$

But since $u(x_0, t_0) = \sup\limits_Q u$, $Ru(x_0, t_0) \geq \frac{\partial u}{\partial t}(x_0, t_0) > 0$. This is a contradiction. □

Theorem 4. *Let R be defined by $Ru := \partial_t u - Lu$ where $Lu = J * u - u + c\partial_\xi u$. Assume J satisfies hypothesis (H2). Suppose that $Ru \leq 0$ in $Q := \mathbb{R} \times (0; T)$. Also suppose that there is a point $(x_0; t_0)$ such that $0 < t_0 \leq T$, $x_0 \in \mathbb{R}$ and*

$$u(x_0, t_0) = \max_Q u$$

.

Then u is a constant in $\mathbb{R} \times [0; t_0]$.

Proof. Assume, by way of contradiction, that there exists (x_1, t_1) such that $0 < t_1 < t_0$ and $u(x_1, t_1) < M$. Then, by Lemma 3.1,

$$u(x, t_1) < M \text{ for all } x \in \mathbb{R}.$$

Let $\bar{t_0}$ be such that $u(x_0, \bar{t_0}) = M$ and $u(x_0, t) < M$ for all $t \in (0, \bar{t_0})$, so that $t_1 < \bar{t_0} \leq t_0$. Then, $u(x, t) < M$ for all $x \in \mathbb{R}$ and all $t \in (t_1, \bar{t_0})$. Lemma 3.2 implies that $u(x_0, \bar{t_0}) < M$, which is a contradiction with the definition of $\bar{t_0}$. Thus we have the conclusion of the strong maximum principle. □

4 Uniqueness

Note $L^c u := \partial_t u - [J * u - u] - c\partial_\xi u - f(u, t)$. Since we got a nonlinear comparison principle and a strong maximum principle, we can follow the method in six steps presented by Alikakos et al [1]. Let (c, U) and $(\bar{c}, \bar{U})$ be two solutions of (2.1). Assume $\bar{c} \leq c$.

Step 1: Define $\mathcal{M}^\pm := \sup\limits_{\xi \in \mathbb{R}} (\pm U(\xi, 0))$. Notice $w(\pm\mathcal{M}^\pm, 0) = \pm\mathcal{M}^\pm$. Therefore, by the comparison principle, we get $w(-\mathcal{M}^-, t) \leq U(\xi, t) \leq w(\mathcal{M}^+, t)$ for all $\xi \in \mathbb{R}, t \geq 0$. By T-periodicity, we get

$w(-\mathcal{M}^-, kT+t) \leq U(\xi,t) \leq w(\mathcal{M}^+, kT+t)$ for all $\xi \in \mathbb{R}, t \geq 0, k \in \mathbb{N}$. It follows, with the assumptions on P, that $\lim_{k\to+\infty} w(\pm\mathcal{M}^\pm, t+kT) = W^\pm(t)$. So

$$W^-(t) \leq U(\xi,t) \leq W^+(t) \text{ for all } t \geq 0, \xi \in \mathbb{R}.$$

By the strong maximum principle and the periodicity of U and $W^\pm$, we get:

$$W^-(t) < U(\xi,t) < W^+(t) \text{ for all } t \geq 0, \xi \in \mathbb{R}.$$

We can find the same result for $\bar{U}$ with exactly the same reasoning.

Step 2: Define $\nu^\pm := -\frac{1}{T}\int_0^T \partial_u f(W^\pm(t),t)dt$ and $a^\pm(t) := \exp(\frac{\nu^\pm t}{2} + \int_0^t \partial_u f(W^\pm(\tau),\tau)d\tau)$.

Notice $P'(\alpha^\pm) = \exp(\int_0^T \partial_u f(W^\pm(\tau),\tau)d\tau) < 1$, so $\nu^\pm > 0$ and $a^\pm(T) = \exp(-\frac{\nu^\pm T}{2}) < 1$.

Define

$$\delta_0 := \frac{\sup\{\eta > 0 : |\partial_u f(u,t) - \partial_u f(W^\pm(t),t)| \leq \frac{\nu^\pm}{2}\ \forall t \in [0,T], u \in [W^\pm(t)-\eta, W^\pm(t)+\eta]\}}{2\left\|a^+\right\|_{C^0([0,T])} + 2\left\|a^-\right\|_{C^0([0,T])}},$$

$$\xi_0 := \inf\left\{\bar{\xi} \geq 1 : |U(\pm\xi,t) - W^\pm(t)| \leq \frac{\delta_0}{2}\ \xi \in [\bar{\xi}; +\infty) \text{ and } t \in [0;T]\right\}.$$

For all $\delta \in (0;\delta_0]$, we define $U_\delta^\pm(\xi,t) := U(\xi,t) + \delta a^\pm(t)$. Then,

$$L^c U_\delta^\pm = \delta\partial_t a^\pm + f(U,t) - f(U+\delta a^\pm, t) = \delta a^\pm\left[\frac{\nu^\pm}{2} + \partial_u f(W^\pm(t),t) - \int_0^1 \partial_u f(U+\delta\theta a^\pm(t),t)d\theta\right] > 0$$

in $[\xi_0;+\infty)\times[0;T]$ for the sign "+" and in $(-\infty;-\xi_0]\times[0;T]$ for the sign "−". Therefore, $U_\delta^\pm$ are super solutions in these domains.

Step 3: By assumption, $\bar{U}(\pm\infty,t) = W^\pm(t)$ uniformly on $\mathbb{R}$. Therefore, there exists $\tilde{z}_0 > 0$ sufficiently large such that

$$\bar{U}(\xi - z + (c-\bar{c})t, t) \leq \begin{cases} U(\xi,t) \text{ if } \xi \in [-\xi_0,\xi_0] \\ U(\xi,t) + \delta_0 \text{ if } \xi \notin [-\xi_0,\xi_0] \end{cases} \quad \forall t \in [0;T], z \geq \tilde{z}_0. \tag{4.1}$$

Define $\delta_{\tilde{z}_0} = \inf\{\delta > 0 \mid \bar{U}(\xi - z, 0) \leq U(\xi,0) + \delta, \text{ for all } z \geq \tilde{z}_0, \xi \in \mathbb{R}\}$. By definition, we notice $\delta_{\tilde{z}_0} \in [0;\delta_0]$. We want to show that $\delta_{\tilde{z}_0} = 0$.

