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M2 Internship Proposal

Title : Acceleration phenomena for reaction-diffusion equations set on domains with a line of fast diffusion

Internship summary. The overall objective of this internship is to investigate the qualitative properties of a class of systems of parabolic partial differential equations which couple a reaction-diffusion equation on a half-plane (or a band) with another reaction-diffusion equation on its boundary via a non homogeneous Robin boundary condition. Such types of systems naturally appear in the modeling of population dynamics, for example in ecology [1] or epidemiology [2]. More specifically, we will focus on the case where the diffusion on the boundary is much larger than the diffusion in the half-space and investigate some acceleration phenomena induced by this line of fast diffusion.

Prerequisites : some basic PDE courses.

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Location : the internship will take place at the Institut de Mathématiques de Toulouse.

Dates : 5 months from March to July 2024

NB : an internship stipend is possible and this internship may lead to a PhD thesis.

Extended description of the project

It is well known and documented that transportation networks can enhance biological invasions and epidemics. This project aims at studying a class of partial differential equations which precisely models such effects. More specifically, we shall work within the framework of reaction-diffusion equations and investigate during the internship the effects of a line of fast diffusion on the boundary of the domain on the spreading properties of the overall system. The model under study reads as follows

$$\begin{cases} \partial_t u = d\Delta u + f(u), \quad x \in \mathbb{R}, \quad y > 0, \quad t > 0, \\ \partial_t v = D\partial_x^2 v + \nu u(t,x,0) - \mu v, \quad x \in \mathbb{R}, \quad t > 0, \\ -d\partial_y u(t,x,0) = \mu v - \nu u(t,x,0), \quad x \in \mathbb{R}, \quad t > 0, \end{cases} \tag{1}$$

and can be represented schematically by the following diagram

$$\partial_t u = d\Delta u + f(u) \qquad x \in \mathbb{R}, \quad y > 0$$

$$-d\partial_y u(t,x,0) = \mu v - \nu u(t,x,0)$$

$$\partial_t v = D\partial_x^2 v + \nu u(t,x,0) - \mu v \qquad x \in \mathbb{R}$$

for some strictly positive parameters $D, d, \nu, \mu > 0$ and $\Delta = \partial_x^2 + \partial_y^2$. In (1), u can be interpreted as the density of a given species that can move in the half-plane $\Omega = \{x \in \mathbb{R},\ y > 0\}$, and may leave the half-space Ω to the real line $\mathbb{R}$ and become v. The term $f(u)$ traduces some nonlinear effects such as birth, death and competition among the individuals of the species. Note that the diffusion coefficients in the half-space and on the boundary $\mathbb{R}$ are allowed to be different, and actually we will be mainly interested in the regime where $D \gg d$. The non homogeneous Robin boundary condition is really natural in the sense that without nonlinear effects (that is with $f(u) = 0$), it is the only natural boundary condition which ensures the preservation of the total mass. Let us finally note that many generalizations (and sophistications) of the above PDE system (1) have been considered in the literature with different reaction terms, nonlocal forms of diffusion or different types of geometries to name a few.

The first step of the internship will consist in a review of the existing literature on the subject, starting with the seminal work [1] and the follow up papers. As previously emphasized, the key objective will be to understand the long time dynamics of (1) in the regime where $D \gg d$. This is a very relevant question both from the modeling and mathematical point of view since one expects nontrivial transitions in the dynamics typically of slow/fast nature (see [3] in the case of an ignition nonlinearity). Depending on the student interests and strengths the next steps of the internship will be *à la carte* and can take various colorations ranging from the exploration of new models (predator-prey, deadly field, epidemic spreading, etc...), to an in depth numerical analysis of the problem or to the more specific theoretical study of acceleration phenomena in the regime of large diffusion along the boundary.

Références

[1] H. Berestycki, J-M. Roquejoffre and L. Rossi. *The influence of a line of fast diffusion in Fisher-KPP propagation.* J. Math. Biol. 66 (2013), pp. 743-766.

[2] H. Berestycki, J-M. Roquejoffre and L. Rossi. *Propagation of epidemics along lines with fast diffusion.* Bulletin of Mathematical Biology 83.1 (2021), pp. 1-34.

[3] L. Dietrich and J.-M. Roquejoffre. *Front propagation directed by a line of fast diffusion : large diffusion and large time asymptotics.* Journal de l'École Polytechnique, Mathématiques, 4 (2017), pp. 141-176.