Notice that $L^c(\bar{U})(\xi - z + (c-\bar{c})t, t) = 0$. Using the comparison principle, the definition of $\delta_{\tilde{z}_0}$ on $[\xi_0;+\infty)\times\{0\}$ and (4.2) on $\{\xi_0\}\times[0;T]$, we get:

$$\bar{U}(\xi - z + (c-\bar{c})t, t) \leq U^+_{\delta_{\tilde{z}_0}}(\xi,t) \text{ for all } z \geq \tilde{z}_0, (\xi,t) \in (\xi_0;+\infty)\times[0;T].$$

Since z is arbitrarily taken in $[\tilde{z}_0;+\infty)$ and $c \geq \bar{c}$, we get:

$$\bar{U}(\xi - z, T) \leq U^+_{\delta_{\tilde{z}_0}}(\xi,T) \text{ for all } z \geq \tilde{z}_0, \xi \in (\xi_0;+\infty).$$

So, by periodicity, for all $\xi \geq \xi_0, z \geq \tilde{z}_0$, we have

$$\bar{U}(\xi - z, 0) \leq U(\xi,0) + \delta_{\tilde{z}_0} a^+(T),$$

and in a similar manner, for all $\xi \leq -\xi_0, z \geq \tilde{z}_0$, we have

$$\bar{U}(\xi - z, 0) \leq U(\xi, 0) + \delta_{\tilde{z}_0} a^-(T).$$

It is known that $\bar{U}(\xi - z, 0) \leq U(\xi, 0)$ for all $z \geq \tilde{z}_0, \xi \in [-\xi_0, \xi_0]$. Therefore,

$$\bar{U}(\xi - z, 0) \leq U(\xi, 0) + \delta_{\tilde{z}_0} \max\{a^+(T), a^-(T)\} \text{ for all } \xi \in \mathbb{R}.$$

Thus, by definition of $\delta_{\tilde{z}_0}$, $\delta_{\tilde{z}_0} \leq \delta_{\tilde{z}_0} \max\{a^+(T), a^-(T)\}$. Since $a^{\pm}(T) < 1$, we get $\delta_{\tilde{z}_0} = 0$. So $\bar{U}(\xi - z, 0) \leq U(\xi, 0)$ for all $\xi \in \mathbb{R}$ and $z \geq \tilde{z}_0$.

Step 4: From this last result, we conclude with a comparison principle that $\bar{U}(\xi - z + (c - \bar{c})t, t) \leq U(\xi, 0)$ for all $(\xi, t) \in \mathbb{R} \times [0; +\infty)$. So, for all $k \in \mathbb{N}$,

$$\alpha^0 = U(0,0) = U(0, kT) \geq \bar{U}(-\tilde{z}_0 + (c - \bar{c})kT, kT) = \bar{U}(-\tilde{z}_0 + (c - \bar{c})kT, 0).$$

With $k \to +\infty$, noticing $\bar{c} \leq c$ and $\bar{U}(+\infty, 0) = \alpha^+ > \alpha^0$, we conclude $c = \bar{c}$.

Step 5: Define $z_0 = \inf\{y \in \mathbb{R} : \bar{U}(\xi - z, 0) \leq U(\xi, 0) \text{ for all } z \geq y, \xi \in \mathbb{R}\}$. It is clear that z_0 exits and is finite. We want to show that $\bar{U}(\xi - z_0, 0) = U(\xi, 0)$ for all $\xi \in \mathbb{R}$. Assume by way of contradiction that there exist some $\xi \in \mathbb{R}$ such that $\bar{U}(\xi - z_0, 0) < U(\xi, 0)$. By a strong maximum principle, (do not forget $c = \bar{c}$) $\bar{U}(\xi - z, t) < U(\xi, t)$ for all $(\xi, t) \in \mathbb{R}^2, z \geq z_0$. Take $\varepsilon > 0$ small enough so that (4.2) is true for $\tilde{z}_0 = z_0 - \varepsilon$. Then, proceed as in step 3 to get $\bar{U}(\xi - z, 0) \leq U(\xi, 0)$ for all $z \geq \tilde{z}_0$. Thus, by definition of z_0, $\tilde{z}_0 \geq z_0$, but this is impossible. Hence $\bar{U}(\xi - z_0, 0) = U(\xi, 0)$ for all $\xi \in \mathbb{R}$.

Step 6: We now want to show $z_0 = 0$.

By a strong maximum principle, we have that $\bar{U}(\xi - z_0, 0) < U(\xi, 0)$ for all $(\xi, t) \in \mathbb{R}^2, z \geq z_0$. We also have $U(\xi, 0) = \bar{U}(\xi - z_0, 0)$. Therefore, $U(\xi + z_0 - z, 0) = \bar{U}(\xi - z, 0) < U(\xi, 0)$ for all $z > z_0$. Hence $U(\cdot, 0)$ is strictly monotonic. Observe $U(z_0, 0) = \bar{U}(0, 0) = \alpha^0 = U(0, 0)$. Since $U(\cdot, 0)$ is strictly monotonic, we must have $z_0 = 0$. Hence, for all $\xi \in \mathbb{R}, U(\xi, 0) = \bar{U}(\xi, 0)$. Therefore, since U and $\bar{U}$ verify the same parabolic equation with the same initial condition and the same limit conditions, we must have $U(\xi, t) = \bar{U}(\xi, t)$ for all $\xi \in \mathbb{R}, t \in [0; T]$.

5 Existence

We intend to use here the method developed in [1]. First of all, we will prove the existence of solutions in bounded domains $[-M; M] \times [0; T]$ with $M \geq 1$. Then, we will prove that these solutions are uniformly bounded for all M, so that we can find a solution (U^*, c^*) taking $M \to +\infty$.

5.1 Existence in a bounded domain

Let $M \geq 1$ be fixed. Define $\Omega_M := (-M; M)$ and $Q_M := \Omega_M \times [0, T]$. We also define

$$\chi_M = \{g \in C^1([-M; M]) : g(\pm M) = \alpha^{\pm}, g(0) = \alpha^0, \partial_\xi g \geq 0 \text{ in } \Omega_M\}.$$

We aim to construct a solution to the problem in a bounded domain. First of all, for any given initial conditions g in χ_M and any propagation speed c in $\mathbb{R}$, we assure the existence of a function V such that for all $\xi \in [-M; M]$ and all $t \in [0; T]$, $\partial_t V(\xi, t) - c\partial_\xi V(\xi, t) - [J * V - V](\xi, t) = f(V(\xi, t), t)$. Then, we will adjust c so that the solution satisfies $V(0, T) = \alpha^0$. Finally, we are going to choose a good initial condition in χ_M to have a T-periodic in time solution.

Lemma 5.1. *Let c be in $\mathbb{R}$ and g be in χ_M. Then, the system*

$$\begin{cases} L^c(V) := \partial_t V - c\partial_\xi V - [J * V - V] = f(V, t) \text{ for all } (\xi, t) \in Q_M, \\ V(\pm M, t) = W^{\pm}(t) \text{ for all } t \geq 0, \\ V(\xi, 0) = g(\xi) \text{ for all } \xi \in \Omega_M. \end{cases} \tag{5.1}$$

has a solution, noted $V(g, c; \xi, t)$.

Remark 5.2. *Here, we use the notation $J * V(\xi, t) = \int_{\mathbb{R}} J(y)\tilde{V}(\xi - y, t)dy$ where*

$$\tilde{V}(\xi, t) = \begin{cases} V(\xi, t) \text{ if } \xi \in \Omega_M, \\ W^-(t) \text{ if } \xi \leq -M, \\ W^+(t) \text{ if } \xi \geq M. \end{cases}$$

Proof. Notice that if V is a solution to (5.1), V is well defined in Q_M, since we find with a comparison principle that

$$W^-(t) < V(\xi, t) < W^+(t) \text{ for all } (\xi, t) \in Q_M.$$

Let c be fixed in $\mathbb{R}$. Notice that if $V \in C^1([-M; M] \times [0; T])$, then $J * V - V \in C^1([-M; M] \times [0; T])$. Therefore, if $c = 0$, then there clearly exists $V_0 \in C^1([-M; M] \times [0; T])$ satisfying (5.1) since $g \in C^1([-M; M])$.

Now, suppose $c \neq 0$. With the semi-group theory, (5.1) admits a solution noted $V(g, c; \xi, t)$. Indeed, define $A(u) = c\partial_\xi u + J * u - u$ for all $u \in D(A) := C^1([-M; M])$. We notice that:

- $D(A)$ is dense in $C^0([-M; M])$,
- A is closed (with the dominated convergence theorem),
- With the Fourier transform, for all $\lambda > 0$, λ belongs to the resolvent set, and A is dissipative.

Thus the Hille-Yosida theorem gives us that A is the generator of a strongly continuous semi-group. Then, the equation $\partial_t u = Au + F(u)$ where $F(u)(\xi, t) := f(u(x, t), t)$ for all $(\xi, t) \in Q_M$, has a solution, since f is regular enough. □

We now want to define a constant $C(M, g)$ such that $V(g, C(M, g); 0, T) = \alpha^0$.

Lemma 5.3. *Let g be in χ_M. There exists a constant $C(M, g) \in \mathbb{R}$ such that $V(g, C(M, g); 0, T) = \alpha^0$.*

Proof. Notice that if $\xi = \pm M$, then $\partial_\xi V \geq 0$. Moreover, $\partial_\xi V$ satisfies :

$$\mathcal{L}(\partial_\xi V) := \partial_t(\partial_\xi V) - c\partial_\xi(\partial_\xi V) - [J * \partial_\xi V - \partial_\xi V] - \partial_u f(V, t)\partial_\xi V = 0.$$

As in section 3, we can prove that the operator $\mathcal{L}$ has a maximum principle, because f is regular enough to have $\partial_u f(V, \cdot) \in L^\infty(Q_M)$. This maximum principle gives us $\partial_\xi V > 0$ in Q_M. We note $\partial_c V = \partial_c V(g, c; \cdot, \cdot)$.

Then $\partial_c V$ satisfies $\mathcal{L}(\partial_c V) = \partial_\xi V > 0$ in Q_M, and $\partial_c V \equiv 0$ if $t = 0$ or if $\xi = \pm M$. Thus, with the maximum principle for $\mathcal{L}$, we get

$$\partial_c V > 0 \text{ in } Q_M.$$

Moreover, notice that $V(g, c, 0, T)$ depends continuously of c. Let $\zeta : [-M; M] \times [0; T] \to \mathbb{R}$ be such that

$$\begin{cases} \partial_\xi \zeta > 0 \text{ in } [-M; M] \times [0; T], \\ \zeta(\pm M) = \alpha^\pm, \\ \zeta(0) = \alpha^0, \end{cases}$$

and $K > 0$ sufficiently large such that $W(\xi, t) := \zeta(\xi) - K(T - t)$ satisfies $W(\pm M, t) \leq W^\pm(t)$ for all $t \in [0; T]$. Define $C^+(M) = \sup\limits_{(\xi,t)\in[-M;M]\times[0;T]} \frac{\partial_t W - [J*W-W] - f(W,t)}{\partial_\xi W}$. Then, by the definition of $C^+(M)$, we get:

$$\begin{cases} L^{C^+(M)}(W) < 0 \text{ in } \Omega_M, \\ W(\pm M, t) \leq W^\pm(t) \text{ for all } t \in [0; T], \\ W(\cdot, 0) < g \text{ for all } g \in \chi_M. \end{cases}$$

Thus, by the comparison principle, $W(\xi, t) \leq V(g, C^+(M); 0, T)$ in Q_M, for all $g \in \chi_M$. Hence

$$\alpha^0 = W(0, T) \leq V(g, C^+(M); 0, T).$$

Define in the same manner $C^-(M) = \inf\limits_{(\xi,t)\in[-M;M]\times[0;T]} \frac{\partial_t W - [J*W-W] - f(W,t)}{\partial_\xi W}$. Then, we find

$$\alpha^0 = W(0, T) \geq V(g, C^-(M); 0, T).$$

Thus, since $V(g, c, 0, T)$ depends continuously of c, and $\partial_c V > 0$, there exists a unique $C = C(M, g) \in [C^-(M); C^+(M)]$ such that $V(g, C; 0, T) = \alpha^0$. Moreover, with the properties of V, we find that

$$V(g, C(M, g); \cdot, T) \in \chi_M.$$

□

Lemma 5.4. *Let g be in χ_M. We note $V(\xi, t) := V(g, C(M, g); \xi, t)$ for all $(\xi, t) \in \mathbb{R} \times [0; T]$. Then, there exist fixed constants $l > 0$ and $\mathcal{K} \geq 0$ such that:*

$$\partial_\xi V(\xi, t) \leq e^{-lt} \sup_{x\in\mathbb{R}} |g'(x)| + \frac{(1 - e^{-lt})\mathcal{K}}{l}$$

for all $(\xi, t) \in \mathbb{R} \times [0; T]$.

Proof. Notice that $\partial_\xi V$ satisfies:

$$\partial_t(\partial_\xi V) + \partial_\xi V - C(M, g)\ \partial_\xi(\partial_\xi V) - \partial_u f(V, t)\partial_\xi V = J' * V.$$

we define $\mathcal{K} := \sup\limits_{\xi\in\mathbb{R}, t\in[0;T]} |J' * V(\xi, t)|$ and $l := 1 - \sup\{\partial_u f(u, t) \mid u \in [W^-(t); W^+(t)], t \in [0; T]\}$. Then, since we made the hypothesis that $\sup\{\partial_u f(u, t) \mid u \in [W^-(t); W^+(t)], t \in [0; T]\} < 1$, we must have $l > 0$. The comparison principle gives us the expected result.

□

Remark 5.5. *The hypothesis* $\sup\{\partial_u f(u,t) \mid u \in [W^-(t); W^+(t)], t \in [0;T]\} < 1$ *is crucial here. If it is not satisfied, as explained in [3], there can exist some discontinuous solutions to (5.2) (defined below) with the propagation speed* $c = 0$.

Remark 5.6. *The bound we found here still depends on* M.

To conclude the construction of a solution in a bounded domain, we only have to adjust $g \in \chi_M$ so that for all $(\xi,t) \in \mathbb{R} \times [0;T]$, we have $V(g, C(M,g); \xi, t+T) = V(g, C(M,g); \xi, t)$.

Theorem 5. *The following system*

$$\begin{cases} L^C(U) := \partial_t U - C\partial_\xi U - [J * U - U] = f(U,t) \text{ for all } (\xi,t) \in Q_M, \\ U(\pm M, t) = W^\pm(t) \text{ for all } t \geq 0, \\ U(\xi, 0) = U(\xi, T) \text{ for all } \xi \in \Omega_M. \end{cases} \tag{5.2}$$

admits a solution noted (V^M, C^M).

Proof. Define $Z : \chi_M \to \chi_M$ with $Z(g) = V(g, C(M,g); \cdot, T)$.

- Lemmas 5.1 and 5.3 give us that Z is well-defined,
- χ_M is a non-empty, closed and convex set of $C^1([-M;M])$,
- Since $\partial_c V > 0$ and $V(\cdot, c, 0, T)$ depends continuously of g, we have that $C(M,g)$ depends continuously of g, and therefore that T is continuous,
- Lemma 5.4 gives us that $Z(\chi_M)$ is bounded in $C^1([-M;M])$, which implies that Z is a compact operator.

Thus, we can use the Schauder fixed point theorem which gives us that there exists $\bar{g} \in \chi_M$ such that $Z(\bar{g}) = \bar{g}$. Hence, $V(\bar{g}, C(M,\bar{g}); \cdot, \cdot)$ is a solution of (5.2) with $C = C(M, \bar{g})$. We note $V^M :\equiv V(\bar{g}, C(M,\bar{g}); \cdot, \cdot)$ and $C^M := C(M, \bar{g})$.

$\square$

5.2 Estimate of C^M and $\|V^M\|$

Our purpose is now to do the passage to the limit $M \to +\infty$, but C^M or V^M could diverge while M tends to $+\infty$. Thus, we now have to prove that $\{C^M\}_{M\geq 1}$ and $\sup_{M\geq 1} \|V^M\|_{C^1([-M;M]\times[0;T])}$ are uniformly bounded.

Lemma 5.7. *We recall that* (V^M, C^M) *be a solution to (5.2).*

(i) If $(\bar{V}, \bar{c})$ *satisfies*

$$\begin{cases} \partial_t \bar{V} - \bar{c}\partial_\xi \bar{V} - [J * \bar{V} - \bar{V}] = f(\bar{V}, t) \text{ for all } (\xi,t) \in Q_M, \\ \bar{V}(\pm M, t) \leq W^\pm(t) \text{ for all } t \in [0;T], \\ \bar{V}(\xi, 0) \leq \bar{V}(\xi, T) \text{ for all } \xi \in [-M;M], \\ \bar{V}(0,0) \geq \alpha^0, \end{cases}$$

then $C^M \leq \bar{c}$.

(ii) If $\hat{V}$ satisfies

$$\begin{cases} \partial_t \hat{V} - C^M \partial_\xi \hat{V} - [J * \hat{V} - \hat{V}] = f(\hat{V}, t) \text{ for all } (\xi, t) \in [0; M] \times [0; T], \\ \hat{V}(\pm M, t) \leq W^{\pm}(t) \text{ for all } t \in [0; T], \\ \hat{V}(\xi, 0) \leq \max\{\alpha^0, \hat{V}(\xi, T)\} \text{ for all } \xi \in [0; M], \end{cases}$$

then $\hat{V} \leq V^M$ in $[0; M] \times [0; T]$.

Proof.

(i) Assume by contradiction that $C^M > \bar{c}$. We know that $\partial_\xi V^M > 0$ in Q_M. Therefore, in Q_M, we have:

$$L^{\bar{c}}(V^M) = (C^M - \bar{c})\partial_\xi V^M > 0.$$

Define:

$$m_0 = \inf \left\{ m \in (-2M; 2M) \mid V^M(\xi, 0) > \bar{V}(\xi - m, 0) \text{ in } (-M; M) \cap (m - M; m + M) \right\}.$$

We know that $V^M(M, 0) = \alpha^+ > \bar{V}(-M, 0)$ and $V^M(0, 0) = \alpha^0 \leq \bar{V}(0, 0)$. Therefore, $m_0 \in [0; 2M)$. Moreover, by definition of m_0, there exists $\xi_0 \in \Omega^M_{m_0} := (m_0 - M; M)$ such that $V^M(\xi_0, 0) = \bar{V}(\xi_0 - m_0, 0)$. With the assumptions on V^M and $\bar{V}$ on the boundary, we deduce that $V^M(\xi, t) \geq \bar{V}(\xi - m_0, t)$ on the parabolic boundary of $\Omega^M_{m_0} \times (0; T]$. Hence, the comparison principle applied to $V^M(\xi, t)$ and $\bar{V}(\xi - m_0, t)$ on $\Omega^M_{m_0} \times [0; T]$ gives us:

$$V^M(\xi, T) > \bar{V}(\xi - m_0, T) \text{ for all } \xi \in \Omega^M_{m_0}.$$

Yet, we also know that

$$V^M(\xi_0, T) = V^M(\xi_0, 0) = \bar{V}(\xi_0 - m_0, 0) \leq \bar{V}(\xi_0 - m_0, T).$$

There is a contradiction. Hence $C^M \leq \bar{c}$.

(ii) Define

$$m_0 = \inf \left\{ m \in [0; M] \mid V^M(\xi, 0) > \hat{V}(\xi - m, 0) \text{ in } [m; M] \right\}.$$

We follow the same method, using a comparison in $(m_0, M) \times (0; T]$ on $V^M(\xi, t)$ and $\hat{V}(\xi - m_0, t)$. If we define ξ_0 such that $V^M(\xi, 0) > \hat{V}(\xi - m_0, 0)$ for all $\xi \in [m_0; M]$, we find the same contradiction as in (i) if $m_0 \neq 0$. Hence, $m_0 = 0$, and the comparison principle in $[m_0; M]$ gives us that $\hat{V} \leq V^M$ in $[0; M] \times [0; T]$.

□

We now want to estimate C^M using Lemma 5.7 (i), in order to be able to pass to the limit $M \to +\infty$ in the next subsection. We will follow the idea given by [1].

Theorem 6. *$\{C^M\}_{M \geq 1}$ is uniformly bounded.*

Proof. Define for all $s \in \mathbb{R}$, $\zeta(s) = \frac{1}{2}(1 + \tanh(\frac{s}{2}))$. Then, $\zeta' = \zeta(1-\zeta) > 0$ in $\mathbb{R}$. We define $w_1 \equiv W^+$ and $w_2(t) = w(\alpha^- - \varepsilon_0, t)$ with ε_0 small enough so that $w_2(t) \geq W^-(t) - 1$ for all $t \in [0;T]$. We choose:

$$\bar{V}(\xi,t) = w_1(t)\zeta(\xi+\xi_0) + w_2(t)[1-\zeta(\xi+\xi_0)] \text{ for all } (\xi,t) \in [-M;M]\times[0;T],$$

where ξ_0 is chosen so that $\zeta(\xi_0) = \frac{\alpha^0-\alpha^-+\varepsilon_0}{\alpha^+-\alpha^-+\varepsilon_0}$.

Since $w_1(T) = w_1(0)$ and $w_2(T) > w_2(0)$, with the definition of $\bar{V}$ we remark:

- $\bar{V}(\cdot,T) > \bar{V}(\cdot,0)$,
- $\partial_\xi \bar{V} > 0$,
- $\bar{V}(0,0) = \alpha^0$, $\bar{V}(+\infty,0) = \alpha^+$, and $\bar{V}(-\infty,0) = \alpha^- - \varepsilon_0$.

Therefore, we only have to find a good $\bar{c}$ to verify the assumptions of Lemma 5.7 (i).

By Taylor's expansion, we get:

$$\zeta f(w_1,t) - (1-\zeta)f(w_2,t) - f(\zeta w_1 + (1-\zeta)w_2,t) = \frac{1}{2}\zeta(1-\zeta)(w_1-w_2)^2\partial_u^2 f(\theta,t),$$

where θ is a constant in $(w_2;w_1)$ and where ζ is evaluated in $\xi+\xi_0$.

Let $\bar{c}$ be fixed later. We calculate:

$$\begin{aligned} L^{\bar{c}}(\bar{V}) &= (\zeta f(w_1,t) + (1-\zeta)f(w_2,t) - f(\zeta w_1 + (1-\zeta)w_2,t) \\ &= -\zeta(1-\zeta)(w_1-w_2)\left[\bar{c} + \tfrac{J*\zeta-\zeta}{\zeta'} - \tfrac{1}{2}(w_1-w_2)\partial_u^2 f(\theta,t)\right]. \end{aligned}$$

Notice that $\zeta' > 0$ in $\mathbb{R}$, $(w_1 - w_2) > 0$ in $[0;T]$, and $\frac{J*\zeta-\zeta}{\zeta'}$ is bounded in $\mathbb{R}$ by a constant $K > 0$.

We take $\bar{c} = K + 1 + \frac{1}{2}\sup\{(W^+(t) - W^-(t) + 2)|\partial_u^2 f(u,t)| \ : t \in [0;T], u \in [W^-(t)-1; W^+(t)+1]\}$.

Thus, we have $L^{\bar{c}}(\bar{V}) < 0$ in Q_M for all M such that $\zeta(-M) \leq \frac{\varepsilon_0}{\alpha^+-\alpha^--\varepsilon_0}$. This last condition on M implies that $\bar{V}(-M,0) \leq \alpha^-$.

Therefore, we can use Lemma 5.7 (i) to find that $C^M \leq \bar{c}$. □

Theorem 7. $\sup\limits_{M\geq 1} \left\|V^M\right\|_{C^1([-M;M]\times[0;T])}$ *is uniformly bounded.*

Proof. We already know that for all $M \geq 1$, $\|V^M\|_{L^\infty(Q_M)} \leq \sup\limits_{t\in[0;T]} |W^\pm(t)|$ and that for all $\xi \in \mathbb{R}, t \in [0;T], k \in \mathbb{Z}$,

$$\partial_\xi V^M(\xi, t+kT) \leq e^{-l(t+kT)} \sup_{x\in\mathbb{R}} |\partial_\xi V^M(\xi,0)| + \frac{(1-e^{-l(t+kT)})\tilde{\mathcal{K}}}{l}$$

by Lemma 5.4, with $\tilde{\mathcal{K}} := \int_{\mathbb{R}} |J'(y)|dy \sup\limits_{t\in[0;T]} |W^\pm(t)|$. Thus, passing to the limit $k \to +\infty$, since V^M is T-periodic in time, we get for all $\xi \in \mathbb{R}$, $t \in [0;T]$,

$$\partial_\xi V^M(\xi,t+kT) \leq \frac{\tilde{\mathcal{K}}}{l},$$

and this bound does not depend on M. Moreover, for all $M \geq 1$,

$$\partial_t V^M = C^M \partial_\xi V^M + \left[J * V^M - V^M\right] + f(V^M,t)$$

and all the terms on the right side are uniformly bounded. We can then deduce the expected result. □

5.3 Existence on an unbounded domain

We have proved that $\{C^M\}_{M\geq M_0}$ and $\sup_{M\geq M_0}||V^M||_{C^1(Q_M)}$ are uniformly bounded. Thus, there exists a subsequence $\{M_j\}$ such that $M_j\to+\infty$ as $j\to+\infty$, and $C^{M_j}\to c^*\in\mathbb{R}$, and $V^{M_j}\to U^*$ uniformly in any compact of $\mathbb{R}\times[0;T]$. Thus, passing to the limit in (5.2), we get:

$$\begin{cases} L^C(U^*) := \partial_t U^* - c^*\partial_\xi U^* - [J * U^* - U^*] = f(U^*,t) \text{ for all } (\xi,t)\in\mathbb{R}\times[0;T],\\ U^*(0,0)=\alpha^0,\\ U^*(\xi,0)=U^*(\xi,T) \text{ for all } \xi\in\mathbb{R},\\ \partial_\xi U^*\geq 0 \text{ in } (\xi,t)\in\mathbb{R}\times[0;T]. \end{cases} \tag{5.3}$$

We now have to show that $U^*(\pm\infty,t)=W^\pm(t)$ for all $t\in[0;T]$.

Remark 5.8. *(5.3) admits a trivial solution which is $w(\alpha^0,t)$. Since we have not yet proved that the V^{M_j} converges to U^* uniformly on $\mathbb{R}$, the limit conditions are not a priori satisfied by U^*, since it can be the trivial solution.*

Lemma 5.9. *If U^* is a non-trivial solution of (5.3), then $U^*(\pm\infty,t)=W^\pm(t)$ for all $t\in[0;T]$.*

Proof. Since U^* is not trivial, $\partial_\xi U^*(\xi,t)\neq 0$ for some (ξ,t) in $\mathbb{R}\times[0;T]$. Thus, by the strong maximum principle, we get

$$\partial_\xi U^*(\xi,t)>0 \text{ in } \mathbb{R}\times[0;T].$$

Therefore, since U^* is uniformly bounded by $\sup_{t\in[0;T]}|W^\pm(t)|$, $U^*(\pm\infty,t)$ exists for all t, and

$$U^*(-\infty,0)<\alpha^0<U^*(+\infty,0).$$

We know that $U^*(\cdot,t)$ is monotone for all t, and $\partial_\xi U^*$ converges weakly to 0 as $|\xi|\to 0$. Hence, $U^*(\pm\infty,t)$ are periodic solutions of $\partial_t w=f(w,t)$, and therefore $U^*(\pm\infty,t)=W^\pm(t)$. So, we have also proved that (C^M,V^M) converges uniformly to (c^*,U^*) as $M\to+\infty$. □

Lemma 5.10. *U^* is non trivial.*

Proof. Without loss of generality, we can assume that $c^*\geq 0$ and $U^*(0;t)\geq w(\alpha^0,t)$, because $U^*(0;t)<w(\alpha^0,t)$ directly gives us that U^* is not trivial. Thus, we have the following statements:

$$\lim_{j\to+\infty} C^{M_j}\geq 0 \text{ and } \lim_{j\to+\infty}\min_{t\in[0;T]}\{V^{M_j}(0,t)-w(\alpha^0,t)\}\geq 0.$$

We will fix j later. With the definition of w, we have that $\partial_\alpha w=\exp(\int_0^t\partial_u f(w,\tau)d\tau)>0$. We define:

$$K:=\max\left\{\frac{|w_{\alpha\alpha}(\alpha,t)|}{w_\alpha(\alpha,t)}+1 : \alpha\in[\alpha^-;\alpha^+], t\in[0;T]\right\}.$$

Let $\varepsilon>0$ be a constant and define

$$\delta\in\left(0;\min\left\{\frac{1}{16K};\frac{\alpha^+-\alpha^0}{8};\frac{\alpha^0-\alpha^-}{2}\right\}\right),$$

and $\zeta \in C^\infty(\mathbb{R})$ be as:

$$\begin{cases} 0 \le \zeta'(s) < 5\sqrt{\delta}, \text{ and } \alpha^0 + 2\delta \le \zeta(s) < \alpha^0 + 7\delta \text{ if } s \ge \sqrt{\delta}, \\ \zeta(s) = \alpha^0 + (s+\sqrt{\delta})^2 - 2\delta \text{ if } s \in [-\sqrt{\delta}; \sqrt{\delta}], \\ 0 \le \zeta'(s) < 4\sqrt{\delta}, \text{ and } \alpha^0 - 2\delta \le \zeta(s) < \alpha^0 + \delta \text{ if } s \le -\sqrt{\delta}. \end{cases}$$

Fix $\delta_1 > 0$ sufficiently small, and define for all α, $\hat{w}(\alpha, t)$ as the solution to

$$\begin{cases} \partial_t \hat{w} = f(\hat{w}, t) - \delta_1(\max\{0, \hat{w} - \hat{w}(\alpha^0 + \delta, t)\})^3, \\ \hat{w}(\alpha, 0) = \alpha. \end{cases}$$

Thus, we have that $\partial_\alpha w \ge \partial_\alpha \hat{w} > 0$ and $\hat{w}(\alpha, \cdot) = w(\alpha, \cdot)$ if $\alpha \le \alpha^0 + \delta$. Fix $\varepsilon > 0$. Now, our purpose is to apply Lemma 5.7 (ii) to compare the functions $\hat{V}(\xi, t) = \hat{w}(\zeta(\varepsilon\xi), t)$ and V^{M_j} with j sufficiently large. By the definition of ζ and $\hat{w}$, with ε small enough, we can verify that $\hat{V}$ satisfies the second and the third assumptions of the lemma. We only have to show that $L^{C^{M_j}}(\hat{V}) \le 0$ on $[0; M_j] \times [0; T]$. We calculate:

$$L^{C^{M_j}}(\hat{V}) = -\delta_1(\max\{0, \hat{w} - \hat{w}(\alpha^0 + \delta, t)\})^3 - \varepsilon\zeta'(\varepsilon\xi)\partial_\alpha \hat{w}(\zeta(\varepsilon\xi), t) - \int_{\mathbb{R}} J(\xi - y)[\hat{w}(\zeta(\varepsilon y), t) - \hat{w}(\zeta(\varepsilon\xi), t)]dy.$$

Define $\delta_2 := \min_{t \in [0;T]}\{\hat{w}(\alpha^0 + 2\delta, t) - \hat{w}(\alpha^0 + \delta, t)\}$.

- *If $\zeta \ge \alpha^0 + 2\delta$:* then $\varepsilon\xi \ge \sqrt{\delta}$ and $0 \le \zeta' < 5\sqrt{\delta}$, and we have:

$$\begin{aligned} L^{C^{M_j}}(\hat{V}) &\le -\delta_1\delta_2^3 - \varepsilon 5\sqrt{\delta}\ \min\{0; C^{M_j}\}\partial_\alpha \hat{w} - [J * \hat{V} - \hat{V}](\xi, t), \\ &\le -\delta_1\delta_2^3 - \varepsilon 5\sqrt{\delta}\ \min\{0; C^{M_j}\}\partial_\alpha \hat{w} + \int_{\mathbb{R}} J(y)|y|dy\varepsilon 5\sqrt{\delta} \sup_{\alpha \in [\alpha^-; \alpha^+], \tau \in [0;T]} |\partial_\alpha \hat{w}(\alpha, \tau)|, \\ &< 0, \end{aligned}$$

  for ε small enough.

- *If $\zeta < \alpha^0 + 2\delta$:* then $\varepsilon\xi \le \sqrt{\delta}$ but we only have to focus on the case $\varepsilon\xi \ge 0$ to use Lemma 5.7 (ii). We have:

$$L^{C^{M_j}}(\hat{V}) \le -\varepsilon 4\sqrt{\delta}\ \min\{0; C^{M_j}\}\partial_\alpha \hat{w} - [J * \hat{V} - \hat{V}](\xi, t)$$

  We can calculate with the Taylor theorem that:

$$\begin{aligned} \hat{w}(\zeta(\varepsilon(\xi - y)), t) &= \hat{w}(\zeta(\varepsilon\xi), t) + \left(-\varepsilon y\zeta'(\varepsilon\xi) + \tfrac{\varepsilon^2 y^2}{2}\zeta''(\varepsilon\xi) + O(\varepsilon^3)\right)\partial_\alpha \hat{w}(\zeta(\varepsilon\xi), t) \\ &\quad + \tfrac{1}{2}\left(-\varepsilon y\zeta'(\varepsilon\xi) + \tfrac{\varepsilon^2 y^2}{2}\zeta''(\varepsilon\xi) + O(\varepsilon^3)\right)^2 \partial_{\alpha\alpha}\hat{w}(\zeta(\varepsilon\xi), t) + O(\varepsilon^3) \\ &= \hat{w}(\zeta(\varepsilon\xi), t) + \left(-\varepsilon y\zeta'(\varepsilon\xi) + \tfrac{\varepsilon^2 y^2}{2}\zeta''(\varepsilon\xi) + O(\varepsilon^3)\right)\partial_\alpha \hat{w}(\zeta(\varepsilon\xi), t) \\ &\quad + \left(\tfrac{\varepsilon^2 y^2}{2}(\zeta'(\varepsilon\xi))^2 + O(\varepsilon^3)\right)\partial_{\alpha\alpha}\hat{w}(\zeta(\varepsilon\xi), t) + O(\varepsilon^3). \end{aligned}$$

  One can verify with an estimation of the rest in the Taylor development that $O(\varepsilon^3)$ denotes a uniformly bounded function of y. Therefore, since J is nonnegative and even, we can remove the terms at odd orders in y in the Taylor expansion of $[J * \hat{V} - \hat{V}]$, and so we get:

$$\begin{aligned} [J * \hat{V} - \hat{V}](\xi, t) &\ge \int_{\mathbb{R}} J(y)y^2 dy\ \partial_\alpha \hat{w}(\zeta(\varepsilon\xi), t)\left[\varepsilon^2 - 16\delta\varepsilon^2 K\right] + O(\varepsilon^3) \\ &> 0 \end{aligned}$$

  .

with ε small enough, since $\delta < \frac{1}{16K}$. Then, since $\lim_{j\to+\infty} C^{M_j} \geq 0$, we can take j large enough so that $L^{C^{M_j}}(\hat{V})(\xi,t) < 0$.

Therefore, using Lemma 5.4 (ii), we get $\hat{V} \leq V^{M_j}$ in $[0, M_j] \times [0;T]$. Hence $U^* \geq \hat{V}$ in $[0;+\infty) \times [0;T]$, and there exist $(\xi,t) \in [0;+\infty) \times [0;T]$ such that $\hat{V}(\xi,t) > w(\alpha^0,t)$. Therefore $U^* \neq w(\alpha^0,t)$. $\square$

Thus, (U^*, c^*) is a solution to (2.3) and therefore we have proved Theorem 1.

6 Extensions

My internship was too short to prove other valuable results, but many extensions of this problem are worth being studied.

6.1 Exponential Stability of pulsating transition front

We have proved that there exists a unique couple (U,c) solution to (2.3). The motivation to study traveling waves solution to (1.1) is the following theorem, which focus on the behaviour of some solutions to (1.1):

Theorem 8. *Define $g \in L^\infty(\mathbb{R})$ be such that $\limsup_{z\to+\infty} g(z) < \alpha^0$ and $\liminf_{z\to+\infty} g(z) > \alpha^0$. Let u be the solution to (1.1) such that for all $x \in \mathbb{R}$, $u(x,0) = g(x)$. Then , there exist $\mu > 0$, $C \geq 0$ and $\hat{x} \in \mathbb{R}$ such that for all $t \geq 0$:*

$$\|u(x,t) - U(x + \hat{x} - ct, t)\|_{L^\infty(\mathbb{R}} \leq Ce^{\mu t}.$$

Such a theorem has already been proved in [2].

6.2 Bistable on average nonlinearity

Definition 6.1. *f is said bistable on average if it satisfies:*

- *$f(\alpha^-,t) = f(\alpha^+,t) = 0$ for all $t \in \mathbb{R}$,*
- *$\frac{1}{T}\int_0^T \partial_u f(\alpha^-,s)ds < 0$ and $\frac{1}{T}\int_0^T \partial_u f(\alpha^+,s)ds > 0$.*

Let $T > 0$ and f^T such that f^T is bistable on average, and let g be defined such that for all $v \in \mathbb{R}$,

$$g(v) := \frac{1}{T}\int_0^T f^T(v,s)ds.$$

Then, we can remark that g is bistable, whose zeros are α^-, α^+ and a third reel α^0 such that $\alpha^- < \alpha^0 < \alpha^+$. In [4], the existence, uniqueness and asymptotic stability of periodic traveling waves solution to:

$$\partial_t u - \partial_{xx} u = f^T(u,t). \tag{6.1}$$

Thus, we can conjecture that the same results remain true for the equation:

$$\partial_t u - [J * u - u] = f^T(u,t). \tag{6.2}$$

That is to say, we can conjecture the following propositions:

Proposition 6.2. *If T is small enough, then there exists a unique periodic traveling wave (u^T, c^T) solution to*

$$\left\{\begin{array}{l} \partial_t u^T - c^T \partial_\xi u^T - [J * u^T - u^T] = f^T(u^T, t), \\ u^T(\xi, t+T) = u^T(\xi, t) \text{ for all } \xi \in \mathbb{R}, t \geq 0, \\ \lim\limits_{\xi \to +\infty} u^T(\xi, t) = W^-(t), \ \lim\limits_{\xi \to -\infty} u^T(\xi, t) = W^+(t) \text{ for all } t \geq 0, \\ u^T(0,0) = \alpha_0, \end{array}\right. \tag{6.3}$$

We note (u_g, c_g) the unique solution to the following system:

$$\left\{\begin{array}{l} \partial_t u_g - c_g \partial_\xi u_g - [J * u_g - u_g] = g(u_g), \\ \lim\limits_{x \to +\infty} u_g(x) = \alpha^-, \ \lim\limits_{x \to -\infty} u_g(x) = \alpha^+, \\ u_g(0,0) = \alpha_0. \end{array}\right.$$

Proposition 6.3.

$$c^T \underset{T \to 0}{\longrightarrow} c_g,$$

$$\left\| u^T - u_g \right\|_{W^{1,2;p}_{loc}} \underset{T \to 0}{\longrightarrow} 0$$

for all $p \in (1; +\infty)$, and where

$$W^{1,2;p}_{loc} := \left\{ v \in L^p_{loc}(\mathbb{R}^2) \mid \partial_t v, \partial_\xi v, \partial_{\xi\xi} v \in L^p_{loc}(\mathbb{R}^2) \right\}.$$

6.3 Regularity

We have used the hypothesis (H1) *(iv)*, that is to say

$$\sup \left\{ \partial_u f(u,t) \mid u \in [W^-(t), W^+(t)], t \in [0;T] \right\} < 1,$$

in order to find a smooth solution u for the speed c. This hypothesis is crucial for the regularity of the solution. Indeed, as explained in [3] in the autonomous case, if $c = 0$ and if (H1) *(iv)* is not satisfied, and with a well-chosen nonlinearity f, then there can exist a solution with a point of jump discontinuity. In order to ignore these problems of discontinuity if (H1) *(iv)* is not satisfied, we can seek for weak solutions, as in [3], that is to say seek for couples (u, c) verifying for all $\phi \in C_c^\infty(\mathbb{R})$ and for all $t \in \mathbb{R}$:

$$\begin{aligned} \int_{\mathbb{R}} u(\xi,t)\phi(\xi) d\xi = \quad & \int_{\mathbb{R}} u(\xi,0)\phi(\xi) d\xi + \int_0^t \int_{\mathbb{R}} [J * u(\xi,s) - u(\xi,s)]\phi(\xi) d\xi ds \\ & -c \int_0^t \int_{\mathbb{R}} u(\xi,s)\phi'(\xi) d\xi ds + \int_0^t \int_{\mathbb{R}} f(u(\xi,s),s)\phi(\xi) d\xi ds, \end{aligned}$$

and we can study the special case $c = 0$ as done in [3] in the autonomous case, to deal with eventual discontinuous solutions.

6.4 Neural network

We can study the following neural network equation:

$$\partial_t u(x,t) - \left[\int_{\mathbb{R}} J(y)u(x-y,t)dy - u(x,t)\right] = \int_{\mathbb{R}} J(y)f(u(x-y,t),t)dy \text{ for all } x \in \mathbb{R}, \quad t \geq 0, \tag{6.4}$$

which can be seen as an extension of our work, with a nonlocal nonlinearity. The existence of periodic traveling wave solution to (6.4) could be proved with a homotopy from a well-known problem, as done in [6] in the autonomous case.

7 Conclusion

The purpose of my internship was to study the existence and uniqueness of the periodic traveling waves solution to the nonlocal evolution equation (1.1). The exponential stability was proved in [2]. Thus, under some hypothesis on the initial conditions, we know the behaviour of a solution to (1.1), which was the main motivation of this study. It is worth highlighting that the strong maximum principle and the comparison principle that I proved in section 3 are useful and interesting theorems, which could be used again in another context. Although the proof of the uniqueness is very similar to the one in [2], the existence was proved with an alternative approach, using a preliminary study of the problem on a bounded domain. In [2], the authors proved the existence of a periodic traveling wave solution to:

$$\partial_t u - (\theta - 1)\partial_{xx} u - \theta[J * u - u] = f(u,t), \tag{7.1}$$

where $\theta \in [0;1]$. It was proved that for all $\theta \in [0;1]$, there exists a unique traveling wave (u_θ, c_θ) solution to (7.1). Then, since $\{c_\theta\}_{\theta \in [0;1)}$ and $\sup\limits_{\theta \in [0;1)} \|u_\theta\|_{C^1(\mathbb{R}\times[0;T])}$ are uniformly bounded, we can pass to the limit $\theta \to 1$ to find a periodic traveling wave solution to (1.1). These are two general methods for the study of reaction-diffusion systems, that I learnt during my internship. The study of nonlocal evolution equation permits to do a generalization of some results on classical local reaction-diffusion systems. This kind of equation is involved in the study of many fields: we have for example already cited in this report the model of population dynamics or neural network equations, but the study of reaction-diffusion systems is capital in biomathematics in general.

To conclude, during this internship, I have discovered more precisely the world of research in mathematics, thanks to the serious and friendly atmosphere in the IMT. I have learnt several valuable methods for the study of reaction-diffusion systems, and I have started becoming familiar with the concept of traveling wave. Therefore, I think that this internship have brought me precious knowledge, that is why I deeply thank my mentor, his other intern Gwenaël Peltier and the PhD students with whom I shared an office.

